NONSMOOTH SOLUTIONS FOR A CLASS OF FULLY NONLINEAR PDE’S ON LIE GROUPS

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ABSTRACT. In this paper we prove the existence of non smooth viscosity solutions for Dirichlet problems involving a class a fully non-linear operators on Lie groups. In particular we consider the elementary symmetric functions of the eigenvalues of the Hessian built with left-invariant vector fields.

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1. Introduction
In [18] Pogorelov showed that convex generalized solutions of the Monge Ampère equation
\[
\det D^2 u = f(x)
\]
in a domain \( \Omega \subset \mathbb{R}^n, n \geq 3 \), need not be of class \( C^2 \), even if \( f \) is positive and smooth.

Urbas in [20] proved that this absence of classical regularity is not confined to equations of Monge Ampère type, but in fact occurs for the \( k \)-th elementary symmetric functions in the Hessian and the equation of prescribed \( k \)-th curvature, where \( k \geq 3 \). Recently Gutierrez Lanconelli and the second author in [9] proved the absence of regularity of graphs with positive and smooth prescribed Levi Monge Ampère curvature in a domain \( \Omega \subset \mathbb{R}^{2n+1} \), for \( n > 1 \).

Our purpose here is to show that a similar result holds for elementary symmetric functions of the Hessian on Lie groups. Our study is motivated by our recent results in [14] about existence and uniqueness of Lipschitz continuous viscosity solutions for Dirichlet problems involving symmetric functions of the Hessian built with left invariant vector fields on Lie groups. A naturally subsequent problem is that of further regularity of Lipschitz continuous viscosity solutions.

To fix the notation let us recall some well known facts. Let \( G = (\mathbb{R}^n, \circ) \) be a Lie group on \( \mathbb{R}^n \) with \( \circ \) as group law. Let us denote by \( I \) the set of the left-invariant vector fields. If \( E = \{ X_1, \ldots, X_n \} \) is any basis of \( I \) then a Riemannian metric \( g \) on \( G \) is left-invariant if and only if the coefficients \( g_{ij} := g(X_i, X_j) \) are constant functions. Each \( n \)-dimensional Lie group possesses a \( n(n+1)/2 \)-dimensional family of distinct left-invariant metrics, see for instance [15]. Let us fix any left-invariant metric \( g \) and let \( u \) be a smooth function, we will denote by \( D_g u \) the gradient of \( u \) with respect to the metric \( g \), that is: \( g(D_g u, X) = Xu = du(X) \), for every vector field \( X \). Moreover, there exists a \( n \times n \) invertible and smooth matrix \( W \) such that \( D_g u = W(x) Du \), where \( Du \) denotes the Euclidean gradient of \( u \). If \( V \) is the Levi-Civita connection for \( g \) (we recall that the connection coefficients in term of any left-invariant basis are constant functions), then the metric Hessian of \( u \) is the tensor field of type \( (0, 2) \) defined by:
\[
H_g u(X, Y) := XYu - (\nabla_X Y)u
\]
for every pair of vector fields \( (X, Y) \); since \( V \) is the Levi-Civita connection for \( g \) (that is \( \nabla_X Y - \nabla_Y X = [X, Y] \)), we note that \( H_g u \) is always symmetric. We will denote by \( D^2_g u := g^{-1}H_g u \) the associated endomorphism. We explicitly note that
the previous definition is intrinsic, namely the eigenvalues of \( D^2_g u \) do not change in a change of basis. Let us consider a coordinate frame \( \{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \} \), referred to local coordinates (in our setting they are actually global), we have:

\[
H_g u \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma^s_{ij} \frac{\partial u}{\partial x_s}
\]

where \( \Gamma^s_{ij} \) are the Christoffel symbols for the metric \( g \) (they are not constant in general). Hence if \( Du, D^2 u \) denote the usual Euclidian gradient and Hessian, then \( D^2_g u \) reads in local coordinates

\[
D^2_g u := G^{-1}(x) [D^2 u + B(x, Du)]
\]

where we denoted by \( G^{-1} \) the symmetric matrix of coefficients of \( g^{-1} \) expressed in the coordinate frame (they are not constant functions in general) and the coefficients of the matrix \( B \) are given by \( b_{ij} = \Gamma^s_{ij} \frac{\partial u}{\partial x_s} \). Let us note that the matrix \( B \) is symmetric because the Levi-Civita connection has null torsion.

Here we will consider also the case of strictly restrictions to some subspace of \( I \). Let then \( m \leq n \) and let \( E_m^l = \{ X_1, \ldots, X_m \} \), we define the subspace of the left-invariant vector fields

\[ HG := \text{span}\{X_1, \ldots, X_m\} \]

Now we consider a left-invariant metric \( g_m \) on \( HG \), we can “complete” it to the full tangent space by defining the blocks metric:

\[
g := \begin{pmatrix} g_m & 0 \\ 0 & \text{Id}_{n-m} \end{pmatrix}
\]

where, for every integer \( n \), \( \text{Id}_n \) denotes the identity matrix of order \( n \). We define

\[
H_{g,m} u(X, Y) := XYu - (\nabla X Y)u, \quad \forall \ X, Y \in HG
\]

and \( D^2_{g,m} u := g^{-1}_m H_{g,m} u \).

We note that there is another recurrent definition of Hessian on Lie groups, let us call it the symmetrized Hessian \( H^s u \), that is, for every smooth function \( u \) and for every pair of left-invariant vector fields \( X, Y \):

\[
H^s u(X, Y) = \frac{XYu + YXu}{2}
\]

In particular there is a very large literature on questions involving this symmetrized Hessian on Carnot groups (see for instance \[7], \[10], \[11], \[13], \[19], \[21], \[5], \[1\])
and the references therein). An easy computation shows that our metric Hessian \( H_gu \) coincides with \( H^s u \) if and only if it holds:

\[
\nabla_X Y = \frac{1}{2} [X, Y], \quad X, Y \in \mathfrak{I}
\]

In \cite{14} Example 2.1 we proved that if we consider the case of stratified Lie groups (in the sequel Carnot groups) (see Definition 4.4 or \cite{3} Definition 2.2.3), then the two Hessian definitions coincide. Moreover, in \cite{14} Example 2.2 we exhibited an example of a Lie group that does not satisfy (4).

Let \( B_R \subseteq \mathbb{R}^n \) be the Euclidean ball of radius \( R \) and center at the origin and let \( f : B_R \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) be a smooth positive function, here we are interested on the Dirichlet problem associated with equations of the following form

\[
(\sigma_k(D^2 g_u, m))^{\frac{1}{k}} = \tilde{f}(x, u, Du), \quad k = 1, \ldots, m
\]

where, for every symmetric matrix \( M \) of order \( m \), \( \sigma_k(M) \) denotes the \( k \)-th elementary symmetric function in the eigenvalues of \( M \).

Our motivation comes from the geometric theory of several complex variables, where fully nonlinear second order pde’s appear, whose linearizations are non variational operators of H"omander type. (See \cite{16} and references therein). These kinds of operators, also arising in many other theoretical and applied settings, have the form of (5). The dependence on the gradient \( D_g u \) in \( f \) is motivated by various applications. An interesting example is the subelliptic analogue of the prescribed Gauss curvature equation, see \cite{7} and \cite{5}.

