THE DIRICHLET PROBLEM FOR A CLASS OF ULTRAPARABOLIC EQUATIONS

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Abstract. In this paper we study the Dirichlet problem for a class of ultraparabolic equations. More precisely, we prove the existence of a generalized Perron-Wiener solution and we provide a geometric condition for the regularity of the boundary points which extends the classical Zaremba exterior cone criterion to our setting. The main steps for deriving our results are: i) the introduction in \( \mathbb{R}^{n+1} \) of a homogeneous structure; ii) the proof of some interior estimates in a suitable space of Hölder-continuous functions; iii) the construction of a basis of open subsets of \( \mathbb{R}^{n+1} \) for which the Dirichlet problem is univocally solvable.

1. Introduction and main results. We consider in \( \mathbb{R}^{n+1} \) the second-order linear operators

\[
L = \sum_{i,j=1}^{n} a_{ij}(x) \partial^2_{x_i x_j} + \sum_{i=1}^{n} b_i \partial_{x_i} - \partial_t,
\]

where \( x = (x, t) \in \mathbb{R}^{n+1}, 1 \leq q \leq N \) and \( b_j \in \mathbb{R} \) for every \( i, j = 1, \ldots, N \).

\[ A_0(z) = (a_{ij}(z))_{i,j=1}^{n} \] is a symmetric matrix, which is positive definite in \( \mathbb{R}^q \).

\[ B = (b_{ij})_{i,j=1}^{n} \] is a constant matrix of the form

\[
B = \begin{pmatrix}
0 & B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
0 & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_r \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]

where each \( B_i \) is a \( p_i \times p_i \) block matrix of rank \( p_i \), \( i = 1, \ldots, r \) with \( q = p_0 \geq p_1 \geq \cdots \geq p_r \) and \( p_0 + p_1 + \cdots + p_r = N \).

Operators like (1.1) naturally arise in the stochastic theory of diffusion processes. For example, if we choose

\[ A_0 = I_n \quad \text{and} \quad B = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} \]

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we obtain in $\mathbb{R}^{2n+1}$

$$L = \sum_{i=1}^{n} a_{\lambda i}^2 + \sum_{i=1}^{n} x_i \partial_{\lambda x_i} - \partial_\lambda,$$

(1.3)

which is the simplest prototype of the Kolmogorov operator and describes the probability of a physical system with $2n$ degrees of freedom (cf. [20, page 167]).

As in the classical theory of elliptic and parabolic operators, we will study the operators (1.1) as an $H\lambda$ perturbation of the $P\lambda e$ operators $L_{\alpha}$:

$$L_{\alpha} = \sum_{i,j=1}^{q} a_{ij}(z_0) \partial_{x_{i,j}}^2 + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_\lambda, \quad z_0 \in \mathbb{R}^{N+1}. \quad (1.4)$$

It is well known (see for example [8]) that the operators (1.4) are hypoelliptic since they satisfy the Hörmander condition; see [6]. Moreover, they are invariant with respect to a dilation group $(D(\lambda))_{\lambda>0}$ and left invariant with respect to a translation group $(\mathbb{R}^{N+1}, \circ)$. The dilation $D(\lambda)$ is defined as

$$D(\lambda) = \text{diag} \left( \lambda I_{p_0}, \lambda^3 I_{p_1}, \lambda^5 I_{p_2}, \ldots, \lambda^{2n+1} I_{p_n}, I_n \right),$$

(1.5)

where $I_k$ denotes the $k \times k$ identity matrix and $p_1, \ldots, p_n$ are introduced in (1.2).

Hence the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$ is

$$Q + 2 = p_0 + 3p_1 + 5p_2 + \cdots + (2n+1)p_n + 2. \quad (1.6)$$

The composition law of the group $(\mathbb{R}^{N+1}, \circ)$ is defined as

$$(x, t) \circ (y, \tau) = (y + E(\tau)x, t + \tau) \quad \forall (x, t), (y, \tau) \in \mathbb{R}^{N+1}, \quad (1.7)$$

where

$$E(\tau) = \exp(-\tau B^2). \quad (1.8)$$

The identity element of the group is $(0, 0)$ and the inverse element $(x, t)^{-1} = (-E(-t)x, -t)$ for every $(x, t) \in \mathbb{R}^{N+1}$.

Note that neither $D(\lambda)$ nor the translation group depend on the point $z_0$, since they depend on the matrix $B$. Hence the introduction in $\mathbb{R}^{N+1}$ of a homogeneous seminorm of degree 1 with respect to $D(\lambda)$ seems natural.

**Definition 1.1.** For any $z = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$ we define

$$||z|| = ||x|| + |t|^\frac{1}{2} = \sum_{i=1}^{N} |x_i| + |t|^{\frac{1}{2}},$$

where $a_1 = \cdots = a_{p_0} = 1, a_{p_0+1} = \cdots = a_{p_0+p_1} = 3, \ldots, a_{p_0+\cdots+p_{n-1}+1} = \cdots = a_{N} = 2r + 1$.

(See Section 2 below for the proof of some properties of $|| \cdot ||$.) We also define a distance which is invariant with respect to the left translations of $(\mathbb{R}^{N+1}, \circ)$

$$d(\xi, \zeta) = ||\xi^{-1} \circ \zeta||.$$  

We will always work in the following space of Hölder-continuous functions.

**Definition 1.2.** Let $\alpha$ be a positive constant, $\alpha < 1$, and let $\Omega$ be an open subset of $\mathbb{R}^{N+1}$. We will say that a function $f : \Omega \to \mathbb{R}$ is Hölder continuous with exponent $\alpha$ in $\Omega$ with respect to the groups $(\mathbb{R}^{N+1}, \circ)$ and $(D(\lambda))_{\lambda>0}$ (in short: $B$-Hölder continuous with exponent $\alpha$) if there exists a positive constant $k$ such that

$$|f(z) - f(\zeta)| \leq k ||\xi^{-1} \circ z||^\alpha \quad \forall z, \zeta \in \Omega.$$

We will denote by $C^\alpha(\Omega; B)$ the space of the $B$-Hölder-continuous functions and by $| \cdot |_{2, \alpha, \Omega}$ the norm

$$|f|_{2, \alpha, \Omega} = \sup_{\Omega} |f| + \sup_{z, \zeta \in \partial \Omega} \frac{|f(z) - f(\zeta)|}{||\xi^{-1} \circ z||^\alpha}.$$  

We will say that $f \in C^{2,\alpha}(\Omega; B)$ if

$$|f|_{2, \alpha, \Omega} = \sup_{\Omega} |f| + \sum_{i,j=1}^{q} \frac{\sup_{i,j \in \mathbb{Z}} |f_{ij}(z)|}{||\xi^{-1} \circ z||^\alpha}$$

$$+ \sum_{i,j=1}^{q} \frac{\sup_{i,j \in \mathbb{Z}} |f_{ij}(z) - f_{ij}(\zeta)|}{||\xi^{-1} \circ z||^\alpha} + \sup_{z \in \partial \Omega} \frac{|Yf(z)| + |Yf(\zeta)|}{||\xi^{-1} \circ z||^\alpha} < \infty,$$

where $f_{ij} = \partial_{x_i} f_j$, $f_{ij} = \partial_{x_i \partial_{x_j}} f$ and $Y$ is the first-order differential operator:

$$Y = (x, B) - \partial_t = \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t,$$

where $D = (\partial_{x_1}, \ldots, \partial_{x_N})$ and $(\cdot, \cdot)$ denote, respectively, the gradient and the inner product in $\mathbb{R}^n$.

Moreover, we will say that $f$ is a locally $B$-Hölder-continuous function, and we will write $f \in C_{loc}^\alpha(\Omega; B)$ (respectively $f \in C_{loc}^{2,\alpha}(\Omega; B)$) if $f \in C^\alpha(\Omega' ; B)$ ($f \in C^{2,\alpha}(\Omega' ; B)$) for every subset $\Omega' \subset \subset \Omega$.

We observe that, if $f$ is an Hölder-continuous function of exponent $\alpha$ in the usual sense, then $f$ is $B$-Hölder continuous of exponent $\alpha$. Vice versa, if $f \in C^\alpha(\Omega ; B)$ then $f$ is $\alpha$-Hölder-continuous in usual sense, where $\beta$ is the real constant defined by $\beta = \frac{1}{2r + 1}$. Here $r$ denotes the number of nonzero blocks of the matrix $B$ (see (1.2)). This remark follows immediately from inequalities (iii) of Proposition 2.1 in the next section.
Definition 1.3. Let $0 < \alpha < 1$ and let $\Omega$ be an open subset of $\mathbb{R}^{N+1}$. We will say that a bounded function $f : \Omega \to \mathbb{R}$ belongs to $C^{\alpha}_{\mathrm{loc}}(\Omega; B)$ if
\[
\sup_{c \in \Omega, z \in B} \frac{|f(z) - f(\bar{z})|}{||z - \bar{z}||^\alpha} < \infty,
\]
where
\[
d_{\Omega, B} = \min\{d_\Omega, d_B\}, \quad \text{and} \quad d_z = \text{dist}(z, \partial\Omega) = \inf_{z \in \partial\Omega} ||z - \bar{z}||.
\]
We define
\[
|f|_{L^1(\Omega; B)} = \sup_{\Omega} |f| + \sup_{c \in \Omega, z \in B} \frac{|f(z) - f(\bar{z})|}{||z - \bar{z}||^\alpha},
\]
and we will say that $f \in C^{\alpha}_{\mathrm{loc}}(\Omega; B)$ if
\[
|f|_{L^1(\Omega; B)} + \sum_{i=1}^q \sup_{i \in \mathbb{N}, z \in B} |f_i(z)| + \sum_{i=1}^q \sup_{i \in \mathbb{N}, z \in B} |f_{i,z}(z)| + \sum_{i=1}^q \sup_{i \in \mathbb{N}, z \in B} |f_{i,z,z}(z)| + \sum_{i=1}^q \sup_{i \in \mathbb{N}, z \in B} |f_{i,z,z,z}(z)| < \infty.
\]

We next introduce further hypotheses on the operators $L$ in (1.1).

Hypothesis (H). For every $i, j = 1, \ldots, q$ the coefficients $a_{ij}$ belong to $C^{0}_{\mathrm{loc}}(\mathbb{R}^{N+1}; B)$, where $\alpha \in (0, 1)$. Moreover, there exists a positive constant $\mu$ such that
\[
\frac{1}{\mu} \sum_{i=1}^q |\xi_i|^2 \leq \sum_{i=1}^q a_{ij}(z) \xi_i \xi_j \leq \mu \sum_{i=1}^q |\xi_i|^2
\]
for all $\xi \in \mathbb{R}^N$ and for all $z \in \mathbb{R}^{N+1}$.

We remark that if the coefficients $a_{ij}$ of (1.1) are smooth, the operators $L$ belong to the class first introduced by Hörmander in [6], Oleinik-Radkevic in [13], and later studied by Rothschild-Stein in [16].

The operators of "parabolic" type $\sum_{i=1}^q X_i^2 - \Delta$, with $C^0$ coefficients have been widely studied, only few results are known for the operators (1.1) when the coefficients are not smooth (see Weber, [22]; Il'In, [7]; Soinin, [21]; Genviev, [5]).

In particular, in [18] the author proves the existence of a solution of the Cauchy-Dirichlet problem for the operator (1.3) in $\mathbb{R}^3$ in the domain $\{(x_1, x_2, t) \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \in \mathbb{R}, 0 < t < T\}$. Hölder estimates of the solutions are proved only for the operators with constant coefficients; in [19] a local Hölder estimate for the solutions of Kolmogorov operator (1.3) is proved.

In [8] Lanconelli and Polidoro have carried out a systematic study for the operators (1.1) with constant coefficients $a_{ij}$. In [14] the author shows the existence of the fundamental solution $\Gamma$ of (1.1) with the Levi's parametrix method, provides a precise local estimate of $\Gamma$ in terms of the fundamental solution of the frozen operators $L_{a_0}$ and shows the Harnack inequality for nonnegative solutions. The Harnack inequality for a particular subclass of operators like (1.1) has been proved by Qi Zhang ([23]). A global lower bound for the fundamental solution of the operators in divergence form can be found in [15].

Finally, we also quote the recent paper [9] where the author studies the Cauchy problem for the operators (1.1) with constant coefficients $a_{ij}$, and proves a global Hölder estimate with respect to the spatial variable $x$.

In this paper we first prove the following Schauder estimates:

Theorem 1.4 (Schauder estimates). Let $L$ be as in (1.1) satisfying hypothesis (H). Let $f \in C^{2}_{\mathrm{loc}}(\Omega; B)$ and let $u$ be a bounded function belonging to $C^{0,1}_{\mathrm{loc}}(\Omega; B)$ such that $Lu = f$ in $\Omega$. Then $u \in C^{2,1}_{\mathrm{loc}}(\Omega; B)$ and there exists a constant $c > 0$, independent of $u$, such that
\[
|u|_{C^{2,1}(\Omega; B)} \leq c (|u|_{C^{0,1}(\Omega; B)} + |f|_{C^{2}_{\mathrm{loc}}(\Omega; B)}).
\]

(The constant $c$ depends on the constant $\mu$ in (H) and on the Hölder norm of the coefficients $a_{ij}$).

