EXAMPLES OF WELL-PRESENTED SOLUTIONS TO (VERY SIMPLE) MATH PROBLEMS

Problem 1. Find the (natural) domain of $f(x) = \sqrt{4 - x^2}$.

Solution. For $f(x)$ to be well-defined it must be $4 - x^2 \geq 0$ (square root only accepts non-negative inputs). Therefore $x^2 \leq 4 \implies |x| \leq 2$. This means $-2 \leq x \leq 2$. So

$$\boxed{D(f) = [-2, 2]}$$

[Note: One could have also written

$$\boxed{D(f) = \{x \text{ real} \mid -2 \leq x \leq 2\}}$$

Problem 2. Solve the inequality $x^2 - 5x + 6 \geq 0$.

Solution I. I first solve the associated equation $x^2 - 5x + 6 = 0$. The solutions are given by

$$x_{1,2} = \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot 6}}{2} = \frac{5 \pm 1}{2},$$

that is, $x_1 = 2,\ x_2 = 3$.
Since the coefficient of x^2 is positive, I know that the values for which the quadratic polynomial is non-negative are those external to the pair of solutions above.
Hence the answer is

$$\boxed{x \leq 2 \ \text{ or } \ x \geq 3}$$

Solution II. Factorizing the expression, we get

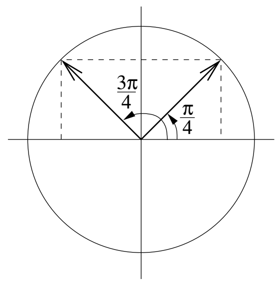
$$(x - 2)(x - 3) \geq 0.$$

Both factors have to be non-negative, or both non-positive.
1st case) $x - 2 \geq 0$ and $x - 3 \geq 0 \implies x \geq 3$.
2nd case) $x - 2 \leq 0$ and $x - 3 \leq 0 \implies x \leq 2$.
The answer is

$$\boxed{x \leq 2 \ \text{ or } \ x \geq 3}$$

Problem 3. Solve $\sin^2 x = \cos^2 x$, for $0 \le x \le \pi$.

Solution I. Taking square roots, I get $|\sin x| = |\cos x|$, that is, the first and second coordinates of the point on the unit circle must be equal in absolute value.



For $0 \le x \le \pi$, this only happens at

$$\boxed{x_1 = \frac{\pi}{4},\ x_2 = \frac{3\pi}{4}}$$

Solution II. Using the fundamental identity of trigonometry, the equation is rewritten as

$$\sin^2 x = 1 - \sin^2 x \quad \Longleftrightarrow \quad 2\sin^2 x = 1 \quad \Longleftrightarrow \quad \sin^2 x = \frac{1}{2}.$$

Hence $\sin x = \dfrac{1}{\sqrt{2}}$ or $\sin x = -\dfrac{1}{\sqrt{2}}$.

But, for $0 \le x \le \pi$, $\sin x \ge 0$, so it has to be $\sin x = \dfrac{1}{\sqrt{2}}$. This happens at

$$\boxed{x_1 = \frac{\pi}{4},\ x_2 = \frac{3\pi}{4}}$$

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