On the Minkowski formula for hypersurfaces in complex space forms

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Abstract In this paper we discuss various Minkowski-type formulas for real hypersurfaces in complex space forms. In particular, we investigate the formulas suggested by the natural splitting of the tangent space. In this direction, our main result concerns a new kind of *second* Minkowski formula.

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1 Introduction

In this paper we establish some Minkowski formulas for general real closed hypersurfaces M embedded in complex space forms.
Minkowski type formulas are widely studied in literature and they have a very large number of applications (see for instance [11, 12, 13, 18, 20, 25, 28, 29, 31, 32, 35, 39]). Typically one considers a *real* space form equipped with a conformal vector field as ambient space. If (K_c, g) is a $(n+1)$-dimensional Riemannian manifold of constant sectional curvature c, one can in fact take the standard position vector $P = s_c(r)\nabla r$ as conformal vector field, namely $\nabla_X P = c_c(r)X$ for $X \in TK_c$; here r denotes the geodesic distance from a given base-point in K_c and the functions $s_c(r)$ and $c_c(r)$ are defined as follows:

$$s_c(r) = \begin{cases} \dfrac{1}{\sqrt{c}}\sin(\sqrt{c}\,r), & c>0 \\ r, & c=0 \\ \dfrac{1}{\sqrt{-c}}\sinh(\sqrt{-c}\,r), & c<0 \end{cases}, \qquad c_c(r) = \begin{cases} \cos(\sqrt{c}\,r), & c>0 \\ 1, & c=0 \\ \cosh(\sqrt{-c}\,r), & c<0 \end{cases}.$$

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Therefore, if M is a closed hypersurface embedded in K_c with ν as unit (outward) normal, the classical Hsiung-Minkowski [11, 12, 34] formulas read as:

$$\int_M c_c(r)\sigma_{k-1}^M - g(P,\nu)\sigma_k^M = 0, \qquad k = 1,\dots,n, \tag{1}$$

where $\sigma_0^M = 1$ and we let, for $k \geq 1$, σ_k^M denote the normalized k-th curvature of M (we will be more precise about such definitions later on).
The situation changes drastically when we consider a complex space form as ambient space. Moreover, the complex structure induces a splitting of the tangent space of the hypersurface, and this leads to consider separately the Levi *horizontal* curvatures and the *vertical* one.
A *first* ($k = 1$) *horizontal* Minkowski formula, i.e. involving the Levi mean curvature, has been proved by Miquel [25] for compact Hopf hypersurfaces, which are very special submanifolds. If M is not Hopf, such a formula is not exact, in the sense that there is a reminder term which is non vanishing in general: as we will show, this term depends strongly on some mixed coefficients of the second fundamental form. Being Hopf is only a sufficient condition for the formula to hold. We will prove, to this aim, that any closed hypersurface in $\mathbb{R}^4$ can be isometrically embedded in $\mathbb{C}^2$ so that the horizontal formula holds; moreover we will show in the Appendix an explicit class of non Hopf hypersurfaces satisfying the first horizontal Minkowski formula.
However, our main result concerns the *second* ($k = 2$) Minkowski formula. Firstly, we will write it in full generality and then in a horizontal version, analogous to the Miquel's one for Hopf hypersurfaces: even in this case we will show in the Appendix some explicit examples of non Hopf hypersurfaces satisfying such second horizontal formula. Finally, we will prove another kind of second Minkowski formula, which is our main theorem indeed, and we will show that, conversely to what happens in the previous situations, it is independent of the mixed terms of the second fundamental form and therefore it is valid for any closed hypersurface embedded in a complex space form. To the best of our knowledge, this formula is new even in the simpler case, namely when considering $\mathbb{C}^{n+1}$ as ambient space.
More recently, further developments of the present study have been carried out in [9], where it is addressed the case $k = 2n$.

1.1 Definitions and statement of the results

Since it requires no extra work, in order to introduce the notations let us briefly recall some basic facts about hypersurfaces in general Kähler manifold. Then we will give the precise statements of the results for the particular case of a complex space form.
Let us denote by K a Kähler manifold of real dimension $2n+2$. We denote by J the complex structure and g the Riemannian metric that are compatible in the following sense:

$$\omega(X,Y) = g(X,JY)$$

for every pair of vector fields $X, Y \in TK$, where ω is the fundamental symplectic 2-form of K. We also denote by ∇ the Levi-Civita connection of K and we recall that both ∇ and g are compatible with the complex structure J, i.e.

$$J\nabla = \nabla J, \qquad g(\cdot,\cdot) = g(J\cdot, J\cdot). \tag{2}$$

We will consider a smooth real orientable and connected embedded manifold M of codimension 1 on K with induced metric g. We will suppose that M is closed, i.e. compact without boundary. We denote by ν the (outward) unit normal to M and by X_0 the characteristic vector field of M, which is defined by $JX_0 = \nu$.
The horizontal distribution or Levi distribution HM is the $2n$-dimensional subspace in TM which is invariant under the action of J:

$$HM = TM \cap JTM,$$

that is a vector field $X \in TM$ belongs to HM if and only if also $JX \in HM$. Then TM splits in the orthogonal direct sum:

$$TM = HM \oplus \mathbb{R}X_0.$$

In addition, we denote by φ the endomorphism

$$JX = \varphi X + g(X, X_0)\nu \qquad \text{for } X \in TM.$$

Now, let A be the Weingarten or shape operator, namely

$$A : TM \to TM, \quad AX := \nabla_X \nu.$$

The Second Fundamental Form of M is defined by

$$h(\cdot,\cdot) := g(A\cdot, \cdot).$$

We recall that the induced connection ∇^M satisfies:

$$\nabla_U V = \nabla^M_U V - h(U, V)\nu \tag{3}$$

for any $U, V \in TM$. We will also need the horizontal part of the second fundamental form, namely:

$$h^H(U, V) = h(U, V), \qquad \forall\, U, V \in HM.$$

The Levi form ℓ can be defined on HM in the following way: for every $X, Y \in HM$, if $Z = \frac{1}{\sqrt{2}}(X - iJX)$ and $W = \frac{1}{\sqrt{2}}(Y - iJY)$, then $\ell(Z, \overline{W}) := \langle \nabla_Z \nu, \overline{W} \rangle$. We can compare the Levi form with the Second Fundamental Form by using the following identity (see [3, Chap.10, Theorem 2]):

$$\forall X \in HM, \quad \ell(Z, \overline{Z}) = \frac{h(X, X) + h(JX, JX)}{2}. \tag{4}$$

We are going to use also some standard complex notation, namely:

$$T_{1,0}M := T^{1,0}K \cap T^{\mathbb{C}}M \qquad \text{and} \qquad T_{0,1}M := \overline{T_{1,0}M},$$

where $T^{1,0}K$ is the holomorphic tangent space of K (i.e. the complex n-dimensional subspace generated by the eigenvalue $+i$ of J) and $T^{\mathbb{C}}M$ is the complexified tangent space of M. Moreover we set

$$H^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M \quad \text{ and we have } \quad T^{\mathbb{C}}M = H^{\mathbb{C}}M \oplus \mathbb{C}X_0.$$

We will still denote by the same symbols the metric, the complex structure, etc., that we will extend by $\mathbb{C}$-linearity, as no ambiguity will occur. Therefore, in complex notations, the Levi form is then the hermitian operator

$$\ell(Z, \overline{W}) := \langle \nabla_Z \nu, \overline{W} \rangle,$$

for any couple of vector fields $Z, W \in T_{1,0}M$. M is said strictly Levi-convex if ℓ is strictly positive definite as quadratic form. We can also extend the Levi form to the whole $H^{\mathbb{C}}M$ by setting

$$\ell(Z, W) := g(\nabla_Z \nu, W) \qquad \text{for } Z, W \in H^{\mathbb{C}}M$$

and we will refer to the symmetric part of the Levi form if both Z and W belong to $T_{1,0}M$ (or both to $T_{0,1}M$).

