SOME RESULTS ON THE DIRAC-EINSTEIN EQUATIONS
ALCUNI RISULTATI SULLE EQUAZIONI DI DIRAC-EINSTEIN

MARTINO VITTORIO

ABSTRACT. In this note we introduce the Dirac-Einstein equations on a spin manifold and we review some recent results, in particular: the compactness of the variational solutions, the classification of the Palais-Smale sequences for the related conformal problem, and finally some existence results.

SUNTO. In questa nota introduciamo le equazioni di Dirac-Einstein su una varietà spin ed illustriamo alcuni recenti risultati, in particolare: la compattezza delle soluzioni variazionali, la classificazione delle successioni di Palais-Smale per il relativo problema conforme, ed infine alcuni risultati di esistenza.

2020 MSC. Primary 53C21, 53C27; Secondary 58E30, 58J05.

KEYWORDS. Spin manifolds, conformal geometry.

1. Introduction and motivation

In high energy Physics the interaction between particles is a wide subject of investigation; in particular the corresponding equations are in general hard to handle and this is due to the different nature of the particles: on one side Bosons, which are force-carrying particles ruled by field type equations, on the other side Fermions, which are matter particles satisfying wave type equations.

Here we will consider the easiest case, namely one single Bosonic-Fermionic interaction, as introduced and described in [8], where the authors consider the interaction between neutrinos (Fermionic part) and the gravitational field (Bosonic part).

Moreover we will adopt the variational approach, as in [13] and we restrict ourselves to the Riemannian setting; let us mention that the same authors treated in [14] also the

Bruno Pini Mathematical Analysis Seminar, Vol. (2023) pp.

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

more general case with the additional coupling of the electromagnetical field, giving rise to the Einstein-Dirac-Maxwell equations.

Let us first recall the well known Einstein and Dirac equations separately. So, let M be a closed (compact, without boundary) manifold of dimension $n \geq 3$, and let us denote by $\mathcal{G}(M)$ the set of all Riemannian metrics on M; therefore the Einstein field equations of gravity read as

$$Ric_g - \frac{R_g}{2} g = T_g, \quad g \in \mathcal{G}(M) \tag{1}$$

where Ric_g is the Ricci tensor, R_g the Scalar curvature and T_g the stress–energy tensor (depending on the particular choice of the matter sources).

On the other side, if we consider M endowed with a fixed Riemannian metric g and a Spin structure $\Sigma_g M$, then the Dirac wave equations are

$$D_g \psi = \lambda \psi, \quad \psi \in \Sigma_g M \tag{2}$$

where D_g is the Dirac operator acting on spinors ψ; here $\lambda > 0$ represents the mass.

Both the previous equations (1), (2) are variational, in the sense that they are critical points equations of two distinct functional. The idea for the coupling is to consider a unique functional, for which the first variation gives rise to a system with the two equations. So, let $n \geq 3$, for $g \in \mathcal{G}(M)$ and $\psi \in \Sigma_g M$, the Dirac-Einstein functional can be written in the following way

$$\mathcal{E}(g, \psi) = \int_M R_g + \langle D_g \psi, \psi \rangle - \lambda |\psi|^2 \, dv_g \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the canonic Hermitian metric defined on $\Sigma_g M$. Therefore, critical points of $\mathcal{E}$ solve the coupled Dirac-Einstein equations

$$\begin{cases} Ric_g - \dfrac{R_g}{2} g = T_{g,\psi} \\ \\ D_g \psi = \lambda \psi \end{cases} \tag{4}$$

with

$$T_{g,\psi}(X, Y) = -\frac{1}{4} \langle X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi \rangle, \quad X, Y \in TM$$

where $\cdot$ and ∇ denote the Clifford multiplication and the metric connection on spinors respectively; we address the reader to [12], [15] and the references therein, for all the additional details regarding the Spin geometry and the analysis of the Dirac operator.

Let us explicitly remark that the system (4) is not obtained merely considering (1) and (2) at the same time, but the two equations are strongly coupled: this can be seen in the first equation in (4), where the presence of the spinor appears in the stress–energy tensor; also, the second equation in (4) differs from the classical Dirac equation, since in (2) the metric g is fixed, whereas in the second equation of the system the metric is a variable and the coefficients of the Dirac operator depend on the metric.

