A Legendre transform on an exotic $S^3$

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Abstract

We consider an exotic contact form $\alpha$ on $S^3$ and we establish explicitly the existence of a non singular vector field $v$ in $\ker(\alpha)$ such that the non-singular one-differential form $\beta(\cdot) := d\alpha(v, \cdot)$ is a contact form on $S^3$ with the same orientation than $\alpha$. In particular this means that a Legendre transform can be completed.

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1 Introduction

In this paper we consider an exotic contact form $\alpha$ on $S^3$, introduced by J.Gonzalo-F.Varela in [2], (case $n = 1$). It is, according to [2] an overtwisted contact structure and in Appendix A we can actually find an explicit disk $D^2$ whose boundary is a Legendrian curve for $\alpha$ and $\ker(\alpha)$ has exactly one point of tangency to $D^2$. This contact structure is therefore not standard. The standard contact form $\alpha_0$ on $S^3$ is a pull-back from the standard contact form on $P(\mathbb{R}^3)$, that is the unit sphere cotangent bundle of $S^2$; therefore it is equipped with its Liouville form. Legendre duality can be completed for this Liouville form. This Legendre transform can be viewed as the data of a vector field $v$ in $\ker(\alpha_0)$ such that $\beta_0(\cdot) := d\alpha_0(v, \cdot)$ is a contact form with the same orientation than $\alpha_0$.

This Legendre transform allows the transformation of a Hamiltonian problem on the cotangent sphere of $S^2$ into a Lagrangian problem. This duality has been extended by A.Bahri-D.Bennequin in [1] to the more general framework of a contact form $\alpha$ on a three-dimensional compact orientable manifold without boundary $M$, leading
to a variational problem on a spaces of curves. In fact if one assumes that:

(i) \( \exists v \in TM, \) a non-vanishing vector field, such that \( v \in \ker(\alpha) \)

(ii) the non-singular one-differential form \( \beta(\cdot) := d\alpha(v, \cdot) \) is a contact form on \( M \) with the same orientation than \( \alpha \)

by defining the action functional

\[
J(x) = \int_0^1 \alpha(\dot{x})dt \tag{1.1}
\]

on the subspace of the \( H^1 \)-loops on \( M \):

\[
C_\beta = \{ x \in H^1(S^1; M) \text{ s.t. } \beta(\dot{x}) = 0; \alpha(\dot{x}) = \text{strictly positive constant} \}
\]

if \( \xi \in TM \) denotes the Reeb vector field of \( \alpha \), i.e.

\[
\alpha(\xi) = 1, \quad d\alpha(\xi, \cdot) = 0 \tag{1.2}
\]

then the following result by A.Bahri-D.Bennequin holds [1]:

**Theorem 1.1.** \( J \) is a \( C^2 \) functional on \( C_\beta \) whose critical points are periodic orbits of \( \xi \).

It is important to observe that this construction is “stable under perturbation”, that is the same \( v \) can be used to complete Legendre duality for forms \( \lambda \alpha \), with \( \lambda \in C^2 \) and \( |\lambda - 1| \) small.

In this work we establish the existence of such a \( v \), which is given explicitly, for the contact structure of J.Gonzalo-F.Varela.

The organization of the paper is the following: in Section 2 we verify the hypothesis (i) giving explicitly the vector field \( v \); in Section 3 we verify the hypothesis (ii); we conclude the paper with four appendices. In Appendix A, we provide an explicit disk that allows to recognize a known fact about the contact structure of \( \alpha \), namely that it is overtwisted. Appendix B is devoted to the graphs of some of the (complicated) functions that we use. Our \( v \) is \( C^\infty \) outside of two curves. It is only \( C^0 \) on \( S^3 \). We regularize it (with a very standard and straightforward regularizing procedure; \( v \) is in fact \( C^\infty \) in the direction of the Reeb vector field \( \xi \) in Appendix C so that it is now \( C^\infty \) and hypotheses (i) and (ii) are still satisfied. We then study in Appendix D the case \( n > 1 \) of the contact forms/structures of Gonzalo-Varela [2]. The definition of \( v \) extends, but hypothesis (ii) is not satisfied anymore by this extension. Another extension might work.

