On the Hopf-Oleinik lemma for degenerate-elliptic equations at characteristic points

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Abstract

In this paper we discuss the validity of the Hopf lemma at boundary points which are characteristic with respect to certain degenerate-elliptic equations. In the literature there are some positive results under the assumption that the boundary of the domain reflects the underlying geometry of the specific operator. We focus here on conditions on the boundary which are suitable for some families of degenerate operators, also in presence of first order terms.

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1 Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth (at least of class C^2) boundary, $y \in \partial\Omega$, and ν the inner unit normal to $\partial\Omega$ at y. We say that a second-order linear partial differential operator $\mathcal{L}$ satisfies the Hopf lemma in Ω at $y \in \partial\Omega$ if, for any $u \in C^2(\Omega \cap W) \cap C^1\left((\Omega \cap W) \cup \{y\}\right)$, we have

$$\begin{cases} \mathcal{L}u \leq 0 \text{ in } \Omega \cap W, \\ u > 0 \text{ in } \Omega \cap W, \\ u(y) = 0 \end{cases} \quad \Rightarrow \quad \frac{\partial u}{\partial \nu}(y) > 0, \tag{1}$$

where W is an open neighborhood of y.

This principle, in its various forms, had enormous influences in the study of linear and

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non-linear second order (most of all elliptic) equations, and the literature around it is really huge. Let us just mention, without any aim of completeness, the pioneering study by Zaremba for the Laplacian [33], the work by Giraud [9], and the celebrated and independent papers by Hopf [10] and Oleinik [29] for uniformly elliptic operators. We also address the reader to the rich historical perspective about the argument present in [2]. Besides the classical works, "Hopf lemmas" have also been studied under several point of views: among the others the issue of the sharpness of the domain [12, 17, 2, 4], and the issues about the presence of a singular drift term [2, 27]. Historically, one of the main applications has probably concerned symmetry results: Hopf lemma is, indeed, a crucial tool in the moving planes technique [1, 30, 15, 16].

In this paper we would like to study the following problem. We want to consider second order degenerate-elliptic linear operators $\mathcal{L}$ (or special classes of such operators), for which the degeneracy is controlled by some vector fields satisfying the Hörmander condition. We want to understand the geometry of the domain Ω around a characteristic point for $\mathcal{L}$, in order to ensure the Hopf property in Ω. In particular we would like to investigate how much the geometry should change if we change the operator inside the same class, and with the possible presence of first-order terms.

Let us fix some notations. For us $\mathcal{L}$ will denote second order linear operators in the following form

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(p)\partial^2_{ij} + \sum_{k=1}^{N} b_k(p)\partial_k :$$

we can think the coefficients a_{ij} and b_k to be continuous functions in some open set $O \supset \overline{\Omega}$, and $A(p) = (a_{ij}(p))_{i,j=1}^N$ a symmetric nonnegative definite $N \times N$ matrix never identically vanishing. These last conditions ensure in particular the validity of a weak maximum principle for $\mathcal{L}$ in small domains (see, e.g, [13]). On the other hand all the operators will be truly degenerate, in the sense that the matrix A will have a non-trivial kernel: in this regard the following definition is crucial.

Definition 1.1. *We say that $y \in \partial\Omega$ is characteristic for $(\mathcal{L}, \Omega)$ if*

$$A(y)\nu = 0.$$

As we mentioned, we are interested in degenerate-elliptic operators whose direction of ellipticity are determined by smooth vector fields X_j's satisfying the Hörmander condition. This means that we want to focus on sum of squares of such vector fields or on non-divergence form operators as $\sum_{i,j} q_{ij} X_i X_j$, with (q_{ij}) uniformly positive definite. We will often assume that the vector fields and their commutators of length 1 recover all the directions of the ambient space. These kind of operators appear in fact as the linearization of nonlinear operators as the Levi operator which is, roughly speaking, the degenerate-elliptic analogue of the classical mean-curvature operator and one of our main motivations of investigation. Boundary comparison principles for the Levi operator are one of the main obstacles to obtain symmetry results via moving planes type techniques (see [11, Section 2]). Nevertheless some Alexandrov-type results for the Levi operator have been proved in the literature [11, 25, 19, 20, 22]. In the last section we will discuss the Hopf property for this specific nonlinear example.

Since the classical works [10, 29], the proof of Hopf lemmas is strictly related to the concept of barrier functions.