A first question about the system (4) is the compactness (or stability) of the set of solutions. In general this is not true, even in the pure Riemannian setting (that is considering only the Einstein equation), since bubbling or asymptotes of solutions can occur. However, we are able to show a compactness property for a class of solutions to the equations (4) under some suitable assumptions, on the sign of the scalar curvature and the diameter of the manifold and the dimension. This type of result is in the spirit of the works [2, 3, 6, 27], regarding the compactness of sequences of Einstein manifolds. So, here we state the following theorem and we address the reader to [25] for the proof.

Theorem 1.1. *Let $n = 3$ and $c, d, k > 0$. We set*

$$\mathcal{A}(c, d, k) = \left\{(g, \psi) \in Crit_c(\mathcal{E}),\ diam(M, g) \leq d,\ -\Delta_g R_g \geq -kR_g\right\}$$

Then the space $\mathcal{A}(c, d, k)$ is compact, that is, if $(g_k, \psi_k) \in \mathcal{A}(c, d, k)$ then there exists a subsequence that converges to $(g_\infty, \psi_\infty) \in \mathcal{A}(c, d, k)$.

In the previous statement the convergence is meant in $C^{r,\alpha}, \forall r > 0,\ \alpha \in (0, 1)$; also we have denoted by $Crit_c(\mathcal{E})$ the set of critical points of $\mathcal{E}$ having critical value c and $diam(M, g)$ stands for the diameter of the manifold M with respect to the metric g.
Let us remark that here we stated the case $n = 3$ for the sake of simplicity, actually in [25] is treated also the case $n = 4$, whose statement is more involved, since the compactness is proved up to a finite number of bubbling terms.

In addition, we note that one could replace the analytic constraint involving the Laplacian by a more geometric condition on the Q-curvature of the manifolds, which reads as follows:

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|Ric|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2.$$

In particular, if Q is bounded from below, then $-\Delta_g R_g$ is also bounded below in terms of R_g^2 and this is enough for carrying the analysis leading to the result in Theorem 1.1.

2. Classification and existence results for the conformal problem

One of the main question about the system of Dirac-Einstein equations (4) is the existence of solutions: in full generality, this is indeed a hard problem manly due to the fact that the space of variations is quite huge (the set of the couples metrics-spinors on the manifold). In this section we will restrict the functional on a fixed conformal class and we will show that in dimension three, even tough a lack of compactness still appears, some existence result can be obtained.

So, for a given metric $g_0 \in \mathcal{G}(M)$, we define its conformal class as

$$[g_0] = \left\{ g \in \mathcal{G}(M),\ g = u^{\frac{4}{n-2}} g_0,\ u \in C^\infty(M),\ u > 0 \right\}.$$

Also, for a spinor $\phi \in \Sigma_{g_0}M$, we set $\psi = u^{\frac{1-n}{n-2}}\phi \in \Sigma_g M$, where in the last equality is understood the action of a canonic isometric isomorphism between the spinor bundles $\Sigma_{g_0}M$ and $\Sigma_g M$ (see for instance [12], Section 2, for more details).

In this way we have the conformal change of the Dirac operator

$$D_g\psi = u^{-\frac{n+1}{n-2}} D_{g_0}\phi. \tag{5}$$

Putting everything in (3), we obtain

$$\mathcal{E}(g,\psi) = \int_M uL_{g_0}u + \langle D_{g_0}\phi, \phi\rangle - \lambda u^{\frac{2}{n-2}}|\phi|^2 \, dv_{g_0} =: E(u,\phi) \tag{6}$$

where $a_n = \frac{4(n-1)}{n-2}$ and

$$L_{g_0}u = -a_n\Delta_{g_0}u + R_{g_0}u$$

is the so-called conformal Laplacian. Therefore critical points of E solve the conformal Dirac-Einstein equations

$$\begin{cases} L_{g_0}u = \dfrac{\lambda}{n-2}|\phi|^2 u^{\frac{4-n}{n-2}} \\ \\ D_{g_0}\phi = \lambda u^{\frac{2}{n-2}}\phi \end{cases}$$

In the previous equations one can see the strong dependence on the dimension; so far we were able to treat the low dimensional case.

