NONSMOOTH SOLUTIONS FOR A CLASS OF FULLY NONLINEAR PDE'S ON LIE GROUPS

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Abstract. In this paper we prove the existence of non smooth viscosity solutions for Dirichlet problems involving a class a fully non-linear operators on Lie groups. In particular we consider the elementary symmetric functions of the eigenvalues of the Hessian built with left-invariant vector fields.

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Contents

1. Introduction

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In [18] Pogorelov showed that convex generalized solutions of the Monge Ampère equation

$$\det D^2 u = f(x) \tag{1}$$

in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, need not be of class C^2, even if f is positive and smooth. Urbas in [20] proved that this absence of classical regularity is not confined to equations of Monge Ampère type, but in fact occurs for the k-th elementary symmetric functions in the Hessian and the equation of prescribed k-th curvature, where $k \geq 3$. Recently Gutierrez Lanconelli and the second author in [9] proved the absence of regularity of graphs with positive and smooth prescribed Levi Monge Ampère curvature in a domain $\Omega \subset \mathbb{R}^{2n+1}$, for $n > 1$.

Our purpose here is to show that a similar result holds for elementary symmetric functions of the Hessian on Lie groups. Our study is motivated by our recent results in [14] about existence and uniqueness of Lipschitz continuous viscosity solutions for Dirichlet problems involving symmetric functions of the Hessian built with left invariant vector fields on Lie groups. A naturally subsequent problem is that of further regularity of Lipschitz continuous viscosity solutions.

To fix the notation let us recall some well known facts. Let $\mathbb{G} = (\mathbb{R}^n, \circ)$ be a Lie group on $\mathbb{R}^n$ with $\circ$ as group law. Let us denote by $\boldsymbol{l}$ the set of the left-invariant vector fields. If $E^l = \{X_1, \ldots, X_n\}$ is any basis of $\boldsymbol{l}$ then a Riemannian metric g on $\mathbb{G}$ is left-invariant if and only if the coefficients $g_{ij} := g(X_i, X_j)$ are constant functions. Each n-dimensional Lie group possesses a $n(n+1)/2$-dimensional family of distinct left-invariant metrics, see for instance [15]. Let us fix any left-invariant metric g and let u be a smooth function, we will denote by $D_g u$ the gradient of u with respect to the metric g, that is: $g(D_g u, X) = Xu = du(X)$, for every vector field X. Moreover, there exists a $n \times n$ invertible and smooth matrix W such that $D_g u = W(x)\, Du$, where Du denotes the Euclidean gradient of u. If ∇ is the Levi-Civita connection for g (we recall that the connection coefficients in term of any left-invariant basis are constant functions), then the metric Hessian of u is the tensor field of type $(0, 2)$ defined by:

$$H_g u(X, Y) := XYu - (\nabla_X Y)u$$

for every pair of vector fields (X, Y); since ∇ is the Levi-Civita connection for g (that is $\nabla_X Y - \nabla_Y X = [X, Y]$), we note that $H_g u$ is always symmetric. We will denote by $D_g^2 u := g^{-1} H_g u$ the associated endomorphism. We explicitly note that the previous definition is intrinsic, namely the eigenvalues of $D^2_g u$ do not change in a change of basis. Let us consider a coordinate frame $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, referred to local coordinates (in our setting they are actually global), we have:

$$H_g u\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) = \frac{\partial^2 u}{\partial x_i x_j} - \Gamma^s_{ij}\frac{\partial u}{\partial x_s}$$

where Γ^s_{ij} are the Christoffel symbols for the metric g (they are not constant in general). Hence if Du, D^2u denote the usual Euclidian gradient and Hessian, then $D^2_g u$ reads in local coordinates

$$D^2_g u := G^{-1}(x)[D^2u + B(x, Du)] \tag{2}$$

where we denoted by G^{-1} the symmetric matrix of coefficients of g^{-1} expressed in the coordinate frame (they are not constant functions in general) and the coefficients of the matrix B are given by $b_{ij} = \Gamma^s_{ij}\frac{\partial u}{\partial x_s}$. Let us note that the matrix B is symmetric because the Levi-Civita connection has null torsion.
Here we will consider also the case of strictly restrictions to some subspace of $\boldsymbol{l}$. Let then $m \le n$ and let $E^l_m = \{X_1, \dots, X_m\}$, we define the subspace of the left-invariant vector fields

$$H\mathbb{G} := span\{X_1, \dots, X_m\}$$

Now we consider a left-invariant metric g_m on $H\mathbb{G}$, we can "complete" it to the full tangent space by defining the blocks metric:

$$g := \begin{pmatrix} g_m & 0 \\ 0 & Id_{n-m} \end{pmatrix}$$

where, for every integer n, Id_n denotes the identity matrix of order n. We define

$$H_{g,m}u(X, Y) := XYu - (\nabla_X Y)u, \qquad \forall\, X, Y \in H\mathbb{G}$$

and $D^2_{g,m}u := g_m^{-1}H_{g,m}u$.

We note that there is another recurrent definition of Hessian on Lie groups, let us call it the symmetrized Hessian $H^s u$, that is, for every smooth function u and for every pair of left-invariant vector fields X, Y:

$$H^s u(X, Y) = \frac{XYu + YXu}{2} \tag{3}$$

In particular there is a very large literature on questions involving this symmetrized Hessian on Carnot groups (see for instance [7], [10], [11], [13], [19], [21], [5], [1] and the references therein). An easy computation shows that our metric Hessian $H_g u$ coincides with $H^s u$ if and only if it holds:

$$\nabla_X Y = \frac{1}{2}[X, Y], \qquad X, Y \in \boldsymbol{l} \tag{4}$$

In [14, Example 2.1] we proved that if we consider the case of stratified Lie groups (in the sequel Carnot groups) (see Definition 4.4 or [3, Definition 2.2.3]), then the two Hessian definitions coincide. Moreover, in [14, Example 2.2] we exhibited an example of a Lie group that does not satisfy (4).

Let $B_R \subseteq \mathbb{R}^n$ be the Euclidean ball of radius R and center at the origin and let $f : B_R \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth positive function, here we are interested on the Dirichlet problem associated with equations of the following form

$$\Big(\sigma_k(D^2_{g,m}u)\Big)^{\frac{1}{k}} = f(x, u, D_g u), \qquad k = 1, \dots, m \tag{5}$$

where, for every symmetric matrix M of order m, $\sigma_k(M)$ denotes the k-th elementary symmetric function in the eigenvalues of M.

Our motivation comes from the geometric theory of several complex variables, where fully nonlinear second order pde's appear, whose linearizations are non variational operators of Hömander type. (See [16] and references therein). These kinds of operators, also arising in many other theoretical and applied settings, have the form of (5). The dependence on the gradient $D_g u$ in f is motivated by various applications. An interesting example is the subelliptic analogue of the prescribed Gauss curvature equation, see [7] and [5].

