LOCAL LIPSCHITZ CONTINUITY OF GRAPHS WITH PRESCRIBED LEVI MEAN CURVATURE

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Abstract. We prove interior gradient estimates of viscosity solutions of the prescribed Levi mean curvature equation.

1. Introduction

Starting from the existence results of Slodkowski Tomassini [17] and Debiard Gaveau [6], Citti Lanconelli and the second author in [5] proved the local smoothness of Lipschitz continuous graphs in $\mathbb{C}^2$, with prescribed smooth and non vanishing Levi curvature. However, these results left open the question of existence of Lipschitz continuous viscosity solutions under sharp conditions on the boundary data. In a recent work [14], by using the techniques of viscosity solutions, we proved the existence and uniqueness of continuous viscosity solution of the prescribed Levi mean curvature equation for every continuous boundary data. In this paper we give a sufficient condition on the prescribed Levi mean curvature which guarantees that such a solution is locally Lipschitz continuous.

Let us introduce some notations. For every $p = (p_1, \dots, p_{N+1}) \in \mathbb{C}^{N+1}$ we denote by $p_{\bar{j}} = \bar{p}_j$ for every $j = 1, \dots, N+1$ and for every Hermitian matrix $r = (r_{i\bar{j}})_{i,j=1}^{N+1}$ we define the Levi determinant (see [1])

$$L_{i,j}(p, \bar{p}, r) = -\det \begin{pmatrix} 0 & p_{\bar{i}} & p_{\bar{j}} \\ p_i & r_{i\bar{i}} & r_{i\bar{j}} \\ p_j & r_{j\bar{i}} & r_{j\bar{j}} \end{pmatrix}. \tag{1.1}$$

With these notations the prescribed Levi mean curvature equation of an oriented hypersurface with a defining function $f : \mathbb{C}^{N+1} \to \mathbb{R}$ and with outward unit normal $\nabla f / |\nabla f|$, writes as (see [15])

$$k(z) = \frac{1}{N} \frac{1}{|\partial f|^3} L(\partial f, \bar{\partial} f, \partial\bar{\partial} f), \quad L(\partial f, \bar{\partial} f, \partial\bar{\partial} f) = \sum_{1 \le i < j \le N+1} L_{i,j}(\partial f, \bar{\partial} f, \partial\bar{\partial} f) \tag{1.2}$$

where $k : \mathbb{C}^{N+1} \to \mathbb{R}$ is a prescribed function.

If the hypersurface is locally the graph of a function $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n, n = 2N+1$ then by identifying $z = (z_1, \dots, z_{N+1}) = (x_1 + ix_{N+1}, \dots, x_N + ix_{2N}, x_{2N+1} + it)$ with

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$(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and writing $f(z) = u(x) - t$, the prescribed Levi mean curvature equation (1.2) writes as

$$k(x,u) = \frac{1}{N}\frac{2^3}{(1+|Du|^2)^{3/2}}F(Du, D^2u) \tag{1.3}$$

where Du and D^2u are the Euclidean gradient and the Hessian matrix of u respectively and F is explicitly defined in (2.1). Let us stress that (1.3) is a quasilinear degenerate elliptic PDE (see Proposition 2.4). However, in spite of the lack of ellipticity of F in one direction, in this paper we prove the following regularity result.

Theorem 1.1. *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $u \in C(\bar{\Omega})$ be a viscosity solution of (1.3) with $k \in C^1(\bar{\Omega} \times \mathbb{R})$ and non negative. If*

$$\frac{\partial k}{\partial u} \geq \alpha > 0$$

then u is locally Lipschitz continuous.

Moreover, if $u \in C^1(\Omega)$ then for all $x \in \Omega$

$$|Du(x)| \leq \frac{c}{d^2(x, \partial\Omega)}$$

where $d(\cdot, \partial\Omega)$ is the boundary distance and c is a positive constant depending on the diameter of Ω and on $M = \sup_{\bar{\Omega}} |u|$, $\alpha = \inf_{\bar{\Omega}\times[-M,M]} \frac{\partial k}{\partial u}$, $M_1 = \sup_{\bar{\Omega}\times[-M,M]} k$, $M_2 = \sup_{\bar{\Omega}\times[-\tilde{M},\tilde{M}]} \left|\frac{\partial k}{\partial x}\right|$ and $\tilde{M}$ only depending on M, M_1.

The key tool in the proof of Theorem 1.1 is a surprising differential property of Levi determinants, which is proved in Section 2. Roughly speaking, this property allows us to apply the maximum principle to $|Du|^2$ times a cut-off function, because it permits to handle the remainder term. In Section 3 we prove Theorem 1.1 by using an approximation argument and uniform Lipschitz estimates.

The idea to use the maximum principle approach to get gradient estimates go back to Bernstein [2], [3]. Bernstein method was then developed by Ladyzhenskaya [10] and Ladyzhenskaya and Ural'tseva [11], [12] to yield both global and interior gradient estimates for uniformly elliptic equations. Later Serrin [16] extended these results to the prescribed classical mean curvature equation.