So, from now on, we let $(M, g_0, \Sigma_{g_0}M)$ be a closed spin manifold of dimension three and we also fix $\lambda = 1$, which is not restrictive. We define

$$E : H^1(M) \times H^{\frac{1}{2}}(\Sigma_{g_0}M) \to \mathbb{R}$$

$$E(u,\phi) = \int_M uL_{g_0}u + \langle D_{g_0}\phi, \phi\rangle - u^2|\phi|^2\, dv_{g_0}$$

where $H^1(M)$ and $H^{\frac{1}{2}}(\Sigma_g M)$ are canonical Sobolev spaces on which the functional E is well defined (see for instance [15]).

This way, critical points of E are (weak) solutions of

$$\begin{cases} L_{g_0}u = |\phi|^2 u \\ \\ D_{g_0}\phi = u^2\phi \end{cases} \tag{7}$$

As we said, one of the issue in finding solutions is the lack of the compactness of the functional, in the sense of Palais-Smale. We recall that given a function $F \in C^1(X)$ on a Hilbert space X, a Palais-Smale sequence for F at level $c \in \mathbb{R}$ is a sequence $\{x_k\} \subseteq X$ such that

$$\begin{cases} F(x_k) \xrightarrow[k\to\infty]{\mathbb{R}} c \\ \\ \nabla F(x_k) \xrightarrow[k\to\infty]{X} 0 \end{cases}$$

and we say that the function F satisfies the Palais-Smale condition if every Palais-Smale sequence at level c admits a converging subsequence, and this holds for any $c \in \mathbb{R}$.

So our first result is a the classification of the Palais-Smale sequences, in the spirit of the classic works [30], [28], [29], [19], [20]. We have then (see [24] for the proof)

Theorem 2.1. *Let $(M, g_0, \Sigma_{g_0}M)$ be a closed spin manifold of dimension three and let us assume that M has a positive Yamabe constant $Y_{g_0}(M)$. Let (u_n, ϕ_n) be a Palais-Smale sequence for E at level c. Then there exist $u_\infty \in C^{2,\alpha}(M)$, $\phi_\infty \in C^{1,\alpha}(\Sigma_{g_0}M)$, $\alpha \in (0,1)$, with (u_∞, ψ_∞) solution of (7), m sequences of points $x_n^1, \cdots, x_n^m \in M$ with $\lim_{n\to\infty} x_n^k = x^k \in M$, for $k = 1, \dots, m$ and m sequences of real numbers $R_n^1, \cdots, R_n^m$ converging to zero, such that:*

i) $u_n = u_\infty + \sum_{k=1}^{m} v_n^k + o(1)$ *in* $H^1(M)$,

ii) $\phi_n = \phi_\infty + \sum_{k=1}^{m} \varphi_n^k + o(1)$ *in* $H^{\frac{1}{2}}(\Sigma_{g_0}M)$,

iii) $E(u_n, \phi_n) = E(u_\infty, \phi_\infty) + \sum_{k=1}^{m} E_{\mathbb{R}^3}(U_\infty^k, \Phi_\infty^k) + o(1)$,

where

$$v_n^k = (R_n^k)^{-\frac{1}{2}} \beta_k \sigma_{n,k}^*(U_\infty^k),$$

$$\varphi_n^k = (R_n^k)^{-1} \beta_k \sigma_{n,k}^*(\Phi_\infty^k),$$

with $\sigma_{n,k} = (\rho_{n,k})^{-1}$ and $\rho_{n,k}(\cdot) = exp_{x_n^k}(R_n^k \cdot)$ is the exponential map defined in a suitable neighborhood of $\mathbb{R}^3$. Also, here β_k is a smooth compactly supported function, such that $\beta_k = 1$ on $B_1(x^k)$, $supp(\beta_k) \subseteq B_2(x^k)$ and $(U_\infty^k, \Phi_\infty^k)$ are solutions to equations (7) on $\mathbb{R}^3$ with its standard Euclidian metric $g_{\mathbb{R}^3}$ and canonical spin bundle $\Sigma_{g_{\mathbb{R}^3}}\mathbb{R}^3$.

