A survey on arithmetic Tutte polynomials : motivations, applications, open problems

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Toric arrangements

A toric arrangement is a set of hypersurfaces in a complex torus $T = (\mathbb{C}^*)^d$ (or in a real torus $\mathbb{S}_1{}^d$) such that every hypersurface is the kernel of a homomorphism $T \to \mathbb{C}^*$.

In recent years, toric arrangement have been intensively studied by several authors.

Among the reasons of this interest, the fact that they may be viewed as *periodic arrangements of hyperplanes* in the universal covering of the torus. Periodic arrangements occur for instance when studying :

- affine Coxeter groups / affine Lie algebras ;
- vector partition functions (via Laplace transform).

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An example of arrangements

Take $V = \mathbb{C}^2$ with coordinates (x, y), $T = (\mathbb{C}^*)^2$ with coordinates (t, s), and

$$X = \{(2,0), (0,3), (-1,1)\} \subset \mathbb{Z}^2.$$

We associate with X three objects :

- a finite hyperplane arrangement given in V by the equations

$$2x = 0, 3y = 0, -x + y = 0;$$

- a periodic hyperplane arrangement given in V by the conditions

$$2x \in \mathbb{Z}, 3y \in \mathbb{Z}, -x + y \in \mathbb{Z};$$

- a toric arrangement given in T by the equations :

$$t^2 = 1; s^3 = 1; t^{-1}s = 1$$

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Models for the toric arrangement

Many results and tools known for hyperplane arrangements admit analogues for toric arrangements. For instance :

- Salvetti complex : a CW complex homotopy equivalent to the complement of the complex arrangement (with a combinatorial description in terms of chambers and faces of the real arrangement). For toric arrangements : (M. -Settepanella 2011), (d'Antonio-Delucchi 2012).
- De Concini-Procesi's wonderful compactification : a smooth complete variety in which the complement of the arrangement is unchanged, while the arrangement is replaced by a normal crossing divisor (combinatorially described by "nested sets"). For toric arrangements : (M. 2011), (De Concini- Gaiffi 2016).

By using these models, many invariants of the complement of a toric arrangement can be computed. It is crucial to have a good combinatorial parametrization of the geometric data.

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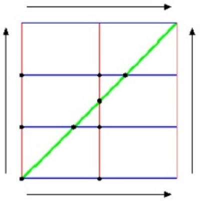
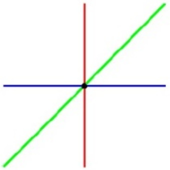
Coloring polynomial and flow polynomial of a graph

Let us look again to the previous example $X = \{(2,0),(0,3),(1,-1)\}$.



toric arrangement



hyperplane arrangement

If we replace the character $(0,3)$ by $(0,1)$ or $(0,5)$, we get the same hyperplane arrangements, but different toric arrangements !

In other words, let X be a list of vectors with integer coordinates. The geometry of the corresponding toric arrangements depends on the linear algebra *and on the arithmetics* of X ; so we need a combinatorial structure keeping track of both.

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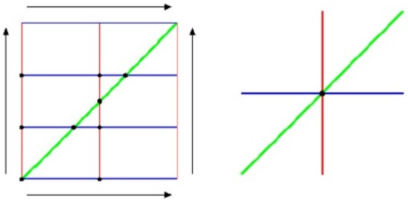

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Axioms

A matroid $\mathcal{M}$ can be defined as a couple (X, rk) where $rk : 2^X \to \mathbb{Z}_{\geq 0}$ is such that $rk(\emptyset) = 0$ and:

(R1) if $A \subseteq X$, then $rk(A) \leq |A|$;

(R2) if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

We say that $[R, S] := \{A : R \subseteq A \subseteq S\}$ is a molecule if S is a disjoint union $S = R \cup F \cup T$ such that for each $A \in [R, S]$, we have $rk(A) = rk(R) + |A \cap F|$.

An arithmetic matroid $\mathcal{A}$ is a matroid and a function $m : 2^X \to \mathbb{N}_{>0}$, s.t.:

(A1) For every $A \subseteq X$, $v \in X$, if $rk(A \cup \{v\}) = rk(A)$, then $m(A \cup \{v\})$ divides $m(A)$; otherwise $m(A)$ divides $m(A \cup \{v\})$;

(A2) if $[R, S]$ is a molecule, then $m(R) \cdot m(S) = m(R \cup F) \cdot m(R \cup T)$;

(P) If $[R, S]$ is a molecule, then

$$\rho(R, S) := (-1)^{|T|} \sum_{A \in [R,S]} (-1)^{|S|-|A|} m(A) \geq 0.$$

Realizabile case

Main example : X is a list of elements in a finitely generated abelian group Λ, and for every $A \subseteq X$, $\mathrm{rk}(A)$ is the rank of the free part of $\langle A \rangle$, while $m(A)$ is the cardinality of the torsion part of $\Lambda / \langle A \rangle$.
We say that a matroid is realizable if it comes from such a list.