Let us stress that for the classical mean curvature equation an interior gradient bound holds without assumptions on the strictly monotonicity of the prescribed curvature function with respect to u. We recall that for the minimal surface equation an interior gradient bound with right hand side in exponential form was discovered, in the case of two variables, by Finn [8] and in the general case by Bombieri, De Giorgi and Miranda [4]. The method of the paper [4] depends upon an isoperimetric inequality of Federer and Fleming [7] and a resulting Sobolev inequality. This method was then used by Ladyzhenskaya and Ural'tseva in [13] to establish an interior gradient bounds for the general prescribed mean curvature equation.

For the prescribed mean Levi curvature equation (1.3) with $k = 0$ we cannot expect that a similar interior gradient bound holds. Indeed, in the appendix we show that for every continuous function $U : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, $u(x) = U(x_n)$ is a viscosity solution of $F(Du, D^2u) = 0$.

2. A differential property of the Levi determinant

In this section we differentiate the Levi determinant (1.1). We start with the case $N = 1$.

Proposition 2.1. *If $N = 1$ then*

$$\left|\frac{\partial L}{\partial p}\right|^2 = \sum_{j,k,l=1}^{2} \frac{\partial L}{\partial r_{j\bar{k}}} r_{j\bar{l}} r_{l\bar{k}}$$

where $\left|\frac{\partial L}{\partial p}\right|^2 := \sum_{j=1}^{N+1} \frac{\partial L}{\partial p_j} \frac{\partial L}{\partial p_{\bar{j}}}$

Proof. If $N = 1$ then

$$L(p, \bar{p}, r) = L_{1,2}(p, \bar{p}, r) = -\det \begin{pmatrix} 0 & p_{\bar{1}} & p_{\bar{2}} \\ p_1 & r_{1\bar{1}} & r_{1\bar{2}} \\ p_2 & r_{2\bar{1}} & r_{2\bar{2}} \end{pmatrix}$$

A direct computation shows that

$$\frac{\partial L}{\partial p_1} = p_{\bar{1}} r_{2\bar{2}} - p_{\bar{2}} r_{2\bar{1}}, \quad \frac{\partial L}{\partial p_2} = p_{\bar{2}} r_{1\bar{1}} - p_{\bar{1}} r_{1\bar{2}}$$

$$\left|\frac{\partial L}{\partial p_{\bar{1}}}\right|^2 = p_{\bar{2}} p_2 r_{2\bar{1}} r_{1\bar{2}} + p_{\bar{1}} p_1 r_{2\bar{2}} r_{2\bar{2}} - p_{\bar{2}} p_1 r_{2\bar{1}} r_{2\bar{2}} - p_{\bar{1}} p_2 r_{2\bar{2}} r_{1\bar{2}}$$

$$\left|\frac{\partial L}{\partial p_{\bar{2}}}\right|^2 = p_{\bar{1}} p_1 r_{1\bar{2}} r_{2\bar{1}} + p_{\bar{2}} p_2 r_{1\bar{1}} r_{1\bar{1}} - p_{\bar{1}} p_2 r_{1\bar{2}} r_{1\bar{1}} - p_{\bar{2}} p_1 r_{1\bar{1}} r_{2\bar{1}}$$

$$\frac{\partial L}{\partial r_{1\bar{1}}} = p_{\bar{2}} p_2, \quad \frac{\partial L}{\partial r_{1\bar{2}}} = -p_{\bar{1}} p_2,$$

$$\frac{\partial L}{\partial r_{2\bar{1}}} = -p_{\bar{2}} p_1, \quad \frac{\partial L}{\partial r_{2\bar{2}}} = p_{\bar{1}} p_1$$

and by substituting we get

$$\begin{aligned}
\left|\frac{\partial L}{\partial p}\right|^2 &= + p_{\bar{2}} p_2 (r_{1\bar{1}} r_{1\bar{1}} + r_{1\bar{2}} r_{2\bar{1}}) - p_{\bar{1}} p_2 (r_{1\bar{1}} r_{1\bar{2}} + r_{1\bar{2}} r_{2\bar{2}}) \\
&\quad - p_{\bar{2}} p_1 (r_{2\bar{1}} r_{1\bar{1}} + r_{2\bar{2}} r_{2\bar{1}}) + p_{\bar{1}} p_1 (r_{2\bar{1}} r_{1\bar{2}} + r_{2\bar{2}} r_{2\bar{2}}) \\
&= + \frac{\partial L}{\partial r_{1\bar{1}}} (r_{1\bar{1}} r_{1\bar{1}} + r_{1\bar{2}} r_{2\bar{1}}) + \frac{\partial L}{\partial r_{1\bar{2}}} (r_{1\bar{1}} r_{1\bar{2}} + r_{1\bar{2}} r_{2\bar{2}}) \\
&\quad + \frac{\partial L}{\partial r_{2\bar{1}}} (r_{2\bar{1}} r_{1\bar{1}} + r_{2\bar{2}} r_{2\bar{1}}) + \frac{\partial L}{\partial r_{2\bar{2}}} (r_{2\bar{1}} r_{1\bar{2}} + r_{2\bar{2}} r_{2\bar{2}}) \\
&= \sum_{j,k,l=1}^{2} \frac{\partial L}{\partial r_{j\bar{k}}} r_{j\bar{l}} r_{l\bar{k}}
\end{aligned}$$

□

We now use Proposition 2.1 to handle the general case $N \in I\!N$.