Afterward, we prove the existence of a generalized solution, in the sense of Perron-Wiener, of the Dirichlet problem
\[
\begin{align*}
Lu &= f \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

Precisely, our main result can be stated as follows:

Theorem 1.5 (Existence of a generalized solution). Let $L$ be as in (1.1) satisfying hypothesis (H). Let $\Omega$ be an open bounded subset of $\mathbb{R}^{N+1}$, $f \in C^{\infty}(\Omega; B)$ and $\phi \in C(\partial\Omega)$. Then, there exists a solution $u \in C^{2,1}_{\mathrm{loc}}(\Omega; B)$ of $Lu = f$ in $\Omega$ such that $\lim_{z \to 0} u(z) = \phi(z)$ for every $z \in \partial\Omega$.

A boundary point $z_0$ of $\Omega$ is $L$-regular if there exists a local barrier in $z_0$, that is, there exist a neighborhood $V$ of $z_0$ and a function $w \in C^{2,1}(V; B)$ such that $w(z_0) = 0$, $w(z) > 0$ for $z \in \Omega \cap V \setminus \{z_0\}$ and $Lu < 0$ in $\Omega \cap V$.

The proof is organized as follows. In Section 2 we show some properties of the homogeneous norm $||\cdot||$ and of the fundamental solution of $L_{a_0}$, and we prove a representation formula for functions $u \in C^{2,1}_{\mathrm{loc}}(\mathbb{R}^{N+1})$. Some interpolation inequalities enable us to deduce Theorem 1.4 (see Section 3).
2. Some preliminary results. In this section we will prove some preliminary results. We first prove some properties of the homogeneous norm \(|\cdot|\) introduced in Definition 1.1, then we study the properties of the fundamental solution \(\Gamma^0\) of \(L_0\). Finally, we write a representation formula for functions \(u \in C^0_\infty(\mathbb{R}^{N+1})\) in terms of \(\Gamma^0\).

**Proposition 2.1.** The function \(z \mapsto |z|\) satisfies

(i) \(|D(\lambda)z| = \lambda|z|\) for every \(z \in \mathbb{R}^{N+1}\) and every \(\lambda > 0\).

(ii) For every \(z, \xi \in \mathbb{R}^{N+1}\)

\[
|z + \xi| \leq c(|z| + |\xi|), \\
|z \circ \xi| \leq c(|z| + |\xi|),
\]

and

\[
\frac{1}{c} |z| \leq |z^{-1}| \leq c|z|,
\]

for some constant \(c = c(\beta) \geq 1\).

(iii) \(\frac{1}{c} |z - \xi| \leq |z^{-1} \circ \xi| \leq c|z - \xi|\) \(\beta\) if \(|z - \xi|, |z^{-1} \circ \xi| \leq 1\),

for some constant \(c = c(\beta) > 0\). Here \(|\cdot|\) denotes the Euclidean norm in \(\mathbb{R}^{N+1}\) and \(\beta = (2\beta + 1) / (2\beta + 1)\) is the number of nonzero blocks of the matrix \(B\), so that \(\beta\) depends on the height of the linear combination which generate the Lie algebra \(\text{Lie}(\mathbb{R}^1 \times \mathbb{R}^N)\).

**Proof.** The proof of (i) follows from Definition 1.1, whereas the proof of (ii) we refer to [10] or [2]. Let us now prove the first inequality in (iii). It is sufficient to show that

\[|z \circ \xi - z| \leq c|\xi|\]

for all \(z, \xi \in K\) and \(|\xi| \leq 1\),

where \(K\) is a bounded subset of \(\mathbb{R}^{N+1}\). If \(z = (x, t)\) and \(\xi = (\xi, \tau)\), then from the definition of the translation \(\psi\) in (1.7) we have

\[|z \circ \xi - z| = |(\xi + E(\tau)x, x + \tau) - (x, t)| = |(\xi + E(\tau) \circ (x, t)| + |\tau|,
\]

where \(I\) indicates the \(N \times N\) identity matrix. On the other hand, since

\[
E(\tau) = \sum_{k=0}^{\infty} (-1)^k \frac{\tau^k}{k!} (B^T)^k,
\]

we get

\[
|E(\tau) - I|x| = \sum_{k=1}^{\infty} \frac{|(B^T)^k x|}{k!} \leq |\tau| \sum_{k=1}^{\infty} \frac{|(B^T)^k x|}{k!} \leq c(K, B)|\tau|.
\]

Then

\[|z \circ \xi - z| \leq c \left( \sum_{i=1}^{N} |\xi_i| + |\tau| \right) \leq (||\xi|| + 1) \leq c(1 + ||\xi||).
\]

Now we prove the second inequality in (iii). If \(\xi = (\xi, \tau)\) and \(z = (x, t)\), we have from (2.1)

\[
|\xi^{-1} \circ z| = \sum_{i=1}^{N} |(x_i - \xi_i) \circ z| |\xi_i|^{-1} \leq |(x_i - t_i) \circ z| + |t_i - \\
+ \sum_{i=1}^{N} |(x_i - \xi_i)| (|\xi_i|^{-1} |x_i - t_i| + |z_i|^{-1} |t_i - z_i||),\n\]

where \(l.c.\{a_1, \ldots, a_s\}\) indicates a linear combination, with constant coefficients, of \(a_1, \ldots, a_s\). Since \(\beta \leq \frac{1}{i} \leq 1\), we obtain

\[
|\xi^{-1} \circ z| \leq c \left( \sum_{i=1}^{N} |x_i - \xi_i| + |t_i - \xi_i| \right) \leq c(1 + |z|^{-1}).
\]

This inequality completes the proof of Proposition 2.1.

Let \(L_0 \in \mathbb{R}^{N+1}\) and \(L_0\) be the frozen operator defined in (1.4). If we denote by \(\Gamma(\tau, \rho) = \Gamma^0(\tau, \rho)\) the fundamental solution of \(L_0\) with pole at zero, then

\[
\Gamma^0(\tau, \rho) = \frac{L_0^{-\frac{1}{2}}}{(4\pi)^{N/2} \text{det}(C(\tau))^{1/2}} \exp\left( -\frac{1}{4} (C^{-1}(\tau) D_0(\tau^{-\frac{1}{2}}) x, D_0(\tau^{-\frac{1}{2}}) x) \right),
\]

if \(\tau > 0\) and \(\Gamma^0(\rho, \tau) = 0\) if \(\tau \leq 0\) (see [8]). Here \(D_0(\lambda)\) denotes the restriction to \(\mathbb{R}^N\) of the dilatation \(D(\lambda)\) defined in (1.5) and \(C(\tau)\) denotes the positive definite matrix

\[
\int_0^1 E(s) A(\tau_0) E^T(s) ds \text{ with } A(\tau_0) \text{ is the } N \times N \text{ constant matrix}
\]

\[
A(\tau_0) = \begin{pmatrix} A(\tau_0) & 0 \\ 0 & 0 \end{pmatrix}, \quad A(\tau_0) = (a_{ij}(\tau_0))_{i,j=1,\ldots,n}.
\]

By the left invariance of \(L_{0\lambda}\), its fundamental solution with pole at \(z\) simply is:

\[
\Gamma^0(\tau, \rho) = \Gamma^0(\tau^{-1} \circ z, \rho) = \Gamma^0(\rho \circ z).
\]
Proposition 2.2. Let \( z_0 \in \mathbb{R}^{N+1} \) and \( \Gamma^0(\cdot) = \Gamma(z_0; \cdot) \). Then

(i) \( \Gamma^0 \in C^\infty(\mathbb{R}^{N+1} \setminus \{0\}) \);
(ii) \( \Gamma^0 \) is \( D(\lambda) \)-homogeneous of degree \(-Q\), \( \Gamma^0_\alpha = \partial_\alpha \Gamma^0 \) and \( \Gamma^0_{\alpha\beta} = \partial_\alpha \partial_\beta \Gamma^0 \) are respectively \( D(\lambda) \)-homogeneous of degree \(-Q - \alpha_1 \) and \(-Q - \alpha_1 - \alpha_2 \) for every \( i, j = 1, \ldots, N \).
(iii) (Vanishing property). For all \( 0 < a < b \) and for all \( i, j = 1, \ldots, q \) we have
\[
\int_{|\xi| \leq \max(b-a,1)} \Gamma^0_{ij}(\xi) \, d\xi = 0.
\]

Proof. (i) and (ii) follow from (2.2). In order to prove (iii) we only have to show that the function
\[
\epsilon \mapsto \int_{|\xi| = \epsilon} \Gamma^0_\alpha(\xi) v_\alpha(\xi) \, d\sigma_\xi,
\]
is constant, where \( v \) is the outer normal to the surface \( \{\xi \in \mathbb{R}^{N+1} : |\xi| = \epsilon\} \). From Definition 1.1, it immediately follows that \( ||(x, t)|| = \epsilon, t > 0 \) if and only if
\[
t = (\epsilon - \sum_{k=1}^{N} |x_k|\epsilon^{\frac{1}{N}})^2;
\]
then, since \( \Gamma^0 = 0 \) if \( t = 0 \) and \( \Gamma^0_\alpha \) is \( D(\lambda) \)-homogeneous of degree \(-Q - 1\), we get
\[
\int_{|\xi| = \epsilon} \Gamma^0_\alpha(\xi) v_\alpha(\xi) \, d\sigma_\xi = \int_{\{t = \epsilon\}} \Gamma^0_\alpha(\xi) v_\alpha(\xi) \, d\sigma_\xi
\]
\[
= \int_{D_\epsilon(0)} \Gamma^0_\alpha(\xi, (\epsilon - \sum_{k=1}^{N} |x_k|\epsilon^{\frac{1}{N}})^2) \frac{2\xi}{\alpha} \epsilon^{Q+2}\alpha^{\frac{1}{N}} \left( \sum_{k=1}^{N} |x_k|\epsilon^{\frac{1}{N}} \right) |x|^{\frac{1}{N}} - 2 \, dx
\]
\[
= (D_\epsilon(0)(\epsilon^{-1}) \times
\]
\[
= \int_{|\xi| = \epsilon} \Gamma^0_\alpha(D_\epsilon(0) \xi), v_\alpha \left( 1 - \frac{\sum_{k=1}^{N} |x_k|\epsilon^{\frac{1}{N}}}{\alpha} \right) \frac{2y}{\alpha} \epsilon^{Q+2}\alpha^{\frac{1}{N}} \left( \sum_{k=1}^{N} |x_k|\epsilon^{\frac{1}{N}} \right) |y|^{\frac{1}{N}} - 2 \, dy
\]
\[
= (\alpha_1 = 1) \int_{|\xi| = \epsilon} \Gamma^0_\alpha(x, (1 - \sum_{k=1}^{N} |y_k|\epsilon^{\frac{1}{N}})^2 \frac{2y}{\alpha} \epsilon^{Q+2}\alpha^{\frac{1}{N}} \left( \sum_{k=1}^{N} |y_k|\epsilon^{\frac{1}{N}} \right) |y|^{\frac{1}{N}} - 2 \, dy.
\]
So that
\[
\int_{|\xi| = \epsilon} \Gamma^0_\alpha(\xi) v_\alpha(\xi) \, d\sigma_\xi = \int_{|\xi| = 1} \Gamma^0_\alpha(\xi) v_\alpha(\xi) \, d\sigma_\xi \quad \forall \epsilon > 0,
\]
which is our assertion.

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Proposition 2.3 (Hörmander-type inequality). Let \( z_0 \in \mathbb{R}^{N+1} \) and \( \Gamma^0(\cdot) = \Gamma(z_0; \cdot) \). Then for every \( i, j = 1, \ldots, q \) there exist two positive constants \( c \) and \( M > 1 \) such that
\[
||\Gamma^0_{ij}(\eta^{-1} \circ z) - \Gamma^0_{ij}(\eta^{-1} \circ z) || \leq c \frac{||z^{-1} \circ \eta||}{||z^{-1} \circ \eta||^{Q+3}},
\]
if \( ||z^{-1} \circ \eta|| \geq M ||z^{-1} \circ \eta|| \).

Proposition 2.3 will be derived from the following general lemma.*

Lemma 2.4. If \( f \in C^1(\mathbb{R}^{N+1} \setminus \{0\}) \), \( f \) is \( D(\lambda) \)-homogeneous of degree \( \alpha \in \mathbb{R} \), then there are two constants \( c > 0 \) and \( M > 1 \) such that
\[
||f(u) - f(v)|| \leq c ||u^{-1} \circ u|| ||u||^{Q+1}
\]
provided \( ||u|| \geq M ||u^{-1} \circ u|| \).