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2 Verification of hypothesis (i)

From now on we consider $S^3$ as embedded submanifold of $\mathbb{R}^4$ where we will carry on most of our computation. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, by denoting
\[ r_1 = x_1^2 + x_2^2, \quad r_2 = x_3^2 + x_4^2 \]
then
\[ S^3 = \{ x \in \mathbb{R}^4 : r_1 + r_2 = 1 \} \]
and on $S^3$ we take the non-standard (or exotic) contact form $\alpha$ defined by J.Gonzalo and F.Varela in [2], (case $n = 1$):
\[ \alpha = -\left( A(x_2dx_1 - x_1dx_2) + B(x_4dx_3 - x_3dx_4) \right) \]
\[ \text{where} \quad \theta = \frac{\pi}{4} + \pi r_2, \quad A = \cos \theta, \quad B = \sin \theta \]
Now we compute $d\alpha$. If we denote by
\[ \tilde{A} = A + \pi r_1 B = \frac{\partial}{\partial r_1} (r_1 A) \quad \tilde{B} = B + \pi r_2 A = \frac{\partial}{\partial r_2} (r_2 B) \]
then by a direct computation
\[ d\alpha = 2 \left( \tilde{A}dx_1 \wedge dx_2 + \tilde{B}dx_3 \wedge dx_4 \right) \quad (2.1) \]
Now, if
\[ \zeta = -\left( \tilde{B}(x_2\partial_{x_1} - x_1\partial_{x_2}) + \tilde{A}(x_4\partial_{x_3} - x_3\partial_{x_4}) \right) \]
one has $\zeta \in T(S^3)$ and it holds\footnote{See Appendix B}
\[ \alpha(\zeta) = A\tilde{B}r_1 + B\tilde{A}r_2 > 0, \quad d\alpha(\zeta, \cdot) = 0 \quad (2.2) \]
Thus the Reeb vector field of $\alpha$ is
\[ \xi = \frac{\zeta}{\alpha(\zeta)} \quad (2.3) \]
Let us define the following non singular\footnote{See Appendix B} vector field in $T(S^3)$
\[ T = -\left( A(x_2\partial_{x_1} - x_1\partial_{x_2}) + B(x_4\partial_{x_3} - x_3\partial_{x_4}) \right) \quad (2.4) \]
so one finds
\[ \alpha(\cdot) = < T, \cdot > \quad (2.5) \]
where $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^4$. In other words, a vector field is in the kernel of $\alpha$ if it is orthogonal to $T$.
Theorem 2.1. Let \( R := |T| \), where
\[
|T|^2 = <T, T> = \alpha(T) = A^2 r_1 + B^2 r_2 > 0
\]
By letting \( C = A/R \) and \( D = B/R \) let us define the vector field
\[
v = v_1 \partial_{x_1} + v_2 \partial_{x_2} + v_3 \partial_{x_3} + v_4 \partial_{x_4}
\]
with
\[
\begin{align*}
v_1 &= x_3 \frac{(x_1^2 - D x_2^2)}{r_1} + \frac{(x_1 x_2 x_4)}{r_1} (1 + D) \\
v_2 &= x_4 \frac{(x_2^2 - D x_1^2)}{r_1} + \frac{(x_1 x_2 x_3)}{r_1} (1 + D) \\
v_3 &= -x_1 \frac{(x_3^2 + C x_2^2)}{r_2} - \frac{(x_2 x_3 x_4)}{r_2} (1 - C) \\
v_4 &= -x_2 \frac{(x_4^2 + C x_3^2)}{r_2} - \frac{(x_1 x_3 x_4)}{r_2} (1 - C)
\end{align*}
\]
(2.6)
Then \( v \in T(S^3) \), \(|v| = 1 \) and \( v \in \ker(\alpha) \), so the condition (i) is satisfied.