Proposition 2.2. *For all $N \in I\!N$ we have*

$$\Big|\frac{\partial L}{\partial p}\Big|^2 \leq N \sum_{m,q,t=1}^{N+1} \frac{\partial L}{\partial r_{m\bar{q}}} r_{m\bar{t}} r_{t\bar{q}}$$

Proof. Since $L(p,\bar{p},r) = \sum_{1\leq i<j\leq N+1} L_{i,j}(p,\bar{p},r)$ and for all $1 \leq \ell \leq N+1$, $L_{i,j}$ depends on p_ℓ iff $i = \ell$ or $j = \ell$, we have

$$\frac{\partial L}{\partial p_\ell} = \sum_{1\leq i<\ell} \frac{\partial L_{i,\ell}}{\partial p_\ell} + \sum_{\ell<j\leq N+1} \frac{\partial L_{\ell,j}}{\partial p_\ell}$$

and by the triangle inequality

$$\Big|\frac{\partial L}{\partial p_\ell}\Big| \leq \sum_{1\leq i<\ell} \Big|\frac{\partial L_{i,\ell}}{\partial p_\ell}\Big| + \sum_{\ell<j\leq N+1} \Big|\frac{\partial L_{\ell,j}}{\partial p_\ell}\Big|.$$

By Proposition 2.1 we have

$$\begin{aligned}
\Big|\frac{\partial L}{\partial p}\Big|^2 = \sum_{\ell=1}^{N+1} \Big|\frac{\partial L}{\partial p_\ell}\Big|^2 &\leq \sum_{\ell=1}^{N+1} \Big(\sum_{1\leq i<\ell} \Big|\frac{\partial L_{i,\ell}}{\partial p_\ell}\Big| + \sum_{\ell<j\leq N+1} \Big|\frac{\partial L_{\ell,j}}{\partial p_\ell}\Big|\Big)^2 \\
&\leq N \sum_{\ell=1}^{N+1} \Big(\sum_{1\leq i<\ell} \Big|\frac{\partial L_{i,\ell}}{\partial p_\ell}\Big|^2 + \sum_{\ell<j\leq N+1} \Big|\frac{\partial L_{\ell,j}}{\partial p_\ell}\Big|^2\Big) \\
&= N \sum_{i=1}^{N+1} \sum_{\ell\neq i} \Big(\Big|\frac{\partial L_{i,\ell}}{\partial p_\ell}\Big|^2 + \Big|\frac{\partial L_{i,\ell}}{\partial p_i}\Big|^2\Big) \\
&= N \sum_{i=1}^{N+1} \sum_{\ell\neq i} \sum_{m,q,t\in\{i,\ell\}} \frac{\partial L_{i,\ell}}{\partial r_{m\bar{q}}} r_{m\bar{t}} r_{t\bar{q}} \\
&= N \sum_{m,q,t=1}^{N+1} \frac{\partial L}{\partial r_{m\bar{q}}} r_{m\bar{t}} r_{t\bar{q}}
\end{aligned}$$

□

Corollary 2.1. *The Hermitian matrix* $\left(\frac{\partial L}{\partial r_{m\bar{q}}}\right)_{m,q=1}^{N+1}$ *is non negative definite.*

Proof. For every $\zeta \in \mathbb{C}^{N+1}$ we have

$$\begin{aligned}
\sum_{m,q=1}^{N+1} \frac{\partial L}{\partial r_{m\bar q}} \zeta_q \bar\zeta_m &= \sum_{m,q=1}^{N+1} \left(\sum_{1\le i<j\le N+1} \frac{\partial L_{i,j}}{\partial r_{m\bar q}} \right) \zeta_q \bar\zeta_m \\
&= \sum_{m,q=1}^{N+1} \left(\sum_{1\le i<j\le N+1} (|p_j|^2 \delta_{im}\delta_{iq} + |p_i|^2 \delta_{jm}\delta_{jq} - p_i p_{\bar j} \delta_{mj}\delta_{iq} - p_j p_{\bar i}\delta_{mi}\delta_{jq}) \right) \zeta_q \bar\zeta_m \\
&= \sum_{1\le i<j\le N+1} \left(|p_j|^2 \zeta_i \bar\zeta_i + |p_i|^2 \zeta_j\bar\zeta_j - p_i p_{\bar j} \zeta_i \bar\zeta_j - p_j p_{\bar i} \zeta_j \bar\zeta_i \right) \\
&= \sum_{1\le i<j\le N+1} \left| p_j \bar\zeta_i - p_i \bar\zeta_j \right|^2 \ge 0
\end{aligned}$$

□

For the function F in (2.1) similar properties hold. We first explicitly write the definition of F in terms of L and of a change of variable. For every $\xi \in \mathbb{R}^n$ and for every $n \times n$ symmetric matrix $X = (X_{ij})_{i,j=1}^n$

$$F(\xi, X) = L(p, \bar p, r) \tag{2.1}$$

where for $1 \le j \le N$

$$p_j = \frac{1}{2}\Big(\xi_j - i\xi_{N+j}\Big), \quad p_{N+1} = \frac{1}{2}\Big(\xi_{2N+1} + i\Big), \tag{2.2}$$

and the matrix r is

$$r = \frac{1}{4} \overline{J}^T X J, \quad \text{with } J = \begin{pmatrix} I_N & 0 \\ iI_N & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3}$$

