MA 182, Honors Mathematical Analysis II, Fall '05.
Extra Homework Sheet # 4.

Sample problem. Find the (absolute) minimum and maximum for $f(x) := x^3 - 3x^2 + 3x - 3$ in $[0, 3]$.

Solution. We first look for the absolute min and max points. It is clear that an absolute min point is either an endpoint of $[0, 3]$ or a relative min point. Same for an absolute max point. Since f is differentiable everywhere we know that, if x_0 is a relative min or max point, then $f'(x_0) = 0$. (A point x_0 for which $f'(x_0) = 0$ is sometimes called a *critical point*, so we can say that relative min or max points are also critical points.)

Let's then find all critical points: We calculate that $f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2$. So, the equation $f(x) = 0$ has the unique solution $x_0 = 1$.

Hence, the candidates to be absolute min or max points are $x = 0$, $x = 1$, $x = 3$ (critical points plus endpoints). We evaluate f on all of them to determine when it takes the lowest and highest values:

$$f(0) = -3, \quad f(1) = -2, \quad f(3) = 6.$$

Therefore

$$\min_{[0,3]} f = -3, \quad \max_{[0,3]} f = 6,$$

which is the sought solution.

The problem didn't ask for this, but we can also say that there is (only) one absolute min point at $x = 0$ and (only) one absolute max point at $x = 3$.

1. Find the (absolute) minimum and maximum for each of these functions in the indicated intervals:

   (a) $f(x) := \sqrt{9 - x^2}, \ \ x \in [-1, 2]$

   (b) $f(x) := \dfrac{x}{x^2 + 4}, \ \ x \in [0, 3]$

   (c) $f(x) := \sin x + \cos x, \ \ x \in \left[0, \frac{\pi}{3}\right]$

   (d) $f(x) := \dfrac{\ln x}{x}, \ \ x \in [1, 3]$

2. Differentiate the following functions:

   (a) $f(x) := a^x$ *(a is a constant)*

   (b) $f(x) := 3^{\sin x + x^2}$

   (c) $f(x) := \log_{10}\left(\dfrac{x}{x - 1}\right)$

   (d) $f(x) := \ln \sqrt[5]{x}$ *(simplify as much as you can here)*

3. **(Cauchy's Theorem)** Consider f, g continuous in $[a, b]$ and differentiable in (a, b). Assume that $g'(x) \neq 0 \; \forall x \in (a, b)$. Prove that there is a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

[Tip: Consider the function

$$h(x) := f(x)\,[g(b) - g(a)] - g(x)\,[f(b) - f(a)]$$

and evaluate it at a *and* b.

Note: if you need extra help, the book has this theorem, but in a slighly modified version. For our problem, at some point we need to divide by something. It's up to you to show that this is legal.]

4. Let $f(x) := 1 - \sqrt[3]{x^2}$. Clearly $f(-1) = f(1) = 0$. On the other hand, $f'(x) \neq 0$ for all $x \in (-1, 1)$. This contradicts Rolle's Theorem. Or does it? Explain.