(We need to enlarge our focus from *lattices* to *abelian groups* in order to be able to perform two basic matroid operations called deletion and contraction, which correspond to the intuitive idea of *removing a vector* from the list and *quotienting by a vector* respectively : in fact, the latter operation can create torsion).

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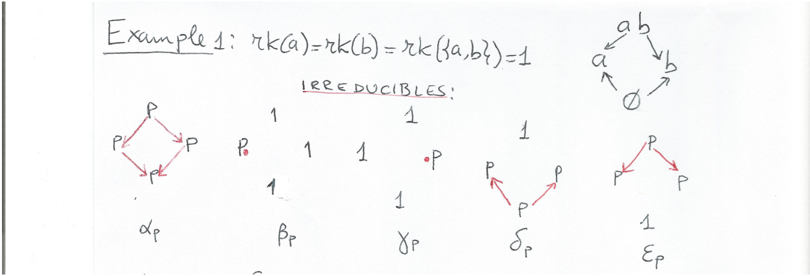
The monoid of arithmetic structures on a matroid.

Theorem (Delucchi- M., 2016)

If (X, rk, m_1) and (X, rk, m_2) are arithmetic matroids, so is $(X, rk, m_1 m_2)$.

Then the set of arithmetic structures on a given matroid is a *commutative monoid*. We are investigating its structure, which seems quite mysterious :



$\alpha_p \beta_p \gamma_p = \delta_p \varepsilon_p$, so that there are no prime elements, and factorization is not unique !

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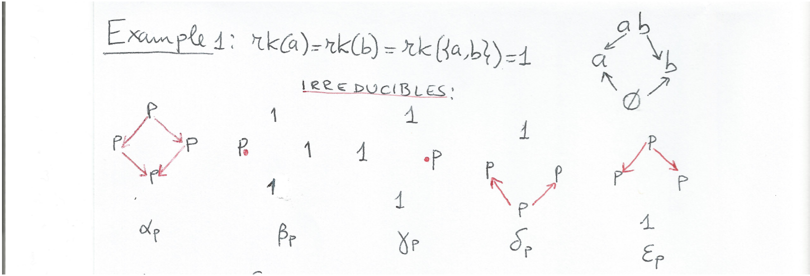

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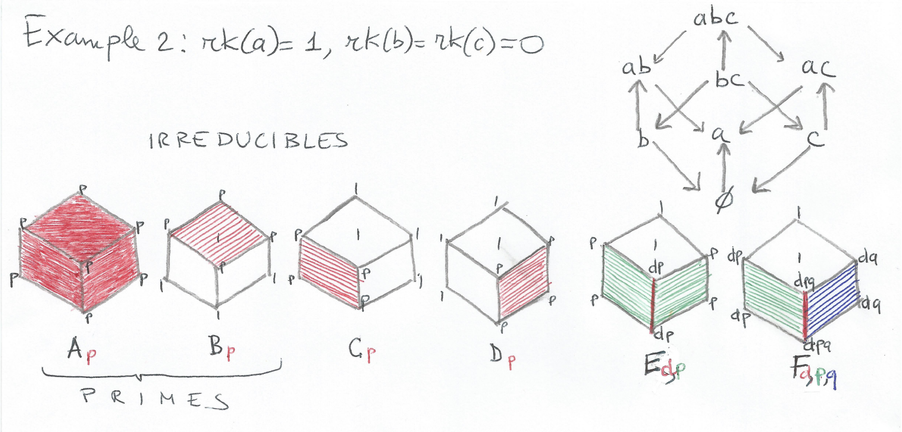
An example with three vectors



Arithmetic Tutte polynomial

The Tutte polyomial of a matroid (X, rk) [Tutte, 1948] is defined as

$$T_X(x,y) = \sum_{A \subseteq X} (x-1)^{\mathrm{rk}\, X - \mathrm{rk}\, A} (y-1)^{|A| - \mathrm{rk}\, A}.$$