Now, for any $N \times N$ symmetric (or Hermitian) matrix A and for any $k = 1, \ldots, N$, we denote by $\sigma_k(A)$ the normalized k-th elementary symmetric function of the eigenvalues of A, that is:

$$\sigma_k(A) = \sigma_k(\lambda_1, \ldots, \lambda_N) := \frac{1}{\binom{N}{k}} \sum_{1 \le i_1 < \ldots < i_k \le N} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of A; we also put $\sigma_0(A) = 1$. Thus we can denote:

$$\begin{aligned} \sigma_k^M &:= \sigma_k(h), \qquad k = 0, 1, \ldots, 2n+1; \\ \sigma_k^H &:= \sigma_k(h^H), \qquad k = 0, 1, \ldots, 2n; \\ \sigma_k^\ell &:= \sigma_k(\ell), \qquad k = 0, 1, \ldots, n. \end{aligned}$$

We will simply denote by

$$H = \sigma_1^M = \frac{\mathrm{trace}(h)}{2n+1}, \qquad L = \sigma_1^\ell = \frac{\mathrm{trace}(\ell)}{n}$$

respectively the classical mean curvature and Levi mean curvature of M. The Levi curvature is a sort of degenerate-elliptic analogue of the classical curvature: it was introduced and studied in [1, 6]. The restriction of the second fundamental form to the holomorphic tangent space involves a lack of information and hence a lack of ellipticity in the relative operator. Under a suitable non-flatness condition, the Levi operator can be seen as a degenerate-elliptic operator of sub-Riemannian type: for further properties and results we address the reader to [5, 22, 23, 26], and the references therein.

Denoting by $\alpha := h(X_0, X_0)$ the vertical or characteristic curvature of M, we get from (4)

$$\begin{aligned} L &= \sigma_1^H = \frac{\operatorname{trace}(h^H)}{2n} \qquad \text{and then} \\ (2n+1)H &= 2nL + \alpha. \end{aligned} \tag{5}$$

Now let us turn our attention to complex space forms. It is known that the models for such manifolds are the standard complex space $\mathbb{C}^{n+1}$ endowed with the standard hermitian metric, the complex projective space $\mathbb{C}P^{n+1}$ with the Fubini-Study metric, and the complex hyperbolic space $\mathbb{C}H^{n+1}$ with the Bergman metric (see for instance [16]). These three prototypes differ in the sign of the (constant) holomorphic sectional curvature (respectively zero, positive, and negative). The reason why we consider the ambient space with constant holomorphic sectional curvature relies on the fact that we are going to use the geodesic distance and the formula for this last one is explicit in these settings. Moreover, also the Codazzi equations become considerably simpler in this situation. Thus, from now on, we assume K_c is a complex space form of real dimension $2n+2$ namely a Kähler manifold with constant holomorphic sectional curvature $4c$, endowed with its aforementioned standard metric (see also [30]).
Let us fix a point in K_c, that we assume as origin, and let as denote by $r(p) = r$ the geodesic distance of any given point $p \in K_c$ from the origin. Next, we need to distinguish the sign of the curvature c. We can handle the three different cases all at once with the help of a *potential* function as follows: we define

$$\psi(r) = \begin{cases} -\dfrac{1}{c}\log\left(\cos(\sqrt{c}\,r)\right), & c > 0 \\[2ex] \dfrac{r^2}{2}, & c = 0 \\[2ex] -\dfrac{1}{c}\log\left(\cosh(\sqrt{-c}\,r)\right), & c < 0. \end{cases}$$

Remark 1.1. *Let us just observe that in the case of $\mathbb{C}P^{n+1}$, i.e. when c is positive, one needs to require the domain of ψ to be contained in the geodesic ball of center the origin and radius smaller that $\pi/\sqrt{4c}$ in order to avoid conjugate points and to ensure that the function ψ be smooth. Without further comments, in the case $c > 0$ we will always tacitly assume the hypersurface M to be contained in the same geodesic ball.*

So, we can define the position vector field $P \in TK_c$ in the following way:

$$P = \nabla\psi = \psi'\nabla r, \tag{6}$$

where ∇r denotes the gradient of r.
In what follows, we will always consider an orthonormal frame for TM of the form

$E := \{X_0, X_k, X_{n+k},\ k = 1, \dots, n\}$, where $X_k \in HM$ is a unit vector field and $X_{n+k} = JX_k$. Hence, we can write the position vector P at a point $p \in M$ as

$$P = \sum_{k=0}^{2n} a_k X_k + \lambda \nu$$

for some smooth functions a_k, with $k = 0, 1, \dots, 2n$. Here we have also denoted by λ the support function

$$\lambda(p) = g(P, \nu), \quad p \in M.$$

We will need also

$$P^T = P - \lambda \nu, \qquad P^H = P - a_0 X_0 - \lambda \nu,$$

being respectively the tangential and the horizontal part of the position vector P. With all these notations, we are in position to write the first Minkowski formula for general hypersurfaces:

Proposition 1.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$\int_M 1 - H\lambda = \frac{c}{2n+1} \int_M (\lambda^2 - a_0^2).$$

The difference with respect to the real case in (1) is apparent. First of all, we notice the different behaviour of the formula with respect to the curvature c, and the dependence on the function a_0 (other than just λ). We also stress that it does not depend on the horizontal part of P.
As we mentioned, Miquel proved in [25] a horizontal version of this formula for Hopf hypersurfaces. We recall here the definition.

Definition 1.1. *A real hypersurface M in a Kähler manifold K is said to be a Hopf hypersurface if the characteristic vector field X_0 is an eigenvector for the shape operator A.*

We address the reader to the papers [2, 4, 10, 14, 17, 19, 24, 27, 30, 36, 37, 38] concerning Hopf hypersurfaces and their classification. However, with our notations this is exactly equivalent to say $h(X_0, X_j) = 0$ for all $j = 1, \dots, 2n$. Moreover, the vanishing of the coefficients $h(X_0, X_j)$ makes the splitting $TM = HM \oplus \mathbb{R}X_0$ orthogonal with respect to h.
If one tries to write such horizontal formula for general hypersurfaces (not just the Hopf ones), it is possible to recognize a suitable combination of the terms $h(X_0, X_j)$ as remainder term (see Proposition 2.1 below). In particular, we can recover the Miquel's formula, namely

if M is a closed embedded real Hopf hypersurface in a complex space form K_c, then it holds

$$\int_M 1 - L\lambda = 0.$$

Let us now turn the attention to the second Minkowski formula. It will also depend on the horizontal part of P. In order to write the formula, we need to define the following terms:

$$
\begin{aligned}
\Theta &= \frac{1}{2n+1}\Big(h(P^T, P^T) - h((JP)^T, (JP)^T)\Big), \\
\Theta^H &= \frac{1}{2n}\Big(h(P^H, P^H) - h(JP^H, JP^H)\Big).
\end{aligned}
$$

Remark 1.2. *Let us explicitly notice that the term Θ^H identically vanishes if one assume that $T_{1,0}M$ is H-parallel, i.e. $\nabla_Z W$ is tangent to M for any $Z, W \in T_{1,0}M$. Equivalently, this means that the symmetric part of the Levi form vanishes. Conditions of this kind could be found for instance in [8, 15, 21, 30]. As an example, the ellipsoids in $\mathbb{C}^2$ of the type $M = \{f = 0\}$ where*

$$
f(z_1, z_2) = A|z_1|^2 + B|z_2|^2 - 1,
$$

with $A, B > 0$, satisfy the previous condition; in particular they are not Hopf, unless $A = B$.