The assumption on M of having a positive Yamabe constant implies that the Dirac operator D_{g_0} has no kernel and this is crucial in the proof. This follows by the conformal invariance of the Dirac operator (5) for which the vanishing of the kernel is preserved by the conformal change, and the Schrödinger-Lichnerowicz formula (which is true for any metric)

$$D_g^2 \psi = -\Delta_{\Sigma_g} \psi + \frac{R_g}{4} \psi,$$

where Δ_{Σ_g} is the Laplacian acting on spinors. So, if the Yamabe constant (of the given metric g_0) is positive, then the conformal class of that given metric contains a conformal metric g with positive scalar curvature R_g, which implies the vanishing of the kernel of D_g, and therefore of D_{g_0}.

With this classification in hands we are able to prove some existence results, the first of which is a direct consequence of the previous theorem. Indeed, as in the classical Yamabe problem one can define a conformal invariant $\widetilde{Y}_{g_0}(M)$ in the following way. For non-trivial $(u,\psi) \in H^1(M) \times H^{\frac{1}{2}}(\Sigma_{g_0}M)$, we set the functionals

$$\tilde{E}(u,\phi) = \frac{\left(\int_M uL_{g_0}u dv_{g_0}\right)\left(\int_M \langle D_{g_0}\phi,\phi\rangle dv_{g_0}\right)}{\int_M |u|^2|\phi|^2 dv_{g_0}}, \quad I(\phi) = \frac{\int_M \langle D_{g_0}\phi,\phi\rangle dv_{g_0}}{\int_M |u|^2|\phi|^2 dv_{g_0}}.$$

Now, we define the conformal invariant

$$\widetilde{Y}_{g_0}(M) = \inf_{(u,\phi)\in\mathcal{H}} \widetilde{E}(u,\phi)$$

where

$$\mathcal{H} = \left\{(u,\phi) \in H^1(M) \times H^{\frac{1}{2}}(\Sigma_{g_0}M) \setminus \{0,0\} \ s.t.\ I(\phi) > 0, P^-\left(D_{g_0}\phi - I(\phi)u^2\phi\right) = 0\right\}.$$

Here we have denoted by P^- the projector on the negative space of $H^{\frac{1}{2}}(\Sigma_{g_0}M)$ according to the natural splitting given by the eigenspinors of the Dirac operator.
So, we have the following Aubin type result (see [4]) by comparing $\widetilde{Y}_{g_0}(M)$ with the invariant on the sphere S^3 with its standard metric g_s and spin structure $\Sigma_{g_s}S^3$; the proof is in [24].

Theorem 2.2. *Let $(M, g_0, \Sigma_{g_0}M)$ be a closed spin manifold of dimension three. It holds:*

$$\widetilde{Y}_{g_0}(M) \le \widetilde{Y}_{g_s}(S^3).$$

Moreover, if

$$\widetilde{Y}_{g_0}(M) < \widetilde{Y}_{g_s}(S^3),$$

then problem (7) *has a non-trivial ground state solution, that is a solution that realize the infimum of the functional.*

Let us recall that in [24] we considered also the case of a three-dimensional closed manifold M with an isometric group action G acting on it, such that the orbits of G have infinite cardinality: again by means of the classification Theorem 2.1, we are able to prove that equations (7) admit two infinite families of solutions.

The next existence result is a perturbation results on the sphere S^3 by using the Bahri-Coron type conditions. So, let us consider the three dimensional sphere S^3 equipped with its standard metric g_s and spin structure $\Sigma_{g_s}S^3$, we define the functional

$$E_s(u,\phi) = \frac{1}{2}\int_{S^3} \left(uL_{g_s}u + \langle D_{g_s}\phi,\phi\rangle - K|u|^2|\phi|^2\right) dv_{g_s} \tag{8}$$

where K is a function of the form $K = 1 + \varepsilon k$, for some positive ε and with k a function on the sphere that will satisfy suitable assumptions to be determined. Critical points of E_s are solutions to the following coupled system:

$$\begin{cases} L_{g_s}u = K|\phi|^2u \\ \\ D_{g_s}\phi = Ku^2\phi \end{cases} \tag{9}$$

and we explicitly notice that, when the parameter ε tends to zero, these solutions converge to solutions of equations (7) on S^3 (or $\mathbb{R}^3$ via stereographic projection).
In this regard, let us denote by $\pi : S^3 \setminus \{s_p\} \to \mathbb{R}^3$ the stereographic projection, where s_p is the south pole. Therefore our second existence result reads as follows (we refer the reader to [17] for the proof)