Proof. We introduce the two objects
\[
M = S^3 \setminus (\{r_1 = 0\} \cup \{r_2 = 0\})
\]
(2.7)
and
\[
T^2 = \{r_1 = c_1, r_2 = c_2, c_1 + c_2 = 1, c_1 \neq 0, c_2 \neq 0\}
\]
(2.8)
So \( T^2 \) are invariant tori for \( \xi \) (i.e. \( \xi \in T(T^2) \)) and \( M \) is the sphere without the two degenerate tori (circles). Moreover, also the vector field \( T \) is tangent to \( T^2 \). We introduce the following two vector fields in \( T(M) \)
\[
X = \frac{1}{\sqrt{r_1 r_2}} \left( D r_2 (x_2 \partial_{x_1} - x_1 \partial_{x_2}) - C r_1 (x_4 \partial_{x_3} - x_3 \partial_{x_4}) \right)
\]
(2.9)
\[
Y = \frac{1}{\sqrt{r_1 r_2}} \left( r_2 (x_1 \partial_{x_1} + x_2 \partial_{x_2}) - r_1 (x_3 \partial_{x_3} + x_4 \partial_{x_4}) \right)
\]
(2.10)
It holds
\[
|X| = |Y| = 1
\]
thus \( X, Y \) are non degenerate on \( M \). Moreover \( X, Y \in \ker(\alpha) \), in particular \( X \in T(T^2) \) and \( Y \in N(T^2) \) (the normal space to \( T^2 \)). With the following coefficients
\[
a = \frac{1}{\sqrt{r_1 r_2}} (x_1 x_3 + x_2 x_4), \quad b = \frac{1}{\sqrt{r_1 r_2}} (x_1 x_4 - x_2 x_3), \quad a^2 + b^2 = 1
\]
(2.11)
let us define
\[
v = aY + bX
\]
(2.12)
So \( v \in ker(\alpha) \), \(|v| = 1\) and by a direct computation one finds the coefficients in (2.6). Let us remark that \( v \) is defined only on \( M \). Since

\[
\lim_{r_2 \to 0} C = -\lim_{r_1 \to 0} D = 1
\]
on \( r_1 = 0 \) one has

\[
v = x_3 \partial_{x_1} + x_4 \partial_{x_2}
\]
whereas on \( r_2 = 0 \) one finds

\[
v = -x_1 \partial_{x_3} - x_2 \partial_{x_4}
\]
so, by continuity, \( v \) is defined on the whole \( S^3 \).

\[\square\]

**Corollary 2.1.** In the same way if we define the vector field

\[
w = w_1 \partial_{x_1} + w_2 \partial_{x_2} + w_3 \partial_{x_3} + w_4 \partial_{x_4}
\]

with

\[
\begin{aligned}
w_1 &= -x_4 \frac{(x_1^2 - Dx_1^2)}{r_1} + \frac{(x_1x_2x_3)}{r_1} (1 + D) \\
w_2 &= x_3 \frac{(x_2^2 - Dx_2^2)}{r_1} - \frac{(x_1x_2x_4)}{r_1} (1 + D) \\
w_3 &= -x_2 \frac{(x_3^2 + Cx_3^2)}{r_2} + \frac{(x_1x_3x_4)}{r_2} (1 - C) \\
w_4 &= x_1 \frac{(x_4^2 + Cx_4^2)}{r_2} - \frac{(x_2x_3x_4)}{r_2} (1 - C)
\end{aligned}
\]  \hfill (2.13)

Then \( w \in T(S^3) \), \(|w| = 1\), \( w \in ker(\alpha) \) and \( w \perp v \).

**Proof.** The proof is the same as in (2.1), with

\[
w = aX - bY
\]  \hfill (2.14)

So \( w \perp v \), \( w \in ker(\alpha) \), \(|w| = 1\). Moreover on \( r_1 = 0 \) one has

\[
w = -x_4 \partial_{x_1} + x_3 \partial_{x_2}
\]
whereas on \( r_2 = 0 \) one finds

\[
w = -x_2 \partial_{x_3} + x_1 \partial_{x_4}
\]  \[\square\]

**Remark 2.1.** We want to point out that the (coefficients of the) vector fields \( v, w \) are by construction only \( C^0 \).
3 Verification of hypothesis (ii)

Let us consider now the non-singular one-differential form

\[ \beta(\cdot) := \alpha(v, \cdot) \]  

(3.1)

By defining \( h := \alpha(\zeta) \), one has

\[ d\alpha(v, w) = d\alpha(aY + bX, aX - bY) = (a^2 + b^2)\alpha(Y, X) = \alpha(Y, X) = -\frac{2}{|T|}h < 0 \]

and

\[ \alpha \wedge d\alpha(\zeta, v, w) = h\alpha(v, w) < 0 \]