A direct computation shows that equation (5) reads then in local coordinates:

$$\Big(\sigma_k\Big(A_m(x)\, D^2 u\, A_m^T(x) + Q_m(x, Du)\Big)\Big)^{\frac{1}{k}} = \widetilde{f}(x, u, Du) \tag{6}$$

where $\widetilde{f}$ is a positive function such that $\widetilde{f}(x, u, Du) = f(x, u, D_g u) = f(x, u, W(x)Du)$ and A_m is a $m \times n$ matrix and Q_m is a square matrix of order m, both with smooth coefficients. Moreover, it is easy to see that (see for instance [3, Section 1.2.2])

$$A_m(0) = \begin{pmatrix} Id_m & 0 \end{pmatrix} \tag{7}$$

and Q_m is symmetric and linear with respect to Du. Precisely, we will consider viscosity solutions of the following Dirichlet problem:

$$\begin{cases} F(x,u,Du,D^2u) = 0, & \text{in } B_R, \\ u = \phi, & \text{on } \partial B_R, \end{cases} \tag{8}$$

where

$$F(x,u,Du,D^2u) := -\Big(\sigma_k\Big(A_m(x)\,D^2u\,A_m^T(x) + Q_m(x,Du)\Big)\Big)^{\frac{1}{k}} + f(x,u,D_g u) \tag{9}$$

and $\phi : \partial B_R \to \mathbb{R}$.
We refer to [12], [6] for a full detailed exposition on the theory of viscosity solutions: we will give the basic definition of sub- and super-solution in the next section.

We explicitly remark here that the partial differential equation $F = 0$ is not elliptic and it is fully nonlinear for $k > 1$.

In analogy with [4] and [14], we define the open cone

$$\Gamma_k^m = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m : \sigma_j(diag(\lambda)) > 0, \text{ for every } j = 1, \ldots, k\},$$

where $diag(\lambda)$ is the $m \times m$ diagonal matrix, and we denote by $\overline{\Gamma_k^m}$ and $\partial\Gamma_k^m$ the closure and the boundary of Γ_k^m respectively.

Remark that F is degenerate elliptic in the cone Γ_k^m, i.e. $F(x,s,p,M) \leq F(x,s,p,N)$, for all $x \in \mathbb{R}^n, s \in \mathbb{R}, p \in \mathbb{R}^n$, and M, N symmetric $n \times n$ matrices whose eigenvalues belongs to the open cone Γ_k^m and such that $M \geq N$.

Therefore, we give the following

Definition 1.1. *Let $x_0 \in \mathbb{R}^n$ and let φ be a C^2 function in a neighborhood of x_0. We will say that φ is strictly F-admissible (respectively F-admissible) at x_0, if the vector $\lambda = (\lambda_1, \ldots \lambda_m)$ of the eigenvalues of $D^2_{g,m}\varphi(x_0)$ belongs to the open cone Γ_k^m (respectively $\overline{\Gamma_k^m}$). Remark that the cone Γ_k^m is invariant with respect to permutation of λ_j.*

We will say that φ is strictly F-admissible (respectively F-admissible) in $\Omega \subset \mathbb{R}^n$ if φ is strictly F-admissible (respectively F-admissible) at x_0 for every $x_0 \in \Omega$.
Moreover if $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth defining function for a bounded open set $\Omega \subset \mathbb{R}^n$, that is

$$\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \qquad \partial\Omega = \{x \in \mathbb{R}^n : \rho(x) = 0\}$$

then we will say that the domain Ω is strictly F-admissible if ρ is strictly F-admissible.

By taking into account (7) it is easy to show that we can always find a small radius $R > 0$ such that the Euclidean ball B_R is strictly F-admissible.

The purpose of this paper is to show that when $2 < k \leq m \leq n$, viscosity solutions may not be regular even if f is positive and smooth. Precisely, we prove the following theorem, which is the main result of the paper.

We will assume that there is $\ell > k/2$ such that F in (9) satisfies the following structure condition

(H_ℓ) $Q_m(x, p) = Q_m(x, p_{\ell+1}, \dots, p_n)$ is independent of $p_1, \dots, p_\ell$.

Let us denote by Lip the space of Lipschitz continuous functions with respect to the Euclidean metric and by $C^{1,\beta}_X$ the space of functions u such that the Lie derivative Xu exists for all $X \in H\mathbb{G}$ and it is β- Hölder continuous with respect to the Euclidean distance (see Section 4 for details).

Theorem 1.1. *Suppose* $2 < k \leq m \leq n$, *and* $f \in C^\infty(B_1 \times \mathbb{R} \times \mathbb{R}^n)$ *is a positive function, monotone increasing with respect to u and satisfying at least one of the following two conditions:*

- *f does not depend on its first argument x*
- $\inf \frac{\partial f}{\partial u} > 0$.

Then there exists $R \in (0, 1)$ *and a F- admissible viscosity solution u to the equation*

$$F(x, u, Du, D^2u) = 0 \quad in \quad B_R, \tag{10}$$

such that $u \in Lip(\bar{B}_R)$.

Moreover, if condition (H_ℓ) *holds true for some* $\ell \in (k/2, k-1)$, *then* $u \notin C^{1,\beta}_X$ *for any* $\beta > \frac{2\ell}{k} - 1$. *If condition* (H_ℓ) *holds true for some* $\ell \geq k-1$, *then* $u \notin C^{1,\beta}_X$ *for any* $\beta > 1 - \frac{2}{k}$.

The proof of this theorem uses Pogorelov's counterexamples, see [18] or [8, Section 5.5], and its extensions developed by Urbas in [20] and by Gutierrez, Lanconelli and Montanari in [9] to show existence of viscosity non classical solutions to real curvature equations and to Gauss-Levi curvature equations, respectively.

A principal tool used to carry out the proof of our theorem are the comparison principles proved in [14, Section 4].

We will show in the Examples section at the end that, in any homogeneous Carnot group, elementary symmetric functions in the eigenvalues of the Hessian of the first layer satisfy (H_ℓ). Moreover, we recall that any stratified Carnot group is isomorphic to a homogeneous Carnot group and that the isomorphism preserves the stratification (see [3, Proposition 2.2.10 and Theorem 2.2.18]). In the same section, we will show that there exist Lie groups, not Carnot, such that elementary symmetric functions in the eigenvalues of the Hessian $H_{g,m}$, for some m still fulfill condition (H_ℓ).