Theorem 2.3. *Let $k \in C^2(S^3)$ be a Morse function on S^3 such that the south pole is not a critical point. Let us set $h = k \circ \pi^{-1}$ and let us assume the following Bahri-Coron type conditions on h:*

$$(i) \quad \Delta h(\xi) \neq 0, \ \forall\, \xi \in Crit(h)\ ,$$

$$(ii) \quad \sum_{\substack{\xi \in Crit(h) \\ \Delta h(\xi) < 0}} (-1)^{m(h,\xi)} \neq -1,$$

where Δ is the standard Laplacian operator on $\mathbb{R}^3$, $Crit(h)$ denotes the set of critical points of h and $m(h,\xi)$ is the morse index of h at a critical point ξ.
Then, there exists $\varepsilon_0 > 0$ such that for $K = 1 + \varepsilon k$ and $\varepsilon \in (0,\varepsilon_0)$, the system (9) has a solution.

Let us remark that the condition on the south pole of the sphere is needed since in the proof we make use of the stereographic projection π, however this condition can be always satisfied by making a unitary transformation which does not affect the generality of the result.

We also recall that the previous result is the analogous of several ones obtained with this hypothesis of Bahri-Coron type: for the standard Riemannian case of prescribing the scalar curvature and its generalization to the Q-curvature see [1, 9, 11]; in the case of prescribing the Webster curvature in the CR setting and its fractional generalization see [26] and [10]; for the spinorial Yamabe type equation involving the Dirac operator on the sphere see [18].

The scheme of the proof is quite standard in the theoretical approach. The main difficulties in dealing with such system of equations are essentially two: one comes from the strongly indefiniteness of one of the operator involved and one is related to the characterization of the critical manifold of the unperturbed problem. The first difficulty does not really affect this type of analysis and it has already been treated in these kind of situations (see [21], [22], [23]). The second difficulty is actually the one that represents a major difference with respect to the problems that we found in the literature: in fact, after characterizing the critical manifold and showing its non-degeneracy, one is led to study the finite-dimensional reduction of the functional and at this stage another kind of degeneracy appears, which is due to the invariance of the equations with respect to one of the parameters induced by the spin structure of the sphere (see Remark 3.6 in [17]).

Finally, let us just mention that in [7] we consider the case of a compact spin manifold of dimension three, with boundary. In this situation the energy functional coupling the Bosonic-Fermionic interaction has an additional boundary term of the type York-Gibbons-Hawking (see [31], [16]), which involves the mean curvature of the boundary. However, even in this case we were able to prove the classification of the Palais-Smale sequences (for which a new boundary behaviour appears) and the existence result of Aubin type.

References

[1] A.Ambrosetti, J.Garcia Azorero, I.Peral. *Perturbation of* $-\Delta u + u^{(N+2)/(N-2)} = 0$*, the scalar curvature problem in* $\mathbb{R}^N$ *and related topics.* J. Funct. Anal. 165 (1999) 117-149.

[2] M. Anderson. *Ricci curvature bounds and Einstein metrics on compact manifolds.* J. Amer. Math. Soc. 2 (1989), no. 3, 455-490

[3] M. Anderson. *Convergence and rigidity of manifolds under Ricci curvature bounds.* Invent. Math. 102 (1990), no. 2, 429-445.

[4] T. Aubin. *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.* J. Math. Pures Appl. (9) 55 (1976), no. 3, 269-296.

[5] A. Bahri, J.M. Coron. *The scalar curvature problem on the standard three dimensional spheres.* J. Funct. Anal., 95 (1991), pp. 106-172

[6] S. Bando, A. Kasue, H. Nakajima. *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth.* Invent. Math. 97 (1989), no. 2, 313-349.

[7] W.Borrelli, A.Maalaoui, V.Martino. *Conformal Dirac-Einstein equations on manifolds with boundary.* Calculus of Variations and Partial Differential Equations, 1, 62:18, 2023

[8] D.R. Brill, J.A. Wheeler. *Interaction of neutrinos and gravitational fields.* Rev. Mod. Phys. 29, 465-479, (1957)

[9] S.A.Chang, P.Yang. *A perturbation result in prescribing scalar curvature on S^n.* Duke Math. J. 64 (1991)27-69.