For exemple if $X = \{(2,0),(0,3),(-1,1)\}$, then

$$T_X(x,y) = (x-1)^2 + 3(x-1) + 3 + (y-1) = x^2 + x + y.$$

The arithmetic Tutte polyomial of an arithmetic matroid (X, rk, m) [M. 2009] is :

$$M_X(x,y) = \sum_{A \subseteq X} |m(A)| (x-1)^{\mathrm{rk}\, X - \mathrm{rk}\, A} (y-1)^{|A| - \mathrm{rk}\, A}.$$

In our example :

$$M_X(x,y) = (x-1)^2 + (2+3+1)(x-1) + (6+2+3) + (y-1) = x^2 + 4x + y + 5.$$

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Applications to toric arrangements

If the arithmetic matroid is realized by a list of vectors X, its arithmetic Tutte polynomial embodies information on the complement of corresponding toric arrangement :

- The number of connected components of the *real* toric arrangement is equal to $M_X(1,0)$;
- the Poincaré polynomial of the *complex* toric arrangements is equal to $q^n M(\frac{2q+1}{q}, 0)$ [M. 2009].

In the previous example, those are equal to 10 and to $17q^2 + 8q + 1$ respectively.

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The zonotope

Let $U_\mathbb{R}$ be the real vector space spanned by the elements of X. Then we define in $U_\mathbb{R}$ the zonotope

$$\mathcal{Z}(X) \doteq \left\{ \sum_{a_i \in X} t_i a_i, 0 \leq t_i \leq 1 \right\}.$$

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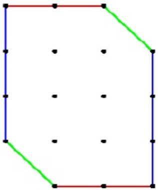

This convex polytope plays a central role both in the theory of arrangements and in that of partition functions.

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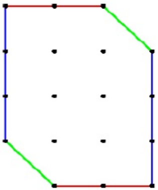

Theorem (M. 2011; D'Adderio- M. 2011)

1. $M_X(1,1)$ *equals the volume of the zonotope* $\mathcal{Z}(X)$*;*
2. $M_X(2,1)$ *is the number of integer points in* $\mathcal{Z}(X)$*;*
3. $M_X(0,1)$ *is the number of integer points in the interior of* $\mathcal{Z}(X)$*;*
4. $M_X(x,1)$ *is the number of integer points in* $\mathcal{Z}(X) - \varepsilon$*, collected according to a suitable stratification.*
5. $q^n M_X(1+1/q,1)$ *equals the Ehrhart polynomial of* $\mathcal{Z}(X)$ *(i.e. the number of integer points in* $q\mathcal{Z}(X), q \in \mathbb{N}$*).*

Combinatorial-geometric interpretation of the coefficients

Theorem (M. -Brändén. 2012)

$$M_X(x,y) = \sum_{B \text{ basis of } X} \left(\sum_{p \in \mathcal{P}(B)} x^{\iota(p)} \right) \left(\sum_{c \in \mathcal{C}(B)} y^{\eta(c)} \right) \text{, where :}$$

- *$I(B) \subseteq B$ and $E(B) \subseteq B^c$ are the sets of "internally active" and "externally active" elements (like in Crapo's formula for $T_X(x,y)$);*
- *$\mathcal{P}(B)$ is the set of integer points into the semi-open zonotope defined by $I(B)$;*
- *given such a point p, $\iota(p)$ is the number of its zero coordinates;*
- *given the toric arrangement defined by $(B \setminus I(B)) \cup E(B)$, $\mathcal{C}(B)$ is the set of connected components of the intersection of the hypersurfaces corresponding to $B \setminus I(B)$;*
- *for every such component c, $\eta(c)$ is the number of elements in $E(B)$ whose hypersurface contains c.*

Other recent developments

Several other properties of the arithmetic Tutte polynomial M_X were discovered in the last years. In fact M_X :

- specializes to the Hilbert series of some graded modules related to the vector partition function, which are called *periodic zonotopal spaces* [Lenz, 2014] ;
- describes the Ehrhart theory of *Lawrence polytopes* [Dall, 2014] ;
- can be recovered from the Tutte polynomials for *group actions on semimatroids* [Delucchi-Riedel, 2015] ;
- has been explicitly computed when X is the set of positive roots of any crystallographic root system [Ardila-Castillo-Henley, 2013].

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Convolution product

Let (X, rk, m_1) and (X, rk, m_2) two arithmetic matroids. We denote by $M_X^{m_1}(x, y)$ and $M_X^{m_2}(x, y)$ the corresponding arithmetic Tutte polynomials. So the usual Tutte polynomial $T_X(x, y)$ is $M_X^{\mathbf{1}}(x, y)$, where $\mathbf{1}$ is the trivial multiplicity.