The two alternative conditions on the function f in Theorem 1.1 and the Lie group structure arise in [14] through the consideration of gradient estimates. We remark that for Monge–Ampère equations (i.e. $k = m$) on a homogeneous Carnot group, Bardi and Mannucci in [2] proved the existence of H-convex (i.e. F-admissible) continuous viscosity solutions of (10) under the only assumption that f is positive and monotone increasing with respect to u. Moreover, as a consequence of the H-convexity, the solution is locally Lipschitz continuous with respect to the Carnot Carathéodory distance (see Definition 4.2) and for R small enough

$$\|Xu\|_{L^\infty(B_{CC}(R))} \leq \frac{C}{R}\|u\|_{L^\infty(B_{CC}(2R))}, \quad \forall X \in H\mathbb{G}$$

where the balls B_{CC} are taken with respect to the Carnot Carathéodory distance and C is a constant independent of u and of R (see [13, Theorem 4.1], [1] and the references therein). Let us denote by $C^{1,\beta}_{X,d_{CC}}$ the space of functions u such that the Lie derivative Xu exists for all $X \in H\mathbb{G}$ and it is β- Hölder continuous with respect to the Carnot Carathéodory distance. Thus, as a corollary of Theorem 1.1 (see also Remark 5.1), we get

Corollary 1.1. *Let us consider a Carnot group on $\mathbb{R}^n$ with $m \leq n$ generators. Suppose $f \in C^\infty(B_1 \times \mathbb{R} \times R^n)$ is a positive function, monotone increasing with respect to u. Then there exists $R \in (0,1)$ and a F- admissible viscosity solution u to the equation*

(11)

$$F(x,u,Du,D^2u) := -\Big(\det\Big(A_m(x)\,D^2u\,A_m^T(x)+Q_m(x,Du)\Big)\Big)^{\frac{1}{m}}+f(x,u,D_gu) = 0 \quad in \quad B_R,$$

such that u is Lipschitz continuous with respect to the Carnot Carathéodory distance in B_R and $u \notin C^{1,\beta}_{X,d_{CC}}(B_R)$ for any $\beta > 1 - \frac{2}{m}$.

Moreover, in that case, by analogy with the classical Monge-Ampère equations (see for instance [8, Section 5.4]), we expect that if the boundary data $\phi \in C^{1,\beta}(\partial B_R)$ for $\beta > 1 - \frac{2}{m}$, then the viscosity solutions u of (11) is strictly H-convex and Xu is β- Hölder continuous with respect to the Carnot Carathéodory distance for all

$X \in H\mathbb{G}$. This would be an optimal regularity property and it will be the topic of future studies.

Our paper is organized as follows. Section 2 contains comparison principles for F-admissible viscosity solutions. In Section 3, we show existence of F-admissible Lipschitz continuous viscosity solutions in small balls. Section 4 contains basic definitions of spaces of Hölder continuous functions and some well known facts in Sub-Riemannian geometry. In Section 5 we prove our main theorem. In Section 6 we exhibit examples.

2. Comparison principle for viscosity solutions

We first recall the definition of sub- and super-solution in the viscosity sense.

Definition 2.1. *Let us consider the equation*

$$F(x, u, Du, D^2u) = 0, \qquad \text{in } \Omega, \tag{12}$$

We say that a function $u \in USC(\Omega)$ is a viscosity sub-solution for (12) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $x_0 \in \Omega$ is a local maximum for the function $u - \varphi$, then φ is F-admissible at x_0 and

$$F\Big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\Big) \le 0 \tag{13}$$

We say that a function $u \in LSC(\Omega)$ is a viscosity super-solution for (12) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $x_0 \in \Omega$ is a local minimum for the function $u - \varphi$, then either φ is F-admissible at x_0 and

$$F\Big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\Big) \ge 0 \tag{14}$$

or φ is not F-admissible at x_0.

A continuous function u is a viscosity solution for (12) if it is either a viscosity sub-solution and a viscosity super-solution for (12).

We say that a function $u \in USC(\overline{\Omega})$ is a viscosity sub-solution for (8) if u is a viscosity sub-solution for (12) and in addition $u \le \phi$ on $\partial\Omega$.

We say that a function $u \in LSC(\overline{\Omega})$ is a viscosity super-solution for (8) if u is a viscosity super-solution for (12) and in addition $u \ge \phi$ on $\partial\Omega$.

A viscosity solution for (8) is either a viscosity sub-solution and a viscosity super-solution for (8).

The following comparison result plays a crucial role in the proof of Theorem 1.1.

Theorem 2.1. *Let u be a viscosity subsolution and let v be a viscosity supersolution to the equation $F = 0$ in a bounded open set $\Omega \subset \mathbb{R}^n$. If f is continuous, positive and strictly increasing with respect to u and*

$$\limsup_{x\to x_0} u(x) \le \liminf_{x\to x_0} v(x) \quad \textit{for every} \quad x_0 \in \partial\Omega,$$

then $u \le v$ in Ω.

This Comparison Principle is proved in [14, Proposition 4.1].

A comparison principle in the class of uniformly horizontal convex sub- and super-solution of the Monge-Ampère equation in homogeneous Carnot groups has been proved in [2].

Here we would like to have a comparison principle for F also when f is only increasing with respect to u. In order to adapt the proof for the strictly monotone case in this situation, one needs to find (for instance) a strictly sub-solution for F, and we can choose $R > 0$ such that the defining function $\rho(x) = (\|x\|^2 - R^2)/2$ of B_R is a a strictly sub-solution for F in B_R. Precisely, we have

Theorem 2.2. *Let f be continuous, positive and increasing with respect to u. Then, there is $R > 0$ such that, if u is a viscosity subsolution and v is a viscosity supersolution to the equation $F = 0$ in B_R and*

$$\limsup_{x\to x_0} u(x) \le \liminf_{x\to x_0} v(x) \quad \textit{for every} \quad x_0 \in \partial B_R,$$

then $u \le v$ in B_R.

3. A preliminary existence result

In this section we assume that f is increasing with respect to u. Let us fix $R > 0$ such that B_R is strictly F-admissible and

$$\left(\sigma_k\Big(A_m(x)\, A_m^T(x) + Q_m(x,x)\Big)\right)^{\frac{1}{k}} > 1/2, \qquad \text{for every } x \in B_R, \tag{15}$$

We define the following function

$$f_\infty : \bar{B}_R \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad f_\infty(x, r, \mathcal{P}) := \lim_{\lambda\to\infty} \frac{f(x, \lambda r, \lambda \mathcal{P})}{\lambda}.$$

Let us suppose that f_∞ exists at every point and that for the defining function $\rho(x) = (\|x\|^2 - R^2)/2$ of B_R it holds

$$f_\infty(x, \rho(x), D_g\rho(x)) < 1/2, \qquad \text{for every } x \in \bar{B}_R. \tag{16}$$

This restriction on the growth of the function f and the Lie group structure arise in [14] through the consideration of gradient estimates.

By [14, Theorem 1.1] we have the following existence result for the Dirichlet problem (8)

Theorem 3.1. *Let us fix $R > 0$ such that B_R is strictly F-admissible and such that* (15) *holds true. Let $f \in C^1$ be a non negative function, increasing with respect to u, and such that that f_∞ exists and it satisfies condition* (16) *and that at least one of the following two conditions holds true:*

- *f does not depend on its first argument x*
- $\inf \frac{\partial f}{\partial u} > 0$.

Then, for every boundary data $\phi \in C^{1,1}$, there exists a viscosity solution $u \in Lip(\overline{B_R})$ of the problem (8).

