MA 182, Honors Mathematical Analysis II, Fall '05.
Extra Homework Sheet # 3.

1. Let p, L be real numbers. Suppose that a neighborhood of p (excluding maybe p) is in the domain of the function f. Prove that

$$\lim_{x \to p} f(x) = L \tag{1}$$

if and only if, for every sequence $\{x_n\}$ converging to p (i.e., $\lim_{n\to+\infty} x_n = p$), one has

$$\lim_{n \to +\infty} f(x_n) = L. \tag{2}$$

[Tips for "top statement implies bottom statement": You have a sequence $\{x_n\}$ converging to p and you want to prove (2), using (1). Given $\varepsilon > 0$, apply the very definition of limit (1): $\exists \delta > 0$ such that ... Then use the definition of limit (2) with δ in the place of ε: thus, for that δ that was obtained earlier, $\exists N \in \mathbb{R}$ such that ... What do you get in the end?

Tips for "bottom statement implies top statement": Assume the contrary of (1) and try to get a contradiction. How do you write the negative (or opposite) statement to (1)? I suggest: $\exists \varepsilon > 0$ such that $\forall \delta > 0$ ...'something'... Since 'something' works for every δ, let's choose $\delta := 1/n$. 'something' tells you that there exists an x such that... Name this number x_n. Since this procedure is done for all $n \geq 1$, you have thus constructed a sequence $\{x_n\}$. Can you use this sequence to contradict your hypothesis, i.e., the bottom statement?]

2. Frank and Gabrielle own two houses, the house in the woods (call it W) and the house at the beach (call it B). There is only one narrow pathway that connects W to B. Frank and Gabrielle are married. Frank and Gabrielle detest each other, and don't like to see much of one another. We know that at 12:00PM today Frank was in W and Gabrielle was in B. We also know that at 1:00PM Gabrielle was in W and Frank was in B. Use math to show that they must have met at some time between 12:00PM and 1:00PM.

P.S. You cannot assume that either spouse walks at constant speed. At any given time they might walk forward, or backward or stand still. (In fact, Frank is often drunk and Gabrielle is a notorious weirdo.)

[Setting up the problem: First of all, measure the position of a point P on the pathway by means of the variable $r \in [0, L]$, where r is the distance from W to P along the pathway (in feet). (Here L is the total length of the pathway.) Also, measure time in seconds, starting the chronometer at 12:00PM (represented thus by $t = 0$). Therefore, at 1:00PM, $t = 3600$. Then set up two functions of time: $f(t)$ for Frank's position along the path (if at time t Frank is in the house, or away from the pathway—for instance at the local bar—set $f(t) := 0$); and $g(t)$ for Gabrielle's position along the path (if at time t Gabrielle is not on the path, set $g(t) := L$). The physics of the problem tells us that f and g are continuous. Consider the function $h(t) := f(t) - g(t)$ both at 12:00PM and at 1:00PM...]