Moreover

\[ \beta \wedge d\beta(\zeta, v, w) = \beta(w)d\beta(\zeta, v) = -d\alpha(v, w)d\alpha(v, [\zeta, v]) \]

Thus

\[ \frac{\beta \wedge d\beta(\zeta, v, w)}{\alpha \wedge d\alpha(\zeta, v, w)} = -\frac{d\alpha(v, [\zeta, v])}{h} \]  

(3.2)

**Theorem 3.1.** \( d\alpha(v, [\zeta, v]) < 0 \), so the condition (ii) is satisfied.

**Proof.** We explicitly write some formulas. For \( 0 < r_1 < 1 \), one finds

\[ Y(r_1) = 2\sqrt{r_1}r_2, \quad Y(r_2) = -2\sqrt{r_1}r_2, \quad Y(\theta) = -2\pi\sqrt{r_1}r_2 \]

\[ Y(A) = 2\pi\sqrt{r_1}r_2 B, \quad Y(B) = -2\pi\sqrt{r_1}r_2 A \]

\[ Y(\tilde{A}) = 2\pi\sqrt{r_1}r_2(2B - \pi r_1 A), \quad Y(\tilde{B}) = -2\pi\sqrt{r_1}r_2(2A - \pi r_2 B) \]

\[ \zeta(a) = -[(\tilde{A} - \tilde{B})a + (\tilde{A} - \tilde{B})b], \quad \zeta(b) = (\tilde{A} - \tilde{B})a \]

\[ [\zeta, X] = 0 \]

\[ [\zeta, Y] = Y(\tilde{B})(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y(\tilde{A})(x_4\partial_{x_3} - x_3\partial_{x_4}) \]

Moreover

\[ [\zeta, v] = [\zeta, aY + bX] = \zeta(a)Y + \zeta(b)X + a[\zeta, Y] + b[\zeta, X] = \]

\[ = (\tilde{A} - \tilde{B})w + a[\zeta, Y] = (\tilde{A} - \tilde{B})w + aY(\tilde{B})(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y(\tilde{A})(x_4\partial_{x_3} - x_3\partial_{x_4}) \]

\[ = (\tilde{A} - \tilde{B})w + 2\pi(x_1x_3 + x_2x_4)\left\{-[(2A - \pi r_2 B)(x_2\partial_{x_1} - x_1\partial_{x_2}) + (2B - \pi r_1 A)(x_4\partial_{x_3} - x_3\partial_{x_4})] \right\} \]

and

\[ \lim_{r_1 \to 0} [\zeta, v] = \frac{\pi}{\sqrt{2}}(-x_4\partial_{x_1} + x_3\partial_{x_2}), \quad \lim_{r_2 \to 0} [\zeta, v] = \frac{\pi}{\sqrt{2}}(-x_2\partial_{x_3} + x_1\partial_{x_4}) \]  

(3.3)

\(^4\)The vector field \( v \) is \( C^0 \) so in order to compute \([\zeta, v]\) we need to regularize \( v \), see Appendix C
By computing
\[ d\alpha(v,[\zeta,v]) = d\alpha(v,(\bar{A} - \bar{B})w + a[\zeta,Y]) = \]
\[ = -2(\bar{A} - \bar{B}) \frac{h}{R} + ad\alpha(aY + bX, \{Y(\bar{B})(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y(\bar{A})(x_4\partial_{x_3} - x_3\partial_{x_4})\}) \]
and by letting
\[ K := \bar{A}(\pi r_2 B - 2A) + \bar{B}(\pi r_1 A - 2B) \]
one has
\[ d\alpha(v,[\zeta,v]) = -2\left\{ (\bar{A} - \bar{B}) \frac{h}{R} + 2\pi a^2 r_1 r_2 K \right\} =: -2Q \]
and\(^4 Q > 0. \]

\[ \Box \]

**Appendix A**

Let us consider on \( S^3 \) the following disk
\[ D^2 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \text{ s.t. } x_1^2 + x_2^2 \leq \frac{3}{4}, x_4 \geq 0, x_3 = \varepsilon \} \]
with \( 0 < \varepsilon \ll 1 \).