Moreover, let us fix a positive constant c such that

$$f(x, (\sup_{\partial B_R} \phi), cD_g\rho(x)) < c/2, \qquad \text{for every } x \in \bar{B}_R. \tag{17}$$

Then

- *if f does not depend on its first argument x, then $\|u\|_{Lip(\overline{B_R})}$ only depends on R, $\|\phi\|_{C^{1,1}(\overline{B_R})}$, c.*
- *If $\inf \frac{\partial f}{\partial u} > 0$, then $\|u\|_{Lip(\overline{B_R})}$ only depends on R, $\|\phi\|_{C^{1,1}(\overline{B_R})}$, c, and on*

$$\sup_{\overline{B_R} \times [\inf_{\partial B_R} \phi + c\rho(x), sup_{\partial B_R} \phi] \times \mathbb{R}^n} \frac{|D_g f|}{f_u}.$$

The following remark shows that if the boundary data is convex, on small balls we can relax the growth condition (16)

Remark 3.1. *If the boundary data $\phi \in Lip(\bar{B}_1)$ is convex in B_1 and $cR < 1$ then*

$$f(x, (\sup_{\partial B_R} \phi), cD_g\rho(x)) = f(x, (\sup_{\partial B_R} \phi), cW(x)x) \leq \sup_{(x,p) \in B_1 \times B_1} f(x, (\sup_{\partial B_1} \phi), W(x)p).$$

Now we choose c such that $\sup_{(x,p)\in B_1\times B_1} f(x,(\sup_{\partial B_1}\phi), W(x)p) < c/2$ *and* $R < 1/c$ *and we get that* (17) *is always satisfied in* B_R *without assuming the growth condition* (16)

In this section we prove the existence of a Lipschitz-continuous viscosity solution to a Dirichlet problem for F on the ball B_R, for R sufficiently small. The boundary data will be the restriction to ∂B_R of a convex function ϕ in $\overline{B_R}$ satisfying the equation $F = 0$ in B_R. The crucial point of this preliminary existence result is the dependence of the gradient of the solution only on the gradient of ϕ. The proof is a refinement of Theorem 3.1

To prove our existence result, Proposition 3.1, we show the following lemma, which provides a strict subsolution to $F = 0$ independent of the second derivatives of the boundary data.

In this section $0 < R < 1$ is fixed such that (15) holds true.

Lemma 3.1. *Let* $\phi \in C^2(B_1) \cap Lip(\bar{B}_1)$ *be a convex function. For each* $\lambda > 0$ *define*

$$u_\lambda(x) := \phi(x) + \lambda\rho(x), \quad x \in B_1,$$

where $\rho(x) = (\|x\|^2 - r^2)/2$*. Then, there exists* $0 < r < R < 1$ *and* $\lambda^* > 0$*, only depending on* $\sup_{B_R} |D\phi|$*, such that*

$$F(x, u_\lambda, Du_\lambda, D^2u_\lambda) < 0 \text{ in } B_r, \quad \text{for every } \lambda > \lambda^*. \tag{18}$$

Proof. Since ϕ is a convex function,

$$A_m(x)D^2u_\lambda A_m^T(x) + Q_m(x,u_\lambda) \geq \lambda\left(A_m(x)A_m^T(x) + Q_m(x,x) + Q_m(x,D\phi/\lambda)\right).$$

Let us choose $\tilde{\lambda}^* = \tilde{\lambda}^*(r, \sup_{\partial B_R}|D\phi|) > 0$ such that for every $\lambda > \tilde{\lambda}^*$ and for any $x \in B_r$

$$\left(A_m(x)A_m^T(x) + Q_m(x,x) + Q_m(x,D\phi/\lambda)\right) > \frac{1}{2}Id_m.$$

In particular u_λ is F-admissible in B_r for every $\lambda > \tilde{\lambda}^*$. Moreover, as a consequence of the monotonicity of the function $A \to \sigma_k^{1/k}(A)$ and of its homogeneity, i.e. $\sigma_k^{1/k}(\lambda A) = \lambda\sigma_k^{1/k}(A)$ for every $\lambda > \tilde{\lambda}^*$, we get

$$\begin{aligned} F(x,u_\lambda,Du_\lambda,D^2u_\lambda) &\leq -\lambda\sigma_k^{1/k}\left(A_m(x)A_m^T(x) + Q_m(x,x) + Q_m(x,D\phi/\lambda)\right) \\ &\quad + f(x,u_\lambda,D_g u_\lambda) \\ &\leq -\frac{1}{2}\lambda + f(x,u_\lambda,D_g u_\lambda) \end{aligned}$$

Then, since $Du_\lambda(x) = \lambda\left(\frac{D\phi}{\lambda} + x\right)$, by recalling Remark 3.1 we can fix $\lambda^* > \tilde{\lambda}^* > 0$ and a small r, only depending on $\sup_{\partial B_1} |D\phi|$, such that

$$\frac{f(x, u_\lambda, D_g u_\lambda)}{\lambda} < \frac{1}{2}$$

for every $\lambda > \lambda^*$.

This inequality easily implies that (18) holds for every $\lambda > \lambda^*$ and for all $x \in B_r$. ◻

Using the previous lemma and by Theorem 3.1, we obtain the main result of this section.

Proposition 3.1. *Assume the conditions of Theorem 1.1. If $\phi \in C^2 \cap Lip(\bar{B}_1)$ is a convex function such that $F \geq 0$ in B_1, then there is $0 < r < R < 1$ such that lthe Dirichlet problem*

$$F = 0, \text{ in } B_r, \qquad u = \phi \text{ on } \partial B_r \tag{19}$$

has a viscosity solution $u \in Lip\,(\overline{B_r})$ satisfying

$$\|u\|_{L^\infty(\overline{B_r})} + \|u\|_{Lip\,(\overline{B_r})} \leq C, \tag{20}$$

where

- *if f does not depend on x, C only depends on r, $\|\phi\|_{L^\infty(\overline{B_R})}$, $\|D\phi\|_{L^\infty(\overline{B_R})}$*
- *if $f_u > 0$, C only depends on r, $\|\phi\|_{L^\infty(\overline{B_R})}$, $\|D\phi\|_{L^\infty(\overline{B_R})}$ and on*

$$\sup_{\overline{B_R}\times[\inf_{\partial B_R}\phi + c\rho(x), sup_{\partial B_R}\phi]\times\mathbb{R}^n} \frac{|D_g f|}{f_u}$$

with c as in Remark 3.1.

Proof. Let $u_\lambda = \phi + \lambda\rho$ be the function given by the previous lemma with $\lambda > \lambda^*$. Then $u_\lambda \in C^2(\overline{B_r})$ and it is a classical subsolution to $F = 0$ in B_r. Moreover, $u_\lambda = \phi$ on ∂B_r. On the other hand, since $F(x, \phi, D\phi, D^2\phi) \geq 0$ in B_1, ϕ is a classical supersolution to $F = 0$ in B_r.