We define the following convolution product

$$(M_X^{m_1} * M_X^{m_2})(x, y) = \sum_{A \subseteq X} M_{X|_A}^{m_1}(0, y) M_{X/A}^{m_2}(x, 0)$$

where $X|A$ and X/A are respectively the restriction (i.e., deletion of the complement) and the contraction by A.

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$$T_X = T_X * T_X.$$

Convolution product

Let (X, rk, m_1) and (X, rk, m_2) two arithmetic matroids. We denote by $M_X^{m_1}(x, y)$ and $M_X^{m_2}(x, y)$ the corresponding arithmetic Tutte polynomials. So the usual Tutte polynomial $T_X(x, y)$ is $M_X^{\mathbf{1}}(x, y)$, where $\mathbf{1}$ is the trivial multiplicity.
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From arithmetic matroids to matroids over a ring

Arithmetic matroids are matroids decorated with an extra function ; this produces a quite long and complicated list of axioms.

Another approach is possible, by defining a structure called "matroids over the integers" [Fink-M. 2012], which has multiple advantages :

- *Simpler* : A theory with only ONE axiom ;
- *More general* : we can replace the integers with any commutative ring.
- *Yields new invariants* : for instance the "Tutte quasi-polynomial".

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Matroids over $\mathbb{Z}$: an example

Let $v_1, \ldots, v_n$ be a configuration of vectors in an R-module N.
Already in the case $R = \mathbb{Z}$ we see that it is convenient to take a system of axioms for the *quotients* $N/\langle v_i | i \in A \rangle$:

Realizable example

$X = \{(2,0), (0,3), (1,-1)\}$

A	$\emptyset$	1	2	12
$M(A)$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}/6$
A	3	13	23	123
$M(A)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	0

Definition

Let R be a commutative ring and E be a finite set.
A matroid over R on the *ground set* E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R-module $M(A)$ satisfying the following axiom:

for all $A \subseteq E$ and $b, c \notin A$, there are elements

$$x = x(b,c), \quad y = y(b,c) \in M(A)$$

such that there is a diagram

$$\begin{array}{ccc} M(A) & \xrightarrow{/x} & M(A \cup \{b\}) \\ \downarrow{\scriptstyle /y} & \lrcorner & \downarrow{\scriptstyle /\overline{y}} \\ M(A \cup \{c\}) & \xrightarrow[/\overline{x}]{} & M(A \cup \{b,c\}). \end{array}$$

In other words, for all $A \subseteq E$ and $b \neq c \notin A$, the diagram above is a pushout square in which all the arrows are surjections with cyclic kernel.

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Realizability

Fundamental example: "vector configurations" in an R-module.

Given a f.g. R-module N and a list $X = x_1, \ldots, x_n$ of elements of N, we have a matroid M_X associating to $A \subseteq X$ the quotient

$$M_X(A) = N \Big/ \left(\sum_{x \in A} Rx \right).$$

For each $x_i \in X$ there is a quotient map

$$M_X(A) \xrightarrow{/\overline{x_i}} M_X(A \cup \{x_i\})$$

and this system of maps obviously satisfies the axiom.

We say that a matroid M over R is realizable if it actually comes from such a list.

Of course not all matroids over R are realizable!

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Classical matroids are matroids over fields

We can, and will, assume that the module $M(E)$ has no nontrivial projective summands, since this makes many results simpler to state.

Theorem 1 (Fink-M.)

Matroids over a field $\mathbb{K}$ are equivalent to matroids.

A f.g. $\mathbb{K}$-module is determined by its *dimension* $\in \mathbb{Z}$.

If $v_1, \ldots, v_n$ are vectors in $\mathbb{K}^r$,
the dimension of $\mathbb{K}^r / \langle v_i : i \in N \rangle$ is $r - \mathrm{rk}(A)$, the corank of A.

Example

$X = \{(2,0), (0,3), (1,-1)\}$

A	$\emptyset$	1	2	12	3	13	23	123
$M(A)$	$\mathbb{R}^2$	$\mathbb{R}$	$\mathbb{R}$	0	$\mathbb{R}$	0	0	0

Note: The definition of matroids over $\mathbb{K}$ is blind to which field $\mathbb{K}$ is, but for *realizability* the choice of $\mathbb{K}$ matters.

