Conformally Invariant Dirac Equation with Non-Local Nonlinearity

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Abstract We study a conformally invariant equation involving the Dirac operator and a non-linearity of convolution type. This non-linearity is inspired from the conformal Einstein-Dirac problem in dimension 4. We first investigate the compactness, bubbling and energy quantization of the associated energy functional then we characterize the ground state solutions of the problem on the standard sphere. As a consequence, we prove an Aubin-type inequality that assures the existence of solutions to our problem and in particular the conformal Einstein-Dirac problem in dimension 4. Moreover, we investigate the effect of a linear perturbation to our problem, leading us to a Brezis-Nirenberg type result.

Keywords: Dirac operator, Convolution non-linearity, Conformal invariance, Brezis-Nirenberg

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1 Introduction and motivation

Let M be a closed (compact, without boundary) manifold of dimension $n \geq 3$, endowed with a fixed Riemannian metric g and a spin structure $\Sigma_g M$. Let D_g be the Dirac operator acting on spinors $\psi \in \Sigma_g M$. Let us introduce the Einstein-Dirac functional

$$\mathcal{E}(g, \psi) = \int_M R_g + \langle D_g \psi, \psi \rangle - \lambda |\psi|^2 \, dv_g \tag{1}$$

where R_g is the Scalar curvature of the metric g and λ is a real parameter. Critical points of $\mathcal{E}$ are solutions of the Einstein-Dirac equations (see for instance [36])

$$\begin{cases} Ric_g - \dfrac{R_g}{2} g = T_{g,\psi} \\[2mm] D_g \psi = \lambda \psi \end{cases} \tag{2}$$

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where Ric_g is the Ricci tensor and T_g is the stress–energy tensor given by

$$T_{g,\psi}(X,Y) = -\frac{1}{4}\langle X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi \rangle, \quad X,Y \in TM \ ,$$

here $\cdot$ and ∇ denote the Clifford multiplication and the connection on $\Sigma_g M$ (see [30], [17]). This functional was investigated in dimensions 3 and 4 in [36], where the authors study the limits of such structures under natural bounds on the diameter and the curvature. We also mention that the first equation in (2) is similar in structure to the semi-classical gravity model coupling gravity with matter in a way that only the matter fields are quantified, we refer to [3] for more details about the model. Now, if we restrict the variations of the metric to a given conformal class, that is $\tilde{g} = u^{\frac{4}{n-2}} g$ and $\tilde{\psi} = u^{\frac{1-n}{n-2}} \psi \in \Sigma_{\tilde{g}} M$, we obtain the following functional

$$\mathcal{E}\big(\tilde{g}, \tilde{\psi}\big) = \int_M u L_g u + \langle D_g \psi, \psi \rangle - \lambda u^{\frac{2}{n-2}} |\psi|^2 \ dv_g =: \mathcal{E}_g(u, \psi) \ , \tag{3}$$

where L_g, here is the conformal Laplacian. The critical points of this functional solve the conformal Einstein-Dirac equations

$$\begin{cases} L_g u = \dfrac{\lambda}{n-2} |\psi|^2 u^{\frac{4-n}{n-2}} \\[4mm] D_g \psi = \lambda u^{\frac{2}{n-2}} \psi \end{cases} . \tag{4}$$

The case $n = 3$ was investigated in [6], [22], [35] and the case $n = 2$ corresponds to the super-Liouville problem investigated in [26, 27, 28]. We also mention the recent work of Sire and Xu [40] where the authors adopt a flow approach to investigate the problem. We notice that in dimension $n = 4$, the system (3) takes a more approachable structure. That is, one can solve the first equation, finding u in terms of $|\psi|^2$ by using the Green's function of the conformal Laplacian, then inserting it in the second equation one has a single equation that can be written as

$$D_g \psi = \Big(\int_M G(x,y) |\psi|^2(y) \ dv_g(y) \Big) \psi,$$

where G is the Green's function of the conformal Laplacian L_g. Due to the singularity of the Green's function $(G(x,y) \sim \frac{1}{|x-y|^2}$, when x is close to y), one can see the similarities with other classical equations in the literature. It is in fact surprising how this type of equations appears naturally in different models in physics. For instance, based on the work in [18], the Schrodinger-Newton model can be derived from the Einstein-Dirac model through a non-relativistic limit and we recall here that the Schrodinger-Newton equation in $\mathbb{R}^3$ takes the form

$$i\frac{\partial \psi}{\partial t} = -\Delta \psi - c\Big(\int \frac{|\psi(t,y)|^2}{|x-y|} \ dy \Big) \psi,$$

where we see clearly the convolution term that appears in the non-linearity. Notice that the static solutions correspond to a version of the Choquard equations. Hence, our problem can be seen as a spinorial version of the Choquard equation, we refer the reader to the survey [37] and the references therein. But also, this equation is similar to the semi-classical Hartree's equation and the Lieb-Yau conjecture for the pseudo-relativistic Boson stars model

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[33, 34, 31]. Another important model where such equation appears is the Dirac-Maxwell system studied in [12] (see also the references therein).

In this work, we propose to study a general problem with the same structure and conformal invariance properties that would capture the solutions to the conformal Einstein-Dirac equation in dimension 4. We then consider the following equation

$$D_g\psi = (G_g^s * |\psi|^2)\psi, \tag{5}$$

and its linear perturbation

$$D_g\psi = \lambda\psi + (G_g^s * |\psi|^2)\psi, \tag{6}$$

where we denoted by

$$(G_g^s * f)(x) := \int_M G_g^s(x,y)f(y) \, dv_g(y)$$

the convolution of a given function f with the Green's function G_g^s of the conformal fractional Laplacian P_g^s, of order $2s = n - 2$.

These equations have a variational structure and the corresponding energy functional for (6) is given by

$$
\begin{aligned}
J_{g,\lambda}(\psi) &= \frac{1}{2}\int_M \langle D_g\psi, \psi\rangle - \lambda|\psi|^2 \, dv_g - \frac{1}{4}\int_M (G_g^s * |\psi|^2)|\psi|^2 \, dv_g \\
&= \frac{1}{2}\int_M \langle D_g\psi, \psi\rangle - \lambda|\psi|^2 \, dv_g - \frac{1}{4}\int_{M\times M} G_g^s(x,y)|\psi(y)|^2 \, |\psi(x)|^2 \, dv_g(y) \, dv_g(x) \,, \quad (7)
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle$ is the canonical Hermitian metric defined on $\Sigma_g M$. Notice that $J_{g,0} =: J_g$ is the energy functional corresponding to (5).

We see that, the particular choice of the parameter s makes the functional J_g invariant under a conformal change of the metric; in order to see this, for any s for which the conformal fractional Laplacian is defined (see [19, 10]), let us consider a conformal change of the metric

$$\widetilde{g} = u^{\frac{4}{n-2s}} g, \ 0 < u \in C^\infty(M) \,. \tag{8}$$

Given a spinor $\psi \in \Sigma_g M$, we set

$$\widetilde{\psi} = u^{\frac{1-n}{n-2s}}\psi \in \Sigma_{\widetilde{g}} M \,,$$

where we implicitly understand the action of a canonical isometric isomorphism between the spinor bundles $\Sigma_{\widetilde{g}} M$ and $\Sigma_g M$ (see [30], Section 2). In this way, we have the conformal change of the Dirac operator

$$D_{\widetilde{g}}\widetilde{\psi} = u^{-\frac{n+1}{n-2s}} D_g\psi \,.$$

Also, by using the conformal covariance property of the fractional Laplacian

$$P_{\widetilde{g}}^s(f) = u^{-\frac{n+2s}{n-2s}} P_g^s(uf) \,,$$

we obtain the conformal change of its Green's function

$$G_{\widetilde{g}}^s(x,y) = u(x)^{-1}u(y)^{-1}G_g^s(x,y) \,.$$

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Now, taking into account the change of the volume

$$dv_{\widetilde{g}} = u^{\frac{2n}{n-2s}} dv_g \ ,$$

we substitute in (5) and find

$$J_{\widetilde{g}}(\widetilde{\psi}) = \frac{1}{2} \int_M \langle D_{\widetilde{g}}\widetilde{\psi}, \widetilde{\psi} \rangle \ dv_{\widetilde{g}} - \frac{1}{4} \int_{M \times M} G_{\widetilde{g}}^s(x,y)|\widetilde{\psi}(y)|^2 \ |\widetilde{\psi}(x)|^2 \ dv_{\widetilde{g}}(y) \ dv_{\widetilde{g}}(x) \ ,$$

$$= \frac{1}{2} \int_M \langle D_g\psi, \psi \rangle \ dv_g - \frac{1}{4} \int_{M \times M} G_g^s(x,y)|\psi(y)|^2 \ |\psi(x)|^2 \ u^{\frac{4+4s-2n}{n-2s}} \ dv_g(y) \ dv_g(x) \ .$$

Therefore, if $2s = n - 2$ we obtain $J_{\widetilde{g}}(\widetilde{\psi}) = J_g(\psi)$. In particular, this says that equation (5) is critical, in the sense of the conformal analysis.

This manuscript is mainly split in two parts. In the first part, we investigate the lack of compactness of the problem, due to the conformal invariance and we prove the following bubbling and energy quantization result for the functional J_g.

Theorem 1.1. *Let us assume that $(M, [g])$ has a positive Yamabe constant $Y_s(M, [g])$ and let $(\psi_k)_{k \in \mathbb{N}}$ be a Palais-Smale sequence for J_g at level $c \geq 0$, where $Y_s(M, [g])$ is the s-Yamabe constant, which we define in the next section.. Then there exist $\psi_\infty \in C^\infty(M, \Sigma_g M)$, a solution of (5), m sequences of points $x_k^1, \cdots, x_k^m \in M$ such that $\lim_{k \to \infty} x_k^j = x^j \in M$, for $j = 1, \ldots, m$ and m sequences of real numbers $R_k^1, \cdots, R_k^m$ converging to zero, such that:*

ii) $\psi_k = \psi_\infty + \sum_{j=1}^m \phi_k^j + o(1)$ *in* $H^{\frac{1}{2}}(\Sigma M)$,

iii) $J_g(\psi_k) = J_g(\psi_\infty) + \sum_{j=1}^m J_{g_{\mathbb{R}^n}}(\Psi_\infty^j) + o(1)$,

where

$$\phi_k^j = (R_k^j)^{-1}\beta_j\sigma_{k,j}^*(\Psi_\infty^j),$$

with $\sigma_{k,j} = (\rho_{k,j})^{-1}$ and $\rho_{k,j}(\cdot) = exp_{x_k^j}(R_k^j\cdot)$ is the exponential map defined in a suitable neighborhood of $\mathbb{R}^n$. Also, here β_j is a smooth compactly supported function, such that $\beta_j = 1$ on $B_1(x^j)$ and $supp(\beta_j) \subset B_2(x^j)$ and Ψ_∞^j is the solution to our equations (5) on $\mathbb{R}^n$ with its Euclidian metric $g_{\mathbb{R}^n}$.

As we will see in the proof, the same result holds for the functional $J_{g,\lambda}$, with the same bubbles at infinity. We also characterize the ground state solutions that appear in the bubbling phenomena in the theorem above.

Theorem 1.2. *Let $\psi \in C^\infty(\Sigma_{g_0}S^n)$ be a non-trivial solution of*

$$D_{g0}\psi = \left(G_{g0}^s * \psi\right)\psi, \ on \ S^n, \tag{9}$$

where g_0 is the round metric on S^n. Then,

$$J_{g0}(\psi) \geq \overline{Y}(S^n, [g_0]) := \frac{\lambda^+(S^n, [g_0])^2 Y_s(S^n, [g_0])}{4}. \tag{10}$$

Moreover, if $J_{g_0}(\psi) = \overline{Y}(S^n, [g_0])$ then, up to a conformal change, ψ is a $-\frac{1}{2}$-Killing spinor. That is, there exists a $-\frac{1}{2}$-Killing spinor $\Psi \in \Sigma_{g_0} S^n$ and a conformal diffeomorphism $f \in Conf(S^n, g_0)$ such that

$$\psi = \left(det(df)\right)^{\frac{n-1}{2n}} F_{f^* g_0, g_0}\left(f^* \Psi\right).$$

As a corollary of this Theorem we have an Aubin-type inequality for the problem (5):

Corollary 1.1. *Under the assumptions of Theorem 1.1, there exists a conformally invariant constant $\overline{Y}(M, [g]) > 0$ with the following properties:*

i) $\overline{Y}(M, [g]) \leq \overline{Y}(S^n, [g_0]) = \frac{\lambda^+(S^n, [g_0])^2 Y_s(S^n, [g_0])}{4}$.

ii) If $\overline{Y}(M, [g]) < \overline{Y}(S^n, [g_0])$ then the problem (5) has a non-trivial solution.

Notice that in particular, when $n = 4$, we can state $ii)$ in the setting of the conformal Einstein-Dirac equation. That is, if $\overline{Y}(M, [g]) < \overline{Y}(S^n, [g_0])$ the conformal Einstein-Dirac problem (4) is solvable.

The second part of this paper deals with the existence of solutions for the linearly perturbed problem (6). Namely, we prove a Brezis-Nirenberg type result associated to the original problem (5).

Theorem 1.3. *Assume that $(M, [g])$ has a positive Yamabe invariant and $Y_s(M, [g]) > 0$. Then for any $\lambda \notin Spec(D_g)$ and $\lambda > 0$, there exists a non-trivial ground-state solution ψ_λ for (6). Moreover, if $\lambda \in (\lambda_k, \lambda_{k+1})$, then $\psi_\lambda \to 0$ as $\lambda \to \lambda_{k+1}$.*

2 Preliminaries

A spin structure on a riemannian manifold (M, g) is a pair $(P_{Spin}(M, g), \sigma)$, where $P_{Spin}(M, g)$ is a $Spin(n)$-principal bundle and $\sigma : P_{Spin}(M, g) \to P_{SO}(M, g)$ is a 2-fold covering map, which restricts to a non-trivial covering $\kappa : Spin(n) \to SO(n)$ on each fiber. That is, the quotient of each fiber by $\mathbb{Z}_2$ is isomorphic to the frame bundle of M and hence, the following diagram commutes:

We denote by $\mathbb{S}_n$ the unique (up to isomorphism) irreducible complex Cl_n-module such that $Cl_n \otimes \mathbb{C} \equiv End_{\mathbb{C}}(\mathbb{S}_n)$ as a $\mathbb{C}$-algebra, where Cl_n denotes the Clifford algebra of $\mathbb{R}^n$. This allows us to define the spinor bundle $\Sigma_g M$ as

$$\Sigma_g M := P_{Spin}(M, g) \times_\sigma \mathbb{S}_n.$$

In fact, $\Sigma_g M$ is a Hermitian bundle equipped with a metric connection induced by the Levi-Civita connection on TM, that we will denote by ∇. Moreover, there is a natural Clifford multiplication defined by the action of TM on $\Sigma_g M$. We can summarize the main properties of the spinor bundle in the following few points:

- For all $X, Y \in C^\infty(M, TM)$ and $\psi \in C^\infty(M, \Sigma_g M)$ we have $X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$. Here, "$\cdot$" denotes the Clifford multiplication.

- If $(\cdot, \cdot)$ denotes the Hermitian metric on $\Sigma_g M$, then for all $X \in C^\infty(M, TM)$ and $\psi, \phi \in C^\infty(M, \Sigma_g M)$ we have $(X \cdot \psi, \phi) = -(\psi, X \cdot \phi)$.

- For all $\psi, \phi \in C^\infty(M, \Sigma_g M)$ and $X \in C^\infty(M, TM)$, then $X(\psi, \phi) = (\nabla_X \psi, \phi) + (\psi, \nabla_X \phi)$.

- For all $X, Y \in C^\infty(M, TM)$ and $\psi \in C^\infty(M, \Sigma_g M)$ we have $\nabla_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X \psi$.