A direct computation shows that equation (5) reads then in local coordinates:

\[
(\sigma_k(A_m(x) D^2 u A_m^T(x) + Q_m(x, Du)))^{\frac{1}{k}} = \tilde{f}(x, u, Du)
\]

where \( \tilde{f} \) is a positive function such that \( \tilde{f}(x, u, Du) = f(x, u, D_g u) = f(x, u, W(x)Du) \) and \( A_m \) is a \( m \times n \) matrix and \( Q_m \) is a square matrix of order \( m \), both with smooth coefficients. Moreover, it is easy to see that (see for instance \cite{3} Section 1.2.2])

\[
A_m(0) = \begin{pmatrix} I_m & 0 \end{pmatrix}
\]
and $Q_m$ is symmetric and linear with respect to $Du$. Precisely, we will consider viscosity solutions of the following Dirichlet problem:

\[
F(x, u, Du, D^2u) = 0, \quad \text{in } B_R, \\
\quad \quad u = \phi, \quad \text{on } \partial B_R,
\]

where

\[
F(x, u, Du, D^2u) := -\left(\sigma(\lambda_m(x) D^2u A_m^T(x) + Q_m(x, Du))\right)^\frac{1}{2} + f(x, u, Dg u)
\]

and $\phi : \partial B_R \to \mathbb{R}$.

We refer to [12], [6] for a full detailed exposition on the theory of viscosity solutions: we will give the basic definition of sub- and super-solution in the next section.

We explicitly remark here that the partial differential equation $F = 0$ is not elliptic and it is fully nonlinear for $k > 1$.

In analogy with [4] and [14], we define the open cone

\[
\Gamma^m_k = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m : \sigma_j(diag(\lambda)) > 0, \text{ for every } j = 1, \ldots, k\},
\]

where $diag(\lambda)$ is the $m \times m$ diagonal matrix, and we denote by $\overline{\Gamma^m_k}$ and $\partial \Gamma^m_k$ the closure and the boundary of $\Gamma^m_k$ respectively.

Remark that $F$ is degenerate elliptic in the cone $\Gamma^m_k$, i.e. $F(x, s, p, M) \leq F(x, s, p, N)$, for all $x \in \mathbb{R}^n$, $s \in \mathbb{R}$, $p \in \mathbb{R}^n$, and $M, N$ symmetric $n \times n$ matrices whose eigenvalues belongs to the open cone $\Gamma^m_k$ and such that $M \geq N$.

Therefore, we give the following

**Definition 1.1.** Let $x_0 \in \mathbb{R}^n$ and let $\phi$ be a $C^2$ function in a neighborhood of $x_0$. We will say that $\phi$ is strictly $F$-admissible (respectively $F$-admissible) at $x_0$, if the vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ of the eigenvalues of $D^2_{g,m} \phi(x_0)$ belongs to the open cone $\Gamma^m_k$ (respectively $\overline{\Gamma^m_k}$). Remark that the cone $\Gamma^m_k$ is invariant with respect to permutation of $\lambda_j$.

We will say that $\phi$ is strictly $F$-admissible (respectively $F$-admissible) in $\Omega \subset \mathbb{R}^n$ if $\phi$ is strictly $F$-admissible (respectively $F$-admissible) at $x_0$ for every $x_0 \in \Omega$.

Moreover if $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth defining function for a bounded open set $\Omega \subset \mathbb{R}^n$, that is

\[
\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \quad \partial \Omega = \{x \in \mathbb{R}^n : \rho(x) = 0\}
\]

then we will say that the domain $\Omega$ is strictly $F$-admissible if $\rho$ is strictly $F$-admissible.
By taking into account (7) it is easy to show that we can always find a small radius $R > 0$ such that the Euclidean ball $B_R$ is strictly $F$-admissible.

The purpose of this paper is to show that when $2 < k \leq m \leq n$, viscosity solutions may not be regular even if $f$ is positive and smooth. Precisely, we prove the following theorem, which is the main result of the paper.

We will assume that there is $\ell > k/2$ such that $F$ in (9) satisfies the following structure condition

$$(H_\ell) \ Q_m(x, p) = Q_m(x, p_{\ell+1}, \ldots, p_n)$$

is independent of $p_1, \ldots, p_\ell$.

Let us denote by $\text{Lip}$ the space of Lipschitz continuous functions with respect to the Euclidean metric and by $C^{1,\beta}_X$ the space of functions $u$ such that the Lie derivative $Xu$ exists for all $X \in HG$ and it is $\beta$- Hölder continuous with respect to the Euclidean distance (see Section 4 for details).

**Theorem 1.1.** Suppose $2 < k \leq m \leq n$, and $f \in C^\infty(B_1 \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function, monotone increasing with respect to $u$ and satisfying at least one of the following two conditions:

- $f$ does not depend on its first argument $x$
- $\inf \frac{\partial f}{\partial u} > 0$.

Then there exists $R \in (0, 1)$ and a $F$-admissible viscosity solution $u$ to the equation

$$(10) \quad F(x, u, Du, D^2u) = 0 \quad \text{in} \quad B_R,$$

such that $u \in \text{Lip}(\overline{B_R})$.

Moreover, if condition $(H_\ell)$ holds true for some $\ell \in (k/2, k - 1)$, then $u \notin C^{1,\beta}_X$ for any $\beta > \frac{2\ell}{k} - 1$. If condition $(H_\ell)$ holds true for some $\ell \geq k - 1$, then $u \notin C^{1,\beta}_X$ for any $\beta > 1 - \frac{2}{k}$.

The proof of this theorem uses Pogorelov’s counterexamples, see [18] or [8] Section 5.5], and its extensions developed by Urbas in [20] and by Gutierrez, Lanconelli and Montanari in [9] to show existence of viscosity non classical solutions to real curvature equations and to Gauss-Levi curvature equations, respectively.

A principal tool used to carry out the proof of our theorem are the comparison principles proved in [14, Section 4].

We will show in the Examples section at the end that, in any homogeneous Carnot group, elementary symmetric functions in the eigenvalues of the Hessian of the first layer satisfy $(H_\ell)$. Moreover, we recall that any stratified Carnot
The group is isomorphic to a homogeneous Carnot group and that the isomorphism preserves the stratification (see [3, Proposition 2.2.10 and Theorem 2.2.18]). In the same section, we will show that there exist Lie groups, not Carnot, such that elementary symmetric functions in the eigenvalues of the Hessian $H_{g,m}$, for some $m$ still fulfill condition $(H)$. 