Definition 1.2. *Fix $y \in \partial\Omega$. We say that a function h is an $\mathcal{L}$-barrier function for Ω at y if*

- *h is a C^2 function defined on an open bounded neighborhood U of y,*
- *$h(y) = 0$,*
- *$\{p \in U \ : \ h(p) \geq 0\} \smallsetminus \{y\} \subseteq \Omega$,*
- *$\mathcal{L}h \geq 0$ in $\{p \in U \ : \ h(p) > 0\}$,*
- *$\nabla h(y) \neq 0$.*

It is well-known (see the beginning of Section 2) that the existence of an $\mathcal{L}$-barrier function for Ω at y ensures the validity of the Hopf lemma for $\mathcal{L}$ in Ω at y. At the non-characteristic points, i.e. at the points y where the normal ν is not in the kernel of $A(y)$, it is easy to find a barrier: since smooth domains have the interior ball property, a barrier is classically given by exponential-type functions. The real issue is at the characteristic points. As a matter of fact in the literature there are some positive and negative results for specific degenerate-elliptic operators. The references [5, 28, 24, 26] deal respectively with the case of the Kohn Laplacian in the Heisenberg group, generalized Greiner operators, and some Grushin-type operators. They pointed out that the boundary of the domain has somehow to reflect the geometry of the operator under consideration if one wants that the Hopf lemma holds true. The Zaremba's interior ball condition is thus replaced with an analogous condition regarding the level sets of the fundamental solution, which allows to find suitable barriers.

In Section 2, we first consider sub-Laplacians in general homogeneous Carnot groups and we prove the validity of Hopf lemma under the condition regarding the level sets of the fundamental solution. Then we consider the class of step-two horizontally elliptic operators. They are a class of operators in non-divergence form, whose degeneracy is controlled by the generators of the Carnot algebra. We prove that an interior homogeneous ball condition is suitable for all the operators in this class, more in the spirit of the Hopf-Oleinik's result. In Section 3 we consider some operators having non-trivial first order terms along a characteristic direction. We show that, unlikely the classical elliptic case, the validity of the Hopf lemma may change drastically in presence of first order terms. To this aim, we will construct explicit counterexamples and we will prove positive results for some model operators. In Section 4, we will briefly discuss the global behavior of the boundary of a bounded open set where the Hopf lemma is satisfied at any point for all the non-divergence form operators uniformly elliptic with respect to two classes of vector fields. In Section 5 we finally discuss the nonlinear case. As an example we study the Levi operator: we will show that some phenomena of the previous sections for the linear models appear similarly also in this nonlinear situation.

2 $\Delta_{\mathbb{G}}$ and 2-step horizontally elliptic operators

We have already recalled that the existence of a barrier implies the validity of the Hopf via the Weak Maximum Principle. For the sake of completeness we give here the outline of the proof.

Remark 2.1. *Let u as in* (1) *and consider an $\mathcal{L}$-barrier function h for Ω at y, defined on U. Let $\rho > 0$ such that $\overline{B_\rho(y)} \subset U \cap W$. We set $V = \{p \in B_\rho(y) : h(p) > 0\}$, which is contained in Ω. We write $\partial V = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{p \in B_\rho(y) : h(p) = 0\}$ and $\Gamma_2 = \partial V \smallsetminus \Gamma_1$. Since $\Gamma_2 \subset \partial B_\rho(y) \cap \Omega$, we have $m = \min_{\overline{\Gamma}_2} u$ is strictly positive. Let us also put $M = \max_{\overline{V}} h > 0$. For $0 < \epsilon < \frac{m}{M}$, we consider $u - \epsilon h$. By construction we get $u - \epsilon h \geq 0$ on ∂V and $\mathcal{L}(u - \epsilon h) \leq 0$ in V. By the Weak Maximum Principle for $\mathcal{L}$, $u \geq \epsilon h$ in V. Since the inner unit normal to $\partial\Omega$ at y is given by $\nu = \frac{\nabla h(y)}{\|\nabla h(y)\|}$ and $y + t\nu \in V$ for small positive t, we obtain*

$$\frac{\partial u}{\partial \nu}(y) \geq \epsilon \frac{\partial h}{\partial \nu}(y) = \epsilon \|\nabla h(y)\| > 0.$$

Smooth domains have the interior ball property at any point $y \in \partial\Omega$, i.e. there exists a ball $B_{r_0}(p_0)$ such that $\overline{B_{r_0}(p_0)} \smallsetminus \{y\} \subset \Omega$. This is the reason why it is easy to find an $\mathcal{L}$-barrier function h for Ω at y in the case y is non-characteristic for $(\mathcal{L}, \Omega)$. As a matter of fact, we have the following