For the rest of the paper, we let $\langle \cdot, \cdot \rangle := Re(\cdot, \cdot)$. Then $\langle \cdot, \cdot \rangle$ defines a metric on $\Sigma_g M$. The Dirac operator D_g is then defined on $C^\infty(M, \Sigma_g M)$ as the composition of the Clifford multiplication and the connection ∇. Indeed, if $(e_1, \cdots, e_n)$ is a local orthonormal frame around a point $p \in M$ and $\psi \in C^\infty(M, \Sigma_g M)$ then one can locally define D_g by

$$D_g \psi := \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi.$$

The Dirac operator is a natural first order operator acting on smooth sections of $\Sigma_g M$. Moreover, if M is compact, then D_g is essentially self-adjoint on $L^2(\Sigma_g M) := L^2(M, \Sigma_g M)$, with compact resolvent. In particular, there exists a complete orthonormal basis $(\varphi_k)_{k \in \mathbb{Z}}$ of $L^2(\Sigma_g M)$ consisting of eigenspinors of D_g. That is $D_g \varphi_k = \lambda_k \varphi_k$, with $\lambda_k \to \pm\infty$ when $k \to \pm\infty$. We will use the convention that $\lambda_k > 0$ (resp. $\lambda_k < 0$) when $k > 0$ (resp. $k < 0$).

Proposition 2.1 ([5, 17]). *Consider a compact spin manifold $(M, g, \Sigma_g M)$, then*

i) *The Dirac operator D_g is conformally invariant. That is, if $\hat{g} := e^{2u}g$, then there exists a unitary isomorphism $F_{g,\hat{g}} : \Sigma_g M \to \Sigma_{\hat{g}} M$ so that for $\varphi \in C^\infty(M, \Sigma_g M)$,*

$$D_{\hat{g}}(e^{-\frac{n-1}{2}u} F_{g,\hat{g}}(\varphi)) = e^{-\frac{n+1}{2}u} F_{g,\hat{g}}(D_g \varphi).$$

ii) *For $\varphi \in C^\infty(M, \Sigma_g M)$, $D_g^2 \varphi = -\Delta_g \varphi + \frac{R_g}{4}\varphi$, where R_g is the scalar curvature.*

In what follows, we will identify spinors $\varphi \in \Sigma_g M$ with their isomorphic image $F_{g,\hat{g}}(\varphi)$, unless there is a specific distinction. Notice that as a result of the two points of the previous

Proposition we have that D_g is invertible if the Yamabe invariant of (M, g) is positive. We can define now the (unbounded) operator $|D_g|^s : L^2(\Sigma_g M) \to L^2(\Sigma_g M)$, for $s > 0$ by

$$|D_g|^s \psi = \sum_{k \in \mathbb{Z}} |\lambda_k|^s a_k \varphi_k,$$

for $\psi = \sum_{k \in \mathbb{Z}} a_k \varphi_k$. The Sobolev space $H^{\frac{1}{2}}(\Sigma_g M)$ is then defined by

$$H^{\frac{1}{2}}(\Sigma_g M) := \{\psi \in L^2(\Sigma_g M); |D_g|^{\frac{1}{2}} \psi \in L^2(\Sigma_g M)\}.$$

This function space is equivalent to the classical $H^{\frac{1}{2}}$-Sobolev space and will be endowed with the inner product $\langle \cdot, \cdot \rangle_{\frac{1}{2}}$ defined by

$$\langle \psi, \phi \rangle_{\frac{1}{2}} := \int_M \langle |D_g|^{\frac{1}{2}} \psi, |D_g|^{\frac{1}{2}} \phi \rangle \, dv_g, \forall \psi, \phi \in H^{\frac{1}{2}}(\Sigma_g M).$$

Notice that this inner product defines a natural semi-norm on $H^{\frac{1}{2}}(\Sigma_g M)$ by setting

$$\|\psi\|_{\frac{1}{2}} := \||D_g|^{\frac{1}{2}} \psi\|_{L^2}.$$

This semi-norm becomes a norm when D_g is invertible. Using the spectral resolution of D_g, we can split the space $H^{\frac{1}{2}}(\Sigma_g M)$ in a convenient way that fits our analysis. That is, we can write

$$H^{\frac{1}{2}}(\Sigma_g M) = H^{\frac{1}{2},-} \oplus H^{\frac{1}{2},0} \oplus H^{\frac{1}{2},+} , \tag{11}$$

with

$$H^{\frac{1}{2},-} := \overline{\operatorname{span}\{\varphi_i\}_{i<0}}, \quad H^{\frac{1}{2},0} := \ker D_g, \quad H^{\frac{1}{2},+} := \overline{\operatorname{span}\{\varphi_i\}_{i>0}}.$$

This leads to the natural projectors $P^\pm : H^{\frac{1}{2}}(\Sigma_g M) \to H^{\frac{1}{2},\pm}$ and for $\psi \in H^{\frac{1}{2}}(\Sigma_g M)$ we will write $\psi^+ := P^+ \psi$ and $\psi^- := P^- \psi$. Since we will be considering a linear perturbation of the Dirac operator, we introduce the following operator $D_\lambda := D_g - \lambda$, for $\lambda \notin Spec(D_g)$. Notice that D_λ has a similar spectral decomposition and hence we can introduce a similar adapted splitting as in (11), for the space $H^{\frac{1}{2}}$. That is:

$$H^{\frac{1}{2}}(\Sigma_g M) = H_\lambda^- \oplus H_\lambda^+ ,$$

The new adapted inner product and norm are then defined by

$$\langle \psi, \phi \rangle_\lambda = \int_M \langle |D_\lambda|^{\frac{1}{2}} \psi, |D_\lambda|^{\frac{1}{2}} \phi \rangle \, dv_g \quad \text{and} \quad \|\psi\|_\lambda = \||D_\lambda|^{\frac{1}{2}} \psi\|_{L^2}, \forall \psi, \phi \in H^{\frac{1}{2}}(\Sigma_g M).$$

We recall now some of the properties of the conformal fractional Laplacian and GJMS operators. A good reference for the material discussed in this paragraph is [19]. For this purpose, we consider a Poincaré-Einstein manifold (X, g^+) with conformal infinity $(M, [g])$. Therefore, there exists a geodesic defining function ρ such that in a neighborhood of M in X of the form $M \times (0, \varepsilon)$, the metric g^+ takes the form

$$g^+ = \frac{1}{\rho^2}(d\rho^2 + g_\rho),$$

where g_ρ is a one parameter family of metrics on M such that $g_0 = g$. Moreover, we have $Ric_{g^+} = -ng_+$. In fact, one can weaken this last Einstein equality to be up to a term of the form $O(\rho^{n-2})$ if n is even and up to a term of $O(\rho^\infty)$, if n is odd. One then can solve (even formally) the following generalized eigenvalue problem: for $s \in (0, \frac{n}{2})$ and $s \notin \mathbb{N}$, and $u \in C^\infty(M)$

$$\begin{cases} -\Delta_{g^+} U - (\frac{n}{2} + s)(\frac{n}{2} - s)U = 0 \text{ in } X \\ \\ U = \rho^{\frac{n}{2}-s}F(\rho) + \rho^{\frac{n}{2}+s}G(\rho); \end{cases}$$

where $F, G \in C^\infty(\overline{X}, \overline{g} := \rho^2 g^+)$ and $F_{|\rho=0} = u$. For the details about the construction of such solution, we refer the reader to [21]. The operator $S(s) : u \mapsto G_{|\rho=0}$ is called the scattering operator. The fractional conformal Laplacian is then defined by

$$P_g^s u := d_s S(s)u, \quad \text{where } d_s = 2^{2s}\frac{\Gamma(s)}{\Gamma(-s)}.$$

The properties of the operator P_g^s can be summarized as follows:

Proposition 2.2 ([10, 19, 21]). *Using the definition above, we have:*

- P_g^s *is a self-adjoint pseudo-differential operator on M with principal symbol coinciding with the one of $(-\Delta_g)^s$.*

- P_g^s *is a conformally covariant operator. That is, if $\hat{g} = u^{\frac{4}{n-2s}}g$ then*

$$P_{\hat{g}}^s(\cdot) = u^{-\frac{n+2s}{n-2s}}P_g^s(u\cdot).$$

- $(P_g^s)_{s \in (0, \frac{n}{2})}$ *constitutes a meromorphic family of operators that has potential simple poles when $s \in \mathbb{N}$. These poles are compensated by the normalization constant d_s, making the family then holomorphic.*

- *When M is the Euclidean space $\mathbb{R}^n$, we have $P_{g_{\mathbb{R}^n}}^s = (-\Delta_{\mathbb{R}^n})^s$.*

Notice that P_g^s is a non-local operator when $s \notin \mathbb{N}$. But when $s = k$ is an integer, then P_g^k is a differential operator and it coincides with the classical GJMS operators [20]. In fact, one can check that

$$P_g^1 = L_g := -\Delta_g + \frac{n-2}{4(n-1)}R_g,$$

and

$$P_g^2 = -\Delta_g^2 + div(a_n R_g g + b_n Ric_g)d + \frac{n-4}{2}Q_2,$$

where a_n and b_n are two constants depending on n and Q_2 is, up to a multiplicative constant, the classical Q-curvature. In a similar way, one can define the fractional Q-curvature by:

$$Q_g^s := \frac{P_g^s(1)}{(n-2s)}.$$

For example, $Q_g^1 = \frac{R_g}{4(n-1)}$. We will restrict ourselves to the case $0 < 2s < n$. One now can formulate the fractional Yamabe problem, which addresses the question of prescribing constant Q_g^s-curvature. This is equivalent to solving the problem of finding $u > 0$ such that

$$P_g^s u = cu^{\frac{n+2s}{n-2s}}. \tag{12}$$

As in the classical Yamabe problem, the sign of the constant c is a conformal invariant and it is determined by the sign of $\int_M Q_g^s \, dv_g$. We will focus on the positive case, that is, when $\int_M Q_g^s \, dv_g > 0$. We consider then the functional $I_s : [g] \to \mathbb{R}$ defined by

$$I_s(h) := \frac{\int_M Q_h^s \, dv_h}{\left(\int_M dv_h \right)^{\frac{n-2s}{n}}}.$$

Taking $h := u^{\frac{4}{n-2s}} g$ yields

$$I_s(u,g) := I_s(h) = \frac{\int_M u P_g^s u \, dv_g}{\left(\int_M u^{\frac{2n}{n-2s}} \, dv_g \right)^{\frac{n-2s}{n}}}.$$

Therefore, finding a critical point of I_s is equivalent to finding a solution to (12). We can then define the s-Yamabe constant by

$$Y_s(M, [g]) := \inf\{I_s(h); h \in [g]\} = \inf\{I_s(u,g); u > 0 \text{ and } u \in H^s(M)\}. \tag{13}$$

Notice that when $Y_s(M, [g]) > 0$ (as in the case of $(S^n, [g_0])$), one can define an equivalent H^s-norm on M by setting

$$\|u\|_{H^s} := \left(\int_M u P_g^s u \, dv_g \right)^{\frac{1}{2}}.$$

In this case, the best constant in the Sobolev embedding $H^s(M) \hookrightarrow L^{\frac{2n}{n-2s}}(M)$ coincides with $Y_s(M, [g])^{-\frac{1}{2}}$. We will assume from now on that the Green's function of P_g^s is positive. This is not a necessary condition but it does make the notations in the proofs easier. In fact, there are several conformally invariant assumptions that we can consider if we truly need the positivity of the G_g^s. We refer the reader to [9] where the authors address the positivity of the Green's function in certain ranges of the parameter s. We point out that when $Y_s(M, [g]) > 0$ and G_g^s is the Green's function of P_g^s, then for any $f \in C^\infty(M)$, we have

$$\int_{M \times M} G_g^s(x,y) f(x) f(y) \, dv_g(y) dv_g(x) \geq 0.$$

We summarize here some of the useful properties of the G_g^s that we will be using in the next sections.

Proposition 2.3. *We consider a compact Riemannian manifold (M, g) as above and fix $0 < s < \frac{n}{2}$. Assume that $Y_s(M, [g]) > 0$, then the Green's function of P_g^s satisfies:*

i) G_g^s is continuous and bounded away from the diagonal $\triangle_{M \times M} := \{(x,x) \in M \times M\}$.

ii) For $p_0 \in M$ there exists a small neighborhood U_{p_0} around p_0 in M such that in normal coordinates around p_0,

$$G_g^s(x,y) = G_{g_{\mathbb{R}^n}}^s(x,y) + r(x,y), \forall x, y \in U_{p_0},$$

where $G_{g_{\mathbb{R}^n}}^s(x,y) = \frac{c_{n,s}}{|x-y|^{n-2s}}$ is the Green's function of $(-\Delta_{\mathbb{R}^n})^s$ and there exists $C > 0$ such that

$$|r(x,y)| \leq \frac{C}{|x-y|^{n-2s-1}}, \forall x, y \in U_{p_0}.$$

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3 Regularity

In this section, we will focus on the study of regularity of solutions of (5), actually the same results hold for (6). In the sequel, for the sake of simplicity, we will omit the dependence on the metric; for instance $\Sigma M = \Sigma_g M$ and so on. For the same reason, we will denote the functional spaces depending only M; for instance $L^p(M) = L^p(M, \Sigma M)$. The $H^{\frac{1}{2}}$-norm will also be denoted simply by $\|\cdot\|$.

Our objective here is to show that weak solutions of (5) are indeed classical solutions. First of all, we consider the Sobolev space $H^{\frac{1}{2}}(M)$ as defined in the previous section. Here we just recall that there exists a continuous Sobolev embedding

$$H^{\frac{1}{2}}(M) \hookrightarrow L^p(M), \quad 1 \leq p \leq \frac{2n}{n-1},$$

this is also compact if $1 \leq p < \frac{2n}{n-1}$.

We will say that $\psi \in L^{\frac{2n}{n-1}}(M)$ is a weak solution of (5) if

$$\int_M \langle D\phi, \psi \rangle \, dv = \int_M (G^s * |\psi|^2) \langle \phi, \psi \rangle \, dv,$$

for all $\phi \in C^\infty(M)$. Notice that for a fixed metric g, the critical points of J_g are weak solutions of (5). We then have the following result:

Theorem 3.1. *Let $\psi \in L^{\frac{2n}{n-1}}(M)$ be a weak solution of (5). Then $\psi \in C^\infty(M)$.*

The idea of the proof is somehow similar to that in [25] (see also [35]), but we will provide here the full details since the non-linearity, in this case, is non-local.

Proof. Given a small $r > 0$, we consider two cut-off functions η_1 and η_2 such that η_1 is supported in B_{3r} and equals 1 on B_{2r}. Similarly, $\eta_2 = 1$ on $B_{\frac{r}{2}}$ and supported in B_r. Now, one has

$$D(\eta_2 \psi) = (G^s * |\psi|^2)\eta_2 \psi + \nabla \eta_2 \cdot \psi. \tag{14}$$

On the other hand, we will write

$$G^s * |\psi|^2 = G^s * (\eta_1 |\psi|^2 + (1 - \eta_1)|\psi|^2) = u_1 + u_2, \tag{15}$$

so that

$$D(\eta_2 \psi) = u_1 \eta_2 \psi + \eta_2 u_2 \psi + \nabla \eta_2 \cdot \psi.$$

Now, for $1 \leq p < n$, let $P : W^{1,p}(M) \to L^p(M)$ defined by

$$Pv = u_1 v.$$

We notice that

$$\|u_1 v\|_{L^p} \leq \|u_1\|_{L^n} \|v\|_{L^{\frac{np}{n-p}}}$$
$$\leq C\|\psi\|^2_{L^{\frac{2n}{n-1}}(B_{3r})} \|v\|_{W^{1,p}(M)}. \tag{16}$$

Thus, we have

$$\|P\|_{Op} \leq C \|\psi\|^2_{L^{\frac{2n}{n-1}}(B_{3r})} ,$$

where $\| \cdot \|_{Op}$ stands for the operator norm. Since $D : W^{1,p}(M) \to L^p(M)$ is invertible, we have for r small enough, that $D - P : W^{1,p}(M) \to L^p(M)$ is invertible. Noticing that $\nabla \eta_2 \psi + \eta_2 u_2 \psi \in L^{\frac{2n}{n-1}}(M)$, there exists a unique solution $v_0 \in W^{1,p}(M)$ of

$$Dv_0 = u_1 v_0 + \nabla \eta_2 \psi + \eta_2 u_2 \psi,$$

for all $1 \leq p \leq \frac{2n}{n-1}$.