The two alternative conditions on the function $f$ in Theorem 1.1 and the Lie group structure arise in [14] through the consideration of gradient estimates. We remark that for Monge–Ampère equations (i.e. $k = m$) on a homogeneous Carnot group, Bardi and Mannucci in [2] proved the existence of H-convex (i.e. $F$-admissible) continuous viscosity solutions of (10) under the only assumption that $f$ is positive and monotone increasing with respect to $u$. Moreover, as a consequence of the H-convexity, the solution is locally Lipschitz continuous with respect to the Carnot Carathéodory distance (see Definition 4.2) and for $R$ small enough

$$||Xu||_{L^\infty(B_{CC}(R))} \leq \frac{C}{R}||u||_{L^\infty(B_{CC}(2R))}, \quad \forall X \in HG$$

where the balls $B_{CC}$ are taken with respect to the Carnot Carathéodory distance and $C$ is a constant independent of $u$ and of $R$ (see [13, Theorem 4.1], [1] and the references therein). Let us denote by $C^{1,\beta}_{X_{dCC}}$ the space of functions $u$ such that the Lie derivative $Xu$ exists for all $X \in HG$ and it is $\beta$- Hölder continuous with respect to the Carnot Carathéodory distance. Thus, as a corollary of Theorem 1.1 (see also Remark 5.1), we get

**Corollary 1.1.** Let us consider a Carnot group on $\mathbb{R}^n$ with $m \leq n$ generators. Suppose $f \in C^\infty(B_1 \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function, monotone increasing with respect to $u$. Then there exists $R \in (0,1)$ and a $F$-admissible viscosity solution $u$ to the equation

$$F(x, u, Du, D^2u) := -\left( \det \left( A_m(x) D^2u A_m^T(x) + Q_m(x, Du) \right) \right)^{\frac{1}{m}} + f(x, u, D_gu) = 0 \quad \text{in} \quad B_R,$$

such that $u$ is Lipschitz continuous with respect to the Carnot Carathéodory distance in $B_R$ and $u \notin C^{1,\beta}_{X_{dCC}}(B_R)$ for any $\beta > 1 - \frac{2}{m}$.

Moreover, in that case, by analogy with the classical Monge–Ampère equations (see for instance [8, Section 5.4]), we expect that if the boundary data $\phi \in C^{1,\beta}(\partial B_R)$ for $\beta > 1 - \frac{2}{m}$, then the viscosity solutions $u$ of (11) is strictly H-convex and $Xu$ is $\beta$- Hölder continuous with respect to the Carnot Carathéodory distance for all
$X \in HG$. This would be an optimal regularity property and it will be the topic of future studies.

Our paper is organized as follows. Section 2 contains comparison principles for $F$-admissible viscosity solutions. In Section 3 we show existence of $F$-admissible Lipschitz continuous viscosity solutions in small balls. Section 4 contains basic definitions of spaces of Hölder continuous functions and some well known facts in Sub-Riemannian geometry. In Section 5 we prove our main theorem. In Section 6 we exhibit examples.

2. Comparison principle for viscosity solutions

We first recall the definition of sub- and super-solution in the viscosity sense.

**Definition 2.1.** Let us consider the equation

\[(12) \quad F(x, u, Du, D^2u) = 0, \quad \text{in } \Omega,\]

We say that a function $u \in \text{USC}(\Omega)$ is a viscosity sub-solution for (12) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $x_0 \in \Omega$ is a local maximum for the function $u - \varphi$, then $\varphi$ is $F$-admissible at $x_0$ and

\[(13) \quad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0\]

We say that a function $u \in \text{LSC}(\Omega)$ is a viscosity super-solution for (12) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $x_0 \in \Omega$ is a local minimum for the function $u - \varphi$, then either $\varphi$ is $F$-admissible at $x_0$ and

\[(14) \quad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0\]

or $\varphi$ is not $F$-admissible at $x_0$.

A continuous function $u$ is a viscosity solution for (12) if it is either a viscosity sub-solution and a viscosity super-solution for (12).

We say that a function $u \in \text{USC}(\Omega)$ is a viscosity sub-solution for (8) if $u$ is a viscosity sub-solution for (12) and in addition $u \leq \phi$ on $\partial \Omega$.

We say that a function $u \in \text{LSC}(\Omega)$ is a viscosity super-solution for (8) if $u$ is a viscosity super-solution for (12) and in addition $u \geq \phi$ on $\partial \Omega$.

A viscosity solution for (8) is either a viscosity sub-solution and a viscosity super-solution for (8).
The following comparison result plays a crucial role in the proof of Theorem 1.1.

**Theorem 2.1.** Let $u$ be a viscosity subsolution and let $v$ be a viscosity supersolution to the equation $F = 0$ in a bounded open set $\Omega \subset \mathbb{R}^n$. If $f$ is continuous, positive and strictly increasing with respect to $u$ and

$$\limsup_{x \to x_0} u(x) \leq \liminf_{x \to x_0} v(x) \quad \text{for every} \quad x_0 \in \partial \Omega,$$

then $u \leq v$ in $\Omega$.

This Comparison Principle is proved in [14, Proposition 4.1].

A comparison principle in the class of uniformly horizontal convex sub- and super-solution of the Monge-Ampère equation in homogeneous Carnot groups has been proved in [2].

Here we would like to have a comparison principle for $F$ also when $f$ is only increasing with respect to $u$. In order to adapt the proof for the strictly monotone case in this situation, one needs to find (for instance) a strictly sub-solution for $F$, and we can choose $R > 0$ such that the defining function $\rho(x) = (\|x\|^2 - R^2)/2$ of $B_R$ is a a strictly sub-solution for $F$ in $B_R$. Precisely, we have

**Theorem 2.2.** Let $f$ be continuous, positive and increasing with respect to $u$. Then, there is $R > 0$ such that, if $u$ is a viscosity subsolution and $v$ is a viscosity supersolution to the equation $F = 0$ in $B_R$ and

$$\limsup_{x \to x_0} u(x) \leq \liminf_{x \to x_0} v(x) \quad \text{for every} \quad x_0 \in \partial B_R,$$

then $u \leq v$ in $B_R$.

### 3. A preliminary existence result

In this section we assume that $f$ is increasing with respect to $u$. Let us fix $R > 0$ such that $B_R$ is strictly $F$-admissible and

$$\left(\sigma_k(A_m(x)A_m^T(x) + Q_m(x,x))\right)^{\frac{1}{2}} > 1/2, \quad \text{for every} \quad x \in B_R,$$

We define the following function

$$f_{\infty} : B_R \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad f_{\infty}(x, r, \mathcal{P}) := \lim_{\lambda \to \infty} \frac{f(x, \lambda r, \lambda \mathcal{P})}{\lambda}.$$
Let us suppose that \( f_\infty \) exists at every point and that for the defining function \( \rho(x) = (\|x\|^2 - R^2)/2 \) of \( B_R \) it holds

\[
 f_\infty(x, \rho(x), D_\beta \rho(x)) < 1/2, \quad \text{for every } x \in \bar{B}_R.
\]

This restriction on the growth of the function \( f \) and the Lie group structure arise in [14] through the consideration of gradient estimates.

By [14, Theorem 1.1] we have the following existence result for the Dirichlet problem (8)

**Theorem 3.1.** Let us fix \( R > 0 \) such that \( B_R \) is strictly \( F \)-admissible and such that (15) holds true. Let \( f \in C^1 \) be a non negative function, increasing with respect to \( u \), and such that that \( f_\infty \) exists and it satisfies condition (16) and that at least one of the following two conditions holds true:

- \( f \) does not depend on its first argument \( x \)
- \( \inf \frac{\partial f}{\partial u} > 0. \)

Then, for every boundary data \( \phi \in C^{1,1} \), there exists a viscosity solution \( u \in \text{Lip}(\bar{B}_R) \) of the problem (8).