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Exemples and basic operations

In the same way, we can extract an arithmetic matroid from a matroid over $\mathbb{Z}$ by setting $m(A) = |M(A)_t|$. (However here some information is lost !)

Similarly, if R is a discrete valuation ring, we obtain a structure called valuated matroid, which was introduced by Dress and Wenzel.

Usual matroid operation such as *direct sum*, *deletion*, *contraction* can be defined in the framework of matroids over R.

Moreover, one new operation can be performed...

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Tensor product. Localizations and generic matroid

Let $R \to S$ be a map of rings. Then the tensor product $- \otimes_R S$ is a functor $R\text{-Mod} \to S\text{-Mod}$. If M is a matroid over R, then

$$(M \otimes_R S)(A) \doteq M(A) \otimes_R S.$$

defines a matroid over S.

Ler R be a Dedekind Two special cases will be fundamental for us:

1. For every prime ideal $\mathfrak{m}$ of R, let $R_{\mathfrak{m}}$ be the localization of R at $\mathfrak{m}$. We call $M \otimes_R R_{\mathfrak{m}}$ the localization of M at $\mathfrak{m}$.
2. If R is a domain, let $\mathrm{Frac}(R)$ be the fraction field of R. Then we call $M \otimes_R \mathrm{Frac}(R)$ the generic matroid of M.

Notice that every matroid over $R_{\mathfrak{m}}$ induces a matroid over the residue field $R_{\mathfrak{m}}/(\mathfrak{m})$.

We can study the matroid M via all these "classical" matroids.

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Local and global theory

In [Fink- M 2012], a complete combinatorial description is given for matroids over a discrete valuation ring.
This involves surprising relations with tropical geometry, that will be further investigated in a future paper.

Passing from the local to the global theory is trivial if R is a unique factorization domain : in this case a family of modules is a matroid over R if and only if it is a matroid over each localization of R.
In general, however, we show that there is some extra condition involving the Picard group of R.

Finally we describe the Tutte-Grothendieck ring of matroids over R. This allows to produce new invariants ; in the next slides we will present one of them.
To do so, we have to go back to the origins of the Tutte polynomial...

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The four color problem

Coloring polynomial and flow polynomial of a graph

Let $G = (V, E)$ be a graph.
A $q-$*coloring* is a map $c : V \to \mathbb{Z}_q$. It is *proper* if $c(i) \neq c(j)\, \forall (i,j) \in E$.
The fuction assigning to every q the number $\chi_G(q)$ of proper q-colorings is a polynomial in q, called the chromatic polynomial.

Given an orientation of G, a $q-$*flow* is a map $f : E \to \mathbb{Z}_q$ such that

$$\forall v \in V, \sum_{t(e)=v} f(e) = \sum_{s(e)=v} f(e).$$

A flow is *nowhere zero* if $f(e) \neq 0\, \forall e \in E$.
The fuction assigning to every q the number $\chi^*_G(q)$ of nowhere zero q-flows is a poynomial in q, called the flow polynomial.

If G is planar, we can build a *dual graph* G^* such that $\chi_{G^*}(q) = q^{cc(G)}\chi^*_G$, where $cc(G)$ is the number of connected components of G.

Coloring polynomial and flow polynomial of a graph

Let $G = (V, E)$ be a graph.
A $q-$*coloring* is a map $c : V \to \mathbb{Z}_q$. It is *proper* if $c(i) \neq c(j)\, \forall (i,j) \in E$.
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Given an orientation of G, a $q-$*flow* is a map $f : E \to \mathbb{Z}_q$ such that

$$\forall v \in V, \sum_{t(e)=v} f(e) = \sum_{s(e)=v} f(e).$$

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The Tutte polynomial

Tutte's idea : introduce a polynomial $T_G(x, y)$ that specializes to both the chromatic and the flow polynomial :

$$\begin{aligned} \chi(q) =& (-1)^{|V|-cc(G)} q^{cc(G)} T_G(1-q, 0) \\ \chi^*(q) =& (-1)^{|E|} T_G(0, 1-q). \end{aligned}$$

Here duality exchanges the variables x and y, and $T_G(1, 1)$ is the number of *spanning trees* of G.

We can associate with every G a *matroid* called the graphical matroid.

If G is not planar, there is no dual graph but there is still a dual matroid.

The polyomials $\chi_G(q), \chi^*_G(q)$ and $T_G(x, y)$ are in fact invariants of the matroid (for example they are the same for graphs A_4 and D_4).

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Coloring and flows of cellular complexes

Let C be a $d-$dimensional CW complex; for every $i = 0, 1, \dots, d$ we denote by C_i the set of its $i-$dimensonal cells.