Similarly, we can consider the invertible operator $D : L^{\frac{2n}{n-1}}(M) \to W^{-1,\frac{2n}{n-1}}(M)$, and define

$$\tilde{P} : L^{\frac{2n}{n-1}}(M) \to W^{-1,\frac{2n}{n-1}}(M)$$

by

$$\tilde{P}\tilde{v} = u_1 \tilde{v} .$$

We see that in this case, we have

$$\|\tilde{P}v\|_{L^{\frac{2n}{n+1}}} \leq \|u_1\|_{L^n} \|v\|_{L^{\frac{2n}{n-1}}}.$$

Therefore, since $L^{\frac{2n}{n+1}}(M) \hookrightarrow W^{-1,\frac{2n}{n-1}}(M)$, we have

$$\|\tilde{P}\|_{Op} \leq C \|\psi\|^2_{L^{\frac{2n}{n-1}}(B_{3r})}.$$

For the same reason as above, there exists a unique solution $\tilde{v}_0 \in L^{\frac{2n}{n-1}}(M)$, of

$$Dv = u_1 v + \nabla \eta_2 \cdot \psi + \eta_2 u_2 \psi.$$

Therefore, since $W^{1,p}(M) \hookrightarrow L^{\frac{2n}{n-1}}(M)$, for $\frac{2n}{n+1} \leq p < n$, we have that

$$v_0 = \tilde{v}_0 = \eta_2 \psi \in W^{1,p}(M), \frac{2n}{n+1} \leq p < n.$$

Thus, $\psi \in W^{1,p}(M)$ for $\frac{2n}{n+1} \leq p < n$, in particular, $\psi \in L^p(M)$ for all $p \geq 1$. Therefore, by the elliptic regularity for D and a standard bootstrap argument, we have that $\psi \in C^\infty(M)$.
$\square$

4   Bubbling and energy quantization

In this section, we will analyze the behaviour of Palais-Smale sequences for J_g. This type of asymptotic study is quite standard when dealing with concentration phenomena (see for instance the books [41, 13]); in particular, for our equation, the estimates that we will need are analogous to those in [35] (Section 4; for the Dirac-Einstein problem, in dimension three) and in [25] (Section 5, for the pure Dirac operator, in any dimension). We start with the first result.

Lemma 4.1. *Let $(\psi_k) \subseteq H^{\frac{1}{2}}(M)$ be a (PS) sequence for J_g. Then (ψ_k) is bounded.*

Proof. Let (ψ_k) be a (PS) sequence for J_g, at level $c \in \mathbb{R}$. Then

$$J_g(\psi_k) = c + o(1)$$

and

$$D_g \psi_k = (G_g^s * |\psi_k|^2)\psi + \varepsilon_k,$$

with $\varepsilon_k \to 0$ in $H^{-\frac{1}{2}}(M)$. Now we notice that

$$2J_g(\psi_k) - \langle \nabla J(\psi_k), \psi_k \rangle = \frac{1}{2} \int_M |(G_g^s * |\psi_k|^2)\psi_k|^2 \ dv. \tag{17}$$

Thus,

$$\int_M (G_g^s * |\psi_k|^2)|\psi_k|^2 \ dv = 4c + o(\|\psi_k\|).$$

From the elliptic regularity and the Sobolev embeddings, there exists $C > 0$ such that

$$\|G_g^s * |\psi_k|^2\|_{L^n}^2 \leq C \int_M (G_g^s * |\psi_k|^2)|\psi_k|^2 \ dv.$$

On the other hand,

$$\begin{aligned}
\|\psi_k^+\|^2 &= \int_M \langle \psi, \psi^+ \rangle G_g^s * |\psi_k|^2) \ dv \\
&\leq \left(\int_M (G_g^s * |\psi_k|^2)|\psi_k|^2 \ dv \right)^{\frac{1}{2}} \left(\int_M (G_g^s * |\psi_k|^2)|\psi_k^+|^2 \ dv \right)^{\frac{1}{2}} \\
&\leq \left(\int_M (G_g^s * |\psi_k|^2)|\psi_k|^2 \ dv \right)^{\frac{1}{2}} \|G_g^s * |\psi_k|^2\|_{L^n}^{\frac{1}{2}} \|\psi_k^+\|_{L^{\frac{2n}{n-1}}} \\
&\leq (C + o(\|\psi_k\|))\|\psi_k^+\|.
\end{aligned} \tag{18}$$

A similar inequality holds for $\|\psi_k^-\|^2$, leading to

$$\|\psi_k\| \leq C + o(\|\psi_k\|).$$

Hence, (ψ_k) is bounded in $H^{\frac{1}{2}}(M)$. $\qquad\square$

Remark 4.1. *From the previous Lemma, it follows that there exists $\psi_\infty \in H^{\frac{1}{2}}(M)$ such that (up to sub-sequences) $\psi_k \rightharpoonup \psi_\infty$ weakly in $H^{\frac{1}{2}}(M)$ and $L^{\frac{2n}{n-1}}(M)$ and strongly in $L^p(M)$ for $1 \leq p < \frac{2n}{n-1}$. Moreover, one can easily see that ψ_∞ is a weak solution of (5); in particular from Theorem 3.1 it is smooth.*

Lemma 4.2. *Let $h_k := \psi_k - \psi_\infty$, then we have*

$$J_g(h_k) = J_g(\psi_k) - J_g(\psi_\infty) + o(1) \ ,$$

and

$$\nabla J_g(h_k) \to 0 \ .$$

Proof. We have

$$J_g(\psi_k) = J_g(\psi_\infty) + J_g(h_k) + \langle \nabla J_g(\psi_\infty), h_k \rangle - \frac{1}{2} \int_M (G_g^s * |\psi_\infty|^2)|h_k|^2 \, dv$$

$$- \int_M (G_g^s * |h_k|^2)\langle \psi_\infty, h_k \rangle \, dv - \int_M (G_g^s * \langle \psi_\infty, h_k \rangle)\langle \psi_\infty, h_k \rangle \, dv.$$

Since ψ_∞ is a solution of (5), we have $\langle \nabla J_g(\psi_\infty), h_k \rangle = 0$. Also, since $h_k \to 0$ weakly in $H^{\frac{1}{2}}(M)$ we have that $h_k \to 0$ strongly in $L^p(M)$ for all $p < \frac{2n}{n-1}$. Therefore,

$$\int_M (G_g^s * |\psi_\infty|^2)|h_k|^2 \, dv \to 0, \text{ as } k \to \infty.$$

Similarly, we have by Hölder's inequalities that

$$\left| \int_M (G_g^s * |h_k|^2)\langle \psi_\infty, h_k \rangle \, dv \right| \leq \|G_g^s * |h_k|^2\|_{L^n} \|\psi_\infty\|_{L^\infty} \|h_k\|_{L^{\frac{n}{n-1}}}$$

$$\leq C\|h_k\|_{L^{\frac{2n}{n-1}}}^2 \|h_k\|_{L^{\frac{n}{n-1}}}.$$

Notice that Lemma 4.1 implies that $\|h_k\|_{H^{\frac{1}{2}}}$ is uniformly bounded and $\|h_k\|_{L^{\frac{n}{n-1}}} \to 0$ as $k \to \infty$. Hence, we have

$$\int_M (G_g^s * |h_k|^2)\langle \psi_\infty, h_k \rangle \, dv \to 0 \text{ as } k \to \infty.$$

Similarly,

$$\int_M (G_g^s * \langle \psi_\infty, h_k \rangle)\langle \psi_\infty, h_k \rangle \, dv \to 0, \text{ as } k \to \infty.$$

Thus,

$$J_g(\psi_k) = J_g(\psi_\infty) + J_g(h_k) + o(1).$$

The statement for $\nabla J_g(h_k)$ can be proved in the same way. $\qquad\square$

From now on, we will assume without loss of generality that the (PS) sequence (ψ_k) converges weekly to 0, namely $\psi_\infty = 0$. Given $\varepsilon_0 > 0$, we define the following sets

$$\Sigma_1(\varepsilon_0) = \left\{ x \in M; \liminf_{r \to 0} \liminf_{k \to \infty} \int_{B_r(x)} |\psi_k|^{\frac{2n}{n-1}} \, dv > \varepsilon_0 \right\},$$

$$\Sigma_2(\varepsilon_0) = \left\{ x \in M; \liminf_{r \to 0} \liminf_{k \to \infty} \int_{B_r(x)} \left(G_g^s * |\psi_k|^2 \right)^n \, dv > \varepsilon_0 \right\},$$

and

$$\Sigma_3(\varepsilon_0) = \left\{ x \in M; \liminf_{r \to 0} \liminf_{k \to \infty} \int_{B_r(x)} (G_g^s * |\psi_k|^2)|\psi_k|^2 \, dv > \varepsilon_0 \right\},$$

where $B_r(x)$ is the geodesic ball with center in x and radius r. We can state then the following ε-regularity type result.

Lemma 4.3. *Let (ψ_k) be a (PS) sequence converging weakly to 0. There exists $\varepsilon_0 > 0$ such that if $x \notin \Sigma_1(\varepsilon_0) \cap \Sigma_2(\varepsilon_0) \cap \Sigma_3(\varepsilon_0)$, then there exists $r > 0$ such that $\psi_k \to 0$ in $H^{\frac{1}{2}}(B_r(x_0))$.*

Proof. We will use the same notations as in the proof of Theorem 3.1. We have, as in (14),

$$D_g(\eta_2\psi_k) = \eta_2(G_g^s * |\psi_k|^2)\psi_k + \nabla\eta_2 \cdot \psi_k + \delta_k,$$

where $\delta_k \to 0$ in $H^{-\frac{1}{2}}(M)$. Using elliptic estimates, we have

$$\|\eta_2\psi_k\|_{H^{\frac{1}{2}}} \leq C\|\eta_2(G_g^s * |\psi_k|^2)\psi_k + \nabla\eta_2 \cdot \psi_k + \delta_k\|_{H^{-\frac{1}{2}}}$$
$$\leq C_1\left(\|\eta_2(G_g^s * |\psi_k|^2)\psi_k\|_{L^{\frac{2n}{n+1}}(B_r)} + \|\nabla\eta_2 \cdot \psi_k\|_{L^{\frac{2n}{n+1}}} + o(1)\right).$$

In addition, we have

$$\|\nabla\eta_2 \cdot \psi_k\|_{L^{\frac{2n}{n+1}}} \leq C_2\|\psi_k\|_{L^{\frac{2n}{n+1}}} \to 0 .$$

Now, we assume first that $x_0 \notin \Sigma_2(\varepsilon)$, then by Hölder's inequalities,

$$\|\eta_2(G_g^s * |\psi_k|^2)\psi_k\|_{L^{\frac{2n}{n+1}}} \leq \|G_g^s * |\psi_k|^2\|_{L^n(B_r)}\|\eta_2\psi_k\|_{L^{\frac{2n}{n-1}}}$$
$$\leq C_3\varepsilon^{\frac{1}{n}}\|\eta_2\psi_k\|_{H^{\frac{1}{2}}} .$$

Therefore, if $C_1C_3\varepsilon^{\frac{1}{n}} < \frac{1}{2}$, we have

$$\|\eta_2\psi_k\|_{H^{\frac{1}{2}}} \leq C_4\|\nabla\eta_2 \cdot \psi_k\|_{L^{\frac{2n}{n+1}}} + o(1).$$

Hence, $\eta_2\psi_k \to 0$ in $H^{\frac{1}{2}}(M)$.

On the other hand, let us assume that $x_0 \notin \Sigma_1(\varepsilon)$. Then as in (15), we can write

$$\eta_2(G_g^s * |\psi_k|^2)\psi_k = \eta_2(G_g^s * |\eta_1\psi_k|^2)\psi_k + \eta_2(G_g^s * (1 - \eta_1^2)|\psi_k|^2)\psi_k = A_1(\psi_k) + A_2(\psi_k). \quad (19)$$

Now we notice that

$$\|A_2(\psi_k)\|_{L^{\frac{2n}{n+1}}} \leq C_5\|\psi_k\|_{L^2}^3 \to 0 \text{ as } k \to 0.$$

Also,

$$\|A_1(\psi_k)\|_{L^{\frac{2n}{n+1}}} \leq C_6\|\eta_1\psi_k\|_{L^{\frac{2n}{n-1}}}^2\|\eta_2\psi_k\|_{L^{\frac{2n}{n-1}}}$$
$$\leq C_6C_7C_8\varepsilon^{\frac{n-1}{n}}\|\eta_2\psi_k\|_{H^{\frac{1}{2}}}.$$

Hence, for $C_6C_7C_8\varepsilon^{\frac{n-1}{n}} < \frac{1}{2}$, we get again

$$\|\eta_2\psi_k\|_{H^{\frac{1}{2}}} \to 0.$$

In order to finish the proof, we see from the decomposition (19) that for $x_0 \notin \Sigma_3(\varepsilon)$ we have

$$\|A_1(\psi_k)\|_{L^{\frac{2n}{n+1}}} \leq \left(\int_M \eta_2(G_g^s * |\eta_1\psi_k|^2)|\psi_k|^2 \, dv\right)^{\frac{1}{2}}\|G_g^s * |\eta_1\psi_k|^2\|_{L^n}^{\frac{1}{2}}$$
$$\leq C_9\varepsilon^{\frac{1}{2}}\|\eta_2\psi_k\|_{H^{\frac{1}{2}}}$$

Again, for $C_9\varepsilon^{\frac{1}{2}} < \frac{1}{2}$ we obtain the desired conclusion. $\quad\square$

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As a corollary, we get the following result.

Proposition 4.1. *Let (ψ_k) be a (PS) sequence converging weekly to 0. If the (PS) sequence (ψ_k) does not converge (up to subsequences) strongly to zero in $H^{\frac{1}{2}}(M)$, then there exists $\varepsilon_0 > 0$ (even smaller if necessary) such that*

$$\Sigma_1(\varepsilon_0) = \Sigma_2(\varepsilon_0) = \Sigma_3(\varepsilon_0) \neq \emptyset.$$

Moreover, if (ψ_k) is a (PS) sequence at level c, with $4c < \varepsilon_0$, then (ψ_k) converges strongly to zero in $H^{\frac{1}{2}}(M)$.

The last assertion follows immediately from equation (17) and the definition of $\Sigma_3(\varepsilon_0)$. We need here to take into account again the dependence on the metric. Let us consider now the concentration function

$$Q_k(t) = \sup_{x \in M} \int_{B_t(x)} (G_g^s * |\psi_k|^2)^n \, dv_g.$$

If we assume that $\Sigma_2(\varepsilon_0) \neq 0$, then given $\varepsilon > 0$ so that $3\varepsilon < \varepsilon_0$, there exists $R_k > 0$ such that $R_k \to 0$ and a sequence $x_k \in M$, that we can assume converging to a certain $x_0 \in \Sigma_2(\varepsilon_0)$ so that

$$Q_k(R_k) = \int_{B_{R_k}(x_k)} (G_g^s * |\psi_k|^2)^n dv_g = \varepsilon. \tag{20}$$

We let $\rho_k(x) = \exp_{x_k}(R_k x)$ defined for $R_k|x| < \iota(M)$; here $\iota(M)$ is the injectivity radius of M, that we will assume for the sake of simplicity $\iota(M) \geq 3$. Therefore, if we let B_R^0 denote the Euclidean ball centered at zero and of radius R, then we have that the two spaces $(B_R^0, R_k^{-2}\rho_k^*g)$ and $(B_{R_kR}(x_k), g)$ are conformally equivalent for k large enough. We define then the metric $g_k = R_k^{-2}\rho_k^*g$ on B_R^0. It is easy to see that $g_k \to g_{\mathbb{R}^n}$ in $C^\infty(B_R^0)$. We will use the map ρ_k to also identify the spinor bundles, that is

$$(\rho_k)_* : \Sigma_{x_0}(B_R^0, g_k) \to \Sigma_{\rho_k(x_0)}(M, g).$$

We can then define the spinor $\Psi_k = R_k^{\frac{n-1}{2}}\rho_k^*\psi_k$ on $\Sigma_{x_0}(B_R^0, g_k)$, where $\rho_k^*\psi_k = (\rho_k)_*^{-1} \circ \psi_k \circ (\rho_k)_*$. Therefore, based on the properties of the convolution and the conformal invariance, we have

$$\int_{B_R^0 \times B_R^0} G_{g_k}^s(x,y)|\Psi_k(x)|^2|\Psi_k(y)|^2 \, dv_{g_k}(x) \, dv_{g_k}(y)$$

$$= \int_{B_{R_kR}(x_k) \times B_{R_kR}(x_k)} G_g^s(x,y)|\psi_k(x)|^2|\psi_k(y)|^2 \, dv_g(x) \, dv_g(y),$$

$$\int_{B_R^0} \langle \Psi_k, D_{g_k}\Psi_k \rangle \, dv_{g_k} = \int_{B_{R_nR}(x_k)} \langle \psi_k, D_g\psi_k \rangle \, dv_g,$$

and

$$\int_{B_R^0} |\Psi_k|^{\frac{2n}{n-1}} \, dv_{g_k} = \int_{B_{R_nR}(x_k)} |\psi_k|^{\frac{2n}{n-1}} \, dv_g. \tag{21}$$

We can now, state the following result.