Moreover, let us fix a positive constant \( c \) such that

\[
 f(x, (\sup_{\partial B_R} \phi), cD_\beta \rho(x)) < c/2, \quad \text{for every } x \in B_R.
\]

Then

- if \( f \) does not depend on its first argument \( x \), then \( \|u\|_{\text{Lip}(\bar{B}_R)} \) only depends on \( R \), \( \|\phi\|_{C^{1,1}(\bar{B}_R)} \), and \( c \).
- if \( \inf \frac{\partial f}{\partial u} > 0 \), then \( \|u\|_{\text{Lip}(\bar{B}_R)} \) only depends on \( R \), \( \|\phi\|_{C^{1,1}(\bar{B}_R)} \), \( c \), and on \( \sup_{\bar{B}_R \times [\inf_{\partial B_R} \phi + c\rho(x), \sup_{\partial B_R} \phi]} \frac{|D_\gamma f|}{f_u} \).

The following remark shows that if the boundary data is convex, on small balls we can relax the growth condition (16)

**Remark 3.1.** If the boundary data \( \phi \in \text{Lip}(B_1) \) is convex in \( B_1 \) and \( cR < 1 \) then

\[
 f(x, (\sup_{\partial B_R} \phi), cD_\beta \rho(x)) = f(x, (\sup_{\partial B_R} \phi), cW(x)) \leq \sup_{(x,p) \in B_1 \times B_1} f(x, (\sup_{\partial B_1} \phi), W(x)p).
\]
Now we choose \( c \) such that \( \sup_{(x,p)\in B_1\times B_1} f(x, (\sup_{\partial B_1} \phi), W(x)p) < c/2 \) and \( R < 1/c \) and we get that (17) is always satisfied in \( B_R \) without assuming the growth condition (16).

In this section we prove the existence of a Lipschitz-continuous viscosity solution to a Dirichlet problem for \( F \) on the ball \( B_R \), for \( R \) sufficiently small. The boundary data will be the restriction to \( \partial B_R \) of a convex function \( \phi \) in \( \overline{B_R} \) satisfying the equation \( F = 0 \) in \( B_R \). The crucial point of this preliminary existence result is the dependence of the gradient of the solution only on the gradient of \( \phi \). The proof is a refinement of Theorem 3.1.

To prove our existence result, Proposition 3.1, we show the following lemma, which provides a strict subsolution to \( F = 0 \) independent of the second derivatives of the boundary data.

In this section \( 0 < R < 1 \) is fixed such that (15) holds true.

**Lemma 3.1.** Let \( \phi \in C^2(B_1) \cap \text{lip}(B_1) \) be a convex function. For each \( \lambda > 0 \) define
\[
 u_\lambda(x) := \phi(x) + \lambda \rho(x), \quad x \in B_1,
\]
where \( \rho(x) = (|x|^2 - r^2)/2 \). Then, there exists \( 0 < r < R < 1 \) and \( \lambda^* > 0 \), only depending on \( \sup_{B_R} |D\phi| \), such that
\[
 F(x, u_\lambda, Du_\lambda, D^2u_\lambda) < 0 \quad \text{in } B_r, \quad \text{for every } \lambda > \lambda^*.
\]

**Proof.** Since \( \phi \) is a convex function,
\[
 A_m(x)D^2u_\lambda A_m^T(x) + Q_m(x, u_\lambda) \geq \lambda \left( A_m(x)A_m^T(x) + Q_m(x, x) + Q_m(x, D\phi/\lambda) \right).
\]

Let us choose \( \tilde{\lambda}^* = \lambda^*(r, \sup_{\partial B_R} |D\phi|) > 0 \) such that for every \( \lambda > \tilde{\lambda}^* \) and for any \( x \in B_r \),
\[
 \left( A_m(x)A_m^T(x) + Q_m(x, x) + Q_m(x, D\phi/\lambda) \right) > \frac{1}{2} Id_m.
\]

In particular \( u_\lambda \) is \( F \)-admissible in \( B_r \) for every \( \lambda > \tilde{\lambda}^* \). Moreover, as a consequence of the monotonicity of the function \( A \to \sigma_k^{1/k}(A) \) and of its homogeneity, i.e. \( \sigma_k^{1/k}(\lambda A) = \lambda \sigma_k^{1/k}(A) \) for every \( \lambda > \tilde{\lambda}^* \), we get
\[
 F(x, u_\lambda, Du_\lambda, D^2u_\lambda) \leq -\lambda \sigma_k^{1/k} \left( A_m(x)A_m^T(x) + Q_m(x, x) + Q_m(x, D\phi/\lambda) \right)
 + f(x, u_\lambda, Dgu_\lambda)
 \leq -\frac{1}{2} \lambda + f(x, u_\lambda, Dgu_\lambda)
\]
Then, since $Du_{\lambda}(x) = \lambda \left( \frac{D\phi}{\lambda} + x \right)$, by recalling Remark 3.1 we can fix $\lambda^* > \tilde{\lambda} > 0$ and a small $r$, only depending on $\sup_{\partial B_1} |D\phi|$, such that
\[
\frac{f(x, u_{\lambda}, D_2 u_{\lambda})}{\lambda} < \frac{1}{2}
\]
for every $\lambda > \lambda^*$.

This inequality easily implies that (18) holds for every $\lambda > \lambda^*$ and for all $x \in B_r$. □

Using the previous lemma and by Theorem 3.1, we obtain the main result of
this section.

**Proposition 3.1.** Assume the conditions of Theorem 1.1. If $\phi \in C^2 \cap \text{Lip}(\overline{B}_1)$ is a convex function such that $F \geq 0$ in $B_1$, then there is $0 < r < R < 1$ such that the Dirichlet problem
\[
(19) \quad F = 0, \text{ in } B_r, \quad u = \phi \text{ on } \partial B_r,
\]
has a viscosity solution $u \in \text{Lip}(\overline{B}_r)$ satisfying
\[
(20) \quad \|u\|_{L^\infty(\overline{B}_r)} + \|u\|_{\text{Lip}(\overline{B}_r)} \leq C,
\]
where

- if $f$ does not depend on $x$, $C$ only depends on $r$, $\|\phi\|_{L^\infty(\overline{B}_r)}$, $|D\phi|_{L^\infty(\overline{B}_R)}$;
- if $f_u > 0$, $C$ only depends on $r$, $\|\phi\|_{L^\infty(\overline{B}_R)}$, $|D\phi|_{L^\infty(\overline{B}_R)}$ and on
\[
\sup_{\overline{B}_R \times \{\inf_{B_R} \phi + c \phi(x), \sup_{B_R} |\phi| |x| \leq \mathbb{R}^n}} \frac{|D_k f|}{f_u}
\]
with $c$ as in Remark 3.1.