Proof. At first, we suppose that \( ||u|| = 1 \). Since \( f \in C^1(\mathbb{R}^{N+1} \setminus \{0\}) \), we have
\[
||f(u) - f(u)|| \leq \sup_{m_0 \leq ||u|| \leq m_1} |DF(u)||u - v|| \leq (Proposition 2.1)
\]
\[
\leq c \sup_{m_0 \leq ||u|| \leq m_1} |DF(u)||u^{-1} \circ u||,
\]
where \( m_0 \) and \( m_1 \) are two positive constants chosen as follows. Since
\[
1 = ||u|| \leq c \left( ||u^{-1} \circ u|| + ||u|| \right) \leq c \left( \frac{1}{M} + ||u|| \right),
\]
then by choosing \( M > c \), we get \( ||u|| \geq 1 - \frac{1}{M} = m_0 > 0 \). Moreover,
\[
||u|| \leq c \left( ||u||^{-1} ||u|| + ||u|| \right) \leq c (||u||^{-1} ||u|| + ||u||) \leq c \left( \frac{m_0}{M} + 1 \right) = m_1.
\]
If \( u, v \in \mathbb{R}^{N+1} \setminus \{0\} \), we obtain
\[
|f(u) - f(v)| = ||u||^{Q+1}|f(D(||u||^{-1})u) - f(D(||u||^{-1})v)|
\]
\[
\leq c ||u||^{Q+1} ||D(||u||^{-1}v)|| \leq c |u|^{-1} \circ (D(||u||^{-1})) ||u||^{Q+1}.
\]
Here, we have used the following equalities:
\[
(D(||u||^{-1})v)^{-1} \circ (D(||u||^{-1})u) = (from \ the \ definitions \ of \ \circ \ and \ D(\lambda))
\]
\[
= (D(||u||^{-1}v)^{-1}) \circ (D(||u||^{-1}u)) = D(||u||^{-1})(v^{-1} \circ u).
\]

*We thank S. Polidoro for providing us with the proof of lemma.
Theorem 2.5 (Representation formula). Let \( u \in C_0^\infty(\mathbb{R}^{n+1}) \) and \( z_0 \in \mathbb{R}^{n+1} \). Then, for every \( z \in \mathbb{R}^{n+1} \), we have
\[
\begin{align*}
\partial_{\xi_j}^2 u(z) &= \lim_{\epsilon \to 0} \int_{|\xi_j\epsilon - z| < \epsilon} \Gamma_\Omega^\epsilon(\xi^{-1} \circ z) L_{z_0} u(\xi) d\xi \\
&= L_{z_0} u(z) \int_{|\xi_j\epsilon| = 1} \Gamma_\Omega^\epsilon(\xi) \nu_j(\xi) d\sigma, \quad i, j = 1, \ldots, g. \tag{2.6}
\end{align*}
\]

Here, \( \nu_j \) is the \( j \)-th component of the outer normal to the surface \( \{ \xi \in \mathbb{R}^{n+1} : \| \xi \| = 1 \} \).

Proof. See [10] or [2].

3. Schauder estimates. In this section we shall give the proof of Theorem 1.4. For every \( z \in \mathbb{R}^{n+1} \) and \( r > 0 \) we indicate by \( B(z, r) \) or by \( B_r(z) \) the set
\[
B(z, r) = \{ \xi \in \mathbb{R}^{n+1} : \| \xi^{-1} \circ z \| < r \}.
\]

We explicitly remark that
\[
|B(z, r)| = |B(0, r)| = |B(0, 1)|r^{2g+2}, \tag{3.1}
\]
where \( | \cdot | \) denotes the Lebesgue measure in \( \mathbb{R}^{n+1} \) and \( Q + 2 \) is the homogeneous dimension of \( \mathbb{R}^{n+1} \) with respect to \( (D(\lambda))_{\lambda \in \mathbb{R}} \) as defined in (1.6). Indeed, the matrix \( E \) as defined in (1.6) verifies the identity
\[
E(\lambda^2) = D_0(\lambda) E(t) D_0(\lambda), \quad \text{for all } \lambda > 0 \text{ and for all } t \in \mathbb{R}, \tag{3.2}
\]
(see (2.20) in [8]). Therefore det \( E(\lambda^2) = \text{det} E(t) \) for all \( \lambda > 0 \). Since det \( E(0) = 1 \), by letting \( \lambda \) tend to 0 we get \( \text{det} E(r) = 1 \) for all \( r \in \mathbb{R} \). From this the first equality in (3.1) follows. Moreover, from (1.6), we have \( |B(0, r)| = |B(0, 1)|r^{2g+2} \).

In particular \( |B(z, 2r)| = 2^{2g+2} |B(z, r)| \), which corresponds to the so-called doubling condition.

Proof of Theorem 1.4. We remark that it suffices to prove inequality (1.9) for compact subsets of \( \Omega \). Let \( (\Omega_\alpha)_{\alpha \in A} \) be a sequence of open subsets of \( \Omega \) such that \( \Omega_\alpha \subset \Omega_{\alpha+1} \subset \cdots \subset \Omega \) for all \( k \) and \( \cup_{k \in \mathbb{N}} \Omega_\alpha = \Omega \). From (1.9) we have that, for \( ||z||_\alpha \leq \alpha \), \( \alpha = 2 \) and for sufficiently large \( \alpha \), we infer that
\[
|u(z)| + d_\alpha(\Omega_\alpha)|u(z)| + d_{2g+2}(\alpha_\alpha)|u(z)| + d_{2g+2}(\Omega_\alpha)|u(z)| - u_{ij}(z)| ||z^{-1} \circ z||^2
\]
\[
\quad + d_{2g+2}(\Omega_\alpha)|Y u(z)| + d_{2g+2}(\Omega_\alpha)|Y u(z)| - Y u(z)| ||z^{-1} \circ z||^2
\]
\[
\leq \epsilon (\sup_{\Omega_\alpha} |u| + d_{2g+2}(\alpha_\alpha)) \leq \epsilon (\sup_{\Omega_\alpha} |u| + d_{2g+2}(\alpha_\alpha)),
\]
where \( d_\alpha(\Omega_\alpha) = \min\{d_\alpha(\Omega_\alpha), d_\alpha(\Omega_{\alpha+1})\} \) and \( d_{2g+2}(\Omega_\alpha) = \text{dist}(z, \partial \Omega_\alpha) \). Hence, on letting \( \epsilon \) tend to infinity, we obtain the inequality
\[
|u(z)| + d_\alpha(\Omega_\alpha)|u(z)| + d_{2g+2}(\alpha_\alpha)|u(z)| - u_{ij}(z)| ||z^{-1} \circ z||^2
\]
\[
\quad + d_{2g+2}(\Omega_\alpha)|Y u(z)| + d_{2g+2}(\Omega_\alpha)|Y u(z)| - Y u(z)| ||z^{-1} \circ z||^2
\]
\[
\leq \epsilon (\sup_{\Omega_\alpha} |u| + d_{2g+2}(\alpha_\alpha)),
\]
for every \( z, \tilde{z} \in \Omega \), which implies the result. We may therefore assume that \( u \in C_0^{2g+2}(\Omega) \).

It follows easily to split our proof into three stages. We first prove an estimate in the \( B \)-Hölder spaces for the solutions of \( L_{z_0} u = g \in C_0^0(\Omega, B) \), where \( L_{z_0} \) is defined in (1.4) and \( u \) is a function with compact support. In the second part, we extend this estimate for a general \( u \). Finally, by using interpolation inequalities, we obtain the thesis.

(a) First step. Let \( z_0 \in \Omega \) and let \( L_{z_0} \) be the corresponding frozen operator. We indicate by \( \Gamma^0 \) the fundamental solution of \( L_{z_0} \) and set \( g = L_{z_0} u \). We suppose that \( u \) has compact support, then from the representation formula (2.6), we can directly estimate the difference \( u'_{ij}(z) - u_{ij}(\tilde{z}) \) for all \( z, \tilde{z} \in \Omega \). By the vanishing property (Proposition 2.2), we have
\[
\lim_{\epsilon \to 0} \int_{|z^{-1} \circ z| < \epsilon} \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) d\xi = \lim_{\epsilon \to 0} \int_{|z^{-1} \circ z| < \epsilon} \Gamma_{\Omega_\alpha}^\epsilon(\xi) g(\xi) d\xi
\]
\[
= \int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi - \int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi.
\]

Let \( \delta = ||z^{-1} \circ z|| > 0 \) and let \( M \) be the positive constant in the Hölder inequality (Proposition 2.3). We write
\[
\int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi = \int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi
\]
\[
= \int M_1 \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi + \int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi
\]
\[
- \int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi - \int \Gamma_{\Omega_\alpha}^\epsilon(\xi^{-1} \circ z) g(\xi) - g(\xi) d\xi
\]
\[
= I_1 + I_2 + I_3 + I_4,
\]
where \( M_1 = \{ \xi \in \mathbb{R}^{n+1} : ||z^{-1} \circ z|| < \delta \}, M_2 = \{ \xi \in \mathbb{R}^{n+1} : ||z^{-1} \circ z|| \geq M_6 \} \), and we estimate separately \( I_1, I_2, I_3 \) and \( I_4 \).

Let \( \Omega' \) be the support of \( u \); then by (ii) in Proposition 2.2 we obtain
\[
|I_4| \leq |g|_{\text{loc}} \int_{M_4} ||z^{-1} \circ z||^2 d\xi = c |g|_{\text{loc}} \int_{M_4} \rho^{2g+2} d\rho = |g|_{\text{loc}} M^g 8^g.
\]
An analogous procedure can be used for estimating \(|\mathcal{L}_d|\). From Proposition 2.3, we get

\[
|\mathcal{L}_d + L_d| = \left| \int_{\mathcal{M}_1} (\Gamma^d_0(\zeta^{-1} \circ z) - \Gamma^d_0(\zeta^{-1} \circ z))g(z - g(z))d\xi \right| \tag{3.4}
\]

\[
\leq c \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} \int_{\mathcal{M}_1} \frac{d\xi}{\left| z \right|^{1+\alpha}} = c \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} \int_{\mathcal{M}_1} \delta^d \, dz = c \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} \delta^d.
\]

Then from the representation formula (2.6) and inequalities (3.3) and (3.4) we infer that

\[
|\mathcal{u}_j(z) - \mathcal{u}_j(\bar{z})| \leq c \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} \left| \zeta \right|^{\alpha}, \quad \forall z, \bar{z} \in \Omega,
\]

so that there exists a positive constant \(c\) such that

\[
d_{z, \xi}^2 \left| \frac{\mathcal{u}_j(z) - \mathcal{u}_j(\bar{z})}{\left| \zeta \right|^{\alpha}} \right| \leq c \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} \left| \zeta \right|^{\alpha}, \quad \forall z, \bar{z} \in \Omega, \quad z \neq \bar{z},
\]

where \(d_{z, \xi}\) is as in Definition 1.3.

(b) Second step. For every fixed \(r > 0\) we shall prove that if \(z, \bar{z} \in \Omega, z \neq \bar{z}\) and \(|\zeta| \leq \frac{r}{2}\), then

\[
r^2 \left| \frac{\mathcal{u}_j(z) - \mathcal{u}_j(\bar{z})}{\left| \zeta \right|^{\alpha}} \right| \leq c \left( r^2 \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} + \sup_{B_r(\zeta)} |u| \right),
\]

where \(g = L_zu\).

In order to prove (3.6), we indicate by \(L^\ast\) the adjoint operator of \(L\); that is,

\[
L^\ast = \sum_{j=1}^q a_j(z_0) \delta_{z_0, z_j} - Y.
\]

The function \(\Gamma^\ast\) defined by \(\Gamma^\ast(z_0; z) = \Gamma^0(z_0; z)\) for all \(z, z_0 \in \mathbb{R}^{N+1}, z \neq z_0\) is the fundamental solution of \(L^\ast\). In the sequel, we use Green's identity

\[
vL_{z_0}u - uL_{z_0}v = \sum_{j=1}^q \partial_{z_0} \left( \sum_{j=1}^q v a_j(z_0) u_j - \sum_{j=1}^q a_j(z_0) v_j \right) + Y(uv), \tag{3.7}
\]

with \(v(z) = \phi(z) \Gamma^\ast(z; z_0), \zeta \in \Omega\) and \(\phi\) a function of class \(C_0^\infty(B_1(\zeta))\), such that \(\phi(z) = 1\) in \(B_{1/2}(\zeta)\) and \(\phi(z) = 0\) in \(B_1(\zeta) \setminus B_{1/2}(\zeta)\). Besides, \(|\phi(z)| \leq \frac{1}{\zeta}\) and \(|\Gamma \phi(z)| \leq \frac{1}{\zeta}\) for all \(z \in B_1(\zeta)\) and for all \(i, j = 1, \ldots, q\).