Then the boundary of \( D^2 \) is a Legendrian curve for the contact form \( \alpha \) (i.e. a curve in the kernel of the contact form), in fact
\[ \partial D^2 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \text{ s.t. } x_1^2 + x_2^2 = \frac{3}{4}, x_4 \geq 0, x_3 = \varepsilon \} \]
thus \( \theta|_{\partial D^2} = \frac{\pi}{2}, A|_{\partial D^2} = 0 \) and \( \alpha(\partial D^2) = 0 \). Now let us consider the identically zero form on \( S^3 \)
\[ \omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4 \]
that on \( D^2 \) it reads as \( x_1 dx_1 + x_2 dx_2 + x_4 dx_4 \). To find the points of tangency between \( ker(\alpha) \) and \( D^2 \) we can see whether \( \omega = \lambda \alpha \) for some non zero real \( \lambda \). Then it should be
\[ \left\{ \begin{array}{l} A x_1 = \lambda x_2 \\ A x_2 = -\lambda x_2 \\ B \varepsilon = \lambda x_4 \end{array} \right. \] \hfill (A.1)
that means in particular
\[ A(x_1^2 + x_2^2) = 0 \]
Now if \( A = 0 \) then \( \lambda = 0 \), thus the only possible case is when \( x_1^2 + x_2^2 = 0 \) and the only one point of tangency between \( ker(\alpha) \) and \( D^2 \) is \( (0, 0, \varepsilon, \sqrt{1 - \varepsilon^2}) \).

\(^4\)See Appendix B
Appendix B

In this section we show the behavior (in particular the non-negativity) of some functions we used before. From now on let us put \( x := r_2 \).

First we study \( h : [0, 1] \rightarrow \mathbb{R}, \)

\[
h(x) := \alpha(\zeta(x)) = A(x) \tilde{B}(x)(1 - x) + B(x) \tilde{A}(x) x = \frac{\sin(2\theta(x))}{2} + \pi(x - x^2)
\]

where \( \theta(x) = \pi \left( \frac{1}{4} + x \right) \). Since \( h \) is symmetric respect to \( x = 1/2 \), we can consider it only for \( x \in [0, 1/2] \). So:

\[
h'(x) = \pi(\cos(2\theta(x)) + 1 - 2x)
\]

\[
h''(x) = -2\pi(\pi \sin(2\theta(x)) + 1) = -2\pi(\pi \cos(2\pi x) + 1)
\]

thus there exists \( c_1 \), with \( 1/4 < c_1 < 1/2 \) such that \( h'' \) is positive definite on \( (c_1, 1/2) \) and \( h' \) is increasing on \( (c_1, 1/2) \). Moreover \( h'(1/2) = 0 \). Thus there exists \( c_2 \), with \( 0 < c_2 < c_1 < 1/2 \) such that \( h'(c_2) = 0 \) and \( h \) is increasing on \( (0, c_2) \). Finally, since \( h(0) = 1/2 \), the minimum of \( h \) is \( h(1/2) = -1/2 + \pi/4 > 0 \)

One has also \( R(x) := |T(x)| > 0 \). Indeed

\[
R^2(x) = |T(x)|^2 = \langle T(x), T(x) \rangle = \alpha(T(x)) = A^2(x)(1 - x) + B^2(x)x \geq 0
\]

and the quantities \( A^2(x)(1-x) \) and \( B^2(x)x \) cannot be zero simultaneously. We prove now that \( Q(x) = (\tilde{A} - \tilde{B}) \frac{h(x)}{R(x)} + 2\pi a_1 r_1 r_2 K > 0 \) for \( x \in [0, 1] \) showing the graphs of some function (a straightforward computation is possible, as for the function \( h, R, \) to localize critical points). If

\[
H(x) := (\tilde{A}(x) - \tilde{B}(x)) \frac{h(x)}{R(x)}
\]

Where \( K \) is positive definite then \( Q > 0 \).

Otherwise, if we define

\[
G := 2\pi r_1 r_2, \quad G(x) = 2\pi(x - x^2)
\]
since $a^2 \leq 1$, where $K$ is negative definite

$$2\pi a^2 r_1 r_2 K \geq GK$$

Thus, where $K$ is negative definite, it holds

$$Q(x) \geq H(x) + G(x)K(x) =: W(x)$$

and $Q > 0$, for every $x \in [0, 1]$.