Then, by Theorem 3.1, the Dirichlet problem (19) has a viscosity solution $u \in C(\overline{B_r})$ and by the comparison principle we have $u_\lambda \leq u \leq \phi$ in $\overline{B_r}$. Hence $\sup_{B_r} |u| \leq \sup_{B_r} |\phi| + \lambda r$. On the other hand, by Lemma 3.1, $\sup_{B_r} |Du_\lambda|$ can be bounded by a constant only depending on r and $\sup_{B_R} |D\phi|$. By the interior gradient estimates in Theorem 3.1, we can conclude that $u \in Lip(\overline{B_r})$ with $\|u\|_{Lip(\overline{B_r})}$ bounded by a constant $C > 0$, where

- if f does not depend on x, C only depends on r, $\|\phi\|_{L^\infty(\overline{B_R})}$, $\|D\phi\|_{L^\infty(\overline{B_R})}$
- if $f_u > 0$, C only depends on r, $\|\phi\|_{L^\infty(\overline{B_R})}$, $\|D\phi\|_{L^\infty(\overline{B_R})}$ and on

$$\sup_{\overline{B_R}\times[\inf_{\partial B_R}\phi+c\rho(x),sup_{\partial B_R}\phi]\times\mathbb{R}^n} \frac{|D_g f|}{f_u}.$$

□

4. Basic definitions: Hölder Spaces, Hörmander vector fields, Carnot Carathéodory distance, Carnot group

In this section we fix notation and we briefly recall some well known facts. Let $X_1, \dots, X_m$ be a system of real smooth vector fields defined in some bounded connected open subset Ω of $\mathbb{R}^n$, with $m \le n$.

Definition 4.1. *For any $\beta \in (0,1)$ we set*

$$C^\beta(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{C^\beta(\Omega)} < \infty \right\}$$

where

$$\|u\|_{C^\beta(\Omega)} = \sup \left\{ \frac{|u(x) - u(y)|}{d(x,y)^\beta} : x, y \in \Omega, x \neq y \right\}$$

and d is the Euclidean distance.

Moreover, let

$$C^{1,\beta}_X(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{C^{1,\beta}_X(\Omega)} < \infty \right\}$$

where

$$\|u\|_{C^{1,\beta}_X(\Omega)} = \sum_{j=1}^m \|X_j u\|_{C^\beta(\Omega)}.$$

For any multi index $I = (i_1, i_2, \dots, i_j)$, $1 \le i_j \le m$, we set

$$X_I = [X_{i_1}, [X_{i_2}, \dots [X_{i_{j-1}}, X_{i_j}] \dots]],$$

where $[X, Y] = XY - YX$. We say that X_I is a commutator of length $|I| = j$.

We say that $X_1, \dots, X_m$ satisfy Hörmander condition at step s in Ω if the vector fields, together with their commutators of length $\le s$, span the tangent space at every point in Ω. The vector fields induce on $\mathbb{R}^n$ a metric d_{CC} in the following way (see [17]).

Definition 4.2 (Carnot Carathéodory metric)**.** *A Lipschitz continuous curve* $\gamma : [0, T] \to \mathbb{R}^n$*,* $T \geq 0$*, is subunit if there exists a vector of measurable functions* $h : [0, T] \to \mathbb{R}^n$ *such that* $\gamma'(\tau) = \sum_{j=1}^m h_j(\tau) X_j(\gamma(\tau))$ *and* $\sum_{j=1}^m h_j^2(\tau) \leq 1$ *for a.e.* $\tau \in [0, T]$*. Define the Carnot Carathéodory distance* $d_{CC} : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ *by setting*

$$d_{CC}(x, y) = \inf\{T \geq 0 : \textit{there exists a subunit curve}\, \gamma : [0, T] \to \mathbb{R}^n \;\; \textit{such that}\, \gamma(0) = x\, \textit{and}\, \gamma(T) = y\}.$$

It is well-known that if the vector fields are smooth and satisfy Hörmander condition then $d_{CC}(x, y)$ is finite for all x, y (see [3], [17]). Moreover, if $X_1, \dots, X_m$ satisfy Hörmander condition at step s in Ω, then if $K \subset\subset \Omega$ is any compact set, there are positive constants c, C so that if $x, y \in K$

$$cd(x, y) \leq d_{CC}(x, y) \leq Cd(x, y)^{1/s}. \tag{21}$$

Definition 4.3. *Let* $X_1, \dots, X_m$ *be a system of real smooth vector fields satisfying Hörmander condition in some bounded connected subset* Ω *of* $\mathbb{R}^n$*, with* $m \leq n$*. For any* $\beta \in (0, 1)$ *we set*

$$C^\beta_{d_{CC}}(\Omega) = \left\{u : \Omega \to \mathbb{R} : \|u\|_{C^\beta_{d_{CC}}(\Omega)} < \infty\right\}$$

where

$$\|u\|_{C^\beta_{d_{CC}}(\Omega)} = \sup\left\{\frac{|u(x) - u(y)|}{d_{CC}(x, y)^\beta} : x, y \in \Omega, x \neq y\right\}$$

Moreover, let

$$C^{1,\beta}_{X,d_{CC}}(\Omega) = \left\{u : \Omega \to \mathbb{R} : \|u\|_{C^{1,\beta}_{X,d_{CC}}(\Omega)} < \infty\right\}$$

where

$$\|u\|_{C^{1,\beta}_{X,d_{CC}}(\Omega)} = \sum_{j=1}^m \|X_j u\|_{C^\beta_{d_{CC}}(\Omega)}.$$

We explicitly remark here that, if $X_1, \dots, X_m$ is a system of real smooth vector fields satisfying Hörmander condition at step s in Ω, then by (21) we have

$$C^\beta(\Omega) \subset C^\beta_{d_{CC}}(\Omega) \subset C^{\beta/s}(\Omega).$$

A remarkable example of a system of real smooth vector fields satisfying Hörmander condition at step s is furnished by the Jacobian basis of a stratified Lie group. For reader convenience, we recall here the definition of stratified Lie group.

Definition 4.4. *A stratified Lie group (or Carnot group)* $\mathbb{H}$ *is a simple connected Lie group whose Lie algebra* $\mathbf{l}$ *admits a stratification, i.e. a direct sum decomposition* $\mathbf{l} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ *such that* $[V_1, V_{i-1}] = V_i$ *for* $2 \leq i \leq s$*, and* $[V_1, V_s] = \{0\}$*.*

In this case we say that $\mathbb{H}$ *has step (of nilpotency) s and has m generators, where* $m = dim(V_1)$*.*

5. Existence of nonsmooth solutions

Proof of Theorem 1.1. First of all remark that if condition (H_ℓ) holds true for some $\ell \geq k-1$ then in particular (H_ℓ) holds true for $\ell = k-1$. Throughout this section we then fix $k/2 < \ell \leq k-1$ such that condition (H_ℓ) holds true, and we denote by $x = (x', x''),\ x' = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$, $x'' = (x_{\ell+1}, \ldots, x_k) \in \mathbb{R}^{k-\ell}$ with $x \in \mathbb{R}^k$. We denote by $\xi = (x, t)$ points of $\mathbb{R}^n$, with $x \in \mathbb{R}^k$ and $t \in \mathbb{R}^{n-k}$.