The second Minkowski formula reads in full generality as:

Proposition 1.2. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$
\int_M H - \sigma_2^M \lambda = \frac{c}{2n} \int_M \Big(H\left(\lambda^2 - a_0^2\right) + \Theta\Big).
$$

Even in this case, it is possible to write down horizontal versions of it (see Proposition 3.1 below). In particular, if one assume as in Miquel's formula M to be Hopf, the vanishing of the coefficients $h(X_0, X_j)$ makes the following formula (see also Corollary 3.1) hold true:

if M is a closed embedded real Hopf hypersurface in a complex space form K_c, then

$$
\int_M L - \sigma_2^H \lambda = \frac{c}{2n-1} \int_M \Theta^H.
$$

However, we will show as our main result that it is also possible to take a calibrated convex combination of (L, α) and $(\sigma_k^H, \sigma_k^\ell)$ so that a new kind of Minkowski type formula holds. We denote

$$
\mathcal{H} := \frac{nL + \alpha}{n+1}
$$

$$
\mathcal{S}_2 := \frac{2(2n-1)}{n+1}\sigma_2^H - \frac{3(n-1)}{n+1}\sigma_2^\ell
$$

We explicitly note that these terms are normalized, in the sense that, if the second fundamental form h were the identity, then we would have exactly $\mathcal{H} = \mathcal{S}_2 = 1$. We also recall that there are no umbilical hypersurfaces in complex space forms with curvature $c \neq 0$.

Therefore, we have the following formula valid for general hypersurfaces and, at the same time, not depending on the mixed coefficients $h(X_0, X_j)$.

Theorem 1.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$\int_M \mathcal{H} - \mathcal{S}_2\lambda = \frac{c}{2n+2}\int_M L(\lambda^2 - a_0^2) + \Theta^H - 3\lambda.$$

Remark 1.3. *We want to stress that, even in the flat case ($c = 0$), the Minkowski-type formula in Theorem 1.1 differs in a significant way from the classical one in Proposition 1.2.*

Moreover, the formula considerably simplifies in $\mathbb{C}^2$, where $\mathcal{S}_2 = \sigma_2^H$ is nothing but the determinant of the horizontal part of the second fundamental form.

Corollary 1.1. *Let M be a closed three-dimensional hypersurface in $\mathbb{C}^2$. Then:*

$$\int_M L + \alpha - 2\sigma_2^H\lambda = 0.$$

2 First Minkowski formulas

We denote by

$$\Gamma_{jk}^l = \left\langle \nabla_{X_j} X_k, X_l \right\rangle, \quad \text{and} \quad h_{jk} = h(X_j, X_k), \qquad j,k,l = 0,1,\ldots,2n$$

respectively the coefficients of the Levi-Civita connection and the coefficients of the Second Fundamental Form with respect to the frame E. We recall that

$$\Gamma_{ij}^k = -\Gamma_{ik}^j, \qquad \text{for any } i,j,k = 0,1,\ldots,2n, \tag{7}$$

and also, by the compatibility (2) with the complex structure J,

$$\begin{aligned} \Gamma_{0k}^0 &= \langle \nabla_{X_0} X_k, X_0\rangle = \langle \nabla_{X_0} X_{n+k}, \nu\rangle = -h_{0n+k} \\ \Gamma_{0n+k}^0 &= \langle \nabla_{X_0} X_{n+k}, X_0\rangle = -\langle \nabla_{X_0} X_k, \nu\rangle = h_{0k}, \qquad \text{for any } k = 1,\ldots,n. \end{aligned} \tag{8}$$

By the very definition of geodesic distance we have

$$g(\nabla r, \nabla r) = 1 \qquad \text{and} \qquad \nabla_{\nabla r}\nabla r = 0.$$

Moreover, by a straightforward computation we get:

$$\nabla_{J\nabla r}\nabla r = \frac{1 - c(\psi')^2}{\psi'} J\nabla r,$$

and, for any vector field $V \in TK_c$ such that $g(V, \nabla r) = g(V, J\nabla r) = 0$,

$$\nabla_V \nabla r = \frac{1}{\psi'} V.$$

Finally, by putting together the last three identities with (6), we have

$$\nabla_V P = V + cg(V,P)P - cg(V,JP)JP, \tag{9}$$

for any vector field $V \in TK_c$.
Let us now recall the Codazzi equations for a real hypersurface M in K_c (see [16, 30]): for all $U, V, W \in TM$ we have

$$(\nabla_U h)(V,W) - (\nabla_V h)(U,W) = \tag{10}$$

$$= c\Big(g(V,X_0)g(\varphi(U),W) - g(U,X_0)g(\varphi(V),W) - 2g(U,\varphi(V))g(X_0,W)\Big)$$

where we have denoted the Bortolotti derivative by

$$(\nabla_U h)(V,W) = U(h(V,W)) - h(\nabla_U V,W) - h(V,\nabla_U W).$$

The aim of this section is to obtain the first Minkowski formula for a general real hypersurface M in a complex space form K_c, and its horizontal versions. The idea to obtain these formulas is the same as in the euclidean setting: we need to take the laplacian of the potential function ψ, where the Laplace-Beltrami Δ acting on a function u is the second order operator in divergence form defined by

$$\Delta u = \sum_{j=0}^{2n} \Big(X_j X_j u - (\nabla^M_{X_j} X_j) u \Big).$$

The derivatives of ψ along the vector fields of the basis E are given by

$$X_k(\psi) = g(X_k, \nabla\psi) = a_k, \quad k = 0, 1, \ldots, 2n.$$

Here and in the next section, we will also need the derivatives of the functions λ and a_k.