**Proof.** Let $u_{\lambda} = \phi + \lambda \rho$ be the function given by the previous lemma with $\lambda > \lambda^*$. Then $u_{\lambda} \in C^2(\overline{B}_r)$ and it is a classical subsolution to $F = 0$ in $B_r$. Moreover, $u_{\lambda} = \phi$ on $\partial B_r$. On the other hand, since $F(x, \phi, D\phi, D^2\phi) \geq 0$ in $B_1$, $\phi$ is a classical supersolution to $F = 0$ in $B_r$.

Then, by Theorem 3.1, the Dirichlet problem (19) has a viscosity solution $u \in C(\overline{B}_r)$ and by the comparison principle we have $u_{\lambda} \leq u \leq \phi$ in $\overline{B}_r$. Hence $\sup_{B_r} |u| \leq \sup_{\overline{B}_r} |\phi| + \lambda r$. On the other hand, by Lemma 3.1, $|Du_{\lambda}|$ can be bounded by a constant only depending on $r$ and $\sup_{B_R} |D\phi|$. By the interior gradient estimates
in Theorem 3.1 we can conclude that \( u \in Lip(B_r) \) with \( \|u\|_{Lip(B_r)} \) bounded by a constant \( C > 0 \), where

- if \( f \) does not depend on \( x \), \( C \) only depends on \( r, \|\phi\|_{L^\infty(B_{2r})}, \|D\phi\|_{L^\infty(B_{2r})} \)

- if \( f_u > 0 \), \( C \) only depends on \( r, \|\phi\|_{L^\infty(B_{2r})}, \|D\phi\|_{L^\infty(B_{2r})} \) and on

\[
\sup_{\overline{B_R} \times [\inf_{\partial B_R} \phi + c \rho(x), \sup_{\partial B_R}\phi] \times \mathbb{R}^n} \frac{|D_{\cdot} f|}{f_u}.
\]

\[\square\]

4. Basic definitions: Hölder Spaces, Hörmander vector fields, Carnot Carathéodory distance, Carnot group

In this section we fix notation and we briefly recall some well known facts. Let \( X_1, \ldots, X_m \) be a system of real smooth vector fields defined in some bounded connected open subset \( \Omega \) of \( \mathbb{R}^n \), with \( m \leq n \).

**Definition 4.1.** For any \( \beta \in (0, 1) \) we set

\[ C^\beta(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : \|u\|_{C^\beta(\Omega)} < \infty \right\} \]

where

\[ \|u\|_{C^\beta(\Omega)} = \sup \left\{ \frac{|u(x) - u(y)|}{d(x, y)^\beta} : x, y \in \Omega, x \neq y \right\} \]

and \( d \) is the Euclidean distance.

Moreover, let

\[ C^{1,\beta}_X(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : \|u\|_{C^{1,\beta}_X(\Omega)} < \infty \right\} \]

where

\[ \|u\|_{C^{1,\beta}_X(\Omega)} = \sum_{j=1}^m \|X_j u\|_{C^\beta(\Omega)} \]

For any multi index \( I = (i_1, i_2, \ldots, i_j), 1 \leq i_j \leq m \), we set

\[ X_I = [X_{i_1}, [X_{i_2}, \ldots [X_{i_{j-1}}, X_{i_j}], \ldots]], \]

where \( [X, Y] = XY - YX \). We say that \( X_I \) is a commutator of length \( |I| = j \).

We say that \( X_1, \ldots, X_m \) satisfy Hörmander condition at step \( s \) in \( \Omega \) if the vector fields, together with their commutators of length \( \leq s \), span the tangent space at every point in \( \Omega \). The vector fields induce on \( \mathbb{R}^n \) a metric \( d_{CC} \) in the following way (see [17]).
Definition 4.2 (Carnot Carathéodory metric). A Lipschitz continuous curve \( \gamma : [0, T] \rightarrow \mathbb{R}^n, T \geq 0 \), is subunit if there exists a vector of measurable functions \( h : [0, T] \rightarrow \mathbb{R}^n \) such that \( \gamma'(\tau) = \sum_{j=1}^m h_j(\gamma(\tau)) X_j(\gamma(\tau)) \) and \( \sum_{j=1}^m h_j^2(\tau) \leq 1 \) for a.e. \( \tau \in [0, T] \).

Define the Carnot Carathéodory distance \( d_{CC} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, +\infty) \) by setting
\[
d_{CC}(x, y) = \inf\{T \geq 0 : \text{there exists a subunit curve } \gamma : [0, T] \rightarrow \mathbb{R}^n \text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y\}.
\]

It is well-known that if the vector fields are smooth and satisfy Hörmander condition then \( d_{CC}(x, y) \) is finite for all \( x, y \) (see [3], [17]). Moreover, if \( X_1, \ldots, X_m \) satisfy Hörmander condition at step \( s \) in \( \Omega \), then if \( K \subset \subset \Omega \) is any compact set, there are positive constants \( c, C \) so that if \( x, y \in K \)
\[
(21) \quad c d(x, y) \leq d_{CC}(x, y) \leq C d(x, y)^{1/s}.
\]

Definition 4.3. Let \( X_1, \ldots, X_m \) be a system of real smooth vector fields satisfying Hörmander condition in some bounded connected subset \( \Omega \) of \( \mathbb{R}^n \), with \( m \leq n \). For any \( \beta \in (0, 1) \) we set
\[
C_{d_{CC}}^\beta(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : \|u\|_{C_{d_{CC}}^\beta(\Omega)} < \infty \right\}
\]
where
\[
\|u\|_{C_{d_{CC}}^\beta(\Omega)} = \sup \left\{ \frac{|u(x) - u(y)|}{d_{CC}(x, y)^\beta} : x, y \in \Omega, x \neq y \right\}
\]
Moreover, let
\[
C_{X_{d_{CC}}}^{1,\beta}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : \|u\|_{C_{X_{d_{CC}}}^{1,\beta}(\Omega)} < \infty \right\}
\]
where
\[
\|u\|_{C_{X_{d_{CC}}}^{1,\beta}(\Omega)} = \sum_{j=1}^m \|X_j u\|_{C_{d_{CC}}^\beta(\Omega)}.
\]

We explicitly remark here that, if \( X_1, \ldots, X_m \) is a system of real smooth vector fields satisfying Hörmander condition at step \( s \) in \( \Omega \), then by (21) we have
\[
C^\beta(\Omega) \subset C_{d_{CC}}^\beta(\Omega) \subset C^{\beta/s}(\Omega).
\]

A remarkable example of a system of real smooth vector fields satisfying Hörmander condition at step \( s \) is furnished by the Jacobian basis of a stratified Lie group. For reader convenience, we recall here the definition of stratified Lie group.
Definition 4.4. A stratified Lie group (or Carnot group) $\mathbb{H}$ is a simple connected Lie group whose Lie algebra $\mathfrak{l}$ admits a stratification, i.e. a direct sum decomposition $\mathfrak{l} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ such that $[V_i, V_{i-1}] = V_i$ for $2 \leq i \leq s$, and $[V_1, V_s] = \{0\}$.

In this case we say that $\mathbb{H}$ has step (of nilpotency) $s$ and has $m$ generators, where $m = \dim(V_1)$.

