Complex group actions on the sphere and sign changing solutions for the CR-Yamabe equation

Ali Maalaoui[(1)] & Vittorio Martino[(2)] & Giulio Tralli[(3)]

Abstract In this paper we prove that the CR-Yamabe equation on the sphere has infinitely many sign changing solutions. The problem is variational but the related functional does not satisfy the Palais-Smale condition, therefore the standard topological methods fail to apply directly. To overcome this lack of compactness, we will exploit different group actions on the sphere in order to find suitable closed subspaces, on which the restricted functional is Palais-Smale: this will allow us to use the minimax argument of Ambrosetti-Rabinowitz to find critical points. By a classification of the positive solutions and by considerations on the energy blow-up, we will get the desired result.

AMS Subject Classifications: 35J20, 35B33, 58E40.

1 Introduction

In this paper we prove the existence of infinitely many sign changing solutions of the following sub-Riemannian Yamabe equation on the Heisenberg group $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}} u = |u|^{\frac{4}{Q-2}} u, \qquad u \in \mathcal{S}_0^1(\mathbb{H}^n), \tag{1}$$

[1]Department of mathematics and natural sciences, American University of Ras Al Khaimah, PO Box 10021, Ras Al Khaimah, UAE. E-mail address: ali.maalaoui@aurak.ae

[2]Dipartimento di Matematica, Università di Bologna, piazza di Porta S.Donato 5, 40127 Bologna, Italy. E-mail address: vittorio.martino3@unibo.it

[3]Dipartimento di Matematica, Università di Bologna, piazza di Porta S.Donato 5, 40127 Bologna, Italy. E-mail address: giulio.tralli2@unibo.it

where $\Delta_{\mathbb{H}}$ denotes the sub-Laplacian of the group, $Q = 2n + 2$ is the homogenous dimension of $\mathbb{H}^n$, and $\mathcal{S}_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev type space on $\mathbb{H}^n$.

The problem is variational but, as in the Riemannian case, the functional associated with the equation (1) fails to satisfy the Palais-Smale compactness condition.
For the classical Yamabe equation on $\mathbb{R}^n$, after the classification for the positive solutions in [4], the first result about sign changing solutions was proved by Ding in [8]. Following the analysis by Ambrosetti and Rabinowitz [1], Ding found a suitable subspace X of the space of the variations for the related functional on which he performed the minimax argument.
Later on, many authors proved the existence of infinitely many sign changing solutions using other kinds of variational methods (see [2], [3] and the references therein). Finally in a couple of recent works [6], [7], del Pino, Musso, Pacard, and Pistoia found sign changing solutions, different from those of Ding, by using a superposition of positive and negative *bubbles* arranged on some special sets.
In the CR case, the positive solutions to the equation (1) were completely classified by Jerison and Lee in [14]. Now, using the Cayley transform one can set the problem on the sphere S^{2n+1}.
In [16], two of the authors proved that there exist solutions to (1) using a very particular group of isometries, namely the one generated by the Reeb vector field of the standard contact form on S^{2n+1}. Using the standard Hopf fibration on the sphere, they showed that the restricted functional satisfies the Palais-Smale condition by showing that the new space of variation is identified with a Sobolev space on a complex projective space: in particular, due to the very special symmetry, they were able to switch from a *critical subelliptic* problem to a *subcritical elliptic* one.
Here we will show that there exist many complex group actions that lead to sign changing solutions, each of them having different symmetries. Moreover in these general cases one cannot use any analogue of the Hopf fibration, therefore we will prove the compactness condition by using a general bubbling behavior of the Palais-Smale sequences, that in our situation will lead to a contradiction (see Lemma (3.2)).
Finally, we recall that in literature there are many other existence and multiplicity results about Yamabe type equations in different settings: we address the reader for instance to the papers [11], [22], [18], [17], [19], [20], [12] [15] and the references therein.