Lemma 2.1. *In our notations we have*

$$\begin{aligned}
X_0(\lambda) &= 2ca_0\lambda + \sum_{j=0}^{2n} h_{0j} a_j, \\
X_k(\lambda) &= c(a_k\lambda + a_{n+k}a_0) + \sum_{j=0}^{2n} h_{kj} a_j, \\
X_{n+k}(\lambda) &= c(a_{n+k}\lambda - a_k a_0) + \sum_{j=0}^{2n} h_{n+kj} a_j, \\
X_0(a_0) &= 1 + c(a_0^2 - \lambda^2) - \alpha\lambda + \sum_{j=1}^{n} h_{0n+j} a_j - h_{0j} a_{n+j}, \\
X_k(a_0) &= c(a_k a_0 - a_{n+k}\lambda) - h_{0k}\lambda + \sum_{j=1}^{n} h_{kn+j} a_j - h_{kj} a_{n+j}, \\
X_{n+k}(a_0) &= c(a_{n+k}a_0 + a_k\lambda) - h_{0n+k}\lambda + \sum_{j=1}^{n} h_{n+kn+j} a_j - h_{n+kj} a_{n+j},
\end{aligned}$$

$$
\begin{aligned}
X_0(a_l) &= c(a_l a_0 - a_{n+l}\lambda) - h_{0l}\lambda - h_{0n+l}a_0 + \sum_{j=1}^{2n} \Gamma_{0l}^j a_j, \\
X_k(a_l) &= \delta_{kl} + c(a_k a_l - a_{n+k}a_{n+l}) - h_{kl}\lambda - h_{kn+l}a_0 + \sum_{j=1}^{2n} \Gamma_{kl}^j a_j, \\
X_{n+k}(a_l) &= c(a_{n+k}a_l + a_k a_{n+l}) - h_{n+kl}\lambda - h_{n+kn+l}a_0 + \sum_{j=1}^{2n} \Gamma_{n+kl}^j a_j, \\
X_0(a_{n+l}) &= c(a_{n+l}a_0 + a_l\lambda) - h_{0n+l}\lambda + h_{0l}a_0 + \sum_{j=1}^{2n} \Gamma_{0n+l}^j a_j, \\
X_k(a_{n+l}) &= c(a_{n+k}a_l + a_k a_{n+l}) - h_{kn+l}\lambda + h_{kl}a_0 + \sum_{j=1}^{2n} \Gamma_{kn+l}^j a_j, \\
X_{n+k}(a_{n+l}) &= \delta_{n+kn+l} - c(a_k a_l - a_{n+k}a_{n+l}) - h_{n+kn+l}\lambda + h_{n+kl}a_0 + \sum_{j=1}^{2n} \Gamma_{n+kn+l}^j a_j,
\end{aligned}
$$

for any $k, l = 1, \ldots, n$.

Proof. It follows by direct computation, by using (9), (2), and the symmetries of the coefficients of the Levi-Civita connection in (7),(8). □

It is now easy to deduce the first Minkowski formula.

Proposition 1.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$\int_M 1 - H\lambda = \frac{c}{2n+1} \int_M (\lambda^2 - a_0^2). \tag{11}$$

Proof. From (3) and Lemma 2.1, we can compute the laplacian of the potential function ψ and we get

$$\Delta\psi = \sum_{j=0}^{2n} \left(X_j X_j \psi - (\nabla^M_{X_j} X_j)\psi \right) = 2n + 1 + c(a_0^2 - \lambda^2) - (2n+1)H\lambda. \tag{12}$$

The desired formula is then provided by the divergence theorem, since M is closed and Δ is in divergence form. □

Let us now deal with the *horizontal* Minkowski formula. In order to do that, we need to have a second order subelliptic operator in analogy to the Laplace-Beltrami operator.

Definition 2.1. *We define the horizontal Laplacian or sublaplacian Δ_H on M the operator acting on a function u in the following way:*

$$\Delta_H u = \Delta u - X_0^2\, u.$$

Let us explicitly notice that Δ_H is in divergence form. An easy way to see it is by observing that characteristic vector field X_0 is always divergence free, in fact:

$$\begin{aligned}\mathrm{div} X_0 &= \sum_{j=0}^{2n} g(\nabla^M_{X_j} X_0, X_j) = \sum_{j=1}^{n} g(\nabla^M_{X_j} X_0, X_j) + g(\nabla^M_{X_{n+j}} X_0, X_{n+j}) \\ &= \sum_{j=1}^{n} g(\nabla_{X_j}\nu, X_{n+j}) - g(\nabla_{X_{n+j}}\nu, X_j) = 0, \end{aligned} \tag{13}$$

therefore the operator Δ_H is in divergence form too.
Let us also remark that in literature other definitions of sublaplacians could be found. For instance, one can consider the trace of the horizontal Hessian of a function u:

$$Hess_H(u)(X,Y) = XYu - (\nabla^M_X Y)u, \qquad \forall\, X, Y \in HM.$$

It is worth to notice that the two definitions do not coincide as in the Riemannian setting; the trace of the horizontal Hessian is indeed not in divergence form in general.
For the sake of convenience, let us denote by

$$Q = \sum_{j=1}^{n} h_{0n+j} a_j - h_{0j} a_{n+j} \tag{14}$$

so that we can write

$$X_0^2 \psi = X_0(a_0) = 1 + c(a_0^2 - \lambda^2) - \alpha\lambda + Q$$

and we can deduce the following horizontal first Minkowski formula:

Proposition 2.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$2n \int_M 1 - L\lambda = \int_M Q \tag{15}$$

Proof. It follows from the previous computation for $X_0^2\psi$, from (12), and (5). □

We now introduce an horizontal vector field that will have a crucial role in the next section, namely $V = \nabla^M_{X_0} X_0$. A simple computation shows that $V\psi = Q$, in fact we can write V as follows:

$$\begin{aligned} V &= \sum_{j=0}^{2n} g(\nabla^M_{X_0} X_0, X_j) X_j = \sum_{j=1}^{n} g(\nabla^M_{X_0} X_0, X_j) X_j + g(\nabla^M_{X_0} X_0, X_{n+j}) X_{n+j} = \\ &= \sum_{j=1}^{n} g(\nabla_{X_0}\nu, X_{n+j}) X_j - g(\nabla_{X_0}\nu, X_j) X_{n+j} = \sum_{j=1}^{n} h_{0n+j} X_j - h_{0j} X_{n+j}, \end{aligned} \tag{16}$$

and the claim follows. Therefore, by mean of the divergence theorem, we can also express the remainder term in (15) by using the divergence of V:

$$\int_M Q = -\int_M \psi \mathrm{div} V.$$

We see that the term Q appearing in the formula depends linearly on the mixed term of the second fundamental form of the type h_{0j}, with $j = 1, \ldots, 2n$. Thus, if we require M to be Hopf, then Q is identically 0 and we recover from (15) the formula obtained by Miquel in [25], i.e.

$$\int_M 1 - L\lambda = 0. \tag{17}$$

However, being Hopf is only a sufficient condition for (17) to hold. Here we will show some explicit examples of hypersurface that are not Hopf, but for which (17) still holds true.
We recall that in $\mathbb{C}^{n+1}$ being Hopf is extremely restrictive: the spheres are in fact the only compact Hopf hypersurfaces (see, e.g., [7]).