Proposition 4.2. *Let Ψ_k be the spinor on $\Sigma_{x_0}(B_R^0, g_k)$, defined as before. Let us set*

$$F_k := D_{g_k}\Psi_k - (G_{g_k}^s * |\Psi_k|^2)\Psi_k.$$

Then $F_k \to 0$ in $H_{loc}^{-\frac{1}{2}}(\mathbb{R}^n)$, namely for $R > 0$, it holds

$$\sup\left\{\int_{\mathbb{R}^n}\langle F_k, \Phi\rangle \ dv_{g_k} \ ; \Phi \in H^{\frac{1}{2}}(\mathbb{R}^n) \ , \|\Phi\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \leq 1 \ , supp(\Phi) \subset B_R^0\right\} \to 0 \ as \ k \to \infty. \tag{22}$$

Proof. From the definition of F_k and the conformal invariance, we have

$$F_k = D_{g_k}\Psi_k - (G_{g_k}^s * |\Psi_k|^2)\Psi_k = R_k^{\frac{n+1}{2}}\rho_k^*\left(D_g\psi_k - (G_g^s * |\psi_k|^2)\psi_k\right) = R_k^{\frac{n+1}{2}}\rho_k^*\delta_k.$$

Consider then a test spinor Φ with $supp(\Phi) \subset B_R^0$ and $\|\Phi\|_{H^{\frac{1}{2}}} \leq 1$. Then we have

$$\int_{B_{R_k^{-1}}^0}\langle F_k, \Phi\rangle \ dv_{g_k} = \int_{B_{R_k^{-1}}^0}\langle \rho_k^*\delta_k, R_k^{-\frac{n-1}{2}}\Phi\rangle \ dv_{\rho_k^* g}$$

$$= \int_{B_1(x_k)}\langle \delta_k, R_k^{-\frac{n-1}{2}}(\rho_k^{-1})^*\Phi\rangle \ dv_g.$$

Since $\|\Phi\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \leq 1$, there exists $C > 0$ such that $\|R_k^{-\frac{n-1}{2}}(\rho_k^{-1})^*\Phi\|_{H^{\frac{1}{2}}(M)} \leq C$. Hence we have that (22) holds. $\qquad\square$

We introduce here the following space

$$D^{\frac{1}{2}}(\mathbb{R}^n) = \left\{\Phi \in L^{\frac{2n}{n-1}}(\mathbb{R}^n) \ ; |\xi|^{\frac{1}{2}}|\widehat{\Phi}| \in L^2(\mathbb{R}^n)\right\},$$

where $\widehat{\Phi}$ is the Fourier transform of Φ.

Proposition 4.3. *Let $\varepsilon > 0$ small enough in (20), then there exists $\Psi_\infty \in D^{\frac{1}{2}}(\mathbb{R}^n)$ such that $\Psi_k \to \Psi_\infty$ in $H_{loc}^{\frac{1}{2}}(\mathbb{R}^n)$ and Ψ_∞ satisfies the equation*

$$D_{g_{\mathbb{R}^n}}\Psi_\infty = (G_{g_{\mathbb{R}^n}}^s * |\Psi_\infty|^2)\Psi_\infty, \quad in \ \mathbb{R}^n . \tag{23}$$

Proof. First, the sequence Ψ_k is bounded in $H_{loc}^{\frac{1}{2}}(\mathbb{R}^n)$, hence there exists Ψ_∞ such that, up to subsequence, $\Psi_k \rightharpoonup \Psi_\infty$ in $H_{loc}^{\frac{1}{2}}(\mathbb{R}^n)$ and strongly in $L_{loc}^p(\mathbb{R}^n)$ for $1 \leq p < \frac{2n}{n-1}$.
Now, from the relation (21), we have

$$\limsup_{k\to\infty}\int_{B_R^0}|\Psi_k|^{\frac{2n}{n-1}} \ dv_{g_k} \leq \sup_{k\geq 1}\int_M |\psi_k|^{\frac{2n}{n-1}} \ dv_g < +\infty,$$

hence $\Psi_\infty \in L^{\frac{2n}{n-1}}(\mathbb{R}^n)$.
Next, arguing as in Lemma 4.1 and Remark 4.1, we have that Ψ_∞ is a weak solution of (23), from which we deduce that $\Psi_\infty \in D^{\frac{1}{2}}(\mathbb{R}^n)$.
We can now assume without loss of generality that $\Psi_\infty = 0$, just replacing Ψ_n by $\Psi_n - \Psi_\infty$

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and using Lemma 4.2.

But by assumption we have that, given $x \in \mathbb{R}^3$, for k big enough we get from (20)

$$\int_{B_1^0} (G_{g_k}^s * |\Psi_k|^2)^n dv_{g_k} = \int_{B_{R_k}(x_k)} (G_g^s * |\psi_k|^2)^n dv_g = \varepsilon.$$

Let $\beta \in C_0^\infty(\mathbb{R}^n)$, with $supp(\beta) \in B_1^0$, we get

$$\|\beta \Psi_k\|_{H^{\frac{1}{2}}} \le C \left(\|D_{g_{\mathbb{R}^n}}(\beta \Psi_k)\|_{H^{-\frac{1}{2}}} + \|\beta \Psi_k\|_{L^2} \right) \tag{24}$$
$$\le C \left(\|D_{g_k}(\beta \Psi_k)\|_{H^{-\frac{1}{2}}} + \|(D_{g_{\mathbb{R}^n}} - D_{g_k})(\beta \Psi_k)\|_{H^{-\frac{1}{2}}} + \|\beta \Psi_k\|_{L^2} \right).$$

We have $\|\beta \Psi_k\|_{L^2} \to 0$ and, since $g_k \to g_{\mathbb{R}^n}$ in C^∞, we also have $\|(D_{g_{\mathbb{R}^n}} - D_{g_k})(\beta \Psi_k)\|_{H^{-\frac{1}{2}}} \to 0$. For the last term, using Proposition 4.2, we have

$$\|D_{g_k}(\beta \Psi_k)\|_{H^{-\frac{1}{2}}} \le \|(G_{g_k}^s * |\Psi_k|^2)\beta \Psi_k + \beta F_k)\|_{H^{-\frac{1}{2}}} + o(1),$$

hence

$$\|D_{g_k}(\beta \Psi_k)\|_{H^{-\frac{1}{2}}} \le \|(G_{g_k}^s * |\Psi_k|^2)\beta \Psi_k)\|_{H^{-\frac{1}{2}}} + o(1),$$

since $\beta F_k \to 0$ in $H^{-\frac{1}{2}}$. Finally

$$\|\beta \Psi_k\|_{H^{\frac{1}{2}}} \le C \|(G_{g_k}^s * |\Psi_k|^2)\beta \Psi_k)\|_{L^{\frac{2n}{n+1}}} + o(1)$$
$$\le C \|G_{g_k}^s * |\Psi_k|^2\|_{L^n} \|\beta \Psi_k\|_{L^{\frac{2n}{n-1}}} + o(1)$$
$$\le C \varepsilon^{\frac{1}{n}} \|\beta \Psi_k\|_{L^{\frac{2n}{n-1}}} + o(1) \to 0 \, .$$

$\square$

We observe that by the regularity Theorem 3.1, we have that indeed $\Psi_\infty \in C^\infty(\mathbb{R}^n)$. Now, for $\varepsilon > 0$ and small enough, as before, there exists $R_k > 0$ such that $R_k \to 0$ and a sequence $x_k \in M$, that we can assume converging to $x_0 \in M$. We consider then a cut-off function $\beta = 1$ on $B_1(x_0)$ with $supp(\beta) \subset B_2(x_0)$ and we define $\phi_k \in C^\infty(M)$ by

$$\phi_k = R_k^{-\frac{n-1}{2}} \beta(\rho_k^{-1})^*(\Psi_\infty). \tag{25}$$

We have then the last result of this section:

Lemma 4.4. *Let (ψ_k) be a (PS) sequence and set $\overline{\psi}_k = \psi_k - \phi_k$. Then, up to a subsequences,*

$$\overline{\psi}_k \rightharpoonup 0 \ in \ H^{\frac{1}{2}}(M); \tag{26}$$

$$\nabla J_g(\phi_k) \to 0 \ and \ \nabla J_g(\overline{\psi}_k) \to 0, \ in \ H^{-\frac{1}{2}}(M). \tag{27}$$

Moreover, we have the following energy estimate

$$J_g(\overline{\psi}_k) = J_g(\psi_k) - J_{g_{\mathbb{R}^n}}(\Psi_\infty) + o(1). \tag{28}$$

Proof. The proofs of these last three estimates are similar to those in [35] (Lemma 4.8, 4.9, 4.10) and in [25] (Lemma 5.6, 5.7, 5.8); for this reason, in order to show how to handle our nonlinearity in this situation, we will prove only the first limit (26).

We know that $\psi_k \rightharpoonup 0$, therefore we need to study the weak convergence of ϕ_k; now, we know that this is bounded in $H^{\frac{1}{2}}(M)$, thus up to subsequences, it has a weak limit: we have to prove that the limit is zero. In particular, given a test spinor $h \in C^\infty(M)$, we will show that

$$\int_M \langle \phi_k, h \rangle dv_g \to 0. \tag{29}$$

Let us fix $R > 0$, we will prove two estimates, the first one on $B_{R_k R}(x_k)$ and the second one on $M \setminus B_{R_k R}(x_k)$. By definition of ϕ_k given in (25) and the conformal change, we have

$$\int_{B_{R_k R}(x_k)} \langle \phi_k, h \rangle dv_g = R_k^{-\frac{n-1}{2}} \int_{B_{R_k R}(x_k)} \beta \langle (\rho_k^{-1})^*(\Psi_\infty), h \rangle dv_g$$

$$= R_k^{\frac{n+1}{2}} \int_{B_R^0} \rho_k^*(\beta) \langle \Psi_\infty, \rho_k^*(h) \rangle dv_{g_k}.$$

Therefore,

$$\left| \int_{B_{R_k R}(x_k)} \langle \phi_k, h \rangle dv_g \right| \le C_1 R_k^{\frac{n+1}{2}} \|h\|_\infty \int_{B_R^0} |\Psi_\infty| dv_{g_{\mathbb{R}^n}}.$$

In the same way, if k is large enough, we have

$$\int_{M \setminus B_{R_k R}(x_k)} \langle \phi_k, h \rangle dv_g = R_k^{\frac{n+1}{2}} \int_{B_{3R_k^{-1}}^0 \setminus B_R^0} \rho_k^*(\beta) \langle \Psi_\infty, \rho_k^*(h) \rangle dv_{g_k}.$$

Thus,

$$\left| \int_{M \setminus B_{R_k R}(x_k)} \langle \phi_k, h \rangle dv_g \right| \le C_2 R_k^{\frac{n+1}{2}} \|h\|_\infty \int_{B_{3R_k^{-1}}^0 \setminus B_R^0} |\Psi_\infty| dv_{g_{\mathbb{R}^n}}$$

$$\le C_3 \|h\|_\infty \left(\int_{B_{3R_k^{-1}}^0 \setminus B_R^0} |\Psi_\infty|^{\frac{2n}{n-1}} dv_{g_{\mathbb{R}^n}} \right)^{\frac{n-1}{2n}}.$$

Finally, we put the two estimates together and we get

$$\left| \int_M \langle \phi_k, h \rangle dv_g \right| \le C \|h\|_\infty \left(R_k^{\frac{n+1}{2}} \int_{B_R^0} |\Psi_\infty| dv_{g_{\mathbb{R}^n}} + \left(\int_{B_{3R_k^{-1}}^0 \setminus B_R^0} |\Psi_\infty|^{\frac{2n}{n-1}} dv_{g_{\mathbb{R}^n}} \right)^{\frac{n-1}{2n}} \right).$$

Therefore, if $k \to \infty$ and then $R \to \infty$, we obtain (29). $\qquad \square$

Remark 4.2. *In order to finalize the proof of Theorem 1.1, we need a last estimate regarding the solutions of equation (23). Since Theorem 1.2 addresses an explicit conformal lower bound of the energy of ground state solutions, we will use the result and leave the details to*

the next section. Here we just notice the following fact: if Ψ_∞ satisfies equation (23), then its pull-back by the standard stereographic projection satisfies equation (9). Therefore, by (10) we have that there exists a positive constant $C_{\mathbb{R}^n}$, such that

$$J_{g_{\mathbb{R}^n}}(\Psi_\infty) \geq C_{\mathbb{R}^n}. \tag{30}$$

Proof. (*of Theorem (1.1)*)
Let (ψ_k) be a Palais-Smale sequence for J_g at level c; we will apply a standard iteration procedure. Let

$$\psi_k^1 := \psi_k - \psi_\infty,$$

then by Lemma 4.2 we have

$$J_g(\psi_k^1) = J_g(\psi_k) - J_g(\psi_\infty) + o(1) .$$

As we saw after Proposition 4.1, we can find a sequence of points $x_k^1 \in M$ converging to some point $x^1 \in M$, a sequence of real numbers R_k^1 converging to zero, a function Ψ_∞^1 solution of (23) and its related ϕ_k^1 defined as in (25). Next we define

$$\psi_k^2 := \psi_k^1 - \phi_k^1 = \psi_k - \psi_\infty - \phi_k^1 .$$

By equation (28) in Lemma 4.4, we obtain

$$J_g(\psi_k^2) = J_g(\psi_k^1) - J_{g_{\mathbb{R}^n}}(\Psi_\infty^1) + o(1) = J_g(\psi_k) - J_g(\psi_\infty) - J_{g_{\mathbb{R}^n}}(\Psi_\infty^1) + o(1) .$$

We can repeat this procedure m times, finding m sequences of points $x_k^1, \cdots, x_k^m \in M$ converging to some points $x^1, \cdots, x^m \in M$, m sequences of real numbers $R_k^1, \cdots, R_k^m$ converging to zero, m functions $\Psi_\infty^1, \cdots, \Psi_\infty^m$ solutions of (23) and the related $\phi_k^1, \cdots, \phi_k^m$ defined as in (25), with

$$\psi_k^{m+1} := \psi_k - \psi_\infty - \sum_{j=1}^m \phi_k^j ,$$

$$J_g(\psi_k^{m+1}) = J_g(\psi_k) - J_g(\psi_\infty) - \sum_{j=1}^m J_{g_{\mathbb{R}^n}}(\Psi_\infty^j) + o(1) .$$

Now, from (30) in Remark 4.2, we have that

$$J_{g_{\mathbb{R}^n}}(\Psi_\infty^j) \geq C_{\mathbb{R}^n} \quad j = 1, \ldots, m .$$

Therefore, since from Proposition 4.1 (PS) sequences at levels strictly below $\frac{\varepsilon_0}{4}$ converge strongly to zero in $H^{\frac{1}{2}}(M)$, we stop the iteration when $c - mC_{\mathbb{R}^n} < \frac{\varepsilon_0}{4}$, obtaining the thesis. $\qquad\square$

5 Least Energy Solution on the Sphere

In this section, we will provide the proof of Theorem 1.2.