For $0 \leq \varepsilon < 1$ and $0 < r < R$ such that Proposition 3.1 holds true, we define

$$w_\varepsilon(x) = w_\varepsilon(x', x'') := (r^2 + |x''|^2)(\varepsilon + |x'|^2)^\alpha, \quad \alpha = \frac{\ell}{k}, \tag{22}$$

and

$$\psi_\varepsilon(\xi) = \psi_\varepsilon(x, t) := M w_\varepsilon(x), \quad \phi_\varepsilon(\xi) = \phi_\varepsilon(x, t) := 2M(\varepsilon + |x'|^2)^\alpha,$$

with M a positive constant that will be determined later. We have

$$\psi_0 \leq \psi_\varepsilon \leq \phi_\varepsilon, \quad \text{in } B_1.$$

Since $\ell > k/2$, the exponent $\alpha = \frac{\ell}{k} > \frac{1}{2}$ and so that ϕ_ε is convex in $\mathbb{R}^n$ for $\varepsilon \geq 0$.

Moreover, ϕ_ε is smooth for $\varepsilon > 0$, and independent of x'' and t. From condition (H_ℓ) we then obtain for $\varepsilon > 0$

$$A_m(\xi) D^2\phi_\varepsilon A_m^T(\xi) + Q_m(\xi, D\phi_\varepsilon) = A_m(\xi) D^2\phi_\varepsilon A_m^T(\xi) \geq 0$$

and it has $m - \ell$ null eigenvalues. This means that ϕ_ε is F admissible and since $m - \ell \geq m - k + 1$

$$\sigma_k\left(A_m(\xi) D^2\phi_\varepsilon A_m^T(\xi) + Q_m(\xi, D\phi_\varepsilon)\right) = 0.$$

Therefore:

$$F(\xi, \phi_\varepsilon, D\phi_\varepsilon, D^2\phi_\varepsilon) = f(\xi, \phi_\varepsilon, D_g\phi_\varepsilon) > 0 \quad \text{in } B_1, \quad \forall\, \varepsilon \in]0, 1[. \tag{23}$$

Thus, applying Proposition 3.1, there exists $0 < r < R$ such that the Dirichlet problem

$$F = 0 \quad \text{in } B_r, \qquad u = \phi_\varepsilon \quad \text{on } \partial B_r,$$

with $\varepsilon \in]0,1[$, has a viscosity solution u_ε such that

$$\| u_\varepsilon \|_{L^\infty(\overline{B_r})} + \| u_\varepsilon \|_{Lip\,(\overline{B_r})} \leq C(r,\varepsilon,M)$$

with $C(r,\varepsilon,M)$ depending on ε only through $C(\phi_\varepsilon) :=\| \phi_\varepsilon \|_{L^\infty(\overline{B_r})} + \| D\phi_\varepsilon \|_{L^\infty(\overline{B_r})}$. On the other hand, an elementary computation shows that $C(\phi_\varepsilon) \leq 8M$. Then, we can choose $C(r,\varepsilon,M)$ independent of ε, and so

$$\| u_\varepsilon \|_{L^\infty(\overline{B_r})} + \| u_\varepsilon \|_{Lip\,(\overline{B_r})} \leq C(r,M). \tag{24}$$

Now we claim that, if $0 < r \ll R$, we can fix $M = M(r)$ such that

$$F(\xi, \psi_\varepsilon, D\psi_\varepsilon, D^2\psi_\varepsilon) < 0 \quad \text{in } B_r, \quad \forall\, \varepsilon \in]0, r^2[. \tag{25}$$

Assuming this claim for a moment, we can use the Comparison Principle of Section 2 to compare u_ε with ψ_ε and ϕ_ε. Indeed, by (23) and (25), ϕ_ε and ψ_ε are, respectively, classical supersolution and subsolution to $F = 0$ in B_r. On the other hand $\psi_\varepsilon \leq \phi_\varepsilon$ in B_1, in particular, $\psi_\varepsilon \leq \phi_\varepsilon$ on ∂B_r. Thus, by the Comparison Principle,

$$\psi_\varepsilon \leq u_\varepsilon \leq \phi_\varepsilon \quad \text{in } B_r, \ \ \forall \varepsilon \in]0, r^2[. \tag{26}$$

The uniform estimate (24) implies the existence of a sequence $\varepsilon_j \searrow 0$ such that $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ uniformly converges to a viscosity solution $u \in Lip(\overline{B_r})$ to the Dirichlet problem

$$F = 0 \quad \text{in } B_r, \quad u = \phi_0 \text{ on } \partial B_r;$$

the proof of this fact is given in [14, Lemma 3.1]. Moreover, from the comparison principle, we get

$$\psi_0 \leq u \leq \phi_0 \quad \text{in } B_r. \tag{27}$$

In particular

$$Mr^2|x_1|^{2\alpha} \leq u(x_1, 0, \ldots, 0) \leq 2M|x_1|^{2\alpha}. \tag{28}$$

As in the proof of [9, Theorem 1] inequalities in (28) imply:

$$\partial_{x_1} u \notin C^\beta, \ \text{ for every } \beta > 2\alpha - 1 = \frac{2\ell}{k} - 1 \quad \text{if } 2\alpha > 1 \ \text{ (i.e. } \ell > k/2).$$

Moreover, inequalities (27) imply that

$$X_1 u \notin C^\beta, \ \text{ for every } \beta > 2\alpha - 1 = \frac{2\ell}{k} - 1 \quad \text{if } 2\alpha > 1 \ \text{ (i.e. } \ell > k/2).$$

Indeed, let us denote by $\exp(sX_1)(0) = \gamma(s)$ the integral curve of the vector field X_1 starting at the origin, i.e. $\gamma'(s) = X_1\gamma(s), \gamma(0) = 0$. Remark that γ is a smooth function, whose first order Taylor expansion in a neighborhood of the origin is (see also [3, section 1.12])

$$\gamma(s) = sX_1(0) + o(s) = se_1 + o(s), \quad \text{as}\, s \to 0,$$

where $e_1 = (1, 0, \dots, 0)$ and $o(s)/s \to 0$ as $s \to 0$.

If $2\alpha > 1$, then $(X_1u)(0) = \partial_{x_1}u(0) = 0 = u(0)$ so that, if X_1u was C^β, with $\beta > 2\alpha-1$, we would have $u(\exp(x_1X_1)(0)) \leq C|x_1|^{1+\beta}$ for a suitable $C > 0$ and for every x_1 sufficiently small. Hence, by denoting $\pi' : \mathbb{R}^n \to \mathbb{R}^\ell$ the usual projection operator associating to a vector $\xi = (x', x'', t) \in \mathbb{R}^n$ its first variables $x' \in \mathbb{R}^\ell$, by the first inequality in (27), we would have the existence of a positive constant c such that

$$c|x_1|^{2\alpha} \leq Mr^2|\pi'(\exp(x_1X_1)(0))|^{2\alpha} \leq \psi_0((\exp(x_1X_1)(0)) \leq C|x_1|^{1+\beta} \tag{29}$$

and it would be $\beta \leq 2\alpha - 1$, a contradiction.