Proposition 2.3. *Let F be the function in* (2.1)*. Then*

$$\Big|\frac{\partial F}{\partial \xi}\Big|^2 \le \frac{N}{2} \sum_{i,j,k=1}^n \frac{\partial F}{\partial X_{ij}} X_{ik} X_{kj}$$

with $n = 2N+1$

Proof. By the change of variable (2.2) we have for $1 \le j \le N$

$$\begin{aligned}
\frac{\partial F}{\partial \xi_j} &= \frac{1}{2}\Big(\frac{\partial L}{\partial p_j} + \frac{\partial L}{\partial \bar p_j}\Big) = Re \frac{\partial L}{\partial p_j} \\
\frac{\partial F}{\partial \xi_{j+N}} &= -\frac{i}{2}\Big(\frac{\partial L}{\partial p_j} - \frac{\partial L}{\partial \bar p_j}\Big) = Im \frac{\partial L}{\partial p_j} \\
\frac{\partial F}{\partial \xi_n} &= \frac{1}{2}\Big(\frac{\partial L}{\partial p_N} + \frac{\partial L}{\partial \bar p_N}\Big) = Re \frac{\partial L}{\partial p_N}
\end{aligned}$$

Hence

$$\Big|\frac{\partial F}{\partial \xi}\Big|^2 = \sum_{j=1}^n \Big(\frac{\partial F}{\partial \xi_j}\Big)^2 = \sum_{j=1}^N \Big|\frac{\partial L}{\partial p_j}\Big|^2 + \Big(Re \frac{\partial L}{\partial p_N}\Big)^2 \le \Big|\frac{\partial L}{\partial p}\Big|^2$$

and by Proposition 2.2 and the change of variable (2.3)

$$\Big|\frac{\partial L}{\partial p}\Big|^2 \leq N \sum_{m,q,t=1}^{N+1} \frac{\partial L}{\partial r_{m\bar{q}}} r_{m\bar{t}} r_{t\bar{q}} = N \operatorname{Tr}\left(\left(\frac{\partial L}{\partial \bar{r}}\right)^T rr\right)$$
$$= \frac{N}{16} \operatorname{Tr}\left(\left(\frac{\partial L}{\partial \bar{r}}\right)^T \overline{J}^T XJ\overline{J}^T XJ\right)$$

Since $XJ\left(\frac{\partial L}{\partial \bar{r}}\right)^T \overline{J}^T X$ is a non negative definite Hermitian matrix and

$$0 \leq J\overline{J}^T = \begin{pmatrix} I_N & -iI_N & 0 \\ iI_N & I_N & 0 \\ 0 & 0 & 1 \end{pmatrix} \leq 2I_n$$

we have

$$\begin{aligned} \Big|\frac{\partial F}{\partial \xi}\Big|^2 &\leq \Big|\frac{\partial L}{\partial p}\Big|^2 \leq \frac{N}{16} \operatorname{Tr}\left(XJ\left(\frac{\partial L}{\partial \bar{r}}\right)^T \overline{J}^T XJ\overline{J}^T\right) \\ &\leq \frac{N}{8} \operatorname{Tr}\left(XJ\left(\frac{\partial L}{\partial \bar{r}}\right)^T \overline{J}^T X\right). \end{aligned} \tag{2.4}$$

We now use again (2.3) to estimate the righthand side in (2.4). We have

$$\frac{\partial F}{\partial X_{ij}} = \sum_{l,p=1}^{N+1} \frac{\partial L}{\partial r_{l\bar{p}}} \frac{\partial r_{l\bar{p}}}{\partial X_{ij}} = \frac{1}{4} \sum_{l,p=1}^{N+1} \frac{\partial L}{\partial r_{l\bar{p}}} \overline{J}^T_{li} J_{jp} \tag{2.5}$$

and

$$\begin{aligned} \operatorname{Tr}\left(\left(\frac{\partial F}{\partial X}\right)^T XX\right) &= \sum_{i,j,k=1}^{n} \frac{\partial F}{\partial X_{ij}} X_{ik} X_{kj} = \frac{1}{4} \sum_{i,j,k}^{n} \left(\sum_{l,p=1}^{N+1} \frac{\partial L}{\partial r_{l\bar{p}}} \overline{J}^T_{li} J_{jp} X_{ik} X_{kj}\right) \\ &= \frac{1}{4} \sum_{i,j,k}^{n} \left(\sum_{l,p=1}^{N+1} X_{kj} J_{jp} \frac{\partial L}{\partial r_{l\bar{p}}} \overline{J}^T_{li} X_{ik}\right) \\ &= \frac{1}{4} \operatorname{Tr}\left(XJ\left(\frac{\partial L}{\partial \bar{r}}\right)^T \overline{J}^T X\right). \end{aligned} \tag{2.6}$$

By substituting (2.6) in (2.4) we finally obtain

$$\Big|\frac{\partial F}{\partial \xi}\Big|^2 \leq \frac{N}{2} \sum_{i,j,k=1}^{n} \frac{\partial F}{\partial X_{ij}} X_{ik} X_{kj}.$$