[10] Y.H.Chen, Y.Wang. *Perturbation of the CR fractional Yamabe problem.* Math. Nachr. 290 (2017), no. 4, 534-545.

[11] G.Chen, Y.Zheng, *A perturbation result for the Q_γ curvature problem on S^n.* Nonlinear Anal. 97 (2014), 4-14

[12] E.C. Kim, T. Friedrich. *The Einstein-Dirac Equation on Riemannian Spin Manifolds.* J. of Geometry and Physics, 33(1-2), 128-172, (2000).

[13] F. Finster, J. Smoller, S.T. Yau. *Particle-like solutions of the Einstein-Dirac equations.* Physical Review. D. Particles and Fields. Third Series 59 (1999)

[14] F. Finster, J. Smoller, S.T. Yau. *Particle-like solutions of the Einstein-Dirac-Maxwell equations.* Phys. Lett. A 259, no.6, 431–436 (1999)

[15] T. Friedrich. *Dirac Operators in Riemannian Geometry.* Grad. Stud. Math., vol. 25, Amer. Math. Soc., Providence, RI, (2000).

[16] G. Gibbons, S. Hawking. *Action integrals and partition functions in quantum gravity.* Phys. Rev. D 15, 2752-2756, (1977)

[17] C.Guidi, A.Maalaoui, V.Martino. *Existence results for the conformal Dirac-Einstein system.* Advanced Nonlinear Studies 2021, 21, 1, 107-117

[18] T.Isobe. *A perturbation method for spinorial Yamabe type equations on S^m and its application.* Math. Ann. 355 (2013), no. 4, 1255-1299.

[19] P.L. Lions. *The concentration-compactness principle in the calculus of variations. The limit case. I.* Rev. Mat. Iberoamericana, 1, no. 2, 45-121, (1985).

[20] P.L.Lions. *The concentration-compactness principle in the calculus of variations. The limit case. II.* Rev. Mat. Iberoamericana, 1, no. 2, 45-121, (1985).

[21] A.Maalaoui. *Rabinowitz-Floer Homology for Super-quadratic Dirac Equations On spin Manifolds.* Journal of fixed point Theory and Applications, 13, 175-199, (2013).

[22] A.Maalaoui, V.Martino. *The Rabinowitz-Floer homology for a class of semilinear problems and applications.* Journal of Functional Analysis, 269, 4006-4037, (2015).

[23] A.Maalaoui, V.Martino. *Homological approach to problems with jumping non-linearity.* Nonlinear Analysis, 144, 165-181, (2016).

[24] A.Maalaoui, V.Martino. *Characterization of the Palais-Smale sequences for the conformal Dirac-Einstein problem and applications.* Journal of Differential Equations, 266, 5, 2493-2541, 2019

[25] A. Maalaoui, V. Martino. *Compactness of Dirac-Einstein spin manifolds and horizontal deformations.* The Journal of Geometric Analysis, 32, 7, 201, (2022)

[26] A.Malchiodi, F.Uguzzoni. *A perturbation result for the Webster scalar curvature problem on the CR sphere.* J. Math. Pures Appl. (9) 81 (2002), no. 10, 983-997

[27] H. Nakajima. *Hausdorff convergence of Einstein 4-manifolds.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988), no. 2, 411-424.

[28] J. Sacks, K. Uhlenbeck. *The existence of minimal immersions of 2-spheres.* Ann. of Math. (2) 113, no. 1, 1-24, (1981).

[29] M. Struwe. *A global compactness result for elliptic boundary value problems involving limiting nonlinearities.* Math. Z. 187, no. 4, 511-517, (1984).

[30] H.C. Wente. *Large solutions to the volume constrained Plateau problem.* Arch. Rational Mech. Anal. 75, no. 1, 59-77, (1980/81).

[31] J. York. *Role of conformal three-geometry in the dynamics of gravitation.* Phys. Rev. Lett. 28, 1082-1085, (1972)

Dipartimento di Matematica, Università di Bologna, piazza di Porta S.Donato 5, 40127 Bologna, Italy

Email address: vittorio.martino3@unibo.it