2 Structure of the equation and group actions

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ be the Heisenberg group. If we denote by $\xi = (z,t) = (x+iy,t) \simeq (x,y,t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ then the group law is given by

$$\xi_0 \centerdot \xi = (x+x_0, y+y_0, t+t_0+2(\langle x, y_0\rangle - \langle x_0, y\rangle)),\ \forall \xi, \xi_0 \in \mathbb{H}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. The left translations are defined by

$$\tau_{\xi_0}(\xi) := \xi_0 \centerdot \xi.$$

Finally the dilations of the group are

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$$

for any $\lambda > 0$. The canonical left-invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

The horizontal (or intrinsic) gradient of the group is

$$D_{\mathbb{H}} = (X_1, \dots, X_n, Y_1, \dots, Y_n).$$

Let us now set

$$q^* = \frac{2Q}{Q-2},$$

then the following Sobolev-type inequality holds

$$\|\varphi\|_{q^*}^2 = \Big(\int_{\mathbb{H}^n} |\varphi|^{q^*} \Big)^{\frac{2}{q^*}} \leq C \int_{\mathbb{H}^n} |D_{\mathbb{H}}\varphi|^2 = C \|D_{\mathbb{H}}\varphi\|_2^2, \quad \forall \varphi \in C_0^\infty(\mathbb{H}^n),$$

with C a positive constant.

Definition 2.1. *For every domain $\Omega \subseteq \mathbb{H}^n$, the Folland-Stein Sobolev space $\mathcal{S}_0^1(\Omega)$ (see [9]) is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm*

$$\| \cdot \| = \| D_{\mathbb{H}} \cdot \|_2.$$

The exponent q^* is called critical since the embedding

$$\mathcal{S}_0^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$$

is continuous but not compact for every domain Ω.
The Kohn Laplacian (or sub-Laplacian) on $\mathbb{H}^n$ is the following second order operator invariant with respect to the left translations and homogeneous of degree two with respect to the dilations:

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{n} X_j^2 + Y_j^2.$$

Let us now consider the following Yamabe type problem on the Heisenberg group $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}} u = |u|^{\frac{4}{Q-2}} u, \qquad u \in \mathcal{S}_0^1(\mathbb{H}^n). \tag{2}$$

We recall that a solution of the problem (2) on $\mathbb{H}^n$ can be found as critical point of the following functional

$$\mathcal{J} : \mathcal{S}_0^1(\mathbb{H}^n) \to \mathbb{R}, \qquad \mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{H}^n} |D_{\mathbb{H}} u|^2 - \frac{1}{q^*} \int_{\mathbb{H}^n} |u|^{q^*}.$$

Since q^* is the critical exponent for the Sobolev embedding then $\mathcal{J}$ does not satisfy the Palais-Smale condition.
Moreover any variational solution is actually a classical solution ([9], [10]).
We will prove the following

Theorem 2.2. *There exists a sequence of solutions $\{u_k\}$ of (2), with*

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u_k|^2 \longrightarrow \infty, \qquad as \quad k \to \infty.$$

Theorem 2.2 will imply that equation (2) has infinitely many sign changing solutions: in fact, by a classification result by Jerison and Lee [14], all the positive solutions of the equation (2) are in the form

$$\omega_{\lambda,\xi} = \lambda^{\frac{2-Q}{2}} \omega \circ \delta_{\frac{1}{\lambda}} \circ \tau_{\xi^{-1}}$$

for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$, where

$$\omega(x, y, t) = \frac{c_0}{\left((1 + |x|^2 + |y|^2)^2 + t^2\right)^{\frac{Q-2}{4}}}$$

with c_0 a positive constant. In particular all the solutions $\omega_{\lambda,\xi}$ have the same energy.
The idea is to consider the problem, after the Cayley transform, on the sphere S^{2n+1}. In that setting we will be able to find a suitable closed subspaces on which the restricted functional satisfies Palais-Smale.
Let us consider the sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ defined by

$$S^{2n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \; s.t. \quad |z_1|^2 + \dots + |z_{n+1}|^2 = 1\}.$$