Example 2.1. *Let us consider in $\mathbb{C}^2 \simeq \mathbb{R}^4$ the ellipsoids $M = \{f = 0\}$ of the form*

$$f(z_1, z_2) = A\left(x_1^2 + (\frac{1}{\sqrt{3}}y_1 + \frac{\sqrt{2}}{\sqrt{3}}x_2)^2\right) + B\left((-\frac{\sqrt{2}}{\sqrt{3}}y_1 + \frac{1}{\sqrt{3}}x_2)^2 + y_2^2\right) - 1,$$

with $A, B > 0$, $A \neq B$. Denote $s_1 = x_1^2 + \eta_1^2 = x_1^2 + (cy_1 + sx_2)^2$ and $s_2 = \eta_2^2 + y_2^2 = (-sy_1 + cx_2)^2 + y_2^2$. We have

$$\lambda = \langle (x_1, y_1, x_2, y_2), \nu \rangle = \frac{As_1 + Bs_2}{(A^2 s_1 + B^2 s_2)^{\frac{1}{2}}},$$

$$H = \frac{1}{3}\frac{A^3 s_1 + B^3 s_2 + 2A^2 B s_1 + 2AB^2 s_2}{(A^2 s_1 + B^2 s_2)^{\frac{3}{2}}},$$

$$L = H + \frac{\sqrt{2}}{3}\frac{AB(A-B)}{(A^2 s_1 + B^2 s_2)^{\frac{3}{2}}}(x_1 y_2 + \eta_1 \eta_2).$$

Since the term $(x_1 y_2 + \eta_1 \eta_2)$ is odd with respect to the symmetry $(x_1, \eta_1, \eta_2, y_2) \mapsto (-x_1, -\eta_1, \eta_2, y_2)$, we get

$$\int_M L\lambda = \int_M H\lambda = \int_M 1$$

which is (17)*.*

This is just a particular case of a more general behavior described by the following:

Theorem 2.1. *For any smooth compact hypersurface M in $\mathbb{R}^4 \simeq \mathbb{C}^2$, there exists an isometric embedding (actually, a rotation not preserving the complex structure) for which we have*

$$\int_M 1 - L\lambda = 0.$$

Proof. Given a smooth compact hypersurface $M \subset \mathbb{C}^2$, suppose that $\int_M 1 - L\lambda \neq 0$. Let us fix the orthonormal frame for TM of the usual form $E = \{X_0, X_1, X_2\}$, where X_0 denotes the characteristic vector field, and X_1, X_2 are the vector fields in (26). In this way we can make explicit and simpler computations. We can assume that $|\int_M 1 - h_{11}\lambda| \geq |\int_M 1 - h_{22}\lambda|$. Thus we have

$$\operatorname{sgn}\left(\int_M 1 - L\lambda\right) = \operatorname{sgn}\left(\int_M 1 - h_{11}\lambda\right). \tag{18}$$

We have to perform rigid transformations in $\mathbb{R}^4$ which are going to change the complex structure. We want to apply the rotations given by

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for $\theta \in \mathbb{R}$. The fact that it does not preserve the complex structure can be seen from the non-commutation of R_θ and the standard complex structure J of $\mathbb{C}^2$ (if $\sin(\theta) \neq 0$). Let us consider

$$I(\theta) = \int_{M^\theta} 2 - 2L^\theta \lambda^\theta,$$

where M^θ and L^θ denote the manifold and its Levi curvature under the change. We want to rewrite $I(\theta)$ as an integral on the initial manifold M. This is a straightforward computation: see (27) in the Appendix for the details. We obtain

$$\begin{aligned} I(\theta) &= \cos^2(\theta)\int_M 1 - h_{11}\lambda + \int_M 1 - h_{22}\lambda + \\ &+ \sin^2(\theta)\int_M 1 - \alpha\lambda - 2\cos(\theta)\sin(\theta)\int_M h_{01}\lambda. \end{aligned}$$

By using the classical Minkowski formula ($c = 0$ and $n = 1$ in (11)), we get

$$\begin{aligned} I(\theta) &= (\cos^2(\theta) - \sin^2(\theta))\int_M 1 - h_{11}\lambda + \\ &+ \cos^2(\theta)\int_M 1 - h_{22}\lambda - 2\cos(\theta)\sin(\theta)\int_M h_{01}\lambda. \end{aligned}$$

Hence, by (18), $I(\theta)$ assumes opposite signs at $\theta = 0$ and $\theta = \frac{\pi}{2}$. There exists then a $\theta = \theta_0$ for which $I(\theta_0) = 0$. In the case $|\int_M 1 - h_{11}\lambda| < |\int_M 1 - h_{22}\lambda|$, we can apply instead of R_θ (which basically mixes X_0 and X_1) the rotations involving X_0 and X_2. This proves the statement. $\square$

In general the rotation we need in order to have the horizontal Minkowski formula (17) depends on the initial manifold M. We will see in the Appendix that it is independent of the manifold in the case special symmetries, i.e. if M is the boundary of a Reinhardt domain in $\mathbb{C}^2$.

3 Second Minkowski formulas

In this section we will write several second Minkowski formulas for M, both in classical and horizontal form. We will show at the end a special horizontal version, which is independent of the mixed terms of the type h_{0j}, with $j = 1, \ldots, 2n$. Since the formulas of horizontal type involve σ_2^H, the following algebraic relations are going to be useful

$$\begin{aligned} 2n(2n-1)\sigma_2^H &= 4n^2L^2 - \sum_{j,l=1}^{2n} h_{jl}^2 \\ n(2n+1)\sigma_2^M &= n(2n-1)\sigma_2^H + 2nL\alpha - |V|^2, \end{aligned} \tag{19}$$

where we have denoted by

$$|V|^2 = g(V, V) = \sum_{j=1}^{2n} h_{0j}^2$$

because of the expression of $V = \nabla_{X_0}^M X_0$ in (16).
The idea for obtaining the desired formulas is again the same as in the classical case. We will in fact take the divergence of a suitable vector field and then we will integrate over M (see for instance [33]). The vector field is the following:

$$W = \sum_{j=0}^{2n} \left(\sum_{l=0}^{2n} h_{lj}a_l - h_{ll}a_j \right) X_j.$$

For future computations, it is convenient to split W into the sum of three vector fields. Let

$$\begin{aligned} W_1 &= a_0 \sum_{j=1}^{2n} h_{0j}X_j - 2nLa_0X_0, \\ W_2 &= \left(\sum_{j=1}^{2n} h_{0j}a_j \right) X_0 - \alpha \sum_{j=1}^{2n} a_jX_j, \\ W_3 &= \sum_{j=1}^{2n} \left(\sum_{l=1}^{2n} h_{lj}a_l - h_{ll}a_j \right) X_j, \end{aligned}$$

so that $W = W_1 + W_2 + W_3$. We want to compute the divergences separately.
First, recalling the definition of Q in (14), we denote

$$Q_J = \sum_{j=1}^{2n} h_{0j}a_j.$$

By using (2), (7), (8), (13), and the relations in Lemma 2.1, we get

$$\begin{aligned}
\mathrm{div}(W_2) &= X_0\left(\sum_{j=1}^{2n} h_{0j}a_j\right) - \sum_{j=1}^{2n} X_j(\alpha a_j) - \alpha\sum_{j=1}^{2n} a_j \mathrm{div}(X_j) \\
&= \sum_{l=1}^{2n} a_l \left(X_0(h_{0l}) - X_l(\alpha)\right) + \alpha Q + 2nL\alpha\lambda - 2n\alpha - |V|^2\lambda \\
&+ c\left(Q_J a_0 + Q\lambda\right) + \sum_{k=1}^{n}\sum_{j=1}^{2n} h_{0k}\Gamma_{0k}^{j} a_j + h_{0n+k}\Gamma_{0n+k}^{j} a_j.
\end{aligned}$$