We start by recalling the conformal invariant $\lambda^+(M,[g])$, which was thoroughly studied in[1, 2] in order to study the optimal first eigenvalues of the Dirac operator. It plays the same role as the classical Yamabe invariant but for its spinorial version (we also refer the reader to [40] for recent results on the spinorial Yamabe problem). One way of defining $\lambda^+(M,[g])$ is as follows:

$$\lambda^+(M,[g]) := \inf\{\lambda_1(D_h)Vol(h)^{\frac{1}{n}}, h \in [g]\}.$$

It can also be characterized by

$$\lambda^+(M,[g]) = \inf_{\psi \in C^\infty(\Sigma M); \langle D_g \psi, \psi \rangle \neq 0} \frac{\left(\int_M |D_g\psi|^{\frac{2n}{n+1}} \, dv\right)^{\frac{n+1}{n}}}{|\int_M \langle D_g\psi, \psi \rangle \, dv|}.$$

Proof. (*of Theorem (1.2)*)

If ψ is a non-trivial solution of (9), then we have

$$\lambda^+(S^n,[g_0]) \leq \frac{\left(\int_{S^n} \left(|\psi||G^s * |\psi|^2\right)^{\frac{2n}{n+1}} \, dv\right)^{\frac{n+1}{n}}}{\int_{S^n} G^s * |\psi|^2|\psi|^2 \, dv}$$

$$\leq \left(\int_{S^n} \left(G^s * |\psi|^2\right)^n \, dv\right)^{\frac{1}{n}}.$$

But $u = G^s * |\psi|^2$ satisfies $P^s_{g_0} u = |\psi|^2$. So if we define the H^s-norm by

$$\|u\|_{H^s} = \|(P^s_g)^{\frac{1}{2}} u\|_{L^2} = \|u P^s_g u\|_{L^1}^{\frac{1}{2}},$$

as in Section 2, we have from the Sobolev embedding $H^s(M) \hookrightarrow L^{\frac{2n}{n-2s}}(M) = L^n(M)$, that

$$\|u\|_{L^n} \leq Y_s(S^n,[g_0])^{\frac{1}{2}} \|u\|_{H^s},$$

where $Y_s(S^n,[g_0])$ is defined in (13). Thus,

$$\left(\int_{S^n} \left(G^s * |\psi|^2\right)^n \, dv\right)^{\frac{1}{n}} \leq \frac{1}{Y_s(S^n,[g_0])^{\frac{1}{2}}}\left(\int_{S^n} |\psi|^2 G^s * |\psi|^2 \, dv\right)^{\frac{1}{2}}. \tag{31}$$

In particular, we have that

$$4J_{g_0}(\psi) \geq \lambda^+(S^n,[g_0])^2 Y_s(S^n,[g_0]). \tag{32}$$

We assume now that ψ is a ground state solution on (S^n, g_0). Then we are in the case of equality in Hölder's inequalities. Namely,

$$\int_{S^n} \left(|\psi|^2 G^s * |\psi|^2\right)^{\frac{n}{n+1}} \left(G^s * |\psi|^2\right)^{\frac{n}{n+1}} \, dv = \left(\int_{S^n} |\psi|^2 G^s * |\psi|^2 \, dv\right)^{\frac{n}{n+1}} \left(\int_{S^n} (G^s * |\psi|^2)^n \, dv\right)^{\frac{1}{n}}.$$

Hence, for $c_n = \frac{\int_{S^n}(G^s * |\psi|^2)^n \ dv}{\int_{S^n}|\psi|^2 G^s * |\psi|^2 \ dv}$, we have

$$c_n|\psi|^2 = \left(G^s * |\psi|^2\right)^{n-1}. \tag{33}$$

From the equalities in (31) and (32), we have

$$c_n^{\frac{1}{n}} Y_s(S^n, [g_0])^{\frac{1}{2}} = \left(\lambda^+(S^n, [g_0])^2 Y_s(S^n, [g_0])\right)^{\frac{n-2}{2n}},$$

Thus

$$c_n = \frac{\lambda^+(S^n, [g_0])^{n-2}}{Y_s(S^n, [g_0])}.$$

On the other hand, from (33), we have that the function $u = G^s * |\psi|^2$ satisfies

$$P^s_{G^s_{\mathbb{R}^n}} u = \frac{1}{c_n} u^{n-1}$$

Hence, by the classification results in [11], we have that up to a conformal change, u is constant and hence $|\psi|^2$ is constant. In particular, from the case of equality in Hijazi's inequality [23, 24], ψ is a $-\frac{1}{2}$-Killing Spinor on S^n. $\qquad\square$

From the conformal invariance of (5) and (7), we also have the following

Corollary 5.1. *Let $\psi \in H^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{C}^N)$ be a non-trivial ground state solution for the equation*

$$D_{\mathbb{R}^n}\psi = G^s_{\mathbb{R}^n} * |\psi|^2 \psi,$$

where $G^s_{\mathbb{R}^n}$ is the Green's function of the Laplacian on $\mathbb{R}^n$, with $N = 2^{[\frac{n}{2}]}$. Then there exists $\Phi_0 \in \mathbb{C}^N$, a point $x_0 \in \mathbb{R}^n$ and $\lambda > 0$ so that

$$\psi(x) = c_n \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2}\right)^{\frac{n}{2}} \left(1 - \left(\frac{x - x_0}{\lambda}\right) \cdot \Phi_0\right).$$

6 Brezis-Nirenberg Problem

We focus now on the linearly perturbed problem

$$D_g\psi = \lambda\psi + G^s * |\psi|^2 \psi,$$

where $\lambda > 0$ and not a spectral value of D_g. Also, define H^+_λ, H^-_λ and H^0_λ to be the positive, negative and null space of $D_g - \lambda$ on $H^{\frac{1}{2}}(\Sigma M)$. Notice that zero is a trivial solution for the problem. In fact, one can obtain solutions to the problem if $\lambda < \lambda_{k+1} \in Spec(D_g)$ and λ close to λ_{k+1}. Indeed, this type of solutions can be obtained using bifurcation theory. For instance, if we define the operator $L_\lambda : H^{\frac{1}{2}}(M) \to H^{\frac{1}{2}}(M)$ by

$$L_\lambda\psi = (1 + |D_g|)^{-1}(D_g\psi) - (1 + |D_g|)^{-1}(\lambda\psi + G^s * |\psi|^2 \psi).$$

Then its differential $\nabla L_\lambda[0]$ takes the form $\nabla L_\lambda h = Ah + C(\lambda, h)$, where A is a self-adjoint Fredholm operator and C is compact (actually, it is a linear self-adjoint operator). Moreover,

we have $L_\lambda 0 = 0$ for all $\lambda \in \mathbb{R}$. Now, if we take $\lambda_k < \lambda_- < \lambda_{k+1} < \lambda_+$ then we can easily check that the operators $\nabla L_{\lambda_\pm}[0]$ are invertible. Moreover, the spectral flow of $\nabla L_\lambda[0]$ on $[\lambda_-, \lambda_+]$ is well defined and can be computed explicitly by

$$Sf(\nabla L_\lambda[0], [\lambda_-, \lambda_+]) = dim(H_{\lambda_-}^- \cap H_{\lambda_+}^+) - dim(H_{\lambda_+}^- \cap H_{\lambda_-}^+)$$
$$= -dim\ker(D_g - \lambda_{k+1}) \neq 0.$$

Hence, by Theorem in [16, Theorem 1], we have that 0 is a bifurcation point and hence, there exists a nontrivial solution ϕ_λ of $L_\lambda \phi_\lambda = 0$ for λ close to λ_{k+1}. Moreover, $\phi_\lambda \to 0$ as $\lambda \to \lambda_{k+1}$.

In what follows, we will show the existence of a non-trivial ground state solution ψ_λ without restriction on $\lambda > 0$ as long as $\lambda \notin Spec(D_g)$. Moreover, if $0 < \lambda \in (\lambda_k, \lambda_{k+1})$, the solution ψ_λ can be thought of as the extension of the bifurcation branch ϕ_λ to all the interval $(\lambda_k, \lambda_{k+1})$. We first start by preparing the variational setting allowing the construction of a minimizing sequence. The setting is very close to the work of Sire and Xu in [39] and the bifurcation result is close in nature to the one in [4] but we will not address the case when $\lambda \in Spec(D_g)$, although we expect a similar result to hold in our setting.

We recall that the energy functional of the problem has the following expression

$$J_{g,\lambda}(\psi) =: J_\lambda(\psi) = \frac{1}{2}\int_M \langle D\psi, \psi \rangle - \lambda|\psi|^2 dv - \frac{1}{4}\int_{M \times M} G^s(x,y)|\psi|^2(x)|\psi|^2(y)dv(x)dv(y).$$

For $\psi \in H^{\frac{1}{2}}$ we will write $\psi = \psi^+ + \psi^- \in H_\lambda^+ \oplus H_\lambda^-$, if λ is not an eigenvalue of D_g.

Proposition 6.1. *There exits a C^1-map $\tau : H_\lambda^+ \to H_\lambda^-$ such that for every $\psi \in H_\lambda^+$*

$$J_\lambda(\psi + h) < J_\lambda(\psi + \tau(\psi)), \forall h \in H_\lambda^-, h \neq \tau(h).$$

Moreover, τ satisfies the following properties:

i) $P_\lambda^-\left[D_g\tau(\psi) - \left(\int_M G^s(x,y)|\tau(\psi) + \psi|^2(y)\ dv(y)\right)(\psi + \tau(\psi))\right] = 0.$

ii) $\|\tau(\psi)\|_\lambda^2 \leq \frac{1}{2}\int_{M \times M}|\psi|^2(x)G^s(x,y)|\psi|^2(y)\ dv(y)dv(x).$

iii) *If $K(\psi) := \frac{1}{4}\int_{M \times M}G^s(x,y)|\psi|^2(x)|\psi|^2(y)\ dv(x)dv(y)$, then*

$$\|\nabla\tau(\psi)\|_{Op} \leq \|\nabla^2 K(\psi + \tau(\psi))\|_{Op}.$$

iv) *Let $\tilde{J} : H_\lambda^+ \to \mathbb{R}$ defined by $\tilde{J}(\psi) := J_\lambda(\psi + \tau(\psi))$. If $(\psi_k)_k$ is a (PS) sequence of $\tilde{J}$, then $(\psi_k + \tau(\psi_k))_k$ is a (PS)-sequence for J_λ and*

$$\|\nabla\tilde{J}(\psi)\| = \|\nabla J_\lambda(\psi + \tau(\psi))\|, \forall\psi \in H_\lambda^+.$$

Proof. First notice that the functional

$$h \to J_\lambda(\psi + h) = \frac{1}{2}\|\psi\|_\lambda^2 - \frac{1}{2}\|h\|_\lambda^2 - \frac{1}{4}\int_M\int_M G^s(x,y)|\psi + h|^2(x)|\psi + h|^2(y)\ dv(x)dv(y),$$

defined on H_λ^-, is strictly concave and anti-coercive, hence it has a unique maximizer $\tau(\psi)$ and therefore $i)$ is satisfied. Now, since $\tau(\psi)$ is a maximizer of $J_\lambda(\psi + \cdot)$ on H_λ^-, we have that

$$J_\lambda(\psi + \tau(\psi)) \geq J_\lambda(\psi).$$

It follows that

$$\|\tau(\psi)\|_\lambda^2 \leq \frac{1}{2}\Big(\int_{M \times M} G^s(x,y)|\psi|^2(x)|\psi|^2(y) \ dv(x)dv(y)$$

$$- \int_{M \times M} G^s(x,y)|\psi + \tau(\psi)|^2(x)|\psi + \tau(\psi)|^2(y) \ dv(x)dv(y)\Big)$$

$$\leq \frac{1}{2} \int_{M \times M} G^s(x,y)|\psi|^2(x)|\psi|^2(y) \ dv(x)dv(y).$$

and $ii)$ follows. We consider now the operator $T := -\nabla_h^2 J_\lambda(\psi + \cdot)[\tau(\psi)] : H_\lambda^- \to H_\lambda^-$ that can be expressed as

$$T(h) = -D_\lambda h + P_\lambda^-\Big(2\int_M G^s(x,y)\langle\psi+\tau(\psi), h\rangle \ dv(x) \ (\psi + \tau(\psi)) + \int_M G^s(x,y)|\psi+\tau(\psi)|^2 dv(x)h\Big).$$

Notice that T is positive definite and

$$\langle T(h), h\rangle = \|h\|_\lambda^2 + 2\int_{M \times M} G^s(x,y)\langle\psi + \tau(\psi), h\rangle(x)\langle\psi + \tau(\psi), h\rangle(y) \ dv(x)dv(y)$$

$$+ \int_{M \times M} G^s(x,y)|\psi + \tau(\psi)|^2|h|^2 \ dv(x)dv(y) \geq \|h\|_\lambda^2. \tag{34}$$

Hence, it is invertible and

$$\|T^{-1}\|_{Op} \leq 1.$$

On the other hand, if

$$L(h, \psi) = P_\lambda^-\Big[D_\lambda h - \int_M G^s(x,y)|h + \psi|^2(y) \ dv(y)(\psi + h)\Big],$$

then from $i)$, we have

$$L(\tau(\psi), \psi) = 0.$$

Applying the implicit function theorem yields

$$\nabla\tau(\psi)\phi = -\Big((\nabla_h L)(\tau(\psi), \psi)\Big)^{-1}(\nabla_\psi L)(\tau(\psi), \psi)\phi, \ \text{for all } \phi \in H_\lambda^+.$$

But $(\nabla_h L)(\tau(\psi), \psi) = T$ and $(\nabla_\psi L)(\tau(\psi), \psi) = \nabla^2 K(\psi + \tau(\psi))$. Hence,

$$\|\nabla\tau(\psi)\|_{Op} \leq \|\nabla^2 K(\psi + \tau(\psi))\|_{Op}.$$

and therefore (iii) holds. We finish the proof now by differentiating $\tilde{J}$ in order to get

$$\nabla\tilde{J}(\psi)\phi = \nabla J_\lambda(\psi + \tau(\psi))[\phi + \nabla\tau(\psi)\phi]$$

But $\nabla\tau(\psi)\phi \in H_\lambda^-$ and $\tau(\psi)$ is a critical point of $J_\lambda(\psi + \cdot)$ restricted to H_λ^-. Therefore,

$$\nabla\tilde{J}(\psi)\phi = \nabla J_\lambda(\psi + \tau(\psi))\phi, \forall \phi \in H_\lambda^+.$$

In particular, if $(\psi)_k \subset H_\lambda^+$ is a (PS) sequence for J_λ, then $(\psi_k + \tau(\psi_k))_k$ is a (PS) sequence for $\tilde{J}$. $\qquad\square$

We claim, next, that $\tilde{J}$ has a mountain-pass geometry. Indeed, we have $\tilde{J}(0) = J(0) = 0$. Moreover, if $\psi \in H_\lambda^+$ with $\|\psi\|_\lambda = 1$, we have

$$\tilde{J}(t\psi) \geq J_\lambda(t\psi) = \frac{t^2}{2} - \frac{t^4}{4} \int_M G^s * |\psi|^2 |\psi|^2 \, dv.$$

Therefore, there exists $t_0 > 0$ and $\nu_0 > 0$ such that

$$\tilde{J}(t\psi) \geq 0, \forall 0 \leq t \leq t_0 \text{ and } \tilde{J}(t_0\psi) \geq \nu_0.$$

In order to find a critical point for $\tilde{J}$ (and hence a critical point for J_λ), we define the min-max level δ_λ by setting

$$\delta_\lambda := \inf_{\psi \in H_\lambda^+ \setminus \{0\}} \max_{t > 0} \tilde{J}(t\psi).$$

Notice that $\delta_\lambda \geq \nu_0 > 0$. This critical level, if it exists, corresponds to the ground state of $\tilde{J}$ on the Nehari manifold

$$\mathcal{M} = \{\psi \in H_\lambda^+; \langle \nabla \tilde{J}(\psi), \psi \rangle = 0\}.$$

That is,

$$\delta_\lambda = \inf_{\psi \in \mathcal{M}} \tilde{J}(\psi),$$

as long as $\mathcal{M} \neq \emptyset$. If $\tilde{J}$ satisfies the (PS) condition and $\mathcal{M} \neq \emptyset$, then δ_λ is indeed a critical value for $\tilde{J}$ and hence for J_λ. But, $\tilde{J}$ and J_λ satisfy the (PS) condition only below $\overline{Y}$. So our objective now, is to show that $\delta_\lambda < \overline{Y}$. In the classical setting, one uses a test function (mainly grafting a standard bubble). In our case, some work needs to be done to handle the τ-component of any potential test spinor. To this end, we want to be able to estimate the energy level of a (PS) sequence of $\tilde{J}$ in terms of the energy levels of J_λ.