The Cayley transform F is the analogous of the stereographic projection, namely it is a CR-diffeomorphism between the sphere minus a point and the Heisenberg group

$$F : S^{2n+1} \setminus \{0, \dots, 0, -1\} \to \mathbb{H}^n$$

$$F(z_1, \dots, z_{n+1}) = \Big(\frac{z_1}{1+z_{n+1}}, \dots, \frac{z_n}{1+z_{n+1}}, Re\big(i\frac{1-z_{n+1}}{1+z_{n+1}}\big)\Big).$$

So, if we denote by θ the standard contact form on S^{2n+1}, and by Δ_θ the related sub-Laplacian, a direct computation shows that equation (2) becomes

$$-\Delta_\theta v + c(n)v = |v|^{\frac{4}{Q-2}} v, \qquad v \in \mathcal{S}^1(S^{2n+1}), \tag{3}$$

with $c(n)$ a suitable positive constant related to the (constant) Webster curvature of the sphere (see [13] for a full detailed exposition).
In particular, by setting

$$u = \varphi v \tag{4}$$

(where φ is the function that gives the conformal factor in the change of the contact form), we have that every solution u of (2) corresponds to a solution v of (3) and it holds

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u|^2 = \int_{S^{2n+1}} |v|^{q^*}.$$

We can consider now the variational problem on the sphere

$$I : \mathcal{S}^1(S^{2n+1}) \to \mathbb{R}, \qquad I(v) = \frac{1}{2}\int_{S^{2n+1}} \big(|D_\theta v|^2 + c(n)v^2\big) - \frac{1}{q^*}\int_{S^{2n+1}} |v|^{q^*}.$$

Here $|D_\theta v|$ stands for the Webster norm (that in this particular case it coincides with the Euclidean one) of the contact gradient $D_\theta v$, namely $D_\theta = \{X_1, Y_1, \dots, X_n, Y_n\}$ is an orthonormal basis of $ker(\theta)$; moreover for any

$j = 1, \ldots, n$ we defined $Y_j = JX_j$ where J the standard complex structure on $\mathbb{C}^{n+1}$. If we identify $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$ with

$$z = (z_1, \ldots, z_{n+1}) \simeq (x_1, y_1, \ldots, x_{n+1}, y_{n+1}),$$

then J is the block matrix

$$J = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & \ddots & 0_{2\times 2} \\ \vdots & 0_{2\times 2} & \cdots & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}.$$

Remark 2.3. *We explicitly note that with the notation*

$$z = (z_1, \ldots, z_{n+1}) \simeq (x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) = (x, y)$$

one get J in the canonical form

$$J = \begin{pmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}.$$

We used a different formalism since the notations will be better in the sequel.

Now, let us denote

$$\mathbb{U}(n+1) = \{g \in \mathbb{O}(2n+2),\ gJ = Jg\},$$

where $\mathbb{O}(2n+2)$ is the group of real valued $(2n+2) \times (2n+2)$ orthogonal matrices.

Remark 2.4. *The functional I is invariant under the linear action of the group $\mathbb{U}(n+1)$, i.e.*

$$I(v) = I(v \circ g), \qquad \forall\, g \in \mathbb{U}(n+1).$$

As a matter of fact, the matrices in $\mathbb{U}(n+1)$, being orthogonal, define isometries of the sphere. On the other hand, they also bring the orthonormal bases for $ker(\theta)$ into each others; in particular the norm of D_θ is independent of the choice of one of such a basis.

If $\mathbb{G}$ is a subgroup of $\mathbb{U}(n+1)$, we define

$$X_{\mathbb{G}} = \{v \in \mathcal{S}^1(S^{2n+1}) \,:\, v \circ g = v,\ \forall\, g \in \mathbb{G}\}.$$

We are going to make the following assumptions on $\mathbb{G}$:

(H1) $X_{\mathbb{G}}$ is an infinite dimensional real vector space;

(H2) for any $z_0 \in S^{2n+1}$, the $\mathbb{G}$-orbit of z_0 has at least one accumulation point.

