NOTES ON ORDERED PAIRS

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In what follows, A and B will denote two generic sets, not necessarily sets of numbers.

Definition. Given $a \in A$ and $b \in B$, the *ordered pair* formed by a and b is defined as

$$(a,b) := \{\, \{a\}, \{a,b\}\, \}; \tag{1}$$

that is, the set whose elements are $\{a\}$ and $\{a,b\}$ (which are themselves sets).

The motivation for this bizarre definition is that we want to create a mathematical object that is composed of two elements *in a given order*, for which it makes a difference how we list the two elements. Using two-element sets will not do, for we know that $\{a,b\} = \{b,a\}$ (it is indeed a fundamental property of sets that it doesn't matter how elements are listed).

With ordered pairs (usually called simply *pairs*) we'll see that, in general, $(a,b) \neq (b,a)$. The next proposition precisely addresses this point.

Theorem. $(a,b) = (a',b') \iff a = a'$ and $b = b'$.

Proof. The "$\Longleftarrow$" direction is trivial. If a and a' are the same object, and b and b' are the same object, then writing (a,b) or writing (a',b') is *exactly* the same thing.

As for the "$\Longrightarrow$" direction, we assume that $(a,b) = (a',b')$ and set out to prove that $a = a$ and $b = b'$. We give two proofs for the two different cases, $a = b$ and $a \neq b$.

First case ($a = b$). In this case the pair (a,b) is equal to $(a,a) = \{\, \{a\}, \{a,a\}\, \}$. But we know by the properties of sets that $\{a,a\} = \{a\}$. Hence $(a,b) = (a,a) = \{\, \{a\}, \{a\}\, \} = \{\, \{a\}\, \}$ (again by the properties of sets). Therefore (a,b) is a set which contains one, and only one, element (indeed, a set).

Since $(a,b) = (a',b')$, then also (a',b') must be a set which contains only one element. But the definition of (a',b') is $\{\, \{a'\}, \{a',b'\}\, \}$. Thus, it must be $\{a'\} = \{a',b'\}$, which forces b' to be equal to a'. So, then, $(a',b') = \{\, \{a'\}\, \}$.

The hypothesis $(a,b) = (a',b')$ then reads $\{\, \{a\}\, \} = \{\, \{a'\}\, \}$, which implies $a = a'$. But we know from before that $a = b$ and $a' = b'$. So, finally, $a = a' = b = b'$, which is the end of the proof in the first case.

Second case ($a \neq b$). Since $a \neq b$, $\{a\} \neq \{a,b\}$ (the former is a one-element set, the latter is a two-element set). So (a,b) is a set that contains exactly two elements (namely, the sets $\{a\}$ and $\{a,b\}$). By hypothesis, $(a,b) = (a',b')$, so (a',b') too contains exactly two elements, the sets $\{a'\}$ and $\{a',b'\}$, which must therefore be unequal. This implies that $a' \neq b'$.

The hypothesis $(a,b) = (a',b')$ reads

$$\{\,\{a\},\{a,b\}\,\} = \{\,\{a'\},\{a',b'\}\,\}, \tag{2}$$

which simply means that the two elements in the left-hand set are the same as the two in the right-hand set. Thus, there are only two prossibilities: either

$$\{a\} = \{a',b'\} \qquad \text{and} \qquad \{a,b\} = \{a'\} \tag{3}$$

or

$$\{a\} = \{a'\} \qquad \text{and} \qquad \{a,b\} = \{a',b'\} \tag{4}$$

Clearly (3) cannot be the case (two-element sets cannot equal one-element sets), so (4) holds true. The first equality of (4) implies that $a = a'$. Plug this in the second equality and read $\{a,b\} = \{a,b'\}$. Necessarily, $b = b'$. Q.E.D.

Definition. The set of all ordered pairs (a,b), with $a \in A$ and $b \in B$, is called the *cartesian product* of A and B, and is denoted by $A \times B$.

When $A = B = \mathbb{R}$, the cartesian product $\mathbb{R} \times \mathbb{R}$ is also denoted by $\mathbb{R}^2$ and is called the *cartesian plane* (for this is the rigorous definition of the (x,y)-plane devised by René Descartes (1596-1650) and still used in analytic geometry).

We also define $\mathbb{R}^3 := \mathbb{R}^2 \times \mathbb{R}$, and call this the *cartesian space*. Theoretically speaking, its elements should be denoted by $((x,y),z)$ (each pair is formed by a pair of reals and a real) but, for simplicity, they are usually denoted by (x,y,z) and are called *triples*. It is easy to convince oneself that

$$(x,y,z) = (x',y',z') \iff x = x', y = y', z = z'. \tag{5}$$

(If it isn't so easy, think that the equality on the right-hand side actually reads $((x,y),z) = ((x',y'),z')$. Then apply the previous theorem to get $(x,y) = (x',y')$ and $z = z'$. Apply the theorem again, to the first of the two equalities, and get $x = x'$ and $y = y'$.)

The definitions of $\mathbb{R}^2$ and $\mathbb{R}^3$ can be generalized by means of the following

Definition. If n denotes a positive integer, the *n-dimensional Euclidean space* $\mathbb{R}^n$ is defined recursively by the equations

$$\mathbb{R}^1 := \mathbb{R}; \tag{6}$$

$$(\text{for } n \geq 2) \qquad \mathbb{R}^n := \mathbb{R}^{n-1} \times \mathbb{R}. \tag{7}$$

The elements of $\mathbb{R}^n$ are called *n-tuples* of real numbers and are denoted by $(x_1, x_2, \ldots, x_n)$. (If $n = 4$, we speak of *quadruples*; if $n = 5$, *quintuples*; etc.)