□

Proposition 2.4. *The matrix* $\left(\frac{\partial F}{\partial X_{ij}}\right)_{i,j=1}^{n}$ *is non negative definite and*

$$\sum_{j=1}^{n} \frac{\partial F}{\partial X_{jj}}(\xi, X) \leq \frac{N}{8}(1 + |\xi|^2)$$

Proof. The first assertion follows by (2.5) and by Corollary 2.1. Moreover, by (2.3) and (2.2)

$$\begin{aligned}\sum_{j=1}^{n}\frac{\partial F}{\partial X_{jj}} =&\frac{1}{4}\sum_{j=1}^{n}\sum_{l,q=1}^{N+1}\frac{\partial L}{\partial r_{l\bar q}}\bar J^T_{lj}J_{jq}\\ =&\frac{1}{2}\sum_{l=1}^{N}\frac{\partial L}{\partial r_{l\bar l}}+\frac{1}{4}\frac{\partial L}{\partial r_{N+1\overline{N+1}}}\\ =&\frac{1}{2}\sum_{l=1}^{N}(|p|^2-|p_l|^2)+\frac{1}{4}((|p|^2-|p_{N+1}|^2))\\ \le&\frac{N}{2}|p|^2=\frac{N}{8}(|\xi|^2+1)\end{aligned}$$

□

3. Interior gradient estimate

In this section we shall prove Theorem 1.1 by using an approximation argument and uniform a priori estimates.

We denote by x a point in $\mathbb{R}^n$ and by $\mathcal{S}(n)$ the space of all $n\times n$ symmetric matrices. For every $(\xi,X)\in\mathbb{R}^n\times\mathcal{S}(n)$ and $\varepsilon>0$ we define

$$F^\varepsilon(\xi,X)=F(\xi,X)+\varepsilon\mathrm{Tr}\,X,$$

with F as in (2.1).

Proof of Theorem 1.1. If $u\in C(\bar\Omega)$ is a viscosity solution of (1.3) with $k\in C^1(\Omega\times\mathbb{R})$, then for every ball $B\subset\subset\Omega$ we consider a solution $u^\varepsilon\in C^3(B)$ of the elliptic PDE

$$F^\varepsilon(Du^\varepsilon,D^2u^\varepsilon)=\frac{N}{2^3}(1+|Du^\varepsilon|^2)^{3/2}k(x,u^\varepsilon) \tag{3.1}$$

with boundary data u on ∂B. Let us recall that the existence of a solution $u^\varepsilon\in C^2(B)$ of (3.1) for every continuous boundary data is guaranteed by [9, Theorem 15.18]. The C^3 regularity follows by the uniform ellipticity of F^ε for every $\varepsilon>0$. Moreover, if $M=\sup_{\bar\Omega}|u|$ and k is non negative then $F^\varepsilon(DM,D^2M)=0\le F^\varepsilon(Du^\varepsilon,D^2u^\varepsilon)$ and by the comparison principle $\sup_B u^\varepsilon\le M$. By [14, estimate (4.2)] we also have

$$\sup_B|u^\varepsilon|\le\tilde M=M+C, \tag{3.2}$$

with C only depending on $\sup_{\bar\Omega\times[-M,M]}k$. We shall prove that if $\inf\frac{\partial k}{\partial u}=\alpha>0$ then $w:=\frac{|Du^\varepsilon|^2}{2}$ is locally bounded by a constant independent of ε. For $\varepsilon\to0$, u^ε will uniformly converge to a Lipschitz continuous viscosity solution $\tilde u$ of (1.3), which agrees with u on ∂B. The comparison principle in [14] will guarantee that $\tilde u=u$ is Lipschitz continuous on B and the thesis of Theorem 1.1 will follow.

We start by differentiating the equality (3.1) with respect to x_l. For a sake of simplicity in the sequel we shall denote by $u_j=\frac{\partial u}{\partial x_j}$, $u_{ij}=\frac{\partial^2u}{\partial x_j\partial x_i}$, and use a similar notation for third order partial derivatives. We get

$$\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}u^{\varepsilon}_{ijl}+\sum_{i=1}^{n}\frac{\partial F^{\varepsilon}}{\partial\zeta_i}u^{\varepsilon}_{il}=\frac{3N}{2^3}(2w+1)^{\frac12}\sum_{i=1}^{n}u^{\varepsilon}_{i}u^{\varepsilon}_{il}k(x,u^{\varepsilon})+\frac{N}{2^3}(2w+1)^{\frac32}\Big(\frac{\partial k}{\partial x_l}+\frac{\partial k}{\partial u}u^{\varepsilon}_{l}\Big) \tag{3.3}$$

By multiplying (3.3) by u^{ε}_l and summing up in l we get

$$\begin{aligned}\sum_{i,j,l=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}u^{\varepsilon}_{ijl}u^{\varepsilon}_{l}+\sum_{i,l=1}^{n}\frac{\partial F^{\varepsilon}}{\partial\zeta_i}u^{\varepsilon}_{il}u^{\varepsilon}_{l}=&\frac{3N}{2^3}(2w+1)^{\frac12}k(x,u^{\varepsilon})\sum_{i,l=1}^{n}u^{\varepsilon}_{i}u^{\varepsilon}_{il}u^{\varepsilon}_{l}\\&+\frac{N}{2^3}(2w+1)^{\frac32}\Big(2\frac{\partial k}{\partial u}w+\sum_{l=1}^{n}\frac{\partial k}{\partial x_l}u^{\varepsilon}_{l}\Big)\end{aligned} \tag{3.4}$$