Remark 2.2. *For $\alpha > 0$ big enough, the function*

$$h_\alpha(p) = e^{-\alpha\|p-p_0\|^2} - e^{-\alpha r_0^2}$$

is a barrier in a neighborhood of y (see e.g. [3]). In fact, we have

$$\{p \in \mathbb{R}^N : h_\alpha \geq 0\} \smallsetminus \{y\} = \overline{B_{r_0}(p_0)} \smallsetminus \{y\}, \qquad \text{and } \nabla h_\alpha(y) = 2\alpha(p_0 - y)e^{-\alpha r_0^2} \neq 0.$$

Thus, it is enough to check the condition on $\mathcal{L}h_\alpha$. We have

$$\mathcal{L}h_\alpha(p) = 2\alpha e^{-\alpha\|p-p_0\|^2} \left(2\alpha \langle A(p)(p - p_0), p - p_0 \rangle - \mathrm{Tr}(A(p)) - \sum_{k=1}^{N} b_k(p)(p - p_0)_k \right).$$

Since y is non-characteristic

$$\langle A(y)(y-p_0), y-p_0\rangle = r_0^2 \langle A(y)\nu, \nu\rangle > 0.$$

By the continuity of the coefficients of the operator $\mathcal{L}$, for big α we can find a neighborhood of y where

$$\mathcal{L}h_\alpha(p) > 0. \tag{2}$$

At characteristic points the Hopf property may not hold true. Let us consider for example, in $\mathbb{R}^3$, the Kohn Laplacian on the Heisenberg group

$$\Delta_{\mathbb{H}}u(x_1,x_2,t) = \partial^2_{x_1x_1}u + \partial^2_{x_2x_2}u - x_2\partial^2_{x_1t}u + x_1\partial^2_{x_2t}u + \frac{1}{4}(x_1^2+x_2^2)\partial^2_{tt}u = (X_1^2+X_2^2)u,$$

where $X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_t$, $X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_t$. For this operator we have

$$A(x_1,x_2,t) = \begin{pmatrix} 1 & 0 & -\frac{1}{2}x_2 \\ 0 & 1 & \frac{1}{2}x_1 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & \frac{1}{4}(x_1^2+x_2^2) \end{pmatrix}.$$

If $y = 0 \in \partial\Omega$ and the inner unit normal is $(0,0,1)$ at 0, then 0 is characteristic for $(\Delta_{\mathbb{H}}, \Omega)$. We have the following

Counterexample 2.3. *Suppose Ω locally around 0 is described by $\{(x,t) \in \mathbb{R}^3 : t > \frac{1}{4}(x_1^2+x_2^2)\}$. Let us consider*

$$u(x,t) = t^2 - \frac{1}{16}(x_1^2+x_2^2)^2.$$

Of course, $u(0) = 0$ and $u > 0$ in Ω (we can assume $\Omega \subseteq \{t > \frac{1}{4}(x_1^2+x_2^2)\}$). Moreover $\Delta_{\mathbb{H}}u(x,t) = -\frac{1}{2}(x_1^2+x_2^2) \leq 0$. But

$$\frac{\partial u}{\partial \nu}(0) = \partial_t u(0) = 0,$$

thus $\Delta_{\mathbb{H}}$ does not satisfy the Hopf lemma in Ω at 0.

Despite this counterexample, it is possible to put some natural conditions on Ω to ensure the validity of the Hopf lemma for $\Delta_{\mathbb{H}}$. This was done by Birindelli and Cutrí in [5, Lemma 2.1]: as far as we know, this was the first example in literature of Hopf lemma for a degenerate-elliptic operator at a characteristic point. They proved that an interior Koranyi-ball condition for Ω allows to find a barrier.
The first thing we want to do is to prove such result in generic homogeneous Carnot groups. To this purpose let us recall some notions (more details can be found in [6]).

Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Carnot group, with homogeneous dimension $Q \geq 3$. Let us fix $X_1, \ldots, X_m$ left-invariant vector fields δ_λ-homogeneous of degree 1, which generate the first layer of the Lie algebra of $\mathbb{G}$ ($2 \le m < N$). We want to consider the degenerate-elliptic operator

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2.$$

We denote by $\Gamma(\cdot\,; p_0)$ its fundamental solution with pole at p_0. We recall that

$$\Gamma(p; p_0) = \Gamma(p_0^{-1} \circ p) \quad \text{and} \quad \Gamma(\delta_\lambda(p)) = \lambda^{2-Q}\Gamma(p) \quad \text{for all } \lambda > 0,\, p \in \mathbb{G},$$

where we have used the notation $\Gamma(\cdot) = \Gamma(\cdot\,; 0)$. Let us denote the $\mathbb{G}$-gauge balls centered at $p_0 \in \mathbb{G}$ with radius r by

$$B_r^{\mathbb{G}}(p_0) = \left\{ p \in \mathbb{R}^N \,:\, \Gamma(p; p_0) > \frac{1}{r^{Q-2}} \right\}, \tag{3}$$

i.e. the superlevel sets of the fundamental solution. We call them balls since $\Gamma(p; p_0)^{\frac{1}{2-Q}}$ defines a homogeneous symmetric norm, satisfying a pseudo-triangle inequality (see [6, Chapter 5]).

Proposition 2.4. *Let us assume there exist p_0 and r_0 such that*

$$y \in \partial B_{r_0}^{\mathbb{G}}(p_0), \quad \overline{B_{r_0}^{\mathbb{G}}(p_0)} \smallsetminus \{y\} \subset \Omega.$$

Then $\Delta_{\mathbb{G}}$ satisfies the Hopf lemma in Ω at y.

Proof. We just check that

$$h(p) = \Gamma(p; p_0) - {r_0}^{2-Q} = \Gamma(p_0^{-1} \circ p) - {r_0}^{2-Q},$$

defined on $U = \mathbb{R}^N \smallsetminus \overline{B_{\frac{r_0}{2}}^{\mathbb{G}}(p_0)}$, is a $\Delta_{\mathbb{G}}$-barrier function for Ω at y. The only condition which really needs to be checked is

$$\nabla h(y) \neq 0,$$

and it will follow by homogeneity arguments. As a matter of fact, if $\nabla h(\bar{p}) = 0$ at some $\bar{p}$, then $\nabla\Gamma(p_0^{-1} \circ \bar{p}) = 0$ since the left-translation $p \mapsto p_0^{-1} \circ p$ is a diffeomorphism. But, by the homogeneity properties, $\nabla\Gamma(p_0^{-1} \circ \bar{p})$ cannot vanish because

$$\left\langle \nabla\Gamma(q), \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta_\lambda(q) \right\rangle = (2-Q)\Gamma(q) \quad \text{for } q \neq 0,$$

and Γ never vanishes (see [6, Proposition 5.3.13]). $\square$

Remark 2.5. *The assumption in Proposition 2.4 is meaningful when y is a characteristic point, otherwise the result is known (see Remark 2.2). For operators as $\Delta_{\mathbb{G}}$, a point $y \in \partial\Omega$ is characteristic iff $X_j(y)$ is tangent to $\partial\Omega$ for all $j \in \{1, \dots, m\}$.*

Remark 2.6. *If $\mathbb{G}$ is the Heisenberg group $\mathbb{H}$, the $\mathbb{G}$-ball in* (3) *defines the Koranyi-ball, i.e. the metric balls with respect to the distance*

$$d((x_1, x_2, t), 0) = \left((x_1^2 + x_2^2)^2 + 16t^2\right)^{\frac{1}{4}}.$$

The assumption in Proposition 2.4 is saying that $\partial\Omega$ at the characteristic point has to be flat enough for the Hopf property to hold (so that to avoid behavior as in Counterexample 2.3). This is the same condition as in [5]. They used a different barrier, i.e. some exponential barrier of Hopf type.

Other examples of Hopf lemmas in literature for degenerate-elliptic operators at characteristic points are the following: [28] for generalized Greiner operators, [24, 26] for some 'so-called' Grushin-type operators. In all these examples, the condition on Ω is a flatness condition with respect to some homogeneous norm relevant for the operator (related to the fundamental solution).

In the cited references and in our Proposition 2.4 the differential operator is fixed, and somehow also the related geometry. We want now to discuss the issue of the stability of the assumptions on Ω if we change the operator. This is a meaningful issue if we consider a class of operators with the same characteristic points. This is one of the reasons why we want to consider the following operators

$$L_Q = \sum_{i,j=1}^{m} q_{ij}(p) X_i X_j \tag{4}$$

where $Q(p)$ is symmetric and uniformly positive definite, i.e. $\lambda \mathbb{I}_m \leq Q(p) \leq \Lambda \mathbb{I}_m$ for some $\Lambda \geq \lambda > 0$. If $X_1, \ldots, X_m$ are the generators of the first (horizontal) layer of a homogeneous Carnot group, they are called *horizontally elliptic operators*. The condition of being characteristic is determined by the vector fields: it is thus independent of the choice of the positive definite matrix Q. We are going to prove stability for the Hopf lemma in the case when the step of nilpotence of the Lie algebra is two.