Example 2.5. *By mimicking Ding we can consider, for any $k \in \{1, \dots, n\}$, the subgroups $\mathbb{G}_k = \mathbb{U}(k) \times \mathbb{U}(n+1-k)$ formed by the matrices*

$$\begin{pmatrix} g_1 & 0_{2k\times 2(n+1-k)} \\ 0_{2(n+1-k)\times 2k} & g_2 \end{pmatrix}$$

with $g_1 \in \mathbb{U}(k)$ and $g_2 \in \mathbb{U}(n+1-k)$.
The functions in $\mathcal{S}^1(S^{2n+1})$ depending only on $|z_1|, |z_2|$ (with $z = (z_1, z_2)$, $z_1 \in \mathbb{C}^k$, $z_2 \in \mathbb{C}^{n+1-k}$) belong to $X_{\mathbb{G}_k}$. Thus we immediately get that $X_{\mathbb{G}_k}$ is infinite dimensional. Moreover, the $\mathbb{G}_k$-orbits of any point contain at least a circle. Therefore $\mathbb{G}_k$ satisfies $(H1)$ and $(H2)$.

We explicitly notice that, differently from Ding, we allow the case $k = 1$: basically this is related to the fact that the orbit of any point under the action of $\mathbb{U}(1)$ is the circle S^1, instead the orbits related to $\mathbb{O}(1)$ are $\mathbb{Z}_2$.
The following is a more general situation than can happen in this direction.

Counterexample 2.6. *For any $m \in \mathbb{N}$, let us consider the subgroups $\mathbb{G}_m = \mathbb{Z}_m \times \mathbb{U}(n)$ formed by the matrices*

$$\begin{pmatrix} \begin{pmatrix} \cos(\frac{2\pi j}{m}) & \sin(-\frac{2\pi j}{m}) \\ \sin(\frac{2\pi j}{m}) & \cos(\frac{2\pi j}{m}) \end{pmatrix} & 0_{2\times 2n} \\ 0_{2n\times 2} & g \end{pmatrix},$$

with $j \in \{0, \dots, m-1\}$ and $g \in \mathbb{U}(n)$. These are subgroups of the group $\mathbb{G}_1$ defined in the previous example. Thus, $X_{\mathbb{G}_m}$ are infinite dimensional. On the other hand, if we fix a point $z_0 = (e^{it_0}, 0) \in \mathbb{C}^{n+1}$, its $\mathbb{G}_m$-orbit contains exactly m points. Therefore, the groups $\mathbb{G}_m$ don't satisfy our main assumption $(H2)$.

Example 2.7. *In [16] it has been considered the case of the one-parameter group $\mathbb{G}_T$ generated by the flow of the Reeb vector field T of θ. In our notations, $\mathbb{G}_T$ is formed by the matrices $\exp(tJ)$, $t \in \mathbb{R}$, and it is a sub-group of any $\mathbb{G}_k$. The orbits are always great circles and our assumptions $(H1)$ and $(H2)$ are thus satisfied for $\mathbb{G}_T$: in particular considering the following Hopf fibration*

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

where the fibers are exactly the orbits of T, we have the identification $X_{\mathbb{G}_T} \simeq \mathcal{S}^1(\mathbb{C}P^n)$.

So far, all the groups we have showed are formed by block diagonal matrices. We can provide other examples in which this does not occur.

Example 2.8. *Let us consider the case $n = 1$, i.e. the case of S^3, and let us define the vector fields*

$$\tilde{X} = x_2\partial x_1 + y_2\partial y_1 - x_1\partial x_2 - y_1\partial y_2$$

and

$$\tilde{Y} = -y_2\partial x_1 + x_2\partial y_1 - y_1\partial x_2 + x_1\partial y_2.$$

Now we consider the one-parameter groups ($\mathbb{G}_{\tilde{X}}$ and $\mathbb{G}_{\tilde{Y}}$, respectively) generated by $\tilde{X}$ and $\tilde{Y}$: in other words,

$$\mathbb{G}_{\tilde{X}} = \{\exp(t\tilde{X}) \,:\, t \in \mathbb{R}\}, \qquad \mathbb{G}_{\tilde{Y}} = \{\exp(t\tilde{Y}) \,:\, t \in \mathbb{R}\}$$

where with some abuse of notation we can identify the vector fields with the matrices

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Such groups are contained in $\mathbb{U}(2)$ since $\tilde{X}$ and $\tilde{Y}$ are skew-symmetric and they commute with J. Moreover, the vector fields are well-defined and non-vanishing everywhere in S^3, and their integral curves are always great circles. This proves in particular that $\mathbb{G}_{\tilde{X}}$ and $\mathbb{G}_{\tilde{Y}}$ satisfy our hypotheses $(H1)$ and $(H2)$.