To complete the proof of the theorem, we are left with the proof of Claim (25). By condition (H_ℓ) and by the smooth regularity of the functions $\xi \to A_m(\xi), \xi \to Q_m(\xi, \cdot)$ and by the linearity of $p \to Q_m(\cdot, p)$ we get

(30)
$$\begin{aligned} A_m(\xi)D^2w_\varepsilon A_m^T(\xi) + Q_m(\xi, Dw_\varepsilon) &= A_m(\xi)D^2w_\varepsilon A_m^T(\xi) + Q_m(\xi, D_{x''}w_\varepsilon) \\ &= A_m(0)D^2w_\varepsilon A_m^T(0)(1 + \omega_1(\xi)) + 2(\varepsilon + |x'|^2)^\alpha Q_m(\xi, x'') \\ &= Id_mD^2w_\varepsilon(x)Id_m(1 + \omega_1(\xi)) + \omega_2(\xi)(\varepsilon + |x'|^2)^\alpha Id_m \end{aligned}$$

where, for $j = 1, 2$, $\omega_j(\xi) \to 0$ uniformly in ε as $\xi \to 0$. Moreover, $Id_mD^2w_\varepsilon(x)Id_m$ has $m - k$ null rows by construction. Therefore

$$\sigma_k(Id_mD^2w_\varepsilon Id_m) = \det D_x^2w_\varepsilon. \tag{31}$$

Direct computations show that

$$\det D_x^2w_\varepsilon(x) = 2^{2k}f_\varepsilon(x) \tag{32}$$

with

$$f_\varepsilon(x) = \alpha^{\ell+1}(r^2 + |x''|^2)^{\ell-1}\,\frac{r^2(\alpha^{-1}\varepsilon + |x'|^2) + \alpha^{-1}\varepsilon(|x'|^2)}{(\varepsilon + |x'|^2)}.$$

For convenience of the reader we include the proof of (32).

We have

$$D_x^2 w_\varepsilon(x) = 4(\varepsilon + |x'|^2)^{\alpha-1}\begin{pmatrix} (r^2+|x''|^2)\left(\alpha Id_\ell + \alpha(\alpha-1)\frac{x'\otimes x'}{\varepsilon+|x'|^2}\right) & \alpha x'\otimes x'' \\ \alpha x''\otimes x' & (\varepsilon+|x'|^2)Id_{k-\ell}\end{pmatrix}$$

Since $(\alpha-1)k = \ell - k$, we get

$$\begin{aligned}
\det D_x^2 w_\varepsilon(x) =& 4^k(\varepsilon+|x'|^2)^{\ell-k}\det\begin{pmatrix} (r^2+|x''|^2)\left(\alpha Id_\ell + \alpha(\alpha-1)\frac{x'\otimes x'}{\varepsilon+|x'|^2}\right) & \alpha x'\otimes x'' \\ \alpha x''\otimes x' & (\varepsilon+|x'|^2)Id_{k-\ell}\end{pmatrix}\\
=&4^k\det\begin{pmatrix} (r^2+|x''|^2)\left(\alpha Id_\ell + \alpha(\alpha-1)\frac{x'\otimes x'}{\varepsilon+|x'|^2}\right) & \alpha\frac{x'\otimes x''}{(\varepsilon+|x'|^2)^{1/2}} \\ \alpha\frac{x''\otimes x'}{(\varepsilon+|x'|^2)^{1/2}} & Id_{k-\ell}\end{pmatrix}\\
=&4^k\det\begin{pmatrix} (r^2+|x''|^2)\left(\alpha Id_\ell + \alpha(\alpha-1)\frac{x'\otimes x'}{\varepsilon+|x'|^2}\right) - \alpha^2|x''|^2\frac{x'\otimes x'}{(\varepsilon+|x'|^2)^{1/2}} & 0 \\ \alpha\frac{x''\otimes x'}{(\varepsilon+|x'|^2)^{1/2}} & Id_{k-\ell}\end{pmatrix}\\
=&4^k\det\left((r^2+|x''|^2)\left(\alpha Id_\ell + \alpha(\alpha-1)\frac{x'\otimes x'}{\varepsilon+|x'|^2}\right) - \alpha^2|x''|^2\frac{x'\otimes x'}{(\varepsilon+|x'|^2)^{1/2}}\right)\\
:=&4^k\det\Gamma,
\end{aligned}$$

where Γ is a $\ell\times\ell$ symmetric matrix. It is easy to see that $\lambda_1 = \alpha(r^2+|x''|^2)$ is an eigenvalue of Γ with multiplicity $\ell-1$. Now, $\operatorname{trace}\Gamma = (\ell-1)\lambda_1 + \lambda_2$ with

$$\begin{aligned}
\lambda_2 =&(r^2+|x''|^2)\left(\alpha+\alpha(\alpha-1)\frac{|x'|^2}{\varepsilon+|x'|^2}\right) - \alpha^2|x''|^2\frac{|x'|^2}{\varepsilon+|x'|^2}\\
=&\alpha^2\,\frac{r^2\left(\frac{\varepsilon}{\alpha}+|x'|^2\right)+\frac{\varepsilon}{\alpha}|x'|^2}{\varepsilon+|x'|^2}
\end{aligned}$$

Thus, $\det\Gamma = \lambda_1^{\ell-1}\lambda_2 = \alpha^{\ell+1}(r^2+|x''|^2)^{\ell-1}\frac{r^2\left(\frac{\varepsilon}{\alpha}+|x'|^2\right)+\frac{\varepsilon}{\alpha}|x'|^2}{\varepsilon+|x'|^2} = f_\varepsilon$, which completes the proof of (32). In particular, $f_\varepsilon \geq \alpha^{\ell+1}r^{2\ell} > \alpha^\ell r^{2\ell}/2$.