Then if \(\zeta \in B_{1/2}(\zeta)\), upon integration on \(B_1(\zeta)\) of (3.7), we obtain

\[
u(z) = - \int_{B_1(\zeta)} \phi(z) g(z) \Gamma^\ast(z; \zeta) \, dz + \int_{B_1(\zeta)} u(z) L^\ast_\zeta(\phi(z) \Gamma^\ast(z; \zeta)) \, dz, \quad \zeta \in B_{1/2}(\zeta).
\]

Consequently, we have

\[
u_{ij}(\zeta) = - \int_{B_1(\zeta)} \phi(z) g(z) \Gamma^\ast(z; \zeta) \, dz + \int_{B_1(\zeta)} u(z) L^\ast_\zeta(\phi(z) \Gamma^\ast(z; \zeta)) \, dz.
\]

The estimate (3.6) for the first term \(u_{ij}\) in (3.8) can be obtained as in step (a) of the proof, then, we consider the second term \(w_{ij}\). From the definition of \(\phi\) and the fact that \(\delta_{\zeta, \zeta_0}(L^\ast \Gamma^\ast) = L^\ast \delta_{\zeta, \zeta_0}^\ast \Gamma^\ast = 0\), we have

\[
w_{ij}(\zeta) = \int_{B_1(\zeta)} u(z) L^\ast_\zeta(\phi(z) \Gamma^\ast(z; \zeta)) \, dz.
\]

Let \(z \in \Omega\) be such that \(|\zeta| \leq \frac{r}{2}\); then

\[
r^2 \left| \frac{\mathcal{u}(z) - \mathcal{u}(\bar{z})}{\left| \zeta \right|^{\alpha}} \right| \leq c \left( r^2 \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} + \sup_{B_r(\zeta)} |u| \right)
\]

\[
\left( \sup_{B_r(\zeta)} |\phi| \right) \int_{B_{1/2}(\zeta) \setminus B_{1/2}(\zeta)} |\Gamma \phi(z_0; \zeta)| \, dz \leq \frac{c}{\zeta} \left( r^2 \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} + \sup_{B_r(\zeta)} |u| \right)
\]

\[
+ \sup_{B_r(\zeta)} |\phi| \int_{B_{1/2}(\zeta) \setminus B_{1/2}(\zeta)} |\Gamma \phi(z_0; -\zeta)| \, dz \leq \frac{c}{\zeta} \left( r^2 \left| g \right|_{L^2(\mathcal{M}_1, |z|^{1+\alpha})} + \sup_{B_r(\zeta)} |u| \right).
\]

We next determine an estimate for \(I_1, I_2\) and \(I_3\).

Let us write

\[
I_1 = \sup_{B_r(\zeta)} |\phi| \left( \int_{B_{1/2}(\zeta) \setminus B_1(\zeta)} |\Gamma \phi(z_0; -\zeta)| \, dz \right)
\]

\[
+ \int_{A_3} |\Gamma \phi(z_0; -\zeta)| \, dz,
\]

where \(A_1\) and \(A_2\) are the sets

\[
A_1 = (B_{1/2}(\zeta) \setminus B_1(\zeta)) \cap \{ \zeta \in \mathbb{R}^{N+1} : |\zeta| \geq M |\zeta| \}
\]

\[
A_2 = (B_{1/2}(\zeta) \setminus B_1(\zeta)) \cap \{ \zeta \in \mathbb{R}^{N+1} : |\zeta| \leq M |\zeta| \},
\]
and $M$ is the positive constant in Proposition 2.3. We thus have
\[
\frac{\int |\Gamma_{ij}(z^{-1} \circ \zeta) - \Gamma_{ij}(z^{-1} \circ \zeta)| \, d\zeta}{\int |\Gamma_{ij}(z^{-1} \circ \zeta)| \, d\zeta} \leq c \left( \frac{||z^{-1} \circ \zeta||}{r} \right)^{s}.
\]  
(3.12)

On the other hand, from (ii) in Proposition 2.2 we have
\[
\frac{\int |\Gamma_{ij}(z^{-1} \circ \zeta) - \Gamma_{ij}(z^{-1} \circ \zeta)| \, d\zeta}{\int |\Gamma_{ij}(z^{-1} \circ \zeta)| \, d\zeta} \leq c \left( \frac{1}{||z^{-1} \circ \zeta||^{2} + \frac{1}{||z^{-1} \circ \zeta||^{2}}} \right) \frac{d\zeta}{d\zeta},
\]
where
\[
\int \frac{d\zeta}{||z^{-1} \circ \zeta||^{2}} \leq c \int \frac{d\zeta}{||z^{-1} \circ \zeta||^{2}} \leq c \frac{1}{||z^{-1} \circ \zeta||^{2}}.
\]

Here $c_1$ and $c_2$ are two positive constants given by the following two inequalities:
\[
||z^{-1} \circ \zeta|| \leq (\text{Proposition 2.1}) \leq c \left( ||z^{-1} \circ \zeta|| + ||z^{-1} \circ \zeta|| \right) \leq c_2 ||z^{-1} \circ \zeta||.
\]
\[
||z^{-1} \circ \zeta|| \geq \frac{1}{c} ||z^{-1} \circ \zeta|| - c ||z^{-1} \circ \zeta|| \geq \frac{1}{c} ||z^{-1} \circ \zeta|| - \frac{cr}{2a} \geq \frac{r}{2a} c_1 - c \geq c_1, r
\]
and $a$ is a positive constant such that $c_1$ is positive. We then deduce
\[
\frac{\int |\Gamma_{ij}(z^{-1} \circ \zeta) - \Gamma_{ij}(z^{-1} \circ \zeta)| \, d\zeta}{\int |\Gamma_{ij}(z^{-1} \circ \zeta)| \, d\zeta} \leq c \left( \frac{||z^{-1} \circ \zeta||}{r} \right)^{s},
\]
and, from (3.11)-(3.13), we obtain
\[
I_1 \leq c \sup_{B_{\delta,0}} \phi \left( \frac{||z^{-1} \circ \zeta||}{r} \right)^{s}.
\]
(3.14)

The estimate of $I_2$ in (3.10) is similar to that of $I_1$. Indeed, from Lemma 2.4, the function $\Gamma_{ij}$ satisfies the Hörmander inequality
\[
|\Gamma_{ijk}(z^{-1} \circ \zeta) - \Gamma_{ijk}(z^{-1} \circ \zeta)| \leq c \frac{||z^{-1} \circ \zeta||^{4}}{||z^{-1} \circ \zeta||^{2}} \delta_{i,j,k},
\]
provided $||z^{-1} \circ \zeta|| \geq M ||z^{-1} \circ \zeta||$, $c$ and $M$ being suitable positive constants. Hence, arguing as before, we get
\[
I_2 \leq c \sup_{B_{\delta,0}} \phi \left( \frac{||z^{-1} \circ \zeta||}{r} \right)^{s}.
\]
(3.15)

The integral $I_3$ can be estimated like $I_1$; that is,
\[
I_3 \leq c \sup_{B_{\delta,0}} \phi \left( \frac{||z^{-1} \circ \zeta||}{r} \right)^{s}.
\]
(3.16)

Then, from (3.10), (3.14), (3.15), (3.16) and from the conditions on $\phi$ we have
\[
\mathbb{E} \left[ |u_{ij}(z) - u_{ij}(\zeta)| \right] \leq c r^{-2} \sup_{B_{\delta,0}} \phi.
\]
(3.17)

Combining (3.8), (3.5) and (3.17) we thus have
\[
\mathbb{E} \left[ |u_{ij}(z) - u_{ij}(\zeta)| \right] \leq \mathbb{E} \left[ |v_{ij}(z) - v_{ij}(\zeta)| \right] + \mathbb{E} \left[ |u_{ij}(z) - u_{ij}(\zeta)| \right] \leq c \left( \mathbb{E} \left[ |v_{ij}(z)| \right] \right)^{2}.
\]

This inequality proves (3.6).

(c) Third step. Let $z_0$, $\zeta_0$ be any two distinct points of $\Omega$, and assume that $d_{\alpha} \leq d_{\alpha}$. We write
\[
L_{\alpha}u(z) = L_{\alpha}(z) + (L_{\alpha} - L_{\alpha})u(z) = f(z) + \sum_{i,j=1}^{r} (a_{ij}(z_0) - a_{ij}(z))u_{ij}(z) \equiv F(z)
\]
and we consider $\tilde{\alpha} < 1$ to be a constant to be specified later but such that $B_{\alpha}(z_0) \subset \Omega$ with $\tilde{\alpha} = d_{\alpha}$. Suppose first that $\zeta_0 \in B_{\tilde{\alpha}}(z_0)$; then from (3.6)
\[
\left( \tilde{\alpha} d_{\alpha} \right)^{2} \mathbb{E} \left[ |u_{ij}(z_0) - u_{ij}(\zeta_0)| \right] \leq c \left( \mathbb{E} \left[ |d_{\alpha} F|_{B_{\alpha}(z_0)} \right] + \sup_{B_{\alpha}(z_0)} |u| \right).
\]
(3.18)

On the other hand, if $\zeta_0 \notin B_{\tilde{\alpha}}(z_0)$ we have
\[
d_{\alpha}^{2} \mathbb{E} \left[ |u_{ij}(z_0) - u_{ij}(\zeta_0)| \right] \leq c \left( \frac{2}{\tilde{\alpha}} \right)^{2} \mathbb{E} \left[ \mathbb{E} \left[ |u_{ij}(z_0)| \right] \right] \mathbb{E} \left[ |u_{ij}(\zeta_0)| \right]
\]
\[
\leq \frac{4}{\tilde{\alpha}^{2}} \sup_{\zeta \in \Omega} \mathbb{E} \left[ |u_{ij}(\zeta)| \right],
\]
(3.19)

so that, combining (3.18) and (3.19) we obtain
\[
d_{\alpha}^{2} \mathbb{E} \left[ |u_{ij}(z_0) - u_{ij}(\zeta_0)| \right] \leq c \frac{4}{\tilde{\alpha}^{2}} \mathbb{E} \left[ |F|_{B_{\alpha}(z_0)} \right] + \sup_{\zeta \in \Omega} \mathbb{E} \left[ |u| \right] + \frac{4}{\tilde{\alpha}^{2}} \sup_{\zeta \in \Omega} \mathbb{E} \left[ |u_{ij}(\zeta)| \right].
\]
(3.20)
We proceed by estimating \( |d^2 F|_{x,\bar{d},\bar{B},(z_\alpha)} \) in terms of \( |u_i|_{x,\bar{d},\bar{B},(z_\alpha)} \). We have

\[
|d^2 F|_{x,\bar{d},\bar{B},(z_\alpha)} \leq \sum_{i,j=1}^{d} a_{ij}(z_\alpha - a_{ij}) u_i |u_j|_{x,\bar{d},\bar{B},(z_\alpha)} + |d^2 f|_{x,\bar{d},\bar{B},(z_\alpha)}.
\]  

(3.21)

where

\[
|a_{ij}(z_\alpha - a_{ij}) - a_{ij}|_{x,\bar{d},\bar{B},(z_\alpha)} \leq \sup_{x,\bar{d},\bar{B},(z_\alpha)} |a_{ij}(z_\alpha - a_{ij})| + \varepsilon^{\alpha} \sup_{x,\bar{d},\bar{B},(z_\alpha)} \frac{|a_{ij}(z_\alpha - a_{ij})|}{||z_\alpha - \bar{z}||^\alpha} 
\]

\[
\leq 2 \varepsilon^{\alpha} |\mu|^\alpha |a_{ij}|_{x,\bar{d},\bar{B},\Omega}. \leq c |\mu|^\alpha. \tag{3.22} \label{3.22}
\]

It is then easy to prove that if \( c \) is the constant in the quasi-triangular inequality in Proposition 2.1 and \( \mu \) is a positive constant such that \( \mu \leq \frac{1}{2\varepsilon} \), then

\[
|d^2 u_i|_{x,\bar{d},\bar{B},(z_\alpha)} \leq c |\mu|^{\alpha + 2}  |d^2 u_i|_{x,\bar{d},\bar{B},(z_\alpha)} + |d^2 f|_{x,\bar{d},\bar{B},\Omega}. \tag{3.23} \label{3.23}
\]

Hence, from (3.21)–(3.23) we arrive at the following estimate for the principal term in (3.20):

\[
|d^2 F|_{x,\bar{d},\bar{B},(z_\alpha)} \leq c |\mu|^{\alpha + 2} \left( \sup_{x,\bar{d},\bar{B},(z_\alpha)} |d^2 u_i|_{x,\bar{d},\bar{B},(z_\alpha)} + \mu \sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \right) + |d^2 f|_{x,\bar{d},\bar{B},\Omega}. \tag{3.24} \label{3.24}
\]

To proceed further, we need to remove the term \( \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \) from (3.24) for which we need to prove the following interpolation inequalities.

**Interpolation Inequalities.** Let \( u \in C^{2+\alpha}_d(\Omega; B) \); then for every \( \varepsilon > 0 \) and some constant \( c = c(\varepsilon) > 0 \) we have

\[
\sup_{\Omega} d^2|u_i| \leq c \sup_{\Omega} |u_i| + \varepsilon \sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \tag{3.25} \label{3.25}
\]

and

\[
\sup_{\Omega} |u_i| + \varepsilon \sum_{i=1}^{d} \sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \tag{3.26} \label{3.26}
\]

These inequalities will be proved at the end of the proof.

The conclusion of the proof then follows by applying (3.25) and (3.26). Indeed, inequalities (3.20), (3.24) and (3.25), with \( \varepsilon = \mu \), now yield the estimate

\[
\sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \leq c \sup_{\Omega} |u_i| + |d^2 f|_{x,\bar{d},\bar{B},\Omega}. \tag{3.27} \label{3.27}
\]

By using (3.26) with \( \varepsilon \) small enough, we thus have

\[
\sup_{\Omega} |u_i| + \varepsilon \sum_{i=1}^{d} \sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i + \sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \leq c \sup_{\Omega} |u_i| + |d^2 f|_{x,\bar{d},\bar{B},\Omega}. \tag{3.28} \label{3.28}
\]

Finally, observing that \( Y u = f - \sum_{i=1}^{d} a_{ij} u_{ij} \) we have

\[
|d^2 Y u| \leq |d^2 f| + c \sum_{i=1}^{d} |d^2 u_i|. \tag{3.29} \label{3.29}
\]

Combining these estimates with (3.27) we arrive at the desired estimate (1.9): Thus we are left with the proof of the interpolation inequalities (3.25) and (3.26).