Let us still examine the case $n = 1$: we can describe more precisely which are the symmetries we are referring to.
The Lie algebra of $\mathbb{U}(2)$ is a four dimensional real vector space. Let us fix the following basis:

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Any matrix A in the Lie algebra gives rise to a linear isometry ϕ_t^A of S^3 in the following sense

$$\begin{cases} \dfrac{d}{dt}\phi_t^A(z) &= A\phi_t^A(z) \\ \\ \phi_0^A(z) &= z. \end{cases}$$

By the examples 2.7 and 2.8, the one-parameter groups generated by $J, \tilde{X}, \tilde{Y}$ satisfy hypotheses $(H1)$ and $(H2)$. One can prove that also for the group generated by $\tilde{J}$ our assumptions hold true.
Geometrically, the isometry ϕ_t^J is given by the integral curves of the Reeb vector field T and the functions which are constant along T are the ones considered in [16]; whereas the isometry $\phi_t^{\tilde{J}}$ is given by the integral curves of $\tilde{T}$ which is the Reeb vector field of the "dual" contact form $\tilde{\theta}$ (the one with $\tilde{J}$ as complex structure instead of J). We note that also a linear combination of J and $\tilde{J}$ still satisfies $(H1)$ and $(H2)$.
Finally, the isometries $\phi_t^{\tilde{X}}$ and $\phi_t^{\tilde{Y}}$ are given by the integral curves of the vector fields $\tilde{X}$ and $\tilde{Y}$ which are right-invariant with respect to the standard group structure in S^3. In particular they commute with the left-invariant vector fields X and Y: one can think that is exactly this commutation property that makes the functional I invariant under the action of the groups $\mathbb{G}_{\tilde{X}}$ and $\mathbb{G}_{\tilde{Y}}$.

3 Proof of the Theorem 2.2

The proof is based on the following lemma by Ambrosetti and Rabinowitz, which gives a condition on some particular subspaces of the space of variations on which it is allowed to perform the minimax argument; we will omit the proof (see Theorems 3.13 and 3.14 in [1]).

Lemma 3.1. *Let X be a closed subspace of $\mathcal{S}^1(S^{2n+1})$. Suppose that:*

(i) X is infinite-dimensional;

(ii) $I_{|X}$, the restriction of I on X, satisfies Palais-Smale on X.

Then $I_{|X}$ has a sequence of critical points $\{v_k\}$ in X, such that

$$\int_{S^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad as \quad k \to \infty.$$

Now, suppose we are given $\mathbb{G}$ such that $X_{\mathbb{G}}$ satisfies $(H1)$ and $(H2)$. In order to apply the previous Lemma we need to show that the restricted functional $I_{|X_{\mathbb{G}}}$ is Palais-Smale. We will argue by contradiction, namely: we will consider a general Palais-Smale sequence and, since there is a precise characterization for these last ones, we will see that if Palais-Smale is violated then bubbling occurs, and the concentration set is finite and discrete, therefore the hypothesis $(H2)$ and the invariance given by the group action will lead to a contradiction on the boundedness of the energy.
Hence, we have the following

Lemma 3.2. *Let $\mathbb{G}$ be a subgroup of $\mathbb{U}(n+1)$ that satisfies $(H2)$. Then $I_{|X_{\mathbb{G}}}$, the restriction of I on $X_{\mathbb{G}}$, satisfies the Palais-Smale compactness condition on $X_{\mathbb{G}}$.*