Keeping in mind that $\psi_\varepsilon = Mw_\varepsilon$ and (30), (31), we can choose r small, $0<r<R$, such that

$$\varepsilon_k^{1/k}\left(A_m(\xi)D^2\psi_\varepsilon A_m^T(\xi) + Q_m(\xi, D\psi_\varepsilon)\right) > \alpha^\alpha r^{2\alpha}M.$$

On the other side, direct computations show that

$$|Dw_\varepsilon|^2 = 4\Big(|x''|^2(\varepsilon+|x'|^2)^{2\alpha} + \alpha^2|x'|^2(r^2+|x''|^2)^2(\varepsilon+|x'|^2)^{2(\alpha-1)}\Big)$$

and for every $\varepsilon\in]0,r^2[$,

$$|Dw_\varepsilon|^2 \leq 2^{2\alpha+3}r^{4\alpha+2} \quad \text{in } B_r. \tag{33}$$

From (33), we obtain

$$|D\psi_\varepsilon| \leq 2^{\alpha+3/2} M r^{4\alpha+2} \quad \text{in } B_r. \tag{34}$$

Choosing $M = 2^{-\alpha-(3/2)} r^{-2\alpha-1}$, the right hand side of (34) equals 1, and $\psi_\varepsilon \leq 2^{-\alpha-1/2} r < 1$. The strategy now is to take a smaller r such that

$$\sup_{(\xi,p)\in B_1\times B_1} f(\xi, 1, W(\xi)p) < \alpha^\alpha 2^{-\alpha-(3/2)} r^{-1}.$$

Then, by the increasing monotonicity of $s \to f(\cdot, s, \cdot)$, in B_r we obtain

$$\begin{aligned} F(\xi, \psi_\varepsilon, D\psi_\varepsilon, D^2\psi_\varepsilon) &< -\alpha^\alpha 2^{-\alpha-(3/2)} r^{-1} + f(\xi, \psi_\varepsilon, W(\xi) D\psi_\varepsilon) \\ &< -\alpha^\alpha 2^{-\alpha-(3/2)} r^{-1} + f(\xi, 1, W(\xi) D\psi_\varepsilon) < 0. \end{aligned}$$

This proves claim (25) and completes the proof of the theorem. □

Remark 5.1. *If the vector fields $X_1, \ldots, X_m$ satisfy Hörmander condition in B_R, then $d_{CC}(\exp(x_1 X_1)(0), 0) = x_1$ and from inequality* (29) *we get $u \notin C^{1,\beta}_{X,d_{CC}}(B_R)$.*

6. Examples

Here we will show some examples on which the condition (H_ℓ) is fulfilled. First we consider the case of a homogeneous Carnot group. We refer to [3, Section 1.4] for a full detailed exposition on the theory of homogeneous Carnot groups.

Example 6.1. *Let us consider a homogeneous Carnot group on $\mathbb{R}^n$ with m generators: then condition (H_ℓ) is satisfied with $\ell = m$. Indeed, let us consider the Jacobian basis E^l for Lie algebra $\mathbf{l}$ of the left-invariant vector fields. Let us suppose that the first layer of the stratification V_1^m has dimension $m \leq n$ and it is spanned by the first m vector fields of the basis, namely $E^l_m = \{X_1, \ldots, X_m\}$. We know that such vector fields read in coordinates as*

$$X_i = \frac{\partial}{\partial x_i} + \sum_{k=m+1}^{n} \tau_{ik}(x) \frac{\partial}{\partial x_k}, \quad i = 1, \ldots, m$$

where τ_{ik} are smooth (polynomial) functions defined on the whole $\mathbb{R}^n$. In particular, $Q_m(x,p)$ is independent of $p_1, \cdots, p_m$ and condition (H_ℓ) is satisfied for $\ell = m$.

Next we show an example of a Lie group that is not Carnot, but for which elementary symmetric functions in the eigenvalues of the Hessian $H_{g,m}$, for some m, still satisfy condition (H_ℓ).

Example 6.2. *We consider the Lie group in $\mathbb{R}^{n+1}$ given by the following group law $\circ$: for any $(x, y), (t, s) \in \mathbb{R}^n \times \mathbb{R}$*

$$(x, y) \circ (t, s) = (x_1, \dots, x_n, y) \circ (t_1, \dots, t_n, s) = (x_1 + t_1, \dots, x_n + t_n, s + ye^{t_1+\dots+t_n}).$$

A basis for $\mathbf{l}$ *is given by the left invariant vector fields*

$$X_j = \frac{\partial}{\partial x_j} + y\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}, \quad j = 1, \dots, n$$

having the following commutation properties:

$$[X_i, X_j] = 0, \quad [X_j, Y] = -Y, \quad i, j = 1, \dots, n.$$

Hence, the relevant Lie algebra is not nilpotent and the Lie group is not stratified. We consider the metric g that makes orthonormal the vector fields X_j, Y for any $j = 1, \dots, n$ and we denote by ∇ the Levi-Civita connection for g. We will prove that for any $m \leq n$, the condition (H_ℓ) holds true with $\ell = n$. Indeed, we have that

$$H_{g,m}u(X_i, X_j) = X_iX_ju - (\nabla_{X_i}X_j)u, \quad i, j = 1, \dots, m$$

and

$$\nabla_{X_i}X_j = \sum_{k=1}^{n} g(\nabla_{X_i}X_j, X_k)X_k + g(\nabla_{X_i}X_j, Y)Y$$

Now, since the coefficients of the metric in this basis are constants, we know that for any vector fields V, W, Z in this basis:

$$g(\nabla_V W, Z) = \frac{1}{2}\Big(g([V, W], Z) - g([W, Z], V) + g([Z, V], W)\Big)$$

By the previous formula we get:

$$g(\nabla_{X_i}X_j, X_k) = g(\nabla_{X_i}X_j, Y) = 0, \quad i, j = 1, \dots, m, \ k = 1, \dots, n$$

Therefore the Hessian reads in local coordinates as

$$H_{g,m}u(X_i, X_j) = X_iX_ju = u_{x_ix_j} + yu_{x_iy} + yu_{yx_j} + y^2u_{yy} + yu_y$$

Hence, keeping in mind the formulas (2) and (6), we see that there is no dependance on the first n components of the gradient of u, that is (H_ℓ) holds true with $\ell = n$.

Remark 6.1. *We note that in the previous example, with $m = n + 1$, the condition (H_ℓ) never holds true. Indeed, for any $k = 1, \dots, n$ we have*

$$2g(\nabla_Y Y, X_k) = g([Y, Y], X_k) - g([Y, X_k], Y) + g([X_k, Y], Y) = -2$$

and $g(\nabla_Y Y, Y) = 0$. This means that

$$H_g u(Y,Y) = YYu - (\nabla_Y Y)u = u_{yy} + \sum_{k=1}^{n} X_k u = u_{yy} + nyu_y + \sum_{k=1}^{n} u_{x_k}.$$

In particular, $Q_{n+1}(x, Du)$ depends on all variables $u_{x_1}, \ldots, u_{x_n}, u_y$.
Moreover, if one considers the symmetrized Hessian $H^s u$ (see formula (3)), since there are no Christoffel symbols involved, then one realizes that in $H^s u$ the only gradient term appearing is u_y, for any $m = 1, \ldots, n+1$. Hence, for $H^s u$ the condition (H_ℓ) would be satisfied with any $\ell \leq n$, and for any $m = 1, \ldots, n+1$. However, we explicitly remind that the eigenvalues (and thus their elementary symmetric functions) of $H^s u$ are not intrinsic, namely: they depend on the particular choice of the vector fields; if one changes the coordinates, giving rise to another basis of vector fields, the eigenvalues of the new symmetrized Hessian will change in general.

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