Since $w_i=\sum_{l=1}^{n}u^{\varepsilon}_{l}u^{\varepsilon}_{il}$ and $w_{ij}=\sum_{l=1}^{n}(u^{\varepsilon}_{lj}u^{\varepsilon}_{il}+u^{\varepsilon}_{l}u^{\varepsilon}_{ijl})$ we can rewrite equation (3.4) as

$$\begin{aligned}\sum_{i,j,l=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}(w_{ij}-u^{\varepsilon}_{il}u^{\varepsilon}_{lj})+\sum_{i,l=1}^{n}\frac{\partial F^{\varepsilon}}{\partial\zeta_i}w_i=&\frac{3N}{2^3}(2w+1)^{\frac12}k(x,u^{\varepsilon})\sum_{i}^{n}u^{\varepsilon}_{i}w_i\\&+\frac{N}{2^3}(2w+1)^{\frac32}\Big(2\frac{\partial k}{\partial u}w+\sum_{l=1}^{n}\frac{\partial k}{\partial x_l}u^{\varepsilon}_{l}\Big).\end{aligned} \tag{3.5}$$

Let $\mathcal{L}$ be the following second order elliptic operator

$$\mathcal{L}=\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\partial_{ij}+\sum_{i=1}^{n}\frac{\partial F^{\varepsilon}}{\partial\zeta_i}\partial_i-\frac{3N}{2^3}(2w+1)^{\frac12}k(x,u^{\varepsilon})\sum_{i=1}^{n}u^{\varepsilon}_{i}\partial_i.$$

Then equation (3.5) writes as

$$\mathcal{L}(w)=\sum_{i,j,l=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}u^{\varepsilon}_{il}u^{\varepsilon}_{lj}+\frac{N}{2^3}(2w+1)^{\frac32}\Big(2\frac{\partial k}{\partial u}w+\sum_{l=1}^{n}\frac{\partial k}{\partial x_l}u^{\varepsilon}_{l}\Big)$$

For every $x\in\Omega$ let $R>0$ such that $B=B(x,R)\subseteq\Omega$ and a test function $\varphi\in C^2(\bar{B})$ such that $0\le\varphi\le1$ in $\bar{B}$, $\varphi=0$ on ∂B and $\varphi>0$ in B. We shall choose such a function below. We shall apply $\mathcal{L}$ to the product $v=w\varphi$. We have

$$\begin{aligned}v_i&=w_i\varphi+w\varphi_i\\v_{ij}&=w_{ij}\varphi+w_i\varphi_j+w_j\varphi_i+w\varphi_{ij}\end{aligned}$$

We now apply $\mathcal{L}$ to v. By remarking that $\dfrac{\partial F^{\varepsilon}}{\partial X_{ij}}$ is a symmetric matrix we get

$$\begin{aligned}\mathcal{L}(v)&=\mathcal{L}(w)\varphi+w\mathcal{L}(\varphi)+2\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\varphi_j w_i\\&=\mathcal{L}(w)\varphi+w\mathcal{L}(\varphi)+2\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\varphi_j\frac{(v_i-w\varphi_i)}{\varphi}\end{aligned}$$

We now consider the elliptic PDO

$$H = \mathcal{L} - 2\sum_{i,j=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} \frac{\varphi_j}{\varphi} \partial_i$$

and define

$$\begin{aligned} E = \sum_{i,j,l=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} u^{\varepsilon}_{il} u^{\varepsilon}_{lj} &= \sum_{i,j,l=1}^{n} \frac{\partial F}{\partial X_{ij}} u^{\varepsilon}_{il} u^{\varepsilon}_{lj} + \varepsilon \sum_{i,l=1}^{n} u^{\varepsilon}_{il} u^{\varepsilon}_{li} \\ &\geq \sum_{i,j,l=1}^{n} \frac{\partial F}{\partial X_{ij}} u^{\varepsilon}_{il} u^{\varepsilon}_{lj}. \end{aligned}$$

We have

$$H(v) = \varphi \left(E + \frac{N}{2^3}(2w+1)^{\frac{3}{2}} \Big(2\frac{\partial k}{\partial u} w + \sum_{l=1}^{n} \frac{\partial k}{\partial x_l} u^{\varepsilon}_l \Big) \right) + \left(\mathcal{L}(\varphi) - 2\sum_{i,j=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} \frac{\varphi_i \varphi_j}{\varphi} \right) w.$$

Let x be a local maximum point for v, then $H(v)(x) \leq 0$ and at x

$$0 \geq \varphi \left(E + \frac{N}{2^3}(2w+1)^{\frac{3}{2}} \Big(2\frac{\partial k}{\partial u} w + \sum_{l=1}^{n} \frac{\partial k}{\partial x_l} u^{\varepsilon}_l \Big) \right) + \left(\mathcal{L}(\varphi) - 2\sum_{i,j=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} \frac{\varphi_i \varphi_j}{\varphi} \right) w$$