Proof. Let us first recall a general bubbling behavior of the Palais-Smale sequences (P-S) of the functional I, studied by Citti in [5]. Let $\{v_k\}$ be a $(\text{P-S})_c$ sequence, that is

$$I(v_k) \to c, \quad \text{and} \quad dI(v_k) \to 0, \quad \text{as } k \to \infty.$$

Then there exist $m \geq 0$, m sequences $z_k^j \to z_j \in S^{2n+1}$ (for $1 \leq j \leq m$), a sequence of numbers R_k^j converging to zero, and a solution $v_\infty \in \mathcal{S}^1(S^{2n+1})$, such that up to a subsequence

$$v_k = v_\infty + \sum_{j=1}^{m} v_{k,j} + o(1) \text{ in } \mathcal{S}^1(S^{2n+1}),$$

where, by (4)

$$u_{k,j} = \varphi v_{k,j}, \qquad u_{k,j} = (R_k^j)^{-\frac{Q-2}{2}} \phi_j u_j \circ \delta_{\frac{1}{R_k^j}} \circ \tau_{F(z_k^j)^{-1}}$$

with u_j a solution of

$$-\Delta_{\mathbb{H}} u_j = |u_j|^{\frac{4}{Q-2}} u_j \text{ in } \mathbb{H}^n$$

and ϕ_j is a cut-off function supported in $B_2(z_j)$ and equals to 1 in $B_1(z_j)$. Moreover,

$$I(v_k) = I(v_\infty) + \sum_{j=1}^{m} I(v_j) + o(1) \text{ as } k \to \infty. \tag{5}$$

In [5] the author proved this characterization result in an open set of $\mathbb{H}^n$: the same proof works for what we stated above. In fact, the proof in this case is

easier since here there is no boundary and hence in the blow-up procedure the case of the upper half space cannot occur.
The important claim for what we need is that the blow-up set

$$\Theta = \{z_j \in S^{2n+1}, \quad 1 \leq j \leq m\}$$

is discrete and finite. Now we are looking at the functional $I_{|X_{\mathbb{G}}}$, so we have that our (P-S) sequence is invariant under the action of $\mathbb{G}$ and this means that if $z \in \Theta$ is a concentration point, then the whole orbit of z would be, which is impossible under our assumption. In particular this would contradict the energy quantization (5).
Indeed, let us assume for the sake of simplicity that we have only one concentration point $z_0 \in \Theta$ and let $(g_i)_{1 \leq i \leq l}$ be l elements in $\mathbb{G}$: then $g_i \cdot z_0$ are also concentration points in Θ. In particular

$$c = \lim_{k \to \infty} I(v_k) = I(v_\infty) + \sum_{i=1}^{l} I(v_0) = I(v_\infty) + lI(v_0) \tag{6}$$

with v_0 the bubble concentrating at z_0. Now we notice that $I(v_0) \neq 0$ since from the equation satisfied by v_0 we have that

$$\int_{S^{2n+1}} |D_\theta v_0|^2 + c(n) v_0^2 = \int_{S^{2n+1}} |v_0|^{q^*}.$$

Therefore

$$I(v_0) = \Big(\frac{1}{2} - \frac{1}{q^*}\Big) \int_{S^{2n+1}} |v_0|^{q^*}$$

and this last quantity is different from zero if bubbling occurs.
Finally, since $\mathbb{G}$ satisfies the hypothesis $(H2)$, the orbit of z_0 has at least one accumulation point on the sphere, therefore Θ contains infinitely many points: hence, by letting $l \to \infty$ in (6), we get a contradiction. □

Now we will prove the main theorem.