We consider then a $(PS)_c$ sequence $(\psi)_k$ for J_λ. That is, $J_\lambda(\psi_k) \to c > 0$ and $\|\nabla J_\lambda(\psi_k)\| \to 0$. Based on the study of (PS) sequences above, we know that $\|\psi_k\|_\lambda$ is bounded. Moreover, we have the following properties:

Proposition 6.2. *Given a $(PS)_c$ sequence $(\psi_k)_k$ for J_λ, we have*

i) $\|\psi_k^- - \tau(\psi_k)\|_\lambda = O\Big(\|\nabla J_\lambda(\psi_k)\|\Big).$

ii) $\nabla \tilde{J}(\psi_k^+) \to 0.$

iii) There exists $t_k > 0$ such that $t_k \psi_k^+ \in \mathcal{M}$. Moreover, $|t_k - 1| = O\Big(\|\nabla \tilde{J}(\psi_k^+)\|\Big).$

Proof. We start by the proof of i). We let $z_1 = \psi_k^+ + \tau(\psi_k^+)$ and $z_2 = \psi_k^- - \tau(\psi_k^+)$, so that $z_1 + z_2 = \psi_k$. Recall that

$$\langle \nabla J_\lambda(z_1), z_2 \rangle = 0.$$

Therefore,

$$-\langle z_1^-, z_2 \rangle_\lambda - \int_M G^s * |z_1|^2 \langle z_1, z_2 \rangle \, dv = 0.$$

On the other hand, we have

$$\langle \nabla J_\lambda(\psi_k), z_2 \rangle = -\langle \psi_k^-, z_2 \rangle_\lambda - \int_M G^s * |\psi_k|^2 \langle \psi_k, z_2 \rangle \, dv.$$

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Hence,

$$\langle \nabla J_\lambda(\psi_k), z_2 \rangle = -\|z_2\|_\lambda^2 + \int_M G^s * |z_1|^2 \langle z_1, z_2 \rangle \, dv - \int_M G^s * |\psi_k|^2 \langle \psi_k, z_2 \rangle \, dv.$$

Notice now, that

$$\int_M G^s * |z_1|^2 \langle z_1, z_2 \rangle \, dv - \int_M G^s * |\psi_k|^2 \langle \psi_k, z_2 \rangle \, dv = \langle \nabla K(z_1), z_2 \rangle - \langle \nabla K(\psi_k), z_2 \rangle.$$

Thus, there exists $\mu_k \in [0, 1]$ such that

$$\int_M G^s * |z_1|^2 \langle z_1, z_2 \rangle \, dv - \int_M G^s * |\psi_k|^2 \langle \psi_k, z_2 \rangle = -\langle \nabla^2 K(z_1 + \mu_k z_2)[z_2], z_2 \rangle \leq 0.$$

This yields

$$\|z_2\|_\lambda \leq \|\nabla J_\lambda(\psi_k)\| \to 0,$$

as claimed in i).

For the proof of ii), we start with

$$\nabla J_\lambda(z_1) = \nabla J_\lambda(\psi_k - z_2),$$

and since $z_2 \to 0$, as claimed in i), we have $\nabla J_\lambda(z_1) \to 0$. Therefore,

$$\|\nabla \tilde{J}(\psi_k^+)\| = \|\nabla J_\lambda(z_1)\| \to 0.$$

It remains now to prove iii), which is more involved. First, we claim that there exists $c_0 > 0$ such that

$$\int_M G^s * |z_1|^2 |z_1|^2 \, dv > c_0. \tag{35}$$

Indeed, we have

$$J_\lambda(z_1) - \frac{1}{2} \langle \nabla J_\lambda(z_1), z_1 \rangle = \frac{1}{2} \int_M G^s * |z_1|^2 |z_1|^2 \, dv.$$

On the other hand, since $z_2 \to 0$, we have that $J_\lambda(z_1) = J_\lambda(\psi_k) + o(1) = c + o(1)$ and $\langle \nabla J(z_1), z_1 \rangle = o(1)$. Hence,

$$\frac{1}{2} \int_M G^s * |z_1|^2 |z_1|^2 \, dv = c + o(1),$$

which finishes the proof of the claim. Now we consider the function $f(t) := \langle \nabla \tilde{J}(t\psi_k^+), t\psi_k^+ \rangle$. Notice that $f(1) \to 0$ as $k \to \infty$. Moreover, we have

$$f'(1) = \langle \nabla^2 \tilde{J}(\psi_k^+)[\psi_k^+], \psi_k^+ \rangle + \langle \nabla \tilde{J}(\psi_k^+), \psi_k^+ \rangle.$$

But expanding the first term of the previous equation yields

$$\begin{aligned} \langle \nabla^2 \tilde{J}(\psi_k^+)[\psi_k^+], \psi_k^+ \rangle &= \|\psi_k^+\|_\lambda^2 - \langle \nabla^2 K(\psi_k^+ + \tau(\psi_k^+))[\psi_k^+ + \nabla \tau(\psi_k^+)[\psi_k^+]], \psi_k^+ \rangle \\ &= \langle \nabla \tilde{J}(\psi_k^+), \psi_k^+ \rangle + \langle \nabla K(\psi_k^+ + \tau(\psi_k^+)), \psi_k^+ \rangle \\ &\quad - \langle \nabla^2 K(\psi_k^+ + \tau(\psi_k^+))[\psi_k^+ + \nabla \tau(\psi_k^+)[\psi_k^+]], \psi_k^+ \rangle. \end{aligned}$$

We set then $z_k := \psi_k^+ + \tau(\psi_k^+)$ and $h_k := \nabla\tau(\psi_k^+)[\psi_k^+] - \tau(\psi_k^+)$. Then we have

$$\langle \nabla K(z_k), \psi_k^+ \rangle - \langle \nabla^2 K(z_k)[z_k + h_k], \psi_k^+ \rangle = \langle \nabla K(z_k), z_k \rangle - \langle \nabla K(z_k), \tau(\psi_k^+) \rangle$$
$$- \langle \nabla^2 K(z_k)[z_k + h_k], z_k + h_k \rangle$$
$$+ \langle \nabla^2 K(z_k)[z_k + h_k], \nabla\tau(\psi_k^+)\psi_k^+ \rangle.$$

On the other hand, by differentiating $i)$ in Proposition 6.1 with respect to ψ, we have

$$-\langle \nabla\tau(\psi_k^+)[\psi_k^+], w \rangle_\lambda = \langle \nabla^2 K(z_k)[z_k + h_k], w \rangle.$$

In particular,

$$-\|\nabla\tau(\psi_k^+)[\psi_k^+]\|_\lambda^2 = \langle \nabla^2 K(z_k)[z_k + h_k], \nabla\tau(\psi_k^+)[\psi_k^+] \rangle.$$

Moreover, we have

$$-\|\tau(\psi_k)^+\|^2 = \langle \nabla K(z_k), \tau(\psi_k^+) \rangle \quad \text{and} \quad -\langle \tau(\psi_k^+), \nabla\tau(\psi_k^+)[\psi_k^+] \rangle = \langle \nabla K(z_k), \nabla\tau(\psi_k^+)[\psi_k^+] \rangle.$$

Hence,

$$\langle \nabla K(z_k), \psi_k^+ \rangle - \langle \nabla^2 K(z_k)[z_k + h_k], \psi_k^+ \rangle = -\|h_k\|_\lambda^2 + \langle \nabla K(z_k), z_k \rangle + 2\langle \nabla K(z_k), h_k \rangle$$
$$- \langle \nabla^2 K(z_k)[z_k + h_k], z_k + h_k \rangle.$$

Thus,

$$f'(1) = 2f(1) + \langle \nabla K(z_k), z_k \rangle - \langle \nabla^2 K(z_k)[z_k + h_k], z_k + h_k \rangle + 2\langle \nabla K(z_k), h_k \rangle - \|h_k\|_\lambda^2.$$

In order to evaluate the sign of $f'(1)$, we need to expand $\langle \nabla K(z_k), z_k \rangle - \langle \nabla^2 K(z_k)[z_k + h_k], z_k + h_k \rangle + 2\langle \nabla K(z_k), h_k \rangle$. Indeed,

$$\langle \nabla K(z_k), z_k \rangle - \langle \nabla^2 K(z_k)[z_k + h_k], z_k + h_k \rangle + 2\langle \nabla K(z_k), h_k \rangle = \int_M G^s * |z_k|^2 |z_k|^2 \, dv$$

$$- \int_M G^s * |z_k|^2 |z_k + h_k|^2 \, dv - 2\int_M G^s * \langle z_k + h_k, z_k \rangle \langle z_k + h_k, z_k \rangle \, dv$$

$$+ 2\int_M G^s * |z_k|^2 \langle z_k, h_k \rangle \, dv$$

$$= -2\int_M G^s * \langle z_k + h_k, z_k \rangle \langle z_k + h_k, z_k \rangle \, dv - \int_M G^s * |z_k|^2 |h_k|^2 \, dv$$

$$\leq -2\int_M G^s * \langle z_k + h_k, z_k \rangle \langle z_k + h_k, z_k \rangle \, dv \leq 0.$$

Therefore,

$$f'(1) \leq 2f(1) - 2\int_M G^s * \langle z_k + h_k, z_k \rangle \langle z_k + h_k, z_k \rangle \, dv - \|h_k\|_\lambda^2.$$

But $f(1) \to 0$ as $k \to \infty$, which leads to two cases. Either there exists $\mu_0 > 0$ such that $\|h_k\|_\lambda^2 \geq \mu_0$ for k large enough, and thus for k large enough

$$f'(1) \leq -\frac{\mu_0}{2},$$

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or $\|h_k\|_\lambda^2 \to 0$ and in that case,

$$\int_{M \times M} G^s(x,y) \langle z_k + h_k, z_k \rangle(x) \langle z_k + h_k, z_k \rangle(y) \ dv(x)dv(y) = \int_M G^s * |z_k|^2 |z_k|^2 \ dv + o(1).$$

Now using (35), we have the existence of $\mu_0 > 0$ such that

$$2 \int_M G^s * \langle z_k + h_k, z_k \rangle \langle z_k + h_k, z_k \rangle \ dv \geq \mu_0.$$

In conclusion, we have for k large

$$f'(1) \leq -\frac{\mu_0}{2}.$$

In particular, $f'(t) < -\frac{\mu_0}{4}$ in a small neighborhood of 1, independent of k, of the form $[1 - \mu, 1 + \mu]$ for a certain $\mu > 0$ and small but fixed. Using the mean value theorem, we have

$$f(1 + \mu) \leq f(1) - \frac{\mu_0 \mu}{4} < 0 \text{ and } f(1 - \mu) \geq f(1) + \frac{\mu_0 \mu}{4} > 0.$$

So there exists $t_k \in [1 - \mu, 1 + \mu]$ such that $f(t_k) = 0$. Moreover, since $|\frac{1}{f'(t)}| \leq \frac{4}{\mu_0}$ for $t \in [1 - \mu, 1 + \mu]$, we have

$$|t_k - 1| = |f^{-1}(0) - f^{-1}(f(1))| \leq \frac{4}{\mu_0}|f(1)| = O\left(\|\nabla \tilde{J}(\psi_k^+)\|\right),$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 6.3. *Assume that $(\psi_k)_k$ is a $(PS)_c$ sequence for J_λ with $c > 0$. Then*

$$\delta_\lambda \leq J_\lambda(\psi_k) + O\left(\|\nabla J_\lambda(\psi_k)\|^2\right).$$

In particular, if J_λ satisfies the (PS) condition at the level set δ_λ then it has a critical point ψ at that level.

Proof. We will be using the notations of the previous proof. That is, we let $z_k = \psi_k^+ + \tau(\psi_k^+)$ and $w_k = t_k \psi_k^+ + \tau(t_k \psi_k^+)$. Then we have from Proposition 6.2:

$$\|\psi_k - w_k\|_\lambda \leq \|\psi_k - z_k\|_\lambda + |t_k - 1|\|\psi_k^+\|_\lambda + \|\tau(\psi_k^+) - \tau(t_k \psi_k^+)\|_\lambda = O\left(\|\nabla J_\lambda(\psi_k)\|\right) + O\left(\|\nabla \tilde{J}(\psi_k^+)\|\right).$$

On the other hand,

$$\|\nabla \tilde{J}(\psi_k^+)\| = \|\nabla J_\lambda(z_k)\| = \|\nabla J_\lambda(\psi_k)\| + O\left(\|z_k - \psi_k\|_\lambda\right) = O\left(\|\nabla J_\lambda(\psi_k)\|\right).$$

In particular, we have

$$\|\psi_k - w_k\|_\lambda \leq O\left(\|\nabla J_\lambda(\psi_k)\|\right).$$

Next, we notice that since $t_k \psi_k^+ \in \mathcal{M}$, we have that

$$\langle \nabla J_\lambda(w_k), \psi_k - w_k \rangle = \langle \nabla J_\lambda(w_k), (\psi_k - w_k)^+ \rangle = (1 - t_k)\langle \nabla \tilde{J}(t_k \psi_k^+), \psi_k^+ \rangle = 0.$$

Hence,

$$J_\lambda(\psi_k) = J_\lambda(w_k) + \langle \nabla J_\lambda(w_k), \psi_k - w_k \rangle + O\left(\|\psi_k - w_k\|_\lambda^2\right)$$

$$= J_\lambda(w_k) + O\left(\|\nabla J_\lambda(\psi_k)\|^2\right). \tag{36}$$

Therefore,

$$\delta_\lambda \leq \tilde{J}(t_k \psi_k^+) = J_\lambda(w_k) = J_\lambda(\psi_k) + O\left(\|\nabla J_\lambda(\psi_k)\|^2\right).$$

$\square$

6.1 Test Spinor

We are now ready to construct a test spinor that will allow us to go under the critical energy threshold and hence have compactness of the minimizing Palais-Smale sequence. We will closely follow the construction in [25] and [39].

Consider a constant spinor ψ_0 on $\mathbb{R}^n$ such that $|\psi_0|^2 = a_n$, where a_n is a constant satisfying

$$a_n^{\frac{n}{n-1}} \omega_n = 2^n \overline{Y} c_n^{-\frac{1}{n-1}}.$$

Here, c_n is the constant introduced in (33). We define now the spinor

$$\Psi = \left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}} (1-x) \cdot \psi_0,$$

so that if $f(r) = \frac{1}{1+r^2}$, then $|\Psi|^2 = a_n f(|x|)^{n-1}$. Notice that

$$D_{\mathbb{R}^n} \Psi = \frac{n}{2} f \Psi.$$

We fix $\delta > 0$ so that $2\delta < i(M)$, the injectivity radius of M. We let η to be a smooth function on $\mathbb{R}^n$ with support in $B_{2\delta}(0) =: B_{2\delta}$ such that $\eta = 1$ on $B_\delta(0) =: B_\delta$. Now, we can define the spinor $\psi_\varepsilon(x) = \eta(x)\varepsilon^{-\frac{n-1}{2}} \Psi(\frac{x}{\varepsilon}) = \eta(x)\Psi_\varepsilon(x)$. Next, we use the Bourguignon-Gauduchon [5] trivialization in order to graft the spinor ψ_ε on M. Indeed, we fix $p_0 \in M$ and $(x_1, \cdots, x_n)$ local normal coordinates around p_0 provided by the exponential map $\exp_{p_0}$. That is, there exists a neighborhood $U \subset T_{p_0}M = \mathbb{R}^n$ and a neighborhood $V \subset M$, such that $\exp_{p_0}: U \to V$ is a diffeomorphism.