The Codazzi equations (10) allow us to compute the terms $X_0(h_{0l}) - X_l(\alpha)$ and to deduce that

$$\begin{aligned}
\sum_{l=1}^{2n} a_l \left(X_0(h_{0l}) - X_l(\alpha)\right) &= 3\sum_{l=1}^{2n}\left(\sum_{k=1}^{n} h_{lk}h_{0n+k} - h_{n+kl}h_{0k}\right) a_l + \\
&- \alpha Q + \sum_{k=1}^{n}\sum_{l=1}^{2n} h_{0k}\Gamma_{0l}^{k} a_l + h_{0n+k}\Gamma_{0l}^{n+k} a_l.
\end{aligned}$$

By putting together the last two expressions and denoting

$$Q_1 = \sum_{l=1}^{2n}\left(\sum_{k=1}^{n} h_{n+kl}h_{0k} - h_{lk}h_{0n+k}\right) a_l,$$

we have

$$\mathrm{div}(W_2) = -3Q_1 + 2nL\alpha\lambda - 2n\alpha - |V|^2\lambda + c\left(Q_J a_0 + Q\lambda\right). \tag{20}$$

Let us now deal with W_1. Again by (2), (7), (8), (13), and Lemma 2.1, we get

$$\begin{aligned}
\mathrm{div}(W_1) &= \sum_{j=1}^{2n}\Big(X_j(a_0h_{0j}) - a_0X_0(h_{jj}) + a_0h_{0j}\mathrm{div}(X_j)\Big) - 2nLX_0(a_0) = \\
&= a_0\sum_{j=1}^{2n}\left(X_j(h_{0j}) - X_0(h_{jj})\right) + a_0\sum_{j,l=1}^{2n} h_{0j}\Gamma_{lj}^{l} + \\
&+ 2nL\alpha\lambda - 2nL - 2nLQ - |V|^2\lambda + c\left(2nL(\lambda^2 - a_0^2) + a_0Q_J + \lambda Q\right) + \\
&+ \sum_{i,k=1}^{n} a_i(h_{0k}h_{kn+i} + h_{0n+k}h_{n+kn+i}) - a_{n+i}(h_{0k}h_{ki} + h_{0n+k}h_{in+k}).
\end{aligned}$$

We can compute the terms $X_j(h_{0j}) - X_0(h_{jj})$ with the Codazzi equations (10), and we can then deduce that

$$\sum_{j=1}^{2n}\left(X_j(h_{0j}) - X_0(h_{jj})\right) = -\sum_{j,l=1}^{2n} h_{0j}\Gamma_{lj}^{l}.$$

By putting everything together and denoting

$$Q_2 = \sum_{i,k=1}^{n} a_i(h_{0k}h_{kn+i} + h_{0n+k}h_{n+kn+i}) - a_{n+i}(h_{0k}h_{ki} + h_{0n+k}h_{in+k}),$$

we get

$$\begin{aligned} \mathrm{div}(W_1) &= -2nL + \left(2nL\alpha - |V|^2\right)\lambda + Q_2 - 2nLQ + \\ &+ c\left(2nL(\lambda^2 - a_0^2) + a_0 Q_J + \lambda Q\right). \end{aligned} \tag{21}$$

Finally, it is the turn of W_3. Always by using (2), (7), (8), and Lemma 2.1, we get

$$\begin{aligned} \mathrm{div}(W_3) &= \sum_{j,l=1}^{2n} \left(X_j(h_{lj}a_l - h_{ll}a_j) + (h_{lj}a_l - h_{ll}a_j)\mathrm{div}(X_j)\right) = \\ &= \sum_{j,l=1}^{2n} a_l \left(X_j(h_{lj}) - X_l(h_{jj})\right) + 2n(1-2n)L + 2n(2n-1)\sigma_2^H\lambda + \\ &+ 2nLQ + Q_1 + 2nc\Theta^H + \sum_{j,l,m=1}^{2n} h_{lj}\Gamma_{mj}^m a_l + h_{ml}\Gamma_{ml}^j a_j. \end{aligned}$$

From Codazzi equations (10) we have also

$$\sum_{j,l=1}^{2n} a_l \left(X_j(h_{lj}) - X_l(h_{jj})\right) = 2Q_1 - Q_2 - \sum_{j,l,m=1}^{2n} h_{lj}\Gamma_{mj}^m a_l + h_{ml}\Gamma_{ml}^j a_j.$$

Summing up

$$\mathrm{div}(W_3) = 2n(1-2n)L + 2n(2n-1)\sigma_2^H\lambda + 2nLQ + 3Q_1 - Q_2 + 2nc\Theta^H. \tag{22}$$

We are then ready to obtain the second Minkowski formula.

Proposition 1.2. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$\int_M H - \sigma_2^M\lambda = \frac{c}{2n}\int_M \left(H\left(\lambda^2 - a_0^2\right) + \Theta\right).$$

Proof. If we sum (20), (21), and (22), we can compute $\mathrm{div}(W)$. We have also to keep in mind (5), (19), and the following easy relation

$$(2n+1)\Theta = 2n\Theta^H + 2a_0Q_J + 2\lambda Q + \alpha(a_0^2 - \lambda^2)$$

which comes just from the definition of Θ and Θ^H. Hence we get

$$\mathrm{div}(W) = -2n(2n+1)H + 2n(2n+1)\sigma_2\lambda + c\left((2n+1)H\left(\lambda^2 - a_0^2\right) + (2n+1)\Theta\right).$$

Since M is closed, the divergence theorem gives the desired formula. □

The computation we did for $\mathrm{div}(W_3)$ suggests also an analogous of Proposition 2.1, i.e. the following horizontal second Minkowski formula.

Proposition 3.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$2n(2n-1)\int_M L - \sigma_2^H\lambda - c\int_M 2n\Theta^H = \int_M 2nLQ + 3Q_1 - Q_2. \tag{23}$$

Proof. It follows from integrating the relation in (22). □

All the terms Q, Q_1, Q_2 appearing in the r.h.s of the last formula depend linearly on the mixed terms h_{0j}. Thus, they vanish identically if the hypersurface is of Hopf type.

Corollary 3.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Suppose M is Hopf. Then*

$$\int_M L - \sigma_2^H\lambda = \frac{c}{2n-1}\int_M \Theta^H.$$

Even in this case, being Hopf is just a sufficient condition for such a formula to hold true: see the Appendix concerning the Reinhardt domains.
Our main aim is to prove the second horizontal Minkowski formula we have announced in the Introduction, i.e. Theorem 1.1. It holds for any real hypersuface (not just the Hopf ones), and furthermore, differently from (23), contains no mixed terms. For the convenience of the reader we repeat here the statement.