Proof. (of Theorem 2.2)
Let us take any $\mathbb{G}$ subgroup of $\mathbb{U}(n+1)$ that satisfies assumptions $(H1)$ and $(H2)$: the examples in Section 2 provide the existence of a large class of such groups. By the previous lemma, we have that $I_{|X_{\mathbb{G}}}$ satisfies Palais-Smale on $X_{\mathbb{G}}$. Therefore $X_{\mathbb{G}}$ satisfies conditions (i) and (ii) in the lemma by Ambrosetti and Rabinowitz, so that $I_{|X_{\mathbb{G}}}$ has a sequence of critical points $\{v_k\}$ in $X_{\mathbb{G}}$, such that

$$\int_{S^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad \text{as} \quad k \to \infty.$$

On the other hand, by Remark (2.4) we have that the functional I is invariant under the action of $\mathbb{G}$. By the Principle of Symmetric Criticality [21], this implies that any critical point of the restriction $I_{|X_{\mathbb{G}}}$ is also a critical point of I on the whole space of variations $\mathcal{S}^1(S^{2n+1})$. This ends the proof. □

References

[1] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis 14 (1973), 349-381.

[2] A. Bahri, S. Chanillo, *The difference of topology at infinity in changing-sign Yamabe problems on S^3 (the case of two masses)*, Comm. Pure Appl. Math. 54 (2001), no. 4, 450-478.

[3] A. Bahri, Y. Xu, *Recent progress in conformal geometry*, ICP Advanced Texts in Mathematics, 1. Imperial College Press, London, 2007.

[4] L.A. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), no. 3, 271-297.

[5] G. Citti, *Semilinear Dirichlet Problem Involving Critical Exponent for the Kohn Laplacian*, Ann. Mat. Pura Appl. (4) 169 (1995), 375-392.

[6] M. del Pino, M. Musso, F. Pacard, A. Pistoia, *Torus action on S^n and sign changing solutions for conformally invariant equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 12 (2013), no. 1, 209-237.

[7] M. del Pino, M. Musso, F. Pacard, A. Pistoia, *Large energy entire solutions for the Yamabe equation*, J. Differential Equations 251 (2011), no. 9, 2568-2597.

[8] W.Y. Ding, *On a conformally invariant elliptic equation on $\mathbb{R}^n$*, Comm. Math. Phys. 107 (1986), no. 2, 331-335.

[9] G.B. Folland, E.M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. 27 (1974), 429-522.

[10] N. Garofalo, E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. 41 (1992), no. 1, 71-98.

[11] N. Hirano, *Multiplicity of solutions for nonhomogeneous nonlinear elliptic equations with critical exponents*, Topol. Methods Nonlinear Anal. 18 (2001), 269-281.

[12] T. Isobe, *A perturbation method for spinorial Yamabe type equations on S^m and its application*, Math. Ann. 355 (2013), no. 4, 1255-1299.

[13] D. Jerison, J.M. Lee, *The Yamabe problem on CR manifolds*, J. Differential Geom. 25 (1987), no. 2, 167-197.

[14] D. Jerison, J.M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. 1 (1988), no. 1, 1-13.

[15] A. Maalaoui, *Infinitely many solutions for the Spinorial Yamabe Problem on the Round Sphere*, preprint

[16] A. Maalaoui, V. Martino, *Changing-sign solutions for the CR-Yamabe equation*, Differential Integral Equations 25 (2012), no. 7-8, 601-609.

[17] A. Maalaoui, V. Martino, *Existence and Concentration of Positive Solutions for a Super-critical Fourth Order Equation*, Nonlinear Anal. 75 (2012), 5482-5498

[18] A. Maalaoui, V. Martino, *Existence and Multiplicity Results for a non-Homogeneous Fourth Order Equation*, Topol. Methods Nonlinear Anal. 40 (2012), no. 2, 273-300

[19] A. Maalaoui, V. Martino, *Multiplicity result for a nonhomogeneous Yamabe type equation involving the Kohn Laplacian*, J. Math. Anal. Appl. 399 (2013), no. 1, 333-339

[20] A. Maalaoui, V. Martino, A. Pistoia, *Concentrating Solutions for a Sub-Critical Sub-Elliptic Problem*, Differential and Integral Equations 26 (2013), no. 11-12, 1263-1274

[21] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. 69 (1979), no. 1, 19-30.

[22] N. Saintier, *Changing sign solutions of a conformally invariant fourth-order equation in the Euclidean space*, Comm. Anal. Geom. 14 (2006), no. 4, 613-624.