**Appendix C**

In this section we want to regularize the $C^0$ vector field $v$. The only problems are on the circles $r_1 = 0$ and $r_2 = 0$, otherwise $v$ is a $C^\infty$ vector field. Let us consider the case $r_1 = 0$, the other one is similar. Let $\mathcal{U}$ be a tubular neighborhood of $r_1 = 0$ and consider on $\mathcal{U}$ a basis $\{v_1, v_2\}$ of $ker(\alpha)$ with $v_1, v_2 \in C^\infty$ (it is not difficult to find a local $C^\infty$ vector field). Then

$$v = a_1 v_1 + a_2 v_2$$  \hspace{1cm} (C.1)

with $a_1, a_2 \in C^0$. After a convolution with a quite standard mollifiers we can find $a_1^\varepsilon, a_2^\varepsilon \in C^\infty$ on $\mathcal{U}$, with $\varepsilon > 0$. Thus we define the $C^\infty$ vector field

$$v^\varepsilon = a_1^\varepsilon v_1 + a_2^\varepsilon v_2$$  \hspace{1cm} (C.2)
Now, by formulas (3.3) we already know that the vector field $[\zeta, v] \in C^0$, then using

$$|\zeta(a^1_\varepsilon) - \zeta(a_1)| = o(1), \quad |\zeta(a^2_\varepsilon) - \zeta(a_2)| = o(1), \quad \text{as} \quad \varepsilon \to 0$$

we have on $\mathcal{U}$

$$[\zeta, v] = \lim_{\varepsilon \to 0} [\zeta, v^\varepsilon]$$

Thus, in order to compute $\beta \wedge d\beta$, we can use that

$$d\alpha(v, [\zeta, v]) = \lim_{\varepsilon \to 0} d\alpha(v^\varepsilon, [\zeta, v^\varepsilon])$$

**Appendix D**

The exotic contact form $\alpha$ we considered on $S^3$, actually is the first one of a family of non-standard contact forms introduced by J.Gonzalo-F.Varela in [2]. In fact, for every integer $n \geq 1$ let us define

$$\theta_n = \frac{\pi}{4} + n\pi r_2, \quad A_n = \cos \theta_n, \quad B_n = \sin \theta_n$$

$$\tilde{A}_n = A_n + n\pi r_1 B_n = \frac{\partial}{\partial r_1}(r_1 A_n) \quad \quad \tilde{B}_n = B_n + n\pi r_2 A_n = \frac{\partial}{\partial r_2}(r_2 B_n)$$

then [2]

**Theorem 3.2** (J.Gonzalo - F.Varela). *The non-singular one-differential forms*

$$\alpha_n = - \left( A_n(x_2 dx_1 - x_1 dx_2) + B_n(x_4 dx_3 - x_3 dx_4) \right)$$

are non-standard contact forms on $S^3$, for every $n \geq 1$

If

$$\zeta_n = - \left( \tilde{B}_n(x_2 \partial_{x_1} - x_1 \partial_{x_2}) + \tilde{A}_n(x_4 \partial_{x_3} - x_3 \partial_{x_4}) \right)$$

one has

$$h_n := \alpha_n(\zeta_n) = A_n \tilde{B}_n r_1 + B_n \tilde{A}_n r_2 > 0, \quad d\alpha_n(\zeta_n, \cdot) = 0$$

and then the Reeb vector field of $\alpha_n$ is

$$\xi_n = \frac{\zeta_n}{h_n}$$

Now, using the same arguments as in Section 2 we can see that if

$$T_n = - \left( A_n(x_2 \partial_{x_1} - x_1 \partial_{x_2}) + B_n(x_4 \partial_{x_3} - x_3 \partial_{x_4}) \right)$$