We prove (3.25) first establishing the interpolation inequality

\[
\sup_{\Omega} d^2 u_i \leq c \sup_{\Omega} |u_i| + \varepsilon \sup_{x,\bar{d},\bar{B},(z_\alpha)} d^2 u_i \frac{|u_i(z) - u_i(\bar{z})|}{||z_\alpha - \bar{z}||^\alpha} \tag{3.30} \label{3.30}
\]

where \( c > 0 \) may be arbitrary. Let \( z \in \Omega \) and \( B_r(z) \subset \subset \Omega \) with \( r = \mu d \) and \( \mu \) a positive constant to be specified later.

Let \( z', z'' \in B_r(z) \) be such that \( z' = (x', \bar{t}), z'' = (x'', \bar{t}) \), such that \( ||z'' - z'|| \leq c r \), \( r \) and \( z', z'' \) are parallel to one of the \( x_i \) axes.

We note that \( |z' - z''| = ||(z'' - z')|| \), where \( |.| \) indicates the Euclidean norm in \( \mathbb{R}^{d+1} \). Then, for some \( \bar{z} \in B_r(z) \), we have

\[
|u_i(\bar{z})| = \frac{|u(z') - u(z'')|}{|z' - z''|} \leq \frac{2 \sup_{\Omega} |u|}{||(z'' - z')||} \tag{3.31} \label{3.31}
\]

and

\[
|u_i(z)| = |u_i(\bar{z}) + \int_{\bar{z}}^{z} u_i(\xi) d\xi| \leq 2 \sup_{\Omega} |u| + |z - \bar{z}| \sup_{\Omega} |u| \tag{3.32} \label{3.32}
\]
where \(|z - \bar{z}| \leq c ||z^{-1} \circ z||\). Therefore, from this estimate and (3.29) it follows that
\[
|w_i(z)| \leq \frac{2 \sup_{\Omega} |w|}{||\overline{(z^{-1}) \circ z}||} + c ||z^{-1} \circ z|| \sup_{z \in \partial \Omega} |w_i(z)| \\
\leq \frac{2}{\xi} \sup_{\Omega} |w| + r \sup_{\Omega} |w_i|.
\]
(3.30)

Since \(d_i \geq \frac{1}{\xi} d_1\), we have by (3.30)
\[
d_i |w_i(z)| \leq \frac{2}{\xi} \sup_{\Omega} |w| + \frac{\mu}{4c_d} \sup_{\Omega} d_i^2 |w_i(\zeta)|
\]
provided \(0 < \mu \leq \frac{1}{\xi^2}\) (\(c\) is the positive constant in Proposition 2.1, (ii)). Choosing the smaller value \(\mu\) corresponding to the two cases \(\delta < \frac{1}{\xi}, \mu \leq 4c^2\), and taking the supremum over all \(z \in \Omega\) we obtain inequality (3.28).

If we prove that for every \(\epsilon > 0\) there is a positive constant \(c\) such that
\[
\sup_{z \in \Omega} d_i^2 |w_i(z)| \leq c \sup_{z \in \Omega} d_i |w_i(z)| + \epsilon \sup_{z \in \Omega} d_i^{p+1} \frac{|w_i(z) - w_i(\zeta)|}{||z^{-1} \circ z||^p},
\]
then, from combining (3.28) and (3.31), we arrive at inequality (3.25).

The proof of (3.31) proceeds in much the same way after replacing \(u_i\) with \(u_i;\) the details of proof are therefore omitted.

Finally, interpolation inequality (3.26) follows from combining (3.25) and (3.28).

Theorem 1.4 is thus completely proved.

**Remark 3.1.** If \((u_i)_{n}\) is a sequence of functions in \(C^{2n}_{\alpha}(\Omega; B)\) such that \(Lu_i = 0, u_i \to u\) and \((u_i)_{n}\) is locally bounded, then from the estimates in Theorem 1.4 and by using a diagonal argument \(u\) must belong to \(C^{2n}_{\alpha}(\Omega; B)\) and satisfy \(Lu = 0\) in \(\Omega\).

**4. The Dirichlet problem for strongly regular sets.** In this section we prove that the Dirichlet problem
\[
\begin{align*}
Lu &= f \text{ in } \Omega, \quad f \in C^2(\Omega; B) \\
u &= 0 \text{ on } \partial \Omega, \quad \phi \in C(\partial \Omega)
\end{align*}
\]
(4.1)
has a unique solution \(u \in C^{2n}_{\alpha}(\Omega; B) \cap C(\Omega)\), when \(\Omega\) belongs to a suitable class of open subsets of \(\mathbb{R}^{n+1}\). This result is only a preliminary step in order to apply the Perron method to arbitrary open bounded subsets of \(\mathbb{R}^{n+1}\).

**Definition 4.1.** We say that an open bounded subset \(\Omega\) of \(\mathbb{R}^{n+1}\) is strongly regular for \(L\), in short it is \(L\)-regular, if for every \(z_0 \in \partial \Omega\) there is \(v \in \mathbb{R}^{n+1}\) such that
\[
B_{n_0}(z_0 + v, |v|) \subset \mathbb{R}^{n+1} \setminus \Omega \quad \text{and} \quad \sum_{i,j=1}^{n} a_{ij}(z_0)v_iv_j > 0.
\]
here \(B_{n_0}(z_0 + v, |v|)\) indicates the Euclidean ball of center \(z_0 + v\) and radius \(|v|\).

We can now assert the following existence result.

**Proposition 4.2.** Let \(L\) be as in (1.1) satisfying hypothesis (H) and let \(f \in C^2(\Omega; B)\), \(\phi \in C(\partial \Omega)\). If \(\Omega\) is an \(L\)-regular open subset of \(\mathbb{R}^{n+1}\), then there exists a unique solution \(u\) of the boundary value problem (4.1) and \(u \in C^{2n}_{\alpha}(\Omega; B) \cap C(\Omega)\).

**Proof.** The proof of this proposition is split into three steps. We first prove the unique solvability of problem (4.1); next, using the continuity method, we prove the solvability of the homogeneous problem, \(\phi \equiv 0\); then we study the problem with \(f \equiv 0\). From these steps the proof follows immediately.

**Step 1.** The following weak maximum principle can be proved in a rather standard manner: if \(u\) is a continuous function in \(\Omega, Y_u, \delta_{\partial \Omega} u\) for \(i, j = 1, \ldots, q\) are continuous in \(\Omega\) and
\[
\begin{align*}
Lu &\geq 0 \quad \text{in } \Omega \\
u &\leq 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
then \(u \leq 0\) in \(\Omega\). Furthermore, if \(Lu = f\) in \(\Omega\) and \(u = 0\) on \(\partial \Omega\) then there exists a positive constant \(c\), independent of \(u\), such that
\[
\sup_{\Omega} |u| \leq c \sup_{\Omega} |f|.
\]

Then, the uniqueness in problem (4.1) follows straightforwardly.

**Step 2.** We consider problem (4.1) with homogeneous boundary conditions. Let \(z_0 \in \Omega, L_0 = L_0, \text{and for } \lambda \in [0, 1] \text{ let us now define the operator } L_\lambda \text{ by}
\[
L_\lambda = \lambda L + (1 - \lambda) L_0.
\]
From now on, we shall indicate by \((P_{\lambda, f})\) the Dirichlet problem
\[
\begin{align*}
L_\lambda u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]
(4.3)
and by \(\Lambda\) the set
\[
\Lambda = \{ \lambda \in [0, 1] : \text{the problem } (P_{\lambda, f}) \text{ has a solution} \} \subset C^1(\overline{\Omega}) \text{ for every } f \in C^2(\Omega; B).
\]

We shall prove that
(i) \(\Lambda\) contains \(\lambda = 0\);
(ii) \(\Lambda\) is an open set in the interval \([0, 1]\);
(iii) \(\Lambda\) is a closed set.
It will follow that \(\Lambda = [0, 1]\), and, for \(\lambda = 1\), we will get Step 2.
(i) Let \(f \in C^2(\Omega; B)\) and let \(\Gamma^0\) be the fundamental solution of the operator \(L_0\).
If we set
\[
\psi(z) = - \int_{\Omega} \Gamma^0(z^{-1} \circ z) f(\zeta) d\zeta,
\]
then \( v \in C_{\eta}^{4,\alpha}(-\Omega; B) \) and \( L_0 u = f \) in \( \Omega \) (see the proof of Theorem 1.4).

In order to prove that \( v \) belongs to \( C(\bar{\Omega}) \), let \( \eta \) be a bounded function in \( C^2(\mathbb{R}) \) such that \( \eta(r) = 0 \) if \( |r| \geq 1 \) and \( \eta(r) = 1 \) if \( |r| \leq \frac{1}{2} \). We call \( \eta_\varepsilon(z; \xi) = 1 - \eta(|(D(D^{-1}))(\varepsilon^{-1} o z)|) \) and

\[
\eta_\varepsilon(z) = -\int_{\Omega} \eta_\varepsilon(z; \xi) \xi^0(|\xi^0 o z|) f(\xi) d\xi.
\]

Now, \( u_\varepsilon \in C(\bar{\Omega}) \) and from the translation and dilation invariance of \( \xi^0 \) we get

\[
|u_\varepsilon(z) - v(z)| \leq c \sup_{\Omega} |f| \int_{|\xi^0 o z| \geq 1} |\xi^0(|\xi^0 o z|) d\xi
\]

\[
= c^2 \sup_{\Omega} |f| \int_{|\xi^0| \geq 1} |\xi^0(\xi)| d\xi,
\]

so that \( u_\varepsilon \) uniformly converges to \( v \) in \( \bar{\Omega} \).

Finally, from Theorem 5.2 in [1], there exists a unique solution \( w \in C^\infty(\Omega) \cap C(\bar{\Omega}) \) of the Dirichlet problem

\[
\begin{aligned}
L_0 w &= 0 \quad \text{in} \quad \Omega \\
w &= -v \quad \text{on} \quad \partial \Omega.
\end{aligned}
\]

By virtue of Theorem 1.4 we have \( w \in C_{\eta}^{4,\alpha}(\Omega; B) \); then \( w = w + v \in C_{\eta}^{4,\alpha}(\Omega; B) \) \( \cap C(\bar{\Omega}) \) and it is the solution of \( (P_{\lambda}; f) \).

(ii) Let \( \lambda \in \Lambda \) and let \( f \in C_{\lambda}^{0,\alpha}(\Omega; B) \). We write \( L_\lambda u = f \) in the equivalent form

\[
L_\lambda u = (L_{\lambda_0} - L_{\lambda}) u + f = F(u).
\]

We consider the linear transformation \( A \),

\[
A : C_{\lambda}^{0,\alpha}(\Omega; B) \cap C(\bar{\Omega}) \rightarrow C_{\lambda}^{4,\alpha}(\Omega; B) \cap C(\bar{\Omega}),
\]

defined as follows: for every \( u \), \( Au \) is the (unique) solution of \( (P_{\lambda_0}; F(u)) \).

From Theorem 1.4 and from Step 1 we obtain

\[
|Au|_{2,0,\alpha,\Omega} \leq c \sup_{\Omega} |f| + |d^2 F|_{2,0,\alpha,\Omega} \leq c (K |\lambda - \lambda_0| |w|_{2,0,\alpha,\Omega} + |f|_{2,0,\alpha,\Omega}),
\]

where \( K \) is a positive constant independent of \( \lambda \). Then, if

\[
|w|_{2,0,\alpha,\Omega} \leq 2 c |f|_{2,0,\alpha,\Omega} \quad \text{and} \quad |\lambda - \lambda_0| \leq \frac{1}{2 c K},
\]

we have

\[
|Au|_{2,0,\alpha,\Omega} \leq 2 c |f|_{2,0,\alpha,\Omega}.
\]

If \( \lambda \) satisfies (4.3) then \( A \) is a contraction in the closed set

\[
X = \{ u \in C_{\lambda}^{0,\alpha}(\Omega; B) \cap C(\bar{\Omega}) : |u|_{2,0,\alpha,\Omega} \leq 2 c |f|_{2,0,\alpha,\Omega}, \ u = 0 \text{ on } \partial \Omega \}.
\]

Hence \( A \) has a unique fixed point \( u \in X \) which is the solution of \( (P_{\lambda}; f) \). In other words, the set \( \{ \lambda \in [0, 1] : |\lambda - \lambda_0| \leq \frac{1}{2 c K} \} \) is contained in \( \Lambda \).

(iii) Let \( (\lambda_n) \subset \Lambda \) be such that \( \lambda_n \to \sigma \) as \( n \to \infty \). If \( f \in C_{\lambda}^{0,\alpha}(\Omega; B) \) and \( u_n \) is the solution of \( (P_{\lambda_n}; f) \), by using Theorem 1.4, Step 1 and standard arguments we can prove that \( u_n \) converges to a function \( u \in C_{\lambda}^{0,\alpha}(\Omega; B) \) such that \( L_\lambda u = f \) in \( \Omega \) and \( |u|_{2,0,\alpha,\Omega} \leq c |f|_{2,0,\alpha,\Omega} \).