Since

$$\mathcal{L}(\varphi) = \sum_{i,j=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} \varphi_{ij} + \sum_{i=1}^{n} \frac{\partial F^{\varepsilon}}{\partial \zeta_i} \varphi_i - \frac{3N}{2^3}(2w+1)^{\frac{1}{2}} k(x, u^{\varepsilon}) \sum_{i,l=1}^{n} u^{\varepsilon}_i \varphi_i$$

and by Proposition 2.3

$$\sum_{i=1}^{n} \frac{\partial F^{\varepsilon}}{\partial \zeta_i} \varphi_i \geq -|D\varphi| \Big| \frac{\partial F^{\varepsilon}}{\partial \zeta} \Big| \geq -\frac{\sqrt{N}}{\sqrt{2}} |D\varphi| \sqrt{E}$$

at x we have the inequality

$$\begin{aligned} 0 \geq & \varphi\, E - \frac{\sqrt{N}}{\sqrt{2}} |D\varphi| \sqrt{E} w + \varphi \Big(\frac{N}{2^3}(2w+1)^{\frac{3}{2}} \Big(2\frac{\partial k}{\partial u} w + \sum_{l=1}^{n} \frac{\partial k}{\partial x_l} u^{\varepsilon}_l \Big) \Big) \\ & + w \Big(\sum_{i,j=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} \Big(\varphi_{ij} - 2\frac{\varphi_i \varphi_j}{\varphi} \Big) - \frac{3N}{2^3}(2w+1)^{\frac{1}{2}} k(x, u^{\varepsilon}) \sum_{i,l=1}^{n} u^{\varepsilon}_i \varphi_i \Big) \\ \geq & -\frac{N w^2 |D\varphi|^2}{8\varphi} + \varphi \Big(\frac{N}{2^3}(2w+1)^{\frac{3}{2}} \Big(2\frac{\partial k}{\partial u} w + \sum_{l=1}^{n} \frac{\partial k}{\partial x_l} u^{\varepsilon}_l \Big) \Big) \\ & + w \Big(\sum_{i,j=1}^{n} \frac{\partial F^{\varepsilon}}{\partial X_{ij}} \Big(\varphi_{ij} - 2\frac{\varphi_i \varphi_j}{\varphi} \Big) - \frac{3N}{2^3}(2w+1)^{\frac{1}{2}} k(x, u^{\varepsilon}) \sum_{i,l=1}^{n} u^{\varepsilon}_i \varphi_i \Big) \end{aligned}$$

which reads as

(3.6)
$$\begin{aligned}\frac{N}{4}\varphi(2w+1)^{\frac{3}{2}}\frac{\partial k}{\partial u}w \leq &\frac{Nw^2|D\varphi|^2}{8\varphi} - \frac{N}{2^3}\varphi(2w+1)^{\frac{3}{2}}\sum_{l=1}^{n}\frac{\partial k}{\partial x_l}u_l^{\varepsilon} + \frac{N}{2^3}w(2w+1)^{\frac{1}{2}}k(x,u^{\varepsilon})\sum_{i,l=1}^{n}u_i^{\varepsilon}\varphi_i \\ &+ w\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\Big(2\frac{\varphi_i\varphi_j}{\varphi} - \varphi_{ij}\Big) \\ \leq &\frac{N}{2^3}\left(\frac{w^2|D\varphi|^2}{\varphi} + \varphi(2w+1)^{\frac{3}{2}}\sqrt{w}M_2 + w(2w+1)^{\frac{1}{2}}M_1|D\varphi|\sqrt{w}\right) \\ &+ w\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\Big(2\frac{\varphi_i\varphi_j}{\varphi} - \varphi_{ij}\Big)\end{aligned}$$

where $M_1 = \sqrt{2}\max_{\Omega\times[-M,M]}|k|$, $M_2 = \sqrt{2}\max_{\Omega\times[-\tilde{M},\tilde{M}]}|\partial_x k|$, with $M = \max_{\bar{\Omega}}|u|$ and $\tilde{M}$ as in (3.2). Since $0 < \alpha \leq \frac{\partial k}{\partial u}$ we can divide equation (3.6) by $\frac{N}{4}\sqrt{\varphi}(2w+1)^{\frac{3}{2}}\alpha\sqrt{w}$ and we get

$$\begin{aligned}\sqrt{\varphi w} &\leq C_1\frac{|D\varphi|^2}{\varphi^{\frac{3}{2}}} + C_2\varphi^{\frac{1}{2}} + C_3\frac{|D\varphi|}{\varphi^{\frac{1}{2}}} + C_4(2w+1)^{-1}\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\Big(2\frac{\varphi_i\varphi_j}{\varphi^{\frac{3}{2}}} - \frac{\varphi_{ij}}{\varphi^{\frac{1}{2}}}\Big) \\ &\leq C_1\frac{|D\varphi|^2}{\varphi^{\frac{3}{2}}} + C_2\varphi^{\frac{1}{2}} + C_3\frac{|D\varphi|}{\varphi^{\frac{1}{2}}} + C_4\frac{\sqrt{w}}{(2w+1)^{3/2}}\sum_{i,j=1}^{n}\frac{\partial F^{\varepsilon}}{\partial X_{ij}}\Big(2\frac{\varphi_i\varphi_j}{\varphi^{\frac{3}{2}}} - \frac{\varphi_{ij}}{\varphi^{\frac{1}{2}}}\Big)\end{aligned} \tag{3.7}$$

where C_1, C_2, C_3, C_4 are non negative constants depending on α, M_1, M_2.