Theorem 1.1. *Let M be a closed $(2n+1)$-dimensional real hypersurface in K_c. Then it holds:*

$$\int_M \mathcal{H} - \mathcal{S}_2\lambda = \frac{c}{2n+2}\int_M L(\lambda^2 - a_0^2) + \Theta^H - 3\lambda.$$

Proof. Recalling that $V = \nabla^M_{X_0}X_0 = \sum_{k=1}^n h_{0n+k}X_k - h_{0k}X_{n+k}$, by Lemma 2.1 we have

$$V(\lambda) = -Q_1 + c(a_0Q_J + \lambda Q).$$

Moreover, by (2), (7), (8), we get

$$\mathrm{div}(V) = \sum_{k=1}^n \Big(X_k(h_{0n+k}) - X_{n+k}(h_{0k})\Big) + \sum_{k=1}^n\sum_{l=1}^{2n} h_{0n+k}\Gamma^l_{lk} - h_{0k}\Gamma^l_{ln+k} - |V|^2.$$

Thanks to Codazzi equations (10) we can compute the term

$$\begin{aligned}\sum_{k=1}^n \Big(X_k(h_{0n+k}) - X_{n+k}(h_{0k})\Big) &= 2nc + 2nL\alpha - \sum_{k=1}^n\sum_{l=1}^{2n} h_{0n+k}\Gamma^l_{lk} - h_{0k}\Gamma^l_{ln+k} + \\ &- 2\sum_{i,k=1}^n h_{ik}h_{n+in+k} - h_{kn+i}h_{in+k}.\end{aligned}$$

Denoting by

$$D_2 = \sum_{i,k=1}^{n} h_{ik}h_{n+in+k} - h_{kn+i}h_{in+k},$$

we thus have

$$\mathrm{div}(V) = 2nL\alpha - |V|^2 - 2D_2 + 2nc.$$

By the divergence theorem we infer that $-\int_M V(\lambda) = \int_M \mathrm{div}(V)\lambda$, which says

$$\int_M Q_1 - (2nL\alpha - |V|^2)\lambda - c(a_0 Q_J + \lambda Q) = \int_M (2nc - 2D_2)\lambda. \tag{24}$$

On the other hand, the computation for $\mathrm{div}(W_2)$ in (20) tells us

$$\int_M Q_1 - (2nL\alpha - |V|^2)\lambda - c(a_0 Q_J + \lambda Q) = -\int_M 2Q_1 + 2n\alpha.$$

Hence we deduce

$$\int_M Q_1 = \int_M -n\alpha + (D_2 - nc)\lambda. \tag{25}$$

Furthermore, using (21) and (22) in the identity $\int_M \mathrm{div}(W_1) + \mathrm{div}(W_3) = 0$, we get

$$\begin{aligned} &\int_M Q_1 - (2nL\alpha - |V|^2)\lambda - c(a_0 Q_J + \lambda Q) = \\ = &\int_M 4Q_1 - 4n^2 L + 2n(2n-1)\sigma_2^H\lambda + c(2nL(\lambda^2 - a_0^2) + 2n\Theta^H) = \\ = &\int_M -4n\alpha + 4(D_2 - nc)\lambda - 4n^2 L + 2n(2n-1)\sigma_2^H\lambda + c(2nL(\lambda^2 - a_0^2) + 2n\Theta^H) \end{aligned}$$

where in the last equality we have substituted the identity we found in (25). By comparing this relation with (24) we have

$$\begin{aligned} &\int_M (2nc - 2D_2)\lambda = \\ = &\int_M -4n(nL + \alpha) + (2n(2n-1)\sigma_2^H + 4D_2)\lambda + c(2nL(\lambda^2 - a_0^2) + 2n\Theta^H - 4n\lambda) \end{aligned}$$

which is the same as

$$\int_M 4n(nL + \alpha) - (2n(2n-1)\sigma_2^H + 6D_2)\lambda = c\int_M 2nL(\lambda^2 - a_0^2) + 2n\Theta^H - 6n\lambda.$$

By recalling the definitions of $\mathcal{H}$ and $\mathcal{S}_2$, this proves the desired formula since we have the relation

$$n(2n-1)\sigma_2^H = 2n(n-1)\sigma_2^\ell + D_2.$$

□

If $n = 1$ and $c = 0$, we have as particular case the conclusion of Corollary 1.1.

Appendix: the case of Reinhardt domains

Consider a smooth hypersurface $M \subset \mathbb{C}^2$ described by

$$M = \{p = (z_1, z_2) \in \mathbb{C}^2 \,:\, f(z_1, z_2) = 0\}$$

for a smooth f such that $|\nabla f| \neq 0$ on M. We identify $\mathbb{C}^2 \simeq \mathbb{R}^4$, with $z_j = x_j + iy_j$ (for $j = 1, 2$), and we write $f(z_1, z_2) = f(x_1, y_1, x_2, y_2)$. Since for us $J\partial_{x_j} = \partial_{y_j}$, we have

$$\begin{aligned}
\nu &= \frac{1}{|\nabla f|}\left(f_{x_1}\partial_{x_1} + f_{y_1}\partial_{y_1} + f_{x_2}\partial_{x_2} + f_{y_2}\partial_{y_2}\right) \\
X_0 &= \frac{1}{|\nabla f|}\left(f_{y_1}\partial_{x_1} - f_{x_1}\partial_{y_1} + f_{y_2}\partial_{x_2} - f_{x_2}\partial_{y_2}\right).
\end{aligned}$$

Let us complete the tangent frame with the horizontal vector fields

$$\begin{aligned}
X_1 &= \frac{1}{|\nabla f|}\left(f_{x_2}\partial_{x_1} - f_{y_2}\partial_{y_1} - f_{x_1}\partial_{x_2} + f_{y_1}\partial_{y_2}\right) \\
X_2 &= \frac{1}{|\nabla f|}\left(f_{y_2}\partial_{x_1} + f_{x_2}\partial_{y_1} - f_{y_1}\partial_{x_2} - f_{x_1}\partial_{y_2}\right).
\end{aligned} \tag{26}$$

We want to see what happens to the integral formulas when we rotate the manifold M without fixing the complex structure. We want to perform the rotation

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus we define $M^\theta = \{p_\theta \in \mathbb{C}^2 \,:\, p = R_\theta p_\theta \in M\}$. As a defining function we can surely take $f^\theta = f \circ R_\theta$, for which $|\nabla f^\theta|(p_\theta) = |\nabla f|(R_\theta p_\theta)$. Having f^θ, we can consider the related vector fields $\nu^\theta, X_0^\theta, X_1^\theta, X_2^\theta$. A straightforward computation shows that, for any $p_\theta \in M^\theta$,

$$\begin{aligned}
\lambda^\theta(p_\theta) &= \lambda(R_\theta p_\theta) \\
a_0^\theta(p_\theta) &= \cos(\theta)a_0(R_\theta p_\theta) - \sin(\theta)a_1(R_\theta p_\theta) \\
a_1^\theta(p_\theta) &= \sin(\theta)a_0(R_\theta p_\theta) + \cos(\theta)a_1(R_\theta p_\theta) \\
a_2^\theta(p_\theta) &= a_2(R_\theta p_\theta).
\end{aligned}$$

Moreover

$$h^\theta(p_\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot h(R_\theta p_\theta) \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

where h and h^θ denote (with abuse of notations) the 3×3 matrices representing the second fundamental forms of M and M^θ with respect to the frames $\{X_0, X_1, X_2\}$

and $\{X_0^\theta, X_1^\theta, X_2^\theta\}$.
Now we have all the tools we need in order to keep track of the behavior in θ of the following integrals

$$\int_{M^\theta} 1 - L^\theta \lambda^\theta \qquad \text{and} \qquad \int_{M^\theta} L^\theta - \sigma_2^{H,\theta} \lambda^\theta.$$