In order to prove that \( u = 0 \) on \( \partial \Omega \) we fix \( z_0 \in \partial \Omega \) and set

\[
w_{z_0}(z) = \exp(-Mr^2) - \exp(-M|z - z_0 - rz|^2)
\]

where \( r > 0 \) is chosen in such a way that \( B_{\Omega}(z_0 + rz, r) \cap \Omega = \emptyset \). By construction, \( w_{z_0}(z_0) = 0 \) and if \( \sigma \) is sufficiently large, \( w_{z_0} \) is a local barrier in \( z_0 \) for every \( L_\lambda \); indeed, there is a neighborhood \( V \) of \( z_0 \) such that

\[
w_{z_0} > 0 \text{ in } \Omega \cap V \setminus \{ z_0 \}, \quad w_{z_0}(z_0) = 0 \quad \text{and} \quad L_\lambda w_{z_0}(z) \leq -1 \quad \forall \zeta \in \Omega \cap V
\]

for every \( \lambda \in [0, 1] \). Moreover, from Step 1, it follows that

\[
|u(z)| \leq \sup_{\overline{\Omega}} |w| \quad \text{as } z \to z_0.
\]

This proves that \( \sigma \in \Lambda \), and Step 2 follows.

Step 3 follows.

We study problem (4.1) with \( f = 0 \). Let \( (\phi_n)_{n \in \mathbb{N}} \subset C^\infty(\Omega) \cap C(\bar{\Omega}) \) be uniformly convergent to \( \phi \) in \( \lambda \), and let \( u_n \in C_{\lambda}^{0,\alpha}(\Omega; B) \cap C(\bar{\Omega}) \) be solution of

\[
\begin{aligned}
L_{\lambda_0} u_n &= 0 \quad \text{in} \quad \Omega \\
u_n &= \phi_n \quad \text{on} \quad \partial \Omega.
\end{aligned}
\]

(the existence of \( u_n \) is ensured by Step 2). If \( v_n = u_n - \phi_n \), then

\[
\begin{aligned}
L_\lambda v_n &= 0 \quad \text{in} \quad \Omega \\
v_n &= -\phi_n \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

and from Step 1

\[
\sup_{\overline{\Omega}} |v_n| \leq \sup_{\overline{\Omega}} |\phi_n - \phi_n| \quad \text{for all } m, n \in \mathbb{N}.
\]

From Remark 3.1 we deduce that \( v_n \) converges uniformly in \( \bar{\Omega} \) to a function \( v \) belonging to \( C_{\lambda}^{0,\alpha}(\Omega; B) \cap C(\bar{\Omega}) \) such that \( Lv = 0 \) in \( \Omega \) and \( v = \phi \) on \( \partial \Omega \). Proposition 4.2 is thus proved.
5. The Dirichlet problem for arbitrary bounded open sets: generalized solution. In this section we study the problem
\[
\begin{align*}
Lu &= f \quad \text{in } \Omega \\
u &= \phi \quad \text{on } \partial \Omega,
\end{align*}
\]
(5.1)
where \(\Omega\) is an arbitrary bounded open set, \(f \in C^0(\bar{\Omega}; B)\) and \(\phi \in C(\partial \Omega).\)

If \(f = 0\), a generalized Perron-Wiener solution can be determined by exploiting some results of potential theory on harmonic spaces. If \(f\) is not identically zero, we first need to define a function \(w\) of class \(C_{loc}^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega})\) such that
\[
Lu = f \quad \text{in } \Omega
\]
as follows: Let \(\Gamma(z; \xi)\) be the fundamental solution of the operator \(L\); then we set
\[
w(x, t) = -\int_0^t \int_{\mathbb{R}^n} \Gamma(x, t; y, s) \tilde{f}(y, s) \, dy \, ds,
\]
where \(\tilde{f} = f\) in \(\Omega\) and \(\tilde{f} = 0\) elsewhere.

We call the generalized solution of problem (5.1) the function \(u = v + w\), where \(v\) is the generalized Perron-Wiener solution of the problem \(Lv = 0\) in \(\Omega\), \(v = \phi - w\) on \(\partial \Omega\). We remark that
\[
u \in C_{loc}^{2+\alpha}(\Omega; B), \quad Lu = f \quad \text{in } \Omega,
\]
and \(u\) takes the boundary value \(\phi\) in the \(L\)-regular points of \(\partial \Omega\) (see below).

We define
\[
u = \inf \{ v : v \text{ is } L\text{-superharmonic in } \Omega \text{ and } \lim_{\xi \to z} v(\xi) \geq \phi(z), \forall z \in \partial \Omega \}.
\]
(5.2)
A continuous function \(v\) is \(L\)-superharmonic in \(\Omega\) if \(v \geq H_{vol}^\nu\) for every open \(L\)-regular subset \(V \subset \subset \Omega\), when \(H_{vol}^\nu\) is the unique solution \(C_{loc}^{2+\alpha}(V; B)\) of the problem
\[
\begin{align*}
Lu &= 0 \quad \text{in } V \\
w &= v \quad \text{on } \partial V.
\end{align*}
\]

Proposition 5.1. Let \(\Omega\) be an open bounded subset of \(\mathbb{R}^{n+1}\). If \(\phi\) belongs to \(C(\partial \Omega)\), then the function \(u\) defined in (5.2) belongs to \(C_{loc}^{2+\alpha}(\Omega; B)\), and it is the generalized Perron-Wiener solution of the problem
\[
\begin{align*}
Lu &= 0 \quad \text{in } \Omega \\
u &= \phi \quad \text{on } \partial \Omega.
\end{align*}
\]
Proof. Let \(X\) be an open bounded set in \(\mathbb{R}^{n+1}\). For every \(V \subset X\) we set
\[
\mathcal{H}^L(V) = \{ u \in C_{loc}^{2+\alpha}(V; B) : Lu = 0 \quad \text{in } V \}.
\]
The result follows by showing that \((X, \mathcal{H}^L)\) is a \(\beta\)-harmonic space, according to the classical definition (see for example the monograph of Constantinescu-Cornea, [3]).

The space \((X, \mathcal{H}^L)\) is a \(\beta\)-harmonic space if it satisfies the following properties:
(a) (Property of positivity). For every \(z \in X\) there exist an open neighborhood \(V\) of \(z\) and \(u \in \mathcal{H}^L(V)\) such that \(u(\xi) > 0\) for every \(\xi \in V\).
(b) (Bauer property of convergence). If \(\Omega\) is an open subset of \(X\) and \((u_n)_n\) is a monotone increasing sequence in \(\mathcal{H}^L(\Omega)\) such that \(u = \sup_n u_n\) is locally bounded, then \(u \in \mathcal{H}^L(\Omega)\).
(c) (Property of resolutivity). The Dirichlet problem \(Lu = 0\) in \(\Omega\), \(u = \phi\) in \(\partial \Omega\), is solvable for a basis of open subsets of \(\mathbb{R}^{n+1}\).

(d) (Property of separation). Let \(z_1, z_2 \in X\), \(z_1 \neq z_2\); then there exists a positive \(\mathcal{H}^L\)-superharmonic function \(v\) such that \(v(z_1) \neq v(z_2)\).

The first property is satisfied since constant functions are \(\mathcal{H}^L\)-harmonic. The convergence property holds thanks to Remark 3.1. The third property follows from Proposition 4.2, since the family of the \(L\)-regular sets, according to Definition 4.1, is a basis for the topology on \(\mathbb{R}^{n+1}\) (cf. [1, Corollario 5.2]).

In order to prove (d) we suppose that \(X \subset \mathbb{R}^n \times \{-T, T\}\) and we fix \(z_1 = (x_1, t_1)\), \(z_2 = (x_2, t_2) \in X\). If \(t_1 \neq t_2\), then we set \(v_1(x, t) = t + T\). If \(t_1 = t_2\) we choose \(y \in \mathbb{R}^n\) so that \((x_1, t_1, E^T(-t)y) \neq 0\) and \(M > 0\) so that
\[
v_2(z) = M - (x, E^T(-t)y) > 0, \quad \forall z \in X.
\]

In both cases \(v_1(z_1) \neq v_1(z_2)\), \(v_1\) is a \(\mathcal{H}^L\)-superharmonic function \((Lu_1 = -1)\), while \(v_2\) is a \(\mathcal{H}^L\)-harmonic function in \(X\), since
\[
Lv_2(x, t) = (x, B Dv_2(x)) = -\partial_t v_2(x) = -(x, B E^T(-t) y) + (x, B E^T(-t) y) = 0.
\]

We next prove Theorem 1.5.

Proof of Theorem 1.5. In order to prove the main theorem it is sufficient to get a particular solution \((Lu = f)\), continuous in \(\bar{\Omega}\) and of class \(C_{loc}^{2+\alpha}(\Omega; B)\). If \(\Gamma(z; \xi)\) is the fundamental solution of the operator \(L\) obtained from the parametrix method of Levi in [14], then for every continuous bounded function \(g : \mathbb{R}^n \to \mathbb{R}\), it holds:
\[
\lim_{t \to +1} \int_{\mathbb{R}^n} \Gamma(x, t; \xi, t) g(\xi) \, d\xi = g(x), \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}.
\]

We set
\[
w(x, t) = -\int_0^t \int_{\mathbb{R}^n} \Gamma(x, t; y, s) \tilde{f}(y, s) \, dy \, ds,
\]
(5.3)
where \(\tilde{f} = f\) in \(\Omega\) and \(\tilde{f} = 0\) elsewhere, and we shall prove that \(w \in C(\bar{\Omega}) \cap C_{loc}^{2+\alpha}(\Omega; B)\) and \(Lu = f\) in \(\Omega\).

\*\*We thank A. Montanari for a suggestion on the construction of the function \(v_2\).
Indeed, from (5.6) and from the dilation and translation invariance we have

\[ \int_{B(\epsilon, t)} |Z(z; \zeta)| \, d\zeta \leq c \int_{|z| = \epsilon} \Gamma^+(\zeta^{-1} \circ z) \, d\zeta \]

\[ = c \epsilon^{n+2} \int_{|\xi| \leq 1} \Gamma^+(D(\xi) z) \, d\xi = c \epsilon^{n} \int_{|\xi| \leq 1} \Gamma^+(\xi) \, d\xi \to 0 \quad \text{as} \quad \epsilon \to 0. \]

Let \( \eta \in C(\mathbb{R}^{N+1}) \) be such that \( \eta(z) = 1 \) if \( |z| \geq 1 \) and \( \eta(z) = 0 \) if \( |z| \leq \frac{1}{2} \). We call \( \eta(z) = \eta(D(\epsilon^{-1} \zeta) \xi) \) and set

\[ u_\epsilon(z) = - \int_{0}^{t} \int_{B(\epsilon, s)} \eta(\zeta^{-1} \circ z) Z(z; \zeta) \, d\zeta \, d\xi. \]

Then \( u_\epsilon \in C(\mathbb{R}^{N} \times [0, T]) \) and

\[ |u_\epsilon(z) - w_1(z)| \leq c \sup_{\Omega} |f| \int_{B(\epsilon, t)} |Z(z; \zeta)| \, d\zeta, \]

so that \( u_\epsilon \) uniformly converges to \( w_1 \) as \( \epsilon \to 0 \). We shall study the continuity of \( u_\epsilon \) as defined in (5.5). We rewrite \( u_\epsilon \) in the form

\[ u_\epsilon(x; t) = - \int_{0}^{t} \int_{\mathbb{R}^{N}} J(x, t; y, s) \tilde{f}(y, s) \, dy \, ds \]

\[ - \int_{0}^{t} \int_{\mathbb{R}^{N}} J(x, t; y, s) f(y, s) \, dy \, ds = u_1(x; t) + w_2(x; t). \]

We first prove that \( w_1 \) is continuous in \( \mathbb{R}^{N+1} \). From Proposition 2.4 in [14] we get that there are a symmetric positive constant \( q \times q \) matrix \( A^+_0 \) and a constant \( \tilde{c} > 0 \) such that

\[ Z(z; \zeta) \leq \tilde{c} \Gamma^+(z; \zeta), \quad \forall z, \zeta \in \mathbb{R}^{N+1}, \ z \neq \zeta \]

where \( \Gamma^+(z; \zeta) \) is the fundamental solution of the operator \( \tilde{L} = \text{div}(A^+ D) + Y \) and \( A^+ \) is the \( N \times N \) constant matrix

\[ A^+ = \begin{pmatrix} A^+_0 & 0 \\ 0 & 0 \end{pmatrix}. \]

We remark that

\[ \int_{B(\epsilon, t)} |Z(z; \zeta)| \, d\zeta \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{uniformly in} \quad \mathbb{R}^{N+1}. \]
and the continuity of the function \( \tilde{f} \) follows as before.

To complete the proof it remains to prove that \( \mathcal{L} = f \) in \( \Omega \). We shall show that

\[
L w_1(x, t) = \tilde{f}(x, t) - \int_0^t \int_{\mathbb{R}^n} L Z(x, t; y, s) \tilde{f}(y, s) dy ds \tag{5.8}
\]

and

\[
L w_2(x, t) = \tilde{f}(x, t) - \int_0^t \int_{\mathbb{R}^n} L Z(x, t; y, s) \tilde{f}(y, s) dy ds, \tag{5.9}
\]

therefore we immediately get

\[
L w(x, t) = L w_1(x, t) + L w_2(x, t)
\]

\[
= \int_0^t \int_{\mathbb{R}^n} L Z(x, t; y, s) \Phi(x, t; y, s; t) dy ds.
\]

for all \((x, t) \in \Omega \).