The top-dimensional boundary map $\partial : \mathbb{Z}^{C_d} \to \mathbb{Z}^{C_{d-1}}$ is represented by a matrix with integer coefficients.
By reducing ∂ modulo q, we get a map $\overline{\partial} : \mathbb{Z}_q^{C_d} \to \mathbb{Z}_q^{C_{d-1}}$.
A proper q-coloring of C is an element $c \in \mathbb{Z}_q^{C_{d-1}}$ such that all the entries of the vector $c\overline{\partial}$ are nonzero.
A nowhere zero q-flow on C is an element $f \in \ker \overline{\partial}$ such that the coordinate $f(e)$ is nonzero for every $e \in C_d$.

Example : if $d = 1$, C is a graph and $\partial : \mathbb{Z}^E \to Z^V$ is the signed adjacency matrix, so that we recover the usual definitions.
For $d > 1$, however, the entries of ∂ can be different from $+1, 0, -1$, and the number of proper q-colorings $\chi_C(q)$ and the number of nowhere zero q-flows $\chi^*_C(q)$ in general are not polynomial functions, as the next example will show.

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Coloring $C \simeq \mathbb{P}^2_{\mathbb{R}}$

Let C be the 2-dimensional cellular complex with one 0-dimensional cell, one 1-dimensional cell (attached as a loop), and one 2-dimensional cell attached so that $\partial = [2]$.

Clearly G is homeomorphic to a real projective plane.

A $q-$*coloring* is the assignement to the 1-cell of a color $c \in \mathbb{Z}_q$ such that $2c \neq 0$. So

$$\chi_G(q) = \begin{cases} q-1 & \text{if q is odd} \\ q-2 & \text{if q is even} \end{cases}$$

If you find bizarre this condition for a coloring, cut a Möbius strip into the projective plane...

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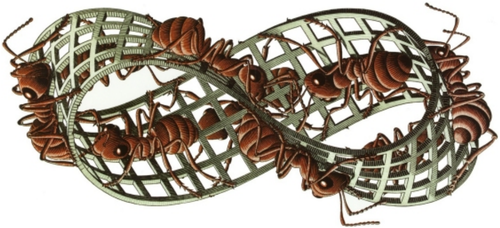
Coloring Escher's ants



...and then imagine to paint the back and the belly of each ant of complementary colors !

Quasi-polynomials and arithmetics

In general, we have :

Theorem (Beck-Breuer-Godkin-Martin, 2012)

The number of proper $q-$colorings $\chi_C(q)$ and the number of nowherezero $q-$flows $\chi^*_C(q)$ are quasi-polynomial functions of q (i.e, there exist a subgroup $m\mathbb{Z}$ such that the restriction to every coset is polynomial).

Of course, there is no hope to obtain these quasi-polynomials as specializations of the usual Tutte polynomial.
However, we will show that its place can be taken by a *Tutte quasi-poynomial*, that was introduced by [Brändén- M. 2012].

In fact, the topology of C does not depend only on the *linear algebra*, but also on the *arithmetics* of the columns of ∂, as the previous example shows.

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Tutte quasi-polynomial

Let X be a list of vectors in $\mathbb{Z}^d$, and for every $A \subseteq X$ let $M(A)_t$ be the torsion part of $\mathbb{Z}^d/\langle A \rangle$.

The Tutte quasi-polyomial of X [Brändén- M. 2012] (or more generally of a matroid over $\mathbb{Z}$) is :

$$Q_X(x,y) = \sum_{A \subseteq X} \frac{|M(A)_t|}{|(x-1)(y-1)M(A)_t|}(x-1)^{\mathrm{rk}\,X - \mathrm{rk}\,A}(y-1)^{|A| - \mathrm{rk}\,A}.$$

This may be seen as a quasi-polynomial interpolating between the (usual) Tutte polyomial

$$T_X(x,y) = \sum_{A \subseteq X} (x-1)^{\mathrm{rk}\,X - \mathrm{rk}\,A}(y-1)^{|A| - \mathrm{rk}\,A}$$

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A higher-dimensional analogue of Tutte's theorem

We now specialize our construction to the case when X is the list of columns of ∂, the top-dimensional boundary matrix of a CW complex C. Then the chromatic quasi-polynomial $\chi_C(q)$ and the flow quasi-polynomial $\chi^*_C(q)$ can be obtained from the Tutte quasi-polynomial :

Theorem (Delucchi-M., 2016)

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