Let $G(p) = (g_{ij}(p))_{ij}$ be the components of the metric at p and $B = G^{-\frac{1}{2}}$. Notice that B is well defined since G is symmetric and positive definite. With these notations, we have that $B^* g = g_{\mathbb{R}^n}$. Therefore, B defines an isometry as a map $B(p): (T_{\exp_{p_0}^{-1} p} U, g_{\mathbb{R}^n}) \to (T_p V, g(p))$. Hence, given an oriented frame $(y_1, \cdots, y_n)$ on U, we obtain a natural oriented frame on V by taking $(By_1, \cdots, By_n)$. Thus, one has an isomorphism of the $SO(n)$-principal bundle induced by the map $\Phi(y_1, \cdots, y_n) = (By_1, \cdots, By_n)$ as described in the diagram below:

$$
\begin{array}{ccc}
P_{SO}(U, g_{\mathbb{R}^n}) & \xrightarrow{\;\;\;\Phi\;\;\;} & P_{SO}(V, g) \subset P_{SO}(M, g) \\
\Big\downarrow & & \Big\downarrow \\
U \subset T_{p_0}M & \xrightarrow[\;\;\exp_{p_0}\;\;]{} & V \subset M
\end{array}
$$

The map Φ commutes with the right action of $SO(n)$ and hence it induces an isomorphism of spin structures:

$$U \times Spin(n) = P_{Spin}(U, g_{\mathbb{R}^n}) \xrightarrow{\tilde{\Phi}} P_{Spin}(V, g) \subset P_{Spin}(M, g)$$

$$U \subset T_{p_0}M \xrightarrow{\exp_{p_0}} V \subset M$$

This leads to an isomorphism between the spin bundles $\Sigma_{g_{\mathbb{R}^n}} U$ and $\Sigma_g V$. If we let $e_i = B(\partial_{x_i})$ we then obtain an orthonormal frame $(e_1, \cdots e_n)$ of (TV, g). We let ∇ and $\overline{\nabla}$, respectively the Levi-Civita connections on $(TU, g_{\mathbb{R}^n})$ and (TV, g). We will keep the same notations for their natural lifts to $\Sigma_{g_{\mathbb{R}^n}} U$ and $\Sigma_g V$. From now on, if $H \to U$ (resp. $H \to V$) is a smooth bundle over U (resp. over V), we let $\Gamma(H)$ be the space of smooth sections of H. The Clifford multiplications then satisfy

$$e_i \cdot \overline{\psi} = B(\partial_{x_i}) \cdot \overline{\psi} = \overline{\partial_{x_i} \cdot \psi},$$

where here we use the identification that any $\psi \in \Gamma(\Sigma_{g_{\mathbb{R}^n}} U)$ corresponds via the previously defined isomorphism to a spinor $\overline{\psi} \in \Gamma(\Sigma_g V)$. If D and $\overline{D}$ are the Dirac operators acting on $\Gamma(\Sigma_{g_{\mathbb{R}^n}} U)$ and $\Gamma(\Sigma_g V)$, then we have for $\psi \in \Gamma(\Sigma U)$

$$\overline{D}\overline{\psi} = \overline{D\psi} + W \cdot \overline{\psi} + X \cdot \overline{\psi} + \sum_{i,j}(b_{ij} - \delta_{ij})\overline{\partial_{x_i} \cdot \nabla_{\partial_{x_j}} \psi},$$

where here, the b_{ij} are such that $e_i = \sum_j b_{ij}\partial_{x_j}$, $W \in \Gamma(Cl(TV))$ and $X \in \Gamma(TV)$ are defined by

$$W = \frac{1}{4} \sum_{i,j,k;i\neq j\neq k\neq i} \sum_{\alpha,\beta} b_{i\alpha}(\partial_{x_\alpha} b_{j\beta}) b_{\beta k}^{-1} e_i \cdot e_j \cdot e_k,$$

and

$$X = \frac{1}{4} \sum_{i,k}(\overline{\Gamma}_{ik}^i - \overline{\Gamma}_{ii}^k)e_k = \frac{1}{2} \sum_{i,k} \overline{\Gamma}_{ik}^i e_k.$$

Using the identification between $x \in \mathbb{R}^n$ and $p = \exp_{p_0} x \in M$, we can write as in [25, 39], that $G = I + O(|x|^2)$ as $|x| \to 0$. Hence, we have

$$b_{ij} = \delta_{ij} + O(|x|^2), \quad W = O(|x|^3) \text{ and } X = O(|x|) \text{ as } |x| \to 0.$$

Our test spinor then, will be $\varphi_\varepsilon := \overline{\psi_\varepsilon}$. Our ultimate goal in here is to apply Proposition 6.3 for the test spinor φ_ε. In order to do that, we need to show that $(\varphi_\varepsilon)_\varepsilon$ is a $(PS)_c$ sequence for J_λ. So we start by estimating the gradient of J_λ at φ_ε:

Lemma 6.1. *For φ_ε defined as above, we have*

$$\|\nabla J_\lambda(\varphi_\varepsilon)\|_{H_\lambda^*} \leq \begin{cases} O(\varepsilon|\ln(\varepsilon)|^{\frac{2}{3}}), & \text{if } n = 3 \\ \\ \varepsilon, & \text{if } n \geq 4 \end{cases}.$$

Proof. We need to estimate the H_λ^*-norm of φ_ε and $R_\varepsilon = \overline{D}\varphi_\varepsilon - \left(G_s * |\varphi_\varepsilon|^2\right)\varphi_\varepsilon$, where H_λ^* here is the dual of the space $H^{\frac{1}{2}}(M)$ equipped with the norm $\|\cdot\|_\lambda$. One notices that since $\lambda \notin Spec(D_g)$, the $\|\cdot\|_\lambda$- norm is equivalent to the usual $H^{\frac{1}{2}}(M)$ norm. Hence, by the continuous embedding $L^{\frac{2n}{n+1}}(M) \hookrightarrow H^{-\frac{1}{2}}(M)$, we have that for all $\psi \in L^{\frac{2n}{n+1}}(M)$,

$$\|\psi\|_{H_\lambda^*} \leq C\|\psi\|_{L^{\frac{2n}{n+1}}}.$$

Therefore we have

$$\|\varphi_\varepsilon\|_{H_\lambda^*} \leq C\|\varphi_\varepsilon\|_{L^{\frac{2n}{n+1}}} = \left(\int_{B_{2\delta}} |\varphi_\varepsilon|^{\frac{2n}{n+1}} \, dv\right)^{\frac{n+1}{2n}}$$

$$\leq C\left(\int_{|x|\leq 2\delta} |\psi_\varepsilon|^{\frac{2n}{n+1}} \, dx\right)^{\frac{n+1}{2n}}$$

$$\leq C\varepsilon\left(\int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{n-1}}{(1+r^2)^{\frac{n(n-1)}{n+1}}} \, dr\right)^{\frac{n+1}{2n}}$$

$$\leq C\begin{cases} \varepsilon|\ln(\varepsilon)|^{\frac{2}{3}} \text{ if } n = 3 \\ \varepsilon \text{ if } n \geq 4 \end{cases}. \tag{37}$$

Next, we move to estimating R_ε. Indeed, we have

$$\overline{D}\varphi_\varepsilon = \overline{D\psi_\varepsilon} + W \cdot \overline{\psi_\varepsilon} + X \cdot \overline{\psi} + \sum_{i,j}(b_{ij} - \delta_{ij})\overline{\partial_{x_i} \cdot \nabla_{\partial_{x_j}}\psi_\varepsilon}$$

$$= \left(\int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y)|\Psi_\varepsilon(y)|^2 \, dy\right)\varphi_\varepsilon + (\nabla\eta(x) + X) \cdot \varphi_\varepsilon + W \cdot \varphi_\varepsilon$$

$$+ \sum_{i,j}(b_{ij} - \delta_{ij})\overline{\partial_{x_i} \cdot \nabla_{\partial_{x_j}}\psi_\varepsilon}. \tag{38}$$

On the other hand,

$$\overline{D\psi_\varepsilon} - (G_g^s * |\varphi_\varepsilon|^2)\varphi_\varepsilon = \left(\int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y)|\Psi_\varepsilon(y)|^2 \, dy\right)\varphi_\varepsilon - \left(\int_M G_g^s(x,y)|\varphi_\varepsilon|^2 \, dy\right)\varphi_\varepsilon + \nabla\eta(x) \cdot \varphi_\varepsilon$$

$$= \left(\int_{|x-y|<\frac{\delta}{2}} [G_{\mathbb{R}^n}^s(x,y) - G_g^s(x,y)]|\varphi_\varepsilon(y)|^2 \, dy\right)\varphi_\varepsilon$$

$$+ \left(\int_{|x-y|>\frac{\delta}{2}} G_{\mathbb{R}^n}^s(x,y)|\Psi_\varepsilon|^2 dy\right)\varphi_\varepsilon - \left(\int_{|x-y|>\frac{\delta}{2}} G_g^s(x,y)|\varphi_\varepsilon|^2 \, dy\right)\varphi_\varepsilon$$

$$+ \nabla\eta \cdot \varphi_\varepsilon.$$

This leads to

$$\overline{D}\varphi_\varepsilon - (G_g^s * |\varphi_\varepsilon|^2)\varphi_\varepsilon = \left(\int_{|x-y|<\frac{\delta}{2}} [G_{\mathbb{R}^n}^s(x,y) - G_g^s(x,y)]|\varphi_\varepsilon(y)|^2 \, dy\right)\varphi_\varepsilon$$

$$+ \left(\int_{|x-y|>\frac{\delta}{2}} G_{\mathbb{R}^n}^s(x,y)|\Psi_\varepsilon|^2 dy\right)\varphi_\varepsilon - \left(\int_{|x-y|>\frac{\delta}{2}} G_g^s(x,y)|\varphi_\varepsilon|^2 \, dy\right)\varphi_\varepsilon$$

$$+ \nabla\eta \cdot \varphi_\varepsilon + W \cdot \overline{\psi_\varepsilon} + X \cdot \overline{\psi} + \sum_{i,j}(b_{ij} - \delta_{ij})\overline{\partial_{x_i} \cdot \nabla_{\partial_{x_j}}\psi_\varepsilon}$$

$$= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.$$

We will estimate now the terms $A_i, i = 1, \cdots, 7$. Indeed, for A_4, we have

$$\|A_4\|_{H^*_\lambda} \le C\|A_4\|_{L^{\frac{2n}{n+1}}} = C\Big(\int_{B_{2\delta}} |\overline{\nabla \eta \cdot \Psi_\varepsilon}|^{\frac{2n}{n+1}} \, dv \Big)^{\frac{n+1}{2n}}$$

$$\le C\Big(\int_{\delta \le |x| \le 2\delta} |\Psi_\varepsilon|^{\frac{2n}{n+1}} \, dx \Big)^{\frac{n+1}{2n}}$$

$$\le C\varepsilon\Big(\int_{\frac{\delta}{\varepsilon}}^{\frac{2\delta}{\varepsilon}} \frac{r^{n-1}}{(1+r^2)^{\frac{n(n-1)}{(n+1)}}} \, dr \Big)^{\frac{n+1}{2n}} \le C\varepsilon^{\frac{n-1}{2}}.$$

For A_1, we use Proposition 2.3 in Section 2 in order to have

$$\|A_1\|_{H^*_\lambda} \le C\|A_1\|_{L^{2n_{n+1}}} \le C\Big(\int_{B_{2\delta}} \Big(\int_{B_{2\delta}} \frac{1}{|x-y|} |\Psi_\varepsilon(y)|^2 \, dy |\Psi_\varepsilon(x)| \Big)^{\frac{2n}{n+1}} dx \Big)^{\frac{n+1}{2n}}$$

$$\le C\|\Psi_\varepsilon\|_{L^{\frac{2n}{n-1}}} \int_{B_{2\delta}} \Big(\int_{B_{2\delta}} \frac{1}{|x-y|} |\Psi_\varepsilon(y)|^2 \, dy \Big)^n \, dx \Big)^{\frac{1}{n}}$$

$$\le C\|\Psi_\varepsilon\|^2_{L^{\frac{2n}{n+1}}(B_{2\delta})}$$

$$\le C \begin{cases} \varepsilon^2 |\ln(\varepsilon)|^{\frac{4}{3}} \text{ if } n = 3 \\ \varepsilon^2 \text{ if } n \ge 4 \end{cases}.$$

For A_2, we use the fact that the Green's function is bounded outside of the diagonal. Thus,

$$\|A_2\|_{H^*_\lambda} \le C\|\Psi_\varepsilon\|^2_{L^2}\|\varphi_\varepsilon\|_{L^{\frac{2n}{n+1}}}$$

$$\le C\varepsilon^2.$$

A similar inequality holds for $\|A_3\|_{H^*_\lambda}$. On the other hand,

$$\|A_5\|_{H^*_\lambda} \le C\Big(\int_{B_{2\delta}} |W|^{\frac{2n}{n+1}} |\varphi_\varepsilon|^{\frac{2n}{n+1}} \, dv \Big)^{\frac{n+1}{n}} \le C\Big(\int_{|x| \le 2\delta} |x|^{\frac{6n}{n+1}} |\Psi_\varepsilon|^{\frac{2n}{n+1}} \, dx \Big)^{\frac{n+1}{2n}}$$

$$\le C\varepsilon^4 \int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{\frac{6n}{n+1}+n-1}}{(1+r^2)^{\frac{n(n-1)}{n+1}}} \, dr \Big)^{\frac{n+1}{2n}}$$

$$\le C \begin{cases} \varepsilon^{\frac{n-1}{2}} \text{ if } 3 \le n \le 8 \\ \varepsilon^4 |\ln(\varepsilon)|^{\frac{5}{9}} \text{ if } n = 9 \\ \varepsilon^4 \text{ if } n \ge 10 \end{cases}.$$

Similarly for A_6 we have

$$\|A_6\|_{H_\lambda^*} \leq C \Big(\int_{B_{2\delta}} |X|^{\frac{2n}{n+1}} |\varphi_\varepsilon|^{\frac{2n}{n+1}} \ dv \Big)^{\frac{n+1}{2n}}$$

$$\leq C \Big(\int_{|x| \leq 2\delta} |x|^{\frac{2n}{n+1}} |\psi_\varepsilon|^{\frac{2n}{n+1}} \ dx \Big)^{\frac{n+1}{2n}}$$

$$\leq C\varepsilon^2 \Big(\int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{\frac{2n}{n+1}+m-1}}{(1+r^2)^{\frac{n(n-1)}{n+1}}} \ dx \Big)^{\frac{n+1}{2n}}$$

$$\leq C \begin{cases} \varepsilon^{\frac{n-1}{2}} & \text{if } n = 3,4 \\ \varepsilon^2 |\ln(\varepsilon)|^{\frac{3}{5}} & \text{if } n = 5 \\ \varepsilon^2 & \text{if } n \geq 6 \end{cases}.$$

It remains now to estimate A_7. We will write $A_7 = B_1 + B_2$, where

$$B_1 := \eta \sum_{i,j} (b_{ij} - \delta_{ij}) \overline{\partial_{x_i} \cdot \nabla_{\partial_{x_j}} \Psi_\varepsilon} \quad \text{and} \quad B_2 := \sum_{i,j} (b_{ij} - \delta_{ij})(\partial_{x_j}\eta)\overline{\partial_{x_i} \cdot \Psi_\varepsilon}.$$

Notice that since $|\nabla\Psi| \leq Cf(r)^{\frac{n}{2}}$, we have

$$\|B_1\|_{H_\lambda^*} \leq C \Big(\int_{|x| \leq 2\delta} |x|^{\frac{4n}{n+1}} |\nabla\Psi_\varepsilon|^{\frac{2n}{n+1}} \ dx \Big)^{\frac{n+1}{2n}}$$

$$\leq C\varepsilon^2 \Big(\int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{\frac{4n}{n+1}+n-1}}{(1+r^2)^{\frac{n^2}{n+1}}} \ dr \Big)^{\frac{n+1}{2n}}$$

$$\leq \begin{cases} \varepsilon^{\frac{n-1}{2}} & \text{if } n = 3,4 \\ \varepsilon^2 |\ln(\varepsilon)|^{\frac{3}{5}} & \text{if } n = 5 \\ \varepsilon^2 & \text{if } n \geq 6 \end{cases}.$$

We finish now by estimating B_2:

$$\|B_2\|_{H_\lambda^*} \leq C \Big(\int_{|x| \leq 2\delta} |x|^{\frac{6n}{n+1}} |\Psi_\varepsilon|^{\frac{2n}{n+1}} \ dx \Big)^{\frac{n+1}{2n}}$$

$$\leq C \begin{cases} \varepsilon^{\frac{n-1}{2}} & \text{if } 3 \leq n \leq 8 \\ \varepsilon^4 |\ln(\varepsilon)|^{\frac{5}{9}} & \text{if } n = 9 \\ \varepsilon^4 & \text{if } n \geq 10 \end{cases}.$$

All the previous estimates can be summarized as follows:

$$\|R_\varepsilon\|_{H_\lambda^*} \leq C \begin{cases} \varepsilon^{\frac{n-1}{2}} & \text{if } n = 3,4 \\ \varepsilon^2 |\ln(\varepsilon)|^{\frac{3}{5}} & \text{if } n = 5 \\ \varepsilon^2 & \text{if } n \geq 6 \end{cases}. \tag{39}$$

Combining (37) and (39) yields the desired result. $\qquad \square$

After observing that $(\varphi_\varepsilon)_\varepsilon$ is indeed a (PS) sequence with a precise estimate on $\|\nabla J_\lambda(\varphi_\varepsilon)\|_{H_\lambda^*}$, we proceed now to estimate the energy.