The proof of (5.8) is split into the proof of the following statements:

(i) \( \partial_\nu w_1 \) exists and is a continuous function for all \( l = 1, \ldots, q \). Moreover

\[
\partial_\nu w_1(z) = -\int_0^t \int_{\mathbb{R}^n} \partial_\nu Z(z; y, s) \tilde{f}(y, s) dy ds, \quad \forall z = (x, t) \in \mathbb{R}^{n+1}.
\]

(ii) \( \partial_\nu, \partial_\nu w_1 \) exists and is a continuous function for all \( l, j = 1, \ldots, q \). Moreover

\[
\partial_\nu \partial_\nu w_1(z) = -\int_0^t \int_{\mathbb{R}^n} \partial_\nu^2 Z(z; y, s) \tilde{f}(y, s) dy ds, \quad \forall z = (x, t) \in \mathbb{R}^{n+1}.
\]

(iii) \( Y w_1 \) exists and is a continuous function. Moreover for all \( z \in \mathbb{R}^{n+1} \)

\[
Y w_1(z) = \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^q a_j(y, s) \partial_\nu^2 Z(z; y, s) \tilde{f}(y, s) dy ds.
\]

We first prove that the integral in (i) converges. From Corollary 2.2 in [14] there exists a positive constant \( c \) such that

\[
|\partial_\nu Z(z; \xi)| \leq \frac{c}{\sqrt{|z - \xi|}}, \quad z, \xi \in \mathbb{R}^{n+1}, z \neq \xi. \tag{5.10}
\]

Let \( \eta \in C^2(\mathbb{R}) \) be bounded and such that \( \eta(r) = 0 \) if \( |r| \geq 1 \), \( \eta(r) = 1 \) if \( |r| \leq \frac{1}{2} \). We call \( \eta_z(z; \xi) = 1 - \eta(||D(\mathcal{L}^z) (\xi^{-1} \circ z)||) \) and set

\[
u_z(z) = -\int_0^t \int_{\mathbb{R}^n} \eta_z(z; \xi) Z(z; \xi) \tilde{f}(\xi) d\xi.
\]

Then \( \nu_z \in C(\mathbb{R}^n \times [0, T]) \) and \( \nu_z \) uniformly converges to \( u \) as \( \epsilon \to 0 \). We thus obtain

\[
\partial_\nu \nu_z(z) = -\int_0^t \int_{\mathbb{R}^n} \partial_\nu (Z \eta_z(z; \xi) \tilde{f}(\xi)) d\xi. \tag{5.11}
\]

Indeed, from (5.10), it follows that

\[
|\partial_\nu (Z \eta_z(z; \xi) \tilde{f}(\xi))| \leq c \sup_{z \in \mathbb{R}^n, t \leq T} \frac{1}{\sqrt{1 - |z - \xi|^2}} = c < \infty.
\]

(The proof of the last estimate is similar to the proof of (3.8) in [14]; the details of proof are therefore omitted.) Hence, the bound

\[
|\partial_\nu (Z \eta_z(z; \xi) \tilde{f}(\xi))| \leq c_1 \tilde{f}(\xi)
\]

holds uniformly with respect to \( z \), the identity (5.11) follows by Lebesgue’s theorem.

In order to prove that

\[
\partial_\nu \nu_z(z) \to -\int_0^t \int_{\mathbb{R}^n} \partial_\nu Z(z; y, s) \tilde{f}(y, s) dy ds \quad \text{as} \quad \epsilon \to 0,
\]

uniformly in \( \mathbb{R}^n \times [0, T] \), we write

\[
\partial_\nu \nu_z(z) - \int_0^t \int_{\mathbb{R}^n} \partial_\nu Z(z; \xi) \tilde{f}(\xi) d\xi
\]

\[
= \int_0^t \int_{\mathbb{R}^n} \partial_\nu Z(z; \xi) (\eta_z(z; \xi) - 1) \tilde{f}(\xi) d\xi
\]

\[
+ \int_0^t \int_{\mathbb{R}^n} Z(z; \xi) \partial_\nu Z(z; \xi) \tilde{f}(\xi) d\xi = I_1(z) + I_2(z),
\]

and we estimate separately \( I_1(z) \) and \( I_2(z) \). From (5.10) it is readily seen that

\[
|I_1(z)| \leq \int_{[0, T] \times \mathbb{R}^n} \frac{c}{\sqrt{1 - s}} \tilde{f}(z; \xi) \tilde{f}(\xi) d\xi d\xi \leq c \sup_{t \leq T} \int_{\mathbb{R}^n} \frac{\tilde{f}(0; \xi)}{\sqrt{1 - \xi^2}} d\xi,
\]

hence

\[
|I_1(z)| \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{5.13}
\]

uniformly in \( \mathbb{R}^n \times [0, T] \).

In order to evaluate \( I_2(z) \) we first note that

\[
\partial_\nu \eta_z(z; \xi) = \frac{1}{\epsilon} \eta'(||D(\mathcal{L}^z)(\xi^{-1} \circ z)||),
\]

and

\[
|\partial_\nu \eta_z(z; \xi)| \leq \frac{1}{\epsilon} \sup_{z \in \mathbb{R}^n} \eta'(z) = \frac{c}{\epsilon} \quad \forall z \in \mathbb{R}^{n+1}, \tag{5.14}
\]

\[
\eta_z(z; \xi) \geq 1 - \eta(||D(\mathcal{L}^z)(\xi^{-1} \circ z)||) \geq 1 - \frac{c}{\epsilon}
\]
besides \( \delta_{\eta, \nu}(z; \xi) = 0 \) for all \( \xi \in \mathbb{R}^{N+1} \setminus B(z, \epsilon) \).  

Using (5.14) and (5.6), we thus have

\[
|I'_2(\xi)| \leq \frac{c}{\epsilon} \int_{B(0, \epsilon)} \Gamma^+(z; \xi) \, d\xi
\]

\[
= c \epsilon \int_{B(0, \epsilon)} \Gamma^+(0; \xi) \, d\xi \to 0 \quad \text{as} \quad \epsilon \to 0. \quad (5.15)
\]

The proof of (i) is obtained from combining (5.12), (5.13) and (5.15).

Note that the integral in (ii) exists as a repeated integral. For every fixed \( x \in (0, t) \) and for every \( y' \in \mathbb{R}^{N} \) it turns out

\[
\int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(z; y, s) \, dy = \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(z; y, s) \left( f(y, s) - f(y', s) \right) \, dy
\]

\[
+ \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 (Z(y, s) - Z(y', s)) \, dy
\]

\[
+ \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(y', s) \, dy
\]

\[
= I_1(z, s) + I_2(z, s) + I_3(z, s), \quad (5.16)
\]

where \( Z(z; \xi) \) is the fundamental solution of the operator \( L_z \) with pole at \( \xi \). We next give some estimates for \( I_1, I_2 \) and \( I_3 \). The condition \( f \in C^0(\Omega, B) \) implies that there also exists a constant \( c > 0 \) such that

\[
|f(y, s) - f(y', s)| \leq c \|y - y'\|^{\alpha} \quad \forall \ (y, s), (y', s) \in \Omega
\]

(|| \cdot ||^{\alpha} \) is introduced in Definition 1.1. Moreover, from Corollary 2.1 in [14], it follows that

\[
\partial_{\eta, \nu}^2 Z(z; \xi) \leq \frac{c}{t - s} \left( 1 + |D_\eta(\sqrt{t - s}(x - E(t - s)\xi))|^{\alpha} \right) \Gamma^+(z; \xi)
\]

for every \( z, \xi \in \mathbb{R}^{N+1}. \) Therefore

\[
|I_1(z, s)| \leq c \int_{\mathbb{R}^{N}} \Gamma^+(z; y, s) \left( 1 + |D_\eta(\sqrt{t - s}(x - E(t - s)y))|^{\alpha} \right) \|y - y'\|^{\alpha} \, dy,
\]

and choosing \( y' = E(t - s)x \) we obtain

\[
\|y - y'\|^{\alpha} = \|y - E(t - s)x\|^{\alpha} = (3.2)
\]

\[
= \frac{1}{(t - s)^{\alpha/2}} \|D_\eta(t - s)^{1/2}(y - E(t - s)x)\|^{\alpha},
\]

and

\[
|I_1(z, s)| \leq \frac{c}{(t - s)^{1-\alpha/2}}. \quad (5.17)
\]

We evaluate \( I_2 \) in (5.16). From Lemma 3.2 in [14], it results

\[
\left| \partial_{\eta, \nu}^2 Z(z; w) - \partial_{\eta, \nu}^2 Z(z; w) \right| \leq \frac{c}{(t - s)^{1-\alpha/2}} \|\eta\|^{\alpha} \Gamma(z; w), \quad \forall \xi, \eta, \zeta \in \mathbb{R}^{N+1}, \quad (5.18)
\]

for some positive constant \( c \). Arguing as above and using (5.18), we obtain

\[
|I_2(z, s)| \leq \frac{c}{(t - s)^{1-\alpha/2}} \left[ \int_{\mathbb{R}^{N}} \Gamma(z; y, t) \|y - y'\|^{\alpha} \, dy \right] \int_{\mathbb{R}^{N}} \Gamma(z; y, t) \|y - y'\|^{\alpha} \, dy \leq \frac{c}{(t - s)^{1-\alpha/2}} \quad (5.19)
\]

Noting that

\[
\int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} Z(z; y, s) \, dy = 1 \quad \forall \ z \in \mathbb{R}^{N+1}, \quad (5.20)
\]

by Lebesgue's theorem \( (s < t) \), we derive

\[
0 = \partial_{\eta, \nu}^2 \int_{\mathbb{R}^{N}} Z(z; y, s) \, dy = \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(z; y, s) \, dy, \quad (5.20)
\]

then \( I_3(z, s) = 0 \). Finally, the existence of the integral in (ii) follows from (5.17), (5.18) and (5.20).

Arguing as in the proof of (i), it can be shown that

\[
\partial_{\eta, \nu} v(\xi) = - \int_{0}^{t} \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 (Z(\eta_\nu(z; \xi) \cdot f(\xi) \, d\xi, \quad (5.21)
\]

and

\[
\partial_{\eta, \nu} v(\xi) = - \int_{0}^{t} \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(\eta_\nu(z; \xi) \cdot f(\xi) \, d\xi \quad \text{as} \quad \epsilon \to 0, \quad (5.22)
\]

uniformly in \( \mathbb{R}^{N} \times [0, T] \).

In order to prove (5.22), we write, for a fixed \( y' \in \mathbb{R}^{N} \),

\[
\partial_{\eta, \nu}^2 Z(\xi) = - \int_{0}^{t} \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(\xi) \cdot f(\xi) \, d\xi
\]

\[
= \int_{0}^{t} \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 (\eta_\nu(z; \xi) - 1) Z(z; y, s) \cdot f(\eta_\nu(z; \xi) - 1) \, dy \, ds
\]

\[
+ \int_{0}^{t} \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 Z(\eta_\nu(z; \xi) - 1) Z(z; y, s) \cdot f(\eta_\nu(z; \xi) - 1) \, dy \, ds
\]

\[
+ \int_{0}^{t} \int_{\mathbb{R}^{N}} \partial_{\eta, \nu}^2 (\eta_\nu(z; \xi) - 1) Z(z; y, s) \cdot f(\eta_\nu(z; \xi) - 1) \, dy \, ds
\]

\[
= I_1'(z, s) + I_2'(z, s) + I_3'(z, s).
\]

We note that \( I_1' \) can be obtained by replacing \( Z \) with \( (\eta_\nu(z; \xi) - 1) Z \) in (5.16) for all \( f = 1, 2, 3 \). Estimates analogous to (5.17) and (5.19) then guarantee that (5.21) and (5.22) hold.
To prove (iii) we can proceed as in the proof of Proposition 3.3 in [14] and with the same arguments as in (c).

Finally, from (i), (ii) and (iii) it follows that

$$L u = - \int_0^t \int_{\Omega \times \Omega} \sum_{i=1}^n \left( a_{ij}(x, t) - a_{ij}(y, s) \right) \frac{\partial^2 u}{\partial x_i \partial x_j} (x, t; y, s) f(y, s) \, dy \, ds,$$

Since $Z(z; \xi)$ is the fundamental solution of $L_{\tau}$, we obtain (5.8).

The identity (5.9) follows as (5.8), observing that $\hat{f}$ satisfies the Hölder estimate

$$|\hat{f}(y, s) - \hat{f}(y', s)| \leq \epsilon |y - y'|^{\gamma}, \quad \forall y, y' \in \mathbb{R}^n, s \in [0, T]$$

(5.23)

for some constant $\epsilon \in (0, 1)$.