About the first integral, we did it in full generality in the proof of Theorem 2.1. Here we want to see what happens when M is the boundary of a Reinhardt domain: we will show that there are specific rotations R_θ for which both the horizontal Minkowski integrals vanish at the same time.
Let us fix a Reinhardt domain Ω (with respect to the origin), i.e. $(z_1, z_2) \in \Omega$ if and only if $(e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in \Omega$ for every $\theta_1, \theta_2 \in \mathbb{R}$. We can then consider a defining function f for $M = \partial\Omega$ depending only on the radii in the following way:

$$f(z_1, z_2) = g(s_1, s_2) = g(s), \qquad s_k = \frac{z_k \bar{z}_k}{2} = \frac{x_k^2 + y_k^2}{2} \quad (k = 1, 2),$$

for some smooth $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. It is useful to explicit everything in terms of g. We shall write g_k instead of $\partial_{s_k} g$. For all $p \in M$ we have

$$\begin{aligned}
|\nabla f|^2 &= 2s_1 g_1^2 + 2s_2 g_2^2 \\
\lambda &= \frac{2s_1 g_1 + 2s_2 g_2}{|\nabla f|} \\
a_0 &= 0 \\
a_1 &= (x_1 x_2 - y_1 y_2) \frac{g_2 - g_1}{|\nabla f|} \\
a_2 &= (x_1 y_2 + x_2 y_1) \frac{g_2 - g_1}{|\nabla f|}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\alpha &= \frac{2s_1 g_1^3 + 2s_2 g_2^3}{|\nabla f|^3} \\
h_{01} &= -(x_1 y_2 + x_2 y_1) \frac{g_1 g_2 (g_2 - g_1)}{|\nabla f|^3} \\
h_{02} &= (x_1 x_2 - y_1 y_2) \frac{g_1 g_2 (g_2 - g_1)}{|\nabla f|^3} \\
h_{11} &= \frac{g_1 g_2 (2s_1 g_1 + 2s_2 g_2)}{|\nabla f|^3} + (x_1 x_2 - y_1 y_2)^2 \frac{g_2^2 g_{11} - 2 g_1 g_2 g_{12} + g_1^2 g_{22}}{|\nabla f|^3} \\
h_{22} &= \frac{g_1 g_2 (2s_1 g_1 + 2s_2 g_2)}{|\nabla f|^3} + (x_1 y_2 + x_2 y_1)^2 \frac{g_2^2 g_{11} - 2 g_1 g_2 g_{12} + g_1^2 g_{22}}{|\nabla f|^3} \\
h_{12} &= (x_1 x_2 - y_1 y_2)(x_1 y_2 + x_2 y_1) \frac{g_2^2 g_{11} - 2 g_1 g_2 g_{12} + g_1^2 g_{22}}{|\nabla f|^3}.
\end{aligned}$$

The functions $(x_1 y_2 + x_2 y_1)$ and $(x_1 x_2 - y_1 y_2)^2 - (x_1 y_2 + x_2 y_1)^2$ are odd with respect to the symmetries $(z_1, z_2) \mapsto (-z_1, z_2)$ and $(x_1, y_1, x_2, y_2) \mapsto (-y_1, x_1, x_2, y_2)$ which leave unchanged the radii s_1 and s_2. That's why we have in particular that

$$\int_M h_{01}\lambda = 0 \qquad \text{and} \qquad \int_M (h_{11}-h_{22})\,\lambda = 0.$$

Hence, if we apply our rotation R_θ, we readily get from (27) that

$$\begin{aligned} 2\int_{M^\theta} 1 - L^\theta\lambda^\theta &= \int_M 2 - \big(\cos^2(\theta)h_{11} + \sin^2(\theta)\alpha + 2\cos(\theta)\sin(\theta)h_{01} + h_{22}\big)\,\lambda \\ &= \int_M 2 - (1+\cos^2(\theta))L\lambda - \sin^2(\theta)\alpha\lambda \\ &= \int_M 2 - (1+\cos^2(\theta))L\lambda - \sin^2(\theta)\,(3-2L\lambda) \\ &= (2-3\sin^2(\theta))\int_M 1 - L\lambda, \end{aligned}$$

where we used the fact that $\int \alpha\lambda = \int 3 - 2L\lambda$ by the classical Minkowski formula. This says that, even if $\int_M 1 - L\lambda$ was not 0 for the Reinhardt domain under consideration at the beginning, it does vanish after a rotation R_{θ_0} for which

$$\sin^2(\theta_0) = \frac{2}{3} = 2\cos^2(\theta_0).$$

Let us check that, for the same θ_0, also $\int L^{\theta_0} - \sigma_2^{H,\theta_0}\lambda^{\theta_0} = 0$. It is easy to see that in $\mathbb{C}^2$ (actually for the case $n=1$ in our notations) we have the relation

$$2LQ + Q_1 - Q_2 = 0.$$

Then Proposition 3.1 ($n=1, c=0$) tells $\int L - \sigma_2^H\lambda = \int Q_1$ for any real hypersurface. Thus, we want to show that

$$\int_{M^{\theta_0}} Q_1^{\theta_0} = 0.$$

From (27) and the fact that $a_0 = 0$ and $a_1h_{01} + a_2h_{02} = 0$, we see that, for any θ,

$$\begin{aligned} \int_{M^\theta} Q_1^\theta &= \cos^2(\theta)\int_M (a_1h_{12}h_{01} - a_1h_{11}h_{02}) + \\ &+ \big(\cos^2(\theta) - \sin^2(\theta)\big)\int_M (a_2h_{22}h_{01} - a_2h_{12}h_{02}) + \\ &+ \cos(\theta)\sin(\theta)\int_M \big(a_1h_{12}\alpha + a_2h_{22}\alpha - a_2h_{11}h_{22} + a_2h_{12}^2\big). \end{aligned}$$

Thanks to the explicit formulas for the involved terms and to the odd simmetry of $(x_1y_2 + x_2y_1)$, it is not difficult to recognize that the last integral at the r.h.s. vanishes. Morever, when $\theta = \theta_0$, we have $\cos^2(\theta_0) = \sin^2(\theta_0) - \cos^2(\theta_0) = \frac{1}{3}$ and

$$\begin{aligned} \int_{M^\theta} Q_1^{\theta_0} &= \frac{1}{3}\int_M (a_1h_{12}h_{01} - a_1h_{11}h_{02} - a_2h_{22}h_{01} + a_2h_{12}h_{02}) \\ &= -\frac{1}{3}\int_M (a_1h_{11}h_{02} + a_2h_{22}h_{01}). \end{aligned}$$

By the odd simmetries of $(x_1x_2 - y_1y_2)^2 - (x_1y_2 + x_2y_1)^2$, also the last integral vanishes. Therefore, summing up what we have just proved, we have

$$\int_{M^{\theta_0}} L^{\theta_0} - \sigma_2^{H,\theta_0}\lambda^{\theta_0} = 0 = \int_{M^{\theta_0}} 1 - L^{\theta_0}\lambda^{\theta_0}.$$

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