Lemma 6.2. *For φ_ε defined as above, we have*

i) $\|\varphi_\varepsilon\|_{L^2} = Q(\varepsilon) + C \begin{cases} O(\varepsilon^{n-1}) \text{ if } n = 3 \\ O(\varepsilon^3|\ln(\varepsilon)|) \text{ if } n = 4 \\ O(\varepsilon^3) \text{ if } n \geq 5 \end{cases}$,

where $Q(\varepsilon) = \varepsilon a_n \omega_{n-1} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-1}} \, dr$.

ii) $J_g(\varphi_\varepsilon) \leq \overline{Y} + O(\varepsilon^2)$.

Proof. Recall that the volume form in normal coordinates takes the form $dv_g = dx + O(|x|^2)$ around p_0. Hence we have

$$
\int_M |\varphi_\varepsilon|^2 \, dv_g = \int_{B_{2\delta}} |\varphi_\varepsilon|^2 \, dv_g
$$

$$
= \int_{|x| \leq \delta} |\Psi_\varepsilon|^2 \, dx + \int_{\delta \leq |x| \leq 2\delta} |\eta(x)\Psi_\varepsilon|^2 \, dx + O\left(\int_{|x| \leq \delta} |x|^2 |\Psi_\varepsilon|^2 \, dx\right)
$$

$$
= \varepsilon a_n \omega_{n-1} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{n-1}}{(1+r^2)^{n-1}} \, dr + O\left(\varepsilon \int_{\frac{\delta}{\varepsilon}}^{\frac{2\delta}{\varepsilon}} \frac{r^{n-1}}{(1+r^2)^{n-1}} \, dr\right) + O\left(\varepsilon^3 \int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{n+1}}{(1+r^2)^{n-1}} \, dr\right)
$$

$$
= Q(\varepsilon) + O(\varepsilon^{n-1}) + C \begin{cases} O(\varepsilon^{n-1}) \text{ if } n = 3 \\ O(\varepsilon^3|\ln(\varepsilon)|) \text{ if } n = 4 \\ O(\varepsilon^3) \text{ if } n \geq 5 \end{cases}
$$

$$
= Q(\varepsilon) + \begin{cases} O(\varepsilon^{n-1}) \text{ if } n = 3 \\ O(\varepsilon^3|\ln(\varepsilon)|) \text{ if } n = 4 \\ O(\varepsilon^3) \text{ if } n \geq 5 \end{cases}. \tag{40}
$$

Here,

$$
Q(\varepsilon) := \varepsilon a_n \omega_{n-1} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-1}} \, dr.
$$

Next, we estimate $\int_M \langle \overline{D}\varphi_\varepsilon, \varphi_\varepsilon \rangle \, dv_g$. Using the same decomposition as in (38), we see that

$$
\int_M \langle \overline{D}\varphi_\varepsilon, \varphi_\varepsilon \rangle \, dv_g = \int_M \int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|\varphi_\varepsilon|^2(x) \, dy \, dv_g(x) + \int_M \eta^2 \langle W \cdot \overline{\Psi_\varepsilon}, \overline{\Psi_\varepsilon} \rangle \, dv_g
$$

$$
+ \int_M \sum_{i,j} (b_{ij} - \delta_{ij})\eta^2 \langle \overline{\partial_{x_i} \cdot \nabla_{\partial_{x_j}} \Psi_\varepsilon}, \overline{\Psi_\varepsilon} \rangle \, dv_g
$$

$$
= F_1 + F_2 + F_3.
$$

We will estimate each term individually starting by F_1. Indeed,

$$
F_1 = \int_M \int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)\eta(x)|\overline{\Psi_\varepsilon}|^2(x) \, dy \, dv_g(x)
$$

$$
= \int_{|x| \leq \delta} \int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|\Psi_\varepsilon|^2(x) \, dy dx
$$

$$
+ O\left(\int_{\delta \leq |x| \leq 2\delta} \int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|\Psi_\varepsilon|^2(x) \, dy dx\right)
$$

$$
+ O\left(\int_{|x| \leq 2\delta} \int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|x|^2|\Psi_\varepsilon|^2(x) \, dy dx\right).
$$

But recall that

$$\int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)\ dy = c_n^{\frac{1}{n-1}}|\Psi_\varepsilon|^{\frac{2}{n-1}}(x),$$

where c_n is the constant defined in (33). Hence,

$$\int_{|x|\leq\delta}\int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|\Psi_\varepsilon|^2(x)\ dydx = c_n^{\frac{1}{n-1}}a_n^{\frac{n}{n-1}}\omega_{n-1}\int_0^{\frac{\delta}{\varepsilon}}\frac{r^{n-1}}{(1+r^2)^n}\ dr$$

$$= c_n^{\frac{1}{n-1}}a_n^{\frac{n}{n-1}}\omega_{n-1}\int_0^{+\infty}\frac{r^{n-1}}{(1+r^2)^n}\ dr + O(\varepsilon^n).$$
$$(41)$$

On the other hand,

$$\int_{\delta\leq|x|\leq2\delta}\int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|\Psi_\varepsilon|^2(x)\ dydx = c_n^{\frac{1}{n-1}}a_n^{\frac{n}{n-1}}\omega_{n-1}\int_{\frac{\delta}{\varepsilon}}^{\frac{2\delta}{\varepsilon}}\frac{r^{n-1}}{(1+r^2)^n}\ dr$$

$$= O(\varepsilon^n).$$
$$(42)$$

And to finish, we have

$$O\left(\int_{|x|\leq2\delta}\int_{\mathbb{R}^n} G^s_{\mathbb{R}^n}(x,y)|\Psi_\varepsilon|^2(y)|x|^2|\Psi_\varepsilon|^2(x)\ dydx\right) = O\left(\varepsilon^2\int_0^{\frac{2\delta}{\varepsilon}}\frac{r^{n+1}}{(1+r^2)^n}\ dr\right)$$

$$= O(\varepsilon^2).$$

Therefore,

$$F_1 = c_n^{\frac{1}{n-1}}a_0^{\frac{n}{n-1}}\omega_{n-1}\int_0^{+\infty}\frac{r^{n-1}}{(1+r^2)^n}\ dr + O(\varepsilon^2).$$

The estimates for F_2 and F_3 are relatively simpler. Indeed,

$$F_2 \leq C\int_{|x|\leq2\delta}|x|^3|\Psi_\varepsilon|^2\ dx \leq C\varepsilon^4\int_0^{\frac{2\delta}{\varepsilon}}\frac{r^{n+2}}{(1+r^2)^n}\ dr$$

$$\leq \begin{cases} O(\varepsilon^{n-1}) \text{ if } n = 3,4 \\ O(\varepsilon^4|\ln(\varepsilon)|) \text{ if } n = 5 \\ O(\varepsilon^4) \text{ if } n \geq 6 \end{cases}.$$

Similarly,

$$F_3 \leq C\int_{|x|\leq2\delta}|x|^2|\nabla\Psi_\varepsilon||\Psi_\varepsilon|\ dx \leq C\varepsilon^2\int_0^{\frac{2\delta}{\varepsilon}}\frac{r^{n+1}}{(1+r^2)^n}\ dr$$

$$\leq \begin{cases} O(\varepsilon^2|\ln(\varepsilon)|) \text{ if } n = 3 \\ O(\varepsilon^2) \text{ if } n \geq 4 \end{cases}.$$

Thus,

$$\int_M\langle\overline{D}\varphi_\varepsilon,\varphi_\varepsilon\rangle\ dv_g \leq c_n^{\frac{1}{n-1}}a_n^{\frac{n}{n-1}}\omega_{n-1}\int_0^{+\infty}\frac{r^{n-1}}{(1+r^2)^n}\ dr + \begin{cases} O(\varepsilon^2|\ln(\varepsilon)|) \text{ if } n = 3 \\ O(\varepsilon^2) \text{ if } n \geq 4 \end{cases}.$$

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Now we need to estimate the second term of the energy functional J_g.

$$\int_{M \times M} |\varphi_\varepsilon|^2(x) G_g^s(x,y) |\varphi_\varepsilon|^2(y) \, dv(x) dv(y) = \int_{|x-y| < \frac{\delta}{2}} |\varphi_\varepsilon|^2(x) G_g^s(x,y) |\varphi_\varepsilon|^2(y) \, dv(x) dv(y)$$

$$+ \int_{|x-y| > \frac{\delta}{2}} |\varphi_\varepsilon|^2(x) G_g^s(x,y) |\varphi_\varepsilon|^2(y) \, dv(x) dv(y)$$

$$= \int_{|x-y| < \frac{\delta}{2}; |x| \leq \frac{\delta}{2}} |\Psi_\varepsilon|^2(x) [G_{\mathbb{R}^n}^s(x,y) + r(x,y)] |\Psi_\varepsilon|^2(y) \, dx dy$$

$$+ O\Big(\int_{|x| \geq \frac{\delta}{2}} \int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) |x|^2 |\Psi_\varepsilon|^2(x) \, dy dx \Big) + O\Big(\Big(\int_M |\varphi_\varepsilon|^2 \, dv_g \Big)^2 \Big)$$

$$= \int_{|x-y| < \frac{\delta}{2}; |x| \leq \frac{\delta}{2}} |\Psi_\varepsilon|^2(x) G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) \, dx dy + O\Big(\int_{|x-y| < \frac{\delta}{2}} |\Psi_\varepsilon|^2(x) \frac{1}{|x-y|} |\Psi_\varepsilon|^2(y) \, dx dy \Big)$$

$$+ O\Big(\int_{|x| \geq \frac{\delta}{2}} \int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) |x|^2 |\Psi_\varepsilon|^2(x) \, dy dx \Big) + O\Big(\Big(\int_M |\varphi_\varepsilon|^2 \, dv_g \Big)^2 \Big)$$

$$= \int_{|x| \leq \frac{\delta}{2}} |\Psi_\varepsilon|^2(x) G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) \, dx dy + O\Big(\int_{|x-y| < \frac{\delta}{2}; |x| \leq \frac{\delta}{2}} |\Psi_\varepsilon|^2(x) \frac{1}{|x-y|} |\Psi_\varepsilon|^2(y) \, dx dy \Big)$$

$$+ O\Big(\int_{|x| \geq \frac{\delta}{2}} \int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) |x|^2 |\Psi_\varepsilon|^2(x) \, dy dx \Big) + O\Big(\Big(\int_M |\varphi_\varepsilon|^2 \, dv_g \Big)^2 \Big)$$

$$+ O\Big(\Big(\int_{\mathbb{R}^n} |\Psi_\varepsilon|^2 \, dx_g \Big)^2 \Big).$$

Using (40), we get

$$O\Big(\Big(\int_M |\varphi_\varepsilon|^2 \, dv_g \Big)^2 \Big) + O\Big(\Big(\int_{\mathbb{R}^n} |\Psi_\varepsilon|^2 \, dx_g \Big)^2 \Big) = O(\varepsilon^2).$$

Moreover, from (41) and (42), we have

$$\int_{|x| \leq \frac{\delta}{2}} \int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) |\Psi_\varepsilon|^2(x) \, dy dx = c_n^{\frac{1}{n-1}} a_0^{\frac{n}{n-1}} \omega_{n-1} \int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^n} \, dr + O(\varepsilon^n),$$

and

$$O\Big(\int_{|x| \geq \frac{\delta}{2}} \int_{\mathbb{R}^n} G_{\mathbb{R}^n}^s(x,y) |\Psi_\varepsilon|^2(y) |x|^2 |\Psi_\varepsilon|^2(x) \, dy dx \Big) = O(\varepsilon^2).$$

It remains to estimate

$$O\Big(\int_{|x-y| < \frac{\delta}{2}; |x| \leq \frac{\delta}{2}} |\Psi_\varepsilon|^2(x) \frac{1}{|x-y|} |\Psi_\varepsilon|^2(y) \, dx dy \Big)$$

$$= O\Big(\int_{|x| \leq \delta} \int_{|y| \leq \delta} |\Psi_\varepsilon|^2(x) \frac{1}{|x-y|} |\Psi_\varepsilon|^2(y) \, dx dy \Big) + O(\varepsilon^2).$$

Using the Hardy-Littlewood-Sobolev inequality, we have

$$\int_{|x| \leq \delta} \int_{|y| \leq \delta} |\Psi_\varepsilon|^2(x) \frac{1}{|x-y|} |\Psi_\varepsilon|^2(y) \, dx dy \leq C \||\Psi_\varepsilon|^2\|_{L^{\frac{2n}{2n-1}}(B_\delta)} \leq C \|\Psi_\varepsilon\|_{L^2}^2 = O(\varepsilon^2).$$

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Hence,

$$\int_{M \times M} |\varphi_\varepsilon|^2(x) G_s(x,y) |\varphi_\varepsilon|^2(y) \, dv(x) dv(y) = c_n^{\frac{1}{n-1}} a_n^{\frac{n}{n-1}} \omega_{n-1} \int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^n} \, dr + O(\varepsilon^2).$$

It follows that

$$\begin{aligned} J_g(\varphi_\varepsilon) &\leq \frac{1}{4} c_n^{\frac{1}{n-1}} a_n^{\frac{n}{n-1}} \omega_{n-1} \int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^n} \, dr + O(\varepsilon^2) \\ &= \overline{Y} + O(\varepsilon^2). \end{aligned} \tag{43}$$

$\square$

Proof. (*of Theorem (1.3)*)
From Lemma 6.1 and 6.2, we have that

$$J_\lambda(\varphi_\varepsilon) \leq \overline{Y} - \lambda Q(\varepsilon) + O(\varepsilon^2)$$

and

$$\|\nabla J_\lambda(\varphi_\varepsilon)\|_{H_\lambda^*} \leq \begin{cases} O(\varepsilon |\ln(\varepsilon)|^{\frac{2}{3}}) \text{ if } n = 3 \\ O(\varepsilon) \text{ if } n \geq 4 \end{cases}$$

Therefore, from Proposition 6.3, we have for $\varepsilon > 0$ and small,

$$\begin{aligned} \delta_\lambda &\leq J_\lambda(\varphi_\varepsilon) + O(\|\nabla J_\lambda(\varphi_\varepsilon)\|^2) \\ &\leq \overline{Y} - \lambda Q(\varepsilon) + \begin{cases} O(\varepsilon^2 |\ln(\varepsilon)|^{\frac{4}{3}}) \text{ if } n = 3 \\ O(\varepsilon^2) \text{ if } n \geq 4 \end{cases} \\ &< \overline{Y}. \end{aligned} \tag{44}$$

Since, J_λ and $\tilde{J}$ satisfy the (PS) condition for energy levels below $\overline{Y}$, we have that J_λ has a non-trivial critical point ψ_λ. $\square$

We finally notice that for $\lambda = 0$, δ_0 is a conformal invariant of $(M, [g])$ and we will denote it by $\delta_0 =: \overline{Y}(M, [g])$. With these notations, we see that Corollary 1.1 is a direct consequence of (44).

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