then one finds $|T_n| > 0$ and it holds

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Theorem 3.3. Let us define \( C_n = A_n/|T_n|, D_n = B_n/|T_n| \).
Let \( n \geq 1 \) be odd.
If \( v_n = v_n^1 \partial x_1 + v_n^2 \partial x_2 + v_n^3 \partial x_3 + v_n^4 \partial x_4 \), with
\[
\begin{align*}
v_n^1 &= x_3 \frac{(x_1^2 - D_n x_2^2)}{r_1} + \frac{(x_1 x_2 x_4)}{r_1} (1 + D_n) \\
v_n^2 &= x_4 \frac{(x_2^2 - D_n x_1^2)}{r_1} + \frac{(x_1 x_2 x_3)}{r_1} (1 + D_n) \\
v_n^3 &= -x_1 \frac{(x_3^2 + C_n x_4^2)}{r_2} - \frac{(x_2 x_3 x_4)}{r_2} (1 - C_n) \\
v_n^4 &= -x_2 \frac{(x_3^2 + C_n x_4^2)}{r_2} - \frac{(x_1 x_3 x_4)}{r_2} (1 - C_n)
\end{align*}
\]
Then \( v_n \) is a non singular \( C^0 \) vector field in \( \ker(\alpha_n) \) and \( |v_n| = 1 \).
Let \( n \geq 1 \) be even.
If \( v_n = v_n^1 \partial x_1 + v_n^2 \partial x_2 + v_n^3 \partial x_3 + v_n^4 \partial x_4 \), with
\[
\begin{align*}
v_n^1 &= x_3 \frac{(x_1^2 + D_n x_2^2)}{r_1} - \frac{(x_1 x_2 x_4)}{r_1} (1 - D_n) \\
v_n^2 &= -x_4 \frac{(x_2^2 + D_n x_1^2)}{r_1} + \frac{(x_1 x_2 x_3)}{r_1} (1 - D_n) \\
v_n^3 &= -x_1 \frac{(x_3^2 + C_n x_4^2)}{r_2} + \frac{(x_2 x_3 x_4)}{r_2} (1 - C_n) \\
v_n^4 &= x_2 \frac{(x_3^2 + C_n x_4^2)}{r_2} - \frac{(x_1 x_3 x_4)}{r_2} (1 - C_n)
\end{align*}
\]
Then \( v_n \) is a non singular \( C^0 \) vector field in \( \ker(\alpha_n) \) and \( |v_n| = 1 \).
Thus the hypothesis (i) holds by using the previous \( v_n \).
Putting \( \beta_n(\cdot) = d\alpha_n(v_n, \cdot) \) then in order to compute \( \beta_n \wedge d\beta_n \) we need to know the sign of \( d\alpha_n(v_n, [\zeta_n, v_n]) \). By a direct computation:
if \( n \geq 1 \) is odd then
\[
d\alpha_n(v_n, [\zeta_n, v_n]) = -2 \left\{ (\tilde{A}_n - \tilde{B}_n) \frac{h_n}{R_n} + 2n\pi a^2 r_1 r_2 K_n \right\} =: -2Q_n
\]
where
\[
K_n := \tilde{A}_n(n\pi r_2 B_n - 2A_n) + \tilde{B}_n(n\pi r_1 A_n - 2B_n), \quad a = \frac{1}{\sqrt{r_1 r_2}} (x_1 x_3 + x_2 x_4)
\]
whereas if \( n \geq 1 \) is even then
\[
d\alpha_n(v_n, [\zeta_n, v_n]) = -2 \left\{ (\tilde{A}_n + \tilde{B}_n) \frac{h_n}{R_n} + 2n\pi a^2 r_1 r_2 K_n \right\} =: -2Q_n
\]
where

\[ K_n := \tilde{A}_n (n\pi r_2 B_n - 2A_n) + \tilde{B}_n (n\pi r_1 A_n - 2B_n), \quad a = \frac{1}{\sqrt{r_1 r_2}} (x_1 x_3 - x_2 x_4) \]

Now, for every \(0 < r_1 < 1\), there exist \(x_1, x_2, x_3, x_4\) (even along the periodic orbits of \(\xi_n\)) such that \(a^2 = 0\). In such a points the sign of \(d\alpha_n(v_n, [\zeta_n, v_n])\) depends on \(\tilde{A}_n \pm \tilde{B}_n\) and we find the following graphs

Thus in general the hypothesis \((ii)\) does not hold in general. Anyway the existence of a ”good” \(v_n\) (for which the hypotheses \((i), (ii)\) are satisfied) is not a priori excluded.

References