5. Existence of nonsmooth solutions

Proof of Theorem 1.1 First of all remark that if condition $(H_t)$ holds true for some $\ell \geq k - 1$ then in particular $(H_t)$ holds true for $\ell = k - 1$. Throughout this section we then fix $k/2 < \ell \leq k - 1$ such that condition $(H_t)$ holds true, and we denote by $x = (x', x'')$, $x' = (x_1, \ldots, x_t) \in \mathbb{R}^\ell$, $x'' = (x_{t+1}, \ldots, x_k) \in \mathbb{R}^{k-\ell}$ with $x \in \mathbb{R}^k$. We denote by $\xi = (x, t)$ points of $\mathbb{R}^n$, with $x \in \mathbb{R}^k$ and $t \in \mathbb{R}^{n-k}$.

For $0 \leq \varepsilon < 1$ and $0 < r < R$ such that Proposition 3.1 holds true, we define

\begin{equation}
\psi_\varepsilon(x) = \psi_\varepsilon(x', x'') := (r^2 + |x''|^2)(\varepsilon + |x'|^2)^\alpha, \quad \alpha = \frac{\ell}{k},
\end{equation}

and

\begin{equation}
\psi_0 \leq \psi_\varepsilon \leq \phi_\varepsilon, \quad \text{in } B_1.
\end{equation}

Since $\ell > k/2$, the exponent $\alpha = \frac{\ell}{k} > \frac{1}{2}$ and so that $\phi_\varepsilon$ is convex in $\mathbb{R}^n$ for $\varepsilon \geq 0$.

Moreover, $\phi_\varepsilon$ is smooth for $\varepsilon > 0$, and independent of $x''$ and $t$. From condition $(H_t)$ we then obtain for $\varepsilon > 0$

\[
A_m(\xi)D^2\phi_\varepsilon A^T_m(\xi) + Q_m(\xi, D\phi_\varepsilon) = A_m(\xi)D^2\phi_\varepsilon A^T_m(\xi) \geq 0
\]

and it has $m - \ell$ null eigenvalues. This means that $\phi_\varepsilon$ is $F$ admissible and since $m - \ell \geq m - k + 1$

\[
\sigma_k\left(A_m(\xi)D^2\phi_\varepsilon A^T_m(\xi) + Q_m(\xi, D\phi_\varepsilon)\right) = 0.
\]

Therefore:

\begin{equation}
F(\xi, \phi_\varepsilon, D\phi_\varepsilon, D^2\phi_\varepsilon) = f(\xi, \phi_\varepsilon, D\phi_\varepsilon) > 0 \quad \text{in } B_1, \quad \forall \varepsilon \in ]0, 1[.
\end{equation}

Thus, applying Proposition 3.1 there exists $0 < r < R$ such that the Dirichlet problem

\[
F = 0 \quad \text{in } B_r, \quad u = \phi_\varepsilon \quad \text{on } \partial B_r,
\]
with $\varepsilon \in ]0, 1[$, has a viscosity solution $u_\varepsilon$ such that
\[
\|u_\varepsilon\|_{L^\infty(B_r)} + \|u_\varepsilon\|_{\text{Lip}(B_r)} \leq C(r, \varepsilon, M)
\]
with $C(r, \varepsilon, M)$ depending on $\varepsilon$ only through $C(\phi_\varepsilon) := \|\phi_\varepsilon\|_{L^\infty(B_r)} + \|D\phi_\varepsilon\|_{L^\infty(B_r)}$.

On the other hand, an elementary computation shows that $C(\phi_\varepsilon) \leq 8M$. Then, we can choose $C(r, \varepsilon, M)$ independent of $\varepsilon$, and so
\[
\|u_\varepsilon\|_{L^\infty(B_r)} + \|u_\varepsilon\|_{\text{Lip}(B_r)} \leq C(r, M).
\]

Now we claim that, if $0 < r \ll R$, we can fix $M = M(r)$ such that
\[
F(\xi, \psi_\varepsilon, D\psi_\varepsilon, D^2\psi_\varepsilon) < 0 \quad \text{in } B_r, \quad \forall \varepsilon \in ]0, r^2[.
\]

Assuming this claim for a moment, we can use the Comparison Principle of Section 2 to compare $u_\varepsilon$ with $\psi_\varepsilon$ and $\phi_\varepsilon$. Indeed, by (23) and (25), $\phi_\varepsilon$ and $\psi_\varepsilon$ are, respectively, classical supersolution and subsolution to $F = 0$ in $B_r$. On the other hand $\psi_\varepsilon \leq \phi_\varepsilon$ in $B_1$, in particular, $\psi_\varepsilon \leq \phi_\varepsilon$ on $\partial B_r$. Thus, by the Comparison Principle,
\[
\psi_\varepsilon \leq u_\varepsilon \leq \phi_\varepsilon \quad \text{in } B_r, \quad \forall \varepsilon \in ]0, r^2[.
\]

The uniform estimate (24) implies the existence of a sequence $\varepsilon_j \searrow 0$ such that $(u_{\varepsilon_j})_{j \in \mathbb{N}}$ uniformly converges to a viscosity solution $u \in \text{Lip}(\overline{B}_r)$ to the Dirichlet problem
\[
F = 0 \quad \text{in } B_r, \quad u = \phi_0 \text{ on } \partial B_r;
\]
the proof of this fact is given in [14, Lemma 3.1]. Moreover, from the comparison principle, we get
\[
\psi_0 \leq u \leq \phi_0 \quad \text{in } B_r.
\]
In particular
\[
M r^2 |x_1|^{2\alpha} \leq u(x_1, 0, \ldots, 0) \leq 2M |x_1|^{2\alpha}.
\]

As in the proof of [9, Theorem 1] inequalities in (28) imply:
\[
\partial_{x_1} u \notin C^\beta, \quad \text{for every } \beta > 2\alpha - 1 = \frac{2\ell}{k} - 1 \quad \text{if } 2\alpha > 1 \quad \text{(i.e. } \ell > k/2).
\]

Moreover, inequalities (27) imply that
\[
X_1 u \notin C^\beta, \quad \text{for every } \beta > 2\alpha - 1 = \frac{2\ell}{k} - 1 \quad \text{if } 2\alpha > 1 \quad \text{(i.e. } \ell > k/2).
\]
Indeed, let us denote by \( \exp(sX_1)(0) = \gamma(s) \) the integral curve of the vector field \( X_1 \) starting at the origin, i.e. \( \gamma'(s) = X_1 \gamma(s), \gamma(0) = 0 \). Remark that \( \gamma \) is a smooth function, whose first order Taylor expansion in a neighborhood of the origin is (see also \([3, \text{section 1.12}]\))

\[
\gamma(s) = sX_1(0) + o(s) = se_1 + o(s), \quad \text{as } s \to 0,
\]

where \( e_1 = (1,0,\ldots,0) \) and \( o(s)/s \to 0 \) as \( s \to 0 \).