Inequality (5.23) will then be a consequence of the following statement (see Lemma 3.1 in [14]). There exist three constants $C > 0$, $\gamma$ and $\epsilon' \in (0, 1)$ such that

$$|\Phi(x, t; \xi) - \Phi(x', t; \xi)| \leq C |x - x'|^{\gamma} (\log(1 + |x|) + \log(1 + |x'|))$$

for every $\xi = (\xi, \tau) \in \mathbb{R}^{n+1}, t, s \in [0, T], t > s$ and for every $x, x' \in \mathbb{R}^n$.

Thus, from the definition of $\hat{f}$ in (5.7), we get

$$|\hat{f}(y, s) - \hat{f}(y', s)| \leq \epsilon |y - y'|^{\gamma} (\log(1 + |y|) + \log(1 + |y'|))$$

(5.24)

and

$$\int_0^s \int_{\mathbb{R}^n} \frac{1}{|s - \tau|^{n+1}} \hat{f}(y, s; \xi, \tau) \, dy \, d\tau = \int_0^s \int_{\mathbb{R}^n} \frac{1}{|s - \tau|^{n+1}} \hat{f}(y, s; \xi, \tau) \, dy \, d\tau < \infty.$$

An analogous scheme can be adopted for the second integral in (5.24). Theorem 1.5 is thus completely proved.

6. Some geometric criteria for the regularity of the boundary points. We consider the Dirichlet problem

$$\begin{cases} L u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial \Omega, \end{cases}$$

where $\Omega$ is an arbitrary bounded open subset of $\mathbb{R}^{n+1}$, and we indicate by $H^1_\phi$ its generalized solution.

We say that a point $z_0 \in \partial \Omega$ is regular with respect to the operator $L$ for $\Omega$ (in short: $L$-regular for $\Omega$), if

$$\lim_{z \to z_0} H^1_\phi(z) = \phi(z_0) \quad \text{for all } \phi \in C(\partial \Omega).$$

In the following Proposition 6.1 we prove that the boundary points of $\Omega$ for which there exists a noncharacteristic exterior normal at $\partial \Omega$ are $L$-regular and we give a geometric condition which assures the regularity for the characteristic boundary points, when the Fichera function is positive definite.

Proposition 6.1 requires some preliminary notations. For every $z \in \mathbb{R}^{n+1}$ we write $z = (z', z) \in \mathbb{R}^n \times \mathbb{R}^{n+1} - \{0\}$ and we set

$$d_z(z') = \frac{|z|^2}{z} + \frac{|z'|^2}{z},$$

where $| \cdot |$ indicates the Euclidean norm in $\mathbb{R}^n$ or in $\mathbb{R}^{n+1} - \{0\}$ and $\epsilon$ is a positive constant. Besides, we set

$$E(z, \epsilon) = \{ z' \in \mathbb{R}^{n+1} : d_z(z') \leq \epsilon \}.$$
is a barrier for $\Omega$ at $z_0$, provided $M$ is a sufficiently large positive constant, so that $z_0$ is an $L$-regular point.

In the second case, the function

$$v_{z_0}(z) = \exp(-e^3) - \exp(-e^3 d_3(z - z_0)), \quad z \in \Omega$$

is a barrier for $\Omega$ at $z_0$. Direct calculation shows that $v_{z_0}(z_0) = 0$, $v_{z_0}(z) > 0$ for every $z \in \Omega \setminus \{z_0\}$ and

$$L v_{z_0}(z_0) = \frac{2}{\xi} \exp(-e^3)(\xi^2 \text{trace} A_0(z_0) - \langle x_0, Bv_0(z_0) \rangle + v_{N+1}(z_0)) < 0.$$

The following theorem furnishes a sufficient geometric criterion which extends the classical Zaremba criterion of exterior cone to our setting.

We now introduce the definition of $L$-cone.

**Definition 6.2.** We define the $L$-cone of vertex $(0, 0)$, base $K$ and height $T$, the subset of $\mathbb{R}^{n+1}$

$$C = \{ D(r)(x, -T) : x \in K, \ 0 \leq r \leq 1 \},$$

where $T > 0$ and $K$ is a closed bounded set of $\mathbb{R}^n$ with finite and positive Lebesgue measure.

If $z_0 \in \mathbb{R}^{n+1}$ we define the $L$-cone of vertex $z_0$ to be the set

$$C_{z_0} = \{ z \circ z_0 : z \in C \}.$$

**Theorem 6.3.** Let $\Omega$ be an open set of $\mathbb{R}^{n+1}$ and let $z_0 \in \partial \Omega$. If there exists an $L$-cone $C_{z_0}$ such that $C_{z_0} \subset \mathbb{R}^{n+1} \setminus \Omega$, then $z_0$ is an $L$-regular point.

The proof of Theorem 6.3 requires some preliminary remarks.

If $V$ is an $L$-regular open set according to Definition 4.1 and $F$ is a compact subset of $V$, then, for all $\xi \in F$, we set $G(z, \xi) = \Gamma(z, \xi) - h_1(z)$, where $\Gamma(z, \xi)$ is the fundamental solution of $L$ with pole at $z$ and $h_1$ is the solution of the problem

$$Lu = 0 \quad \text{in} \quad V$$

$$u = \Gamma(\cdot; \xi) \quad \text{on} \quad \partial V.$$ 

We set

$$W_F(z) = \inf\{ u(\xi) : u \text{ is $L$-superharmonic in } V, 0 \leq u \leq 1, \ u \equiv 1 \text{ in } F \}$$

and we indicate by $V_F^L$ its semicontinuous regularization

$$V_F^L(z) = \liminf_{\xi \to z} W_F^L(\xi), \quad \forall z \in V.$$ 

Then (see [12, pages 97–98]) there exists a positive measure $\mu_F$ such that

$$V_F^L(z) = \int_V G(z; \xi) \, d\mu_F(\xi),$$

and ([12, Proposition 15])

$$\mu_F(F) = \sup \{ \mu(F) : \mu \in \mathcal{M}^+(F), \ \int_V G(z; \xi) \, d\mu(\xi) \leq 1, \ \forall z \in V \}.$$ 

We call $\mu_F$ the $L$-equilibrium measure of $F$ and we call $L$-capacity of $F$ in $V$ the total mass of $\mu_F$; i.e., $\text{cap}_{L}(F, V) = \mu_F(F)$.

**Lemma 6.4.** Let $V$ be an open $L$-regular set and $z_0 \in V$. Then

$$\lim_{r \to 0} \text{cap}_L(B_r(z_0), V) = 0.$$

In particular $\text{cap}_L(z_0, V) = 0$.

**Proof.** By the local estimate on $\Gamma$ ([14, Theorem 2.1]) we infer that

$$\limsup_{r \to 0} \Gamma(z; z_0) = +\infty.$$

Moreover, from the maximum principle (see the proof of Proposition 4.2), we obtain

$$\Gamma(z; \xi) - \gamma \leq G(z; \xi) \leq \Gamma(z; \xi), \quad \forall z \in V \text{ and all } \xi \in F.$$ 

(6.1)

where $\gamma = \sup \{ \Gamma(z; \xi) : z \in \partial V, \xi \in F \}$. Then, for every $k > 0$ there exists $z \in V$ such that $G(z; z_0) > k$. Since $G$ is a smooth function in the complement of the diagonal of $V \times V$, there exists $\varepsilon > 0$ such that $G(z; \xi) > k$ for all $\xi \in B_r(z_0)$ and all $r \leq \gamma$. Thus, if $\mu$ denotes the $L$-equilibrium measure of $B_r(z_0)$, we get

$$0 \leq \int_{B_r(z_0)} G(z; \xi) \, d\mu(\xi) \leq k \mu(\overline{B_r(z_0)}) = k \text{cap}_L(\overline{B_r(z_0)}, V)$$

for every $r \leq \gamma$. Consequently $\text{cap}_L(\overline{B_r(z_0)}, V)$ tends to zero as $r \to 0$ as required.

**Proof of Theorem 6.3.** Let $V$ be an open $L$-regular set such that $z_0 = (x_0, t_0) \in V$. For sufficiently small $r > 0$ the set

$$V_r(z_0) = \{ z \in \mathbb{R}^{n+1} \setminus \Omega : ||z - z_0|| \leq r \}$$

is contained in $V$; then (see [12, Theorem 14]) it is enough to prove that

$$\liminf_{r \to 0} V_r^{L_z}(z_0) > 0.$$ 

(6.2)

Since there exists an $L$-cone $C_{z_0}$ of vertex $z_0$, base $K$ and height $T$, such that $C_{z_0} \subset \mathbb{R}^{n+1} \setminus \Omega$, we have for small $r$

$$V_r(z_0) \supset (D_0(r)K + E(-r^2T)x_0) \times \{ t_0 - r^2T \} = K_r(x_0) \times \{ t_0 - r^2T \}.$$ 

If $\nu_r = m_{\nu_0}(x_0) \otimes \delta_{t_0 - r^2T}$, then from (6.1), for every $z \in V$ it holds

$$\int_V G(z; \xi) \, d\nu_r(\xi) \leq \int_{K_r(x_0)} \Gamma(\xi, r; z_0, t_0 - r^2T) \, d\xi \leq 1;$$

then

$$\int_V G(z_0; \xi) \, d\nu_r(\xi) \leq V_r^{L_z}(z_0).$$ 

(6.3)
Moreover, from Theorem 2.1 in [14], there exists a constant $C > 0$ such that
\[
\Gamma(z_0; \xi) \geq \frac{1}{2} \Gamma^-(z_0; \xi),
\]
provided $\Gamma^-(z_0; \xi) \geq C$. $\Gamma^-$ is the fundamental solution of the operator $L^- = \text{div}(A^- D) + Y$, where $A^-$ is the $N \times N$ matrix
\[
A^- = \begin{pmatrix}
\mu I_n & 0 \\
0 & 0
\end{pmatrix},
\]
and $\mu$ is the positive constant in hypothesis (H) of Section 1.

It is easy to prove, by using $D(\lambda)$-homogeneity of $\Gamma^-$ and identity (3.2), that
\[
V_r(z_0) < \{ \xi \in \mathbb{R}^{N+1} : \Gamma^-(z_0; \xi) \geq C\},
\]
for $r$ small enough. Hence, from (6.1) and (6.3) we obtain
\[
V_r(z_0) \geq c T^{-\alpha/2} \int K_r(z_0; \xi) \Gamma^-(z_0; \xi, t_0 - r^2 T) \, d\xi - \gamma r^Q m_N(K),
\]
(6.4)

Since $\Gamma^-(z_0; \xi) = \Gamma^- (\xi^{-1} \circ z_0)$ we have
\[
V_r(z_0) \geq c r^Q \int K_r(z_0; \xi) \Gamma^-(\xi; t_0 - r^2 T) \, d\xi - \gamma r^Q m_N(K)
\]
(6.5)
The last equality in (6.5) follows from (3.2), since
\[
\Gamma^-(E(r^2 T) D_0(r)x, r^2 T) = \Gamma^-(D_0(r\sqrt{T}) E(1) D_0(r)x, r^2 T)
\]
(5.1) is $D(\lambda)$-homogeneous of degree $-Q$.
\[
= r^{-Q} T^{-\alpha/2} \Gamma^-(E(1) D_0(T^{-1/2}) x, 1) - \gamma r^Q m_N(K).
\]

Finally, from (6.5) we infer that
\[
\liminf_{r \to 0} V_r(z_0) \geq c T^{-\alpha/2} \int K \Gamma^- (E(1) D_0(T^{-1/2}) x, 1) \, dx > 0
\]

since $c$ is a constant independent of $r$, therefore inequality (6.2) holds. This completes the proof of Theorem 6.3.

To conclude this section, we propose a simple example.

Example 6.5. Let $L$ be the Kolmogorov operator in $\mathbb{R}^3$, that is,
\[
L = \partial_x^2 + x \partial_y - \partial_z, \quad (x, y, z) \in \mathbb{R}^3
\]
(see (1.3)). Then, with the notation of Section 1,
\[
B = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad E(t) = \begin{pmatrix}
1 & 0 \\
-t & 1
\end{pmatrix}
\]
and $D(\lambda) = (\lambda, \lambda^2, \lambda^3)$ for every $\lambda > 0$.

Let $\Omega$ be the open set $\{(u, v, t) : |u|, |v|, |t| < 1\}$. In this case the points of the set $\{(1, 1, 1), (-1, 1, 1) : |u|, |v|, |t| < 1\}$ are $L$-regular since they are not characteristic boundary points; the points in the sets
\[
\{(x, y, -1) : |u|, |v|, |t| < 1\}, \quad \{(x, 1, t) : x > 0, |t| < 1\}, \quad \{(x, -1, t) : x < 0, |t| < 1\}
\]
are $L$-regular since they satisfy the condition of Proposition 6.1. Furthermore, the points on the line
\[
A = \{(0, 1, t), (0, -1, t) : |t| < 1\}
\]
are regular points, since for every $z_0 \in A$ the exterior $L$-cone condition is satisfied (see Theorem 6.3). Indeed, if $z_0 = (0, 1, t_0) \in A$ and $K$ is a compact subset of $\{(x, y) \in \mathbb{R}^2 : y > 1\}$, the set
\[
C_{z_0} = \{(rx, r^2 y + 1, -r^2 T + t_0) : (x, y) \in K, 0 \leq r \leq 1\}
\]
is an exterior $L$-cone of vertex $z_0$ for the set $\Omega$.

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