We now choose $\varphi = (R^2 - |x|^2)^4$ and compute its derivatives

$$\begin{aligned}\varphi_i &= -8(R^2 - |x|^2)^3 x_i \\ |D\varphi| &= 8(R^2 - |x|^2)^3|x| \\ \varphi_{ij} &= 48(R^2 - |x|^2)^2 x_j x_i - 8(R^2 - |x|^2)^3\delta_{ij}\end{aligned}$$

and we have the following estimates

$$\begin{aligned}\frac{|D\varphi|^2}{\varphi^{\frac{3}{2}}} &= \frac{64(R^2 - |x|^2)^6|x|^2}{(R^2 - |x|^2)^6} \leq 64R^2 \\ \varphi^{\frac{1}{2}} &= (R^2 - |x|^2)^2 \leq R^4 \\ \frac{|D\varphi|}{\varphi^{\frac{1}{2}}} &= \frac{8(R^2 - |x|^2)^3|x|}{(R^2 - |x|^2)^2} \leq 8R^3 \\ 2\frac{\varphi_i\varphi_j}{\varphi^{\frac{3}{2}}} - \frac{\varphi_{ij}}{\varphi^{\frac{1}{2}}} &= 128\frac{(R^2 - |x|^2)^6 x_i x_j}{(R^2 - |x|^2)^6} - \frac{48(R^2 - |x|^2)^2 x_j x_i - 8(R^2 - |x|^2)^3\delta_{ij}}{(R^2 - |x|^2)^2} \\ &= 80x_i x_j + 8(R^2 - |x|^2)\delta_{ij} \leq 88R^2\delta_{ij}\end{aligned}$$

Since the symmetric matrix $\frac{\partial F^\varepsilon}{\partial X_{ij}}$ is non negative definite and by Proposition 2.4 with $|\xi|^2 = 2w$

$$\sum_{j=1}^{n} \frac{\partial F^\varepsilon}{\partial X_{jj}} = \sum_{j=1}^{n} \frac{\partial F}{\partial X_{jj}} + n\varepsilon \leq \frac{N}{8}(2w+1) + n\varepsilon$$

by redefining constants we get

$$\sqrt{v} = \sqrt{\varphi w} \leq C_1 R^2 + C_2 R^4 + C_3 R^3 + C_4 R^2 \leq C_0 \tag{3.8}$$

where C_0 depends on α, M_1, M_2 and on the diameter of the bounded set Ω. Since x is a local maximum of v, for every $x_0 \in B(x,R)$ we have $v(x_0) \leq v(x)$ and by (3.8)

$$\sqrt{\varphi(x_0) w(x_0)} \leq \sqrt{\varphi(x) w(x)} \leq C_0.$$

Hence

$$\frac{|Du^\varepsilon(x_0)|}{\sqrt{2}} \leq \frac{C_0}{\sqrt{\varphi(x_0)}} = \frac{C_0}{(R^2 - |x_0|^2)^2} = \frac{C_0}{(R - |x_0|)^2 (R + |x_0|)^2}$$

and we can conclude that

$$|Du^\varepsilon(x_0)| \leq \frac{C}{d^2(x_0, \partial B)}$$

where $d(\cdot, \partial B)$ is the boundary distance. □

4. Appendix

In this section we show that, for every continuous function $U : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, the function $u(x) = u(x_1, \ldots, x_{2N}, x_n) = U(x_n)$ is a viscosity solution of $F(Du, D^2u) = 0$ in $\Omega = A \times B$, for every $B \subseteq \mathbb{R}^{2N}$.

For every $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x, we have $\phi_j(x) = u_j(x) = 0$ for every $j = 1, \ldots, 2N$, and for every $\eta \in \mathbb{R}^{2N}$

$$0 = \sum_{i,j=1}^{2N} u_{ij}(x)\eta_i\eta_j \leq \sum_{i,j=1}^{2N} \phi_{ij}(x)\eta_i\eta_j.$$

By (2.1) with $\xi = D\phi(x)$ and $X = D^2\phi(x)$ we find

$$F(D\phi(x), D^2\phi(x)) = \frac{1 + (\phi_n(x))^2}{2^3} \sum_{j=1}^{2N} \phi_{jj}(x) \geq 0,$$

and we can conclude that u is a viscosity subsolution of $F = 0$ at $x \in \Omega$.

Analogously, for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local minimum at x, we have $\phi_j(x) = u_j(x) = 0$ for every $j = 1, \ldots, 2N$, and for every $\eta \in \mathbb{R}^{2N}$

$$0 = \sum_{i,j=1}^{2N} u_{ij}(x)\eta_i\eta_j \geq \sum_{i,j=1}^{2N} \phi_{ij}(x)\eta_i\eta_j,$$

then

$$F(D\phi(x), D^2\phi(x)) = \frac{1 + (\phi_n(x))^2}{2^3} \sum_{j=1}^{2N} \phi_{jj}(x) \leq 0,$$

and we can conclude that u is a viscosity supersolution of $F = 0$ at $x \in \Omega$.

Thus, u is a viscosity solution of $F = 0$ in Ω.

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