If \( 2\alpha > 1 \), then \( (X_1 u)(0) = \partial_{x_1} u(0) = 0 = u(0) \) so that, if \( X_1 u \) was \( C^\beta \), with \( \beta > 2\alpha-1 \), we would have \( u(\exp(x_1 X_1)(0)) \leq C|x_1|^{1+\beta} \) for a suitable \( C > 0 \) and for every \( x_1 \) sufficiently small. Hence, by denoting \( \pi' : \mathbb{R}^n \to \mathbb{R}^\ell \) the usual projection operator associating to a vector \( \xi = (x',x'',t) \in \mathbb{R}^n \) its first variables \( x' \in \mathbb{R}^\ell \), by the first inequality in \([27]\), we would have the existence of a positive constant \( c \) such that

\[
|\gamma'|^{2\alpha} \leq Mr^2|\pi'(\exp(x_1 X_1)(0))|^{2\alpha} \leq \psi_0((\exp(x_1 X_1)(0)) \leq C|x_1|^{1+\beta}
\]

and it would be \( \beta \leq 2\alpha - 1 \), a contradiction.

To complete the proof of the theorem, we are left with the proof of Claim \([25]\).

By condition \((H_j)\) and by the smooth regularity of the functions \( \xi \to A_m(\xi) \), \( \xi \to Q_m(\xi,\cdot) \) and by the linearity of \( p \to Q_m(\cdot, p) \) we get

\[
A_m(\xi)D^2w_e A^T_m(\xi) + Q_m(\xi, Dw_e) = A_m(\xi)D^2w_e A^T_m(\xi) + Q_m(\xi, D_{x''}w_e)
\]

\[
= A_m(0)D^2w_e A^T_m(0)(1 + \omega_1(\xi)) + 2(\varepsilon + |x'|^2)^n Q_m(\xi, x'')
\]

\[
= Id_m D^2w_e(x) Id_m(1 + \omega_1(\xi)) + \omega_2(\xi)(\varepsilon + |x'|^2)^n Id_m
\]

where, for \( j = 1, 2 \), \( \omega_j(\xi) \to 0 \) uniformly in \( \varepsilon \) as \( \xi \to 0 \). Moreover, \( Id_m D^2w_e(x) Id_m \)

has \( m - k \) null rows by construction. Therefore

\[
\sigma_k(Id_m D^2w_e(x) Id_m) = \det D^2w_e.
\]

Direct computations show that

\[
\det D^2w_e(x) = 2^{2k} f_{\ell}(x)
\]

with

\[
f_{\ell}(x) = \alpha^{\ell+1}(r^2 + |x''|^2)^{\ell-1} \frac{r^2(\alpha^{-1}\varepsilon + |x'|^2) + \alpha^{-1}\varepsilon(|x'|^2)}{(\varepsilon + |x'|^2)}.
\]

For convenience of the reader we include the proof of \([32]\).
We have
\[
D^2_{x_0} w_\varepsilon(x) = 4(\varepsilon + |x'|^2)^{\ell-1} \begin{pmatrix}
(r^2 + |x''|^2) & (\alpha \ell \varepsilon + \alpha (\alpha - 1) \frac{x' \otimes x'}{\varepsilon + |x'|^2}) & \alpha x' \otimes x'' \\
ax'' \otimes x' & (\varepsilon + |x'|^2)I_{d-\ell}
\end{pmatrix}
\]

Since \((\alpha - 1)k = \ell - k\), we get
\[
det D^2_{x_0} w_\varepsilon(x) = 4^k(\varepsilon + |x'|^2)^{\ell-k} \begin{pmatrix}
(r^2 + |x''|^2) & (\alpha \ell \varepsilon + \alpha (\alpha - 1) \frac{x' \otimes x'}{\varepsilon + |x'|^2}) & \alpha x' \otimes x'' \\
x'' \otimes x' & (\varepsilon + |x'|^2)I_{d-\ell}
\end{pmatrix}
\]
\[
= 4^k \begin{pmatrix}
(r^2 + |x''|^2) & (\alpha \ell \varepsilon + \alpha (\alpha - 1) \frac{x' \otimes x'}{\varepsilon + |x'|^2}) & \alpha x' \otimes x'' \\
\alpha x'' \otimes x' & (\varepsilon + |x'|^2)I_{d-\ell}
\end{pmatrix}
\]
\[
= 4^k \begin{pmatrix}
(r^2 + |x''|^2) & (\alpha \ell \varepsilon + \alpha (\alpha - 1) \frac{x' \otimes x'}{\varepsilon + |x'|^2}) - \alpha^2 |x''|^2 \frac{x' \otimes x'}{\varepsilon + |x'|^2} & 0 \\
\alpha x'' \otimes x' & (\varepsilon + |x'|^2)I_{d-\ell}
\end{pmatrix}
\]
\[
= 4^k \begin{pmatrix}
(r^2 + |x''|^2) & (\alpha \ell \varepsilon + \alpha (\alpha - 1) \frac{x' \otimes x'}{\varepsilon + |x'|^2}) - \alpha^2 |x''|^2 \frac{x' \otimes x'}{\varepsilon + |x'|^2} \\
\alpha x'' \otimes x' & (\varepsilon + |x'|^2)I_{d-\ell}
\end{pmatrix}
\]
\[
:= 4^k \det \Gamma,
\]
where \(\Gamma\) is a \(\ell \times \ell\) symmetric matrix. It is easy to see that \(\lambda_1 = \alpha (r^2 + |x''|^2)\) is an eigenvalue of \(\Gamma\) with multiplicity \(\ell - 1\). Now, trace \(\Gamma = (\ell - 1)\lambda_1 + \lambda_2\) with
\[
\lambda_2 = (r^2 + |x''|^2) \left( \alpha + \alpha (\alpha - 1) \frac{|x''|^2}{\varepsilon + |x'|^2} \right) - \alpha^2 |x''|^2 \frac{|x'|^2}{\varepsilon + |x'|^2}
\]
\[
= \alpha^2 \frac{r^2}{\varepsilon + |x'|^2} + \frac{\varepsilon}{\alpha} |x'|^2
\]
Thus, \(\det \Gamma = 4^{\ell-1} \lambda_2 = 4^{\ell+1} (r^2 + |x''|^2)^{\ell-1} \frac{r^2(\varepsilon + |x'|^2) + \varepsilon |x'|^2}{\varepsilon + |x'|^2} = f_\varepsilon\), which completes the proof of (32). In particular, \(f_\varepsilon \geq 4^{\ell+1} r^{2\ell} > \alpha^{\ell} r^{2\ell}/2\).

Keeping in mind that \(\psi_\varepsilon = M \psi_\varepsilon\) and (30), (31), we can choose \(r\) small, \(0 < r < R\), such that
\[
\varepsilon^{1/k}_k \left( A_m(\xi) D^2 \psi_\varepsilon A_m(\xi) + Q_m(\xi, D \psi_\varepsilon) \right) > \alpha^{\ell} r^{2\ell} M.
\]
On the other side, direct computations show that
\[
|D w_\varepsilon|^2 = 4 \left( |x''|^2 (\varepsilon + |x'|^2) + \alpha^2 |x'|^2 (r^2 + |x''|^2) (\varepsilon + |x'|^2)^2 (\alpha - 1) \right)
\]
and for every \(\varepsilon \in [0, r^2]\),
\[
|D w_\varepsilon|^2 \leq 2^{2\alpha+3} r^{4\alpha+2} \quad \text{in } B_r.
\]
From (33), we obtain

\[ |D\psi_\varepsilon| \leq 2^{a+3/2}Mr^{\alpha+2} \text{ in } B_r. \]

Choosing \( M = 2^{-a-(3/2)r-2^{a-1}} \), the right hand side of (34) equals 1, and \( \psi_\varepsilon \leq 2^{-a-1/2}r < 1 \). The strategy now is to take a smaller \( r \) such that

\[ \sup_{(\xi, p) \in B_1 \times B_1} f(\xi, 1, W(\xi)p) < \alpha^a 2^{-a-(3/2)r-1}. \]

Then, by the increasing monotonicity of \( s \rightarrow f(\cdot, s, \cdot) \), in \( B_r \) we obtain

\[ F(\xi, \psi_\varepsilon, D\psi_\varepsilon, D^2\psi_\varepsilon) < -\alpha^a 2^{-a-(3/2)r-1} + f(\xi, \psi_\varepsilon, W(\xi)D\psi_\varepsilon) \]

\[ < -\alpha^a 2^{-a-(3/2)r-1} + f(\xi, 1, W(\xi)D\psi_\varepsilon) < 0. \]

This proves claim (25) and completes the proof of the theorem. \qed

**Remark 5.1.** If the vector fields \( X_1, \ldots, X_m \) satisfy Hörmander condition in \( B_R \), then \( d_{CC}(\exp(x_1X_1)(0), 0) = x_1 \) and from inequality (29) we get \( u \notin C^{1,\beta}_{x, d_{CC}}(B_R) \).

### 6. Examples

Here we will show some examples on which the condition \((H_\ell)\) is fulfilled. First we consider the case of a homogeneous Carnot group. We refer to [3, Section 1.4] for a full detailed exposition on the theory of homogeneous Carnot groups.

**Example 6.1.** Let us consider a homogeneous Carnot group on \( \mathbb{R}^n \) with \( m \) generators: then condition \((H_\ell)\) is satisfied with \( \ell = m \). Indeed, let us consider the Jacobian basis \( E^i \) for Lie algebra \( \mathfrak{l} \) of the left-invariant vector fields. Let us suppose that the first layer of the stratification \( V^m_1 \) has dimension \( m \leq n \) and it is spanned by the first \( m \) vector fields of the basis, namely \( E_m = \{X_1, \ldots, X_m\} \). We know that such vector fields read in coordinates as

\[ X_i = \frac{\partial}{\partial x_i} + \sum_{k=m+1}^n \tau_{ik}(x) \frac{\partial}{\partial x_k}, \quad i = 1, \ldots, m \]

where \( \tau_{ik} \) are smooth (polynomial) functions defined on the whole \( \mathbb{R}^n \). In particular, \( Q_m(x, p) \) is independent of \( p_1, \ldots, p_m \) and condition \((H_\ell)\) is satisfied for \( \ell = m \).

Next we show an example of a Lie group that is not Carnot, but for which elementary symmetric functions in the eigenvalues of the Hessian \( H_{g,m} \), for some \( m \), still satisfy condition \((H_\ell)\).
**Example 6.2.** We consider the Lie group in \( \mathbb{R}^{n+1} \) given by the following group law \( \circ \): for any \((x, y), (t, s) \in \mathbb{R}^n \times \mathbb{R}\)

\[
(x, y) \circ (t, s) = (x_1, \ldots, x_n, y) \circ (t_1, \ldots, t_n, s) = (x_1 + t_1, \ldots, x_n + t_n, s + ye^{t_1 + \cdots + t_n}).
\]

A basis for \( \mathfrak{g} \) is given by the left invariant vector fields

\[
X_j = \frac{\partial}{\partial x_j} + y \frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}, \quad j = 1, \ldots, n
\]

having the following commutation properties:

\[
[X_i, X_j] = 0, \quad [X_j, Y] = -Y, \quad i, j = 1, \ldots, n.
\]

Hence, the relevant Lie algebra is not nilpotent and the Lie group is not stratified. We consider the metric \( g \) that makes orthonormal the vector fields \( X_j, Y \) for any \( j = 1, \ldots, n \) and we denote by \( \nabla \) the Levi-Civita connection for \( g \). We will prove that for any \( m \leq n \), the condition \( (H_\ell) \) holds true with \( \ell = n \). Indeed, we have that

\[
H_{g,m,u}(X_i, X_j) = X_i X_j u - (\nabla_{X_i} X_j) u, \quad i, j = 1, \ldots, m
\]

and

\[
\nabla_{X_i} X_j = \sum_{k=1}^{n} g(\nabla_{X_i} X_j, X_k) X_k + g(\nabla_{X_i} X_j, Y) Y
\]

Now, since the coefficients of the metric in this basis are constants, we know that for any vector fields \( V, W, Z \) in this basis:

\[
g(\nabla_V W, Z) = \frac{1}{2} \left( g([V, W], Z) - g([W, Z], V) + g([Z, V], W) \right)
\]

By the previous formula we get:

\[
g(\nabla_{X_i} X_j, X_k) = g(\nabla_{X_i} X_j, Y) = 0, \quad i, j = 1, \ldots, m, \quad k = 1, \ldots, n
\]

Therefore the Hessian reads in local coordinates as

\[
H_{g,m,u}(X_i, X_j) = X_i X_j u = u_{x_i x_j} + y u_{x_i y} + y^2 u_{yy} + y u_y
\]

Hence, keeping in mind the formulas (2) and (6), we see that there is no dependence on the first \( n \) components of the gradient of \( u \), that is \( (H_\ell) \) holds true with \( \ell = n \).

**Remark 6.1.** We note that in the previous example, with \( m = n + 1 \), the condition \( (H_\ell) \) never holds true. Indeed, for any \( k = 1, \ldots, n \) we have

\[
2g(\nabla_Y X_k) = g([Y, Y], X_k) - g([Y, X_k], Y) + g([X_k, Y], Y) = -2
\]
and \( g(\nabla_Y Y, Y) = 0 \). This means that

\[
H_g u(Y, Y) = YYu - (\nabla_Y Y)u = u_{yy} + \sum_{k=1}^{n} X_k u = u_{yy} + n u_y + \sum_{k=1}^{n} u_{x_k}.
\]

In particular, \( Q_{n+1}(x, Du) \) depends on all variables \( u_{x_1}, \ldots, u_{x_n}, u_y \). Moreover, if one considers the symmetrized Hessian \( H^s u \) (see formula (3)), since there are no Christoffel symbols involved, then one realizes that in \( H^s u \) the only gradient term appearing is \( u_y \), for any \( m = 1, \ldots, n + 1 \). Hence, for \( H^s u \) the condition (\( H_\ell \)) would be satisfied with any \( \ell \leq n \), and for any \( m = 1, \ldots, n + 1 \). However, we explicitly remind that the eigenvalues (and thus their elementary symmetric functions) of \( H^s u \) are not intrinsic, namely: they depend on the particular choice of the vector fields; if one changes the coordinates, giving rise to another basis of vector fields, the eigenvalues of the new symmetrized Hessian will change in general.

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**References**


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