Let us fix some notations. Fix $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ such that the composition law $\circ$ is defined by

$$(x, t) \circ (\xi, \tau) = \left(x + \xi, t + \tau + \frac{1}{2}\langle Bx, \xi\rangle\right),$$

for $(x, t), (\xi, \tau) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^N$. Here we have denoted by $\langle Bx, \xi\rangle$ the vector of $\mathbb{R}^n$ whose components are $\left\langle B^k x, \xi\right\rangle$ (for $k = 1, \ldots, n$) and $B^1, \ldots, B^n$ are $m \times m$ linearly independent skew-symmetric matrices. The group of dilations is defined as $\delta_\lambda((x, t)) = (\lambda x, \lambda^2 t)$ and the inverse of (x, t) is $(-x, -t)$. Up to a choice for the stratification of the Lie algebra and a canonical isomorphism (see [6, Theorem 3.2.2]), a generic step-2 Carnot group is of this form.

We can choose as homogeneous symmetric norm the function $d : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that

$$d\left((x,t)\right) = \left(\|x\|^4 + \|t\|^2\right)^{\frac{1}{4}};$$

from here on we denote by $\|\cdot\|$ both the Euclidean norms in $\mathbb{R}^m$ and in $\mathbb{R}^n$. Hence, we have $B_r^2(x_0,t_0) = (x_0,t_0) \circ B_r(0)$ where

$$B_r^2(0) = \{(x,t) \in \mathbb{R}^N \,:\, \|x\|^4 + \|t\|^2 < r^4\}.$$

Let us fix

$$X_i = \partial_{x_i} + \frac{1}{2}\sum_{k=1}^{n}(B^k x)_i \partial_{t_k} \qquad \text{for } i = 1, \dots, m. \tag{5}$$

These m vector fields are left-invariant and δ_λ-homogeneous of degree 1: they generate the first layer of the Lie algebra of $\mathbb{G}$. It is easy to check that the condition of linear independence for the matrices B^k's is equivalent to require that the vector fields and their commutators of length 1 span the whole tangent space. We want to consider the operator L_Q as in (4) with respect to these specific vector fields.

Theorem 2.7. *Let Ω be an open and bounded set in $\mathbb{R}^N$, and let $y = (\xi,\tau) \in \partial\Omega$ be a characteristic point. Let us assume there exist $(\xi_0,\tau_0) \in \Omega$ and $r_0 > 0$ such that*

$$y \in \partial B_{r_0}^2(\xi_0,\tau_0), \quad \overline{B_{r_0}^2(\xi_0,\tau_0)} \smallsetminus \{y\} \subset \Omega.$$

Then L_Q satisfies the Hopf lemma in Ω at y, for any horizontally elliptic operator in the step-2 Carnot group $\mathbb{G}$.

Proof. We are going to exploit the barriers built by one of the authors in [31] (and then exploited in [32]). Let us write

$$B_{r_0}^2(\xi_0,\tau_0) = \{(x,t) \in \mathbb{R}^N \,:\, F(x,t) < r_0^4\},$$

where $F(x,t) = \|x-\xi_0\|^4 + \left\|t - \tau_0 - \frac{1}{2}\langle B\xi_0, x\rangle\right\|^2$. By hypothesis the normal ν at y is parallel to $\nabla F(y)$. The fact that y is characteristic is equivalent to

$$\langle X_i(y), \nu\rangle = 0 \;\; \forall\, i \quad \Leftrightarrow \quad \nabla_X F(y) := (X_1F(y), \dots, X_mF(y)) = 0$$

$$\Leftrightarrow \quad 4(\xi-\xi_0)\,\|\xi-\xi_0\|^2 + \sum_{k=1}^{n}(B^k(\xi-\xi_0))(\tau^k - \tau_0^k - \frac{1}{2}\left\langle B^k\xi_0, \xi\right\rangle) = 0. \tag{6}$$

By skew-symmetry $(\xi-\xi_0)$ and $B^k(\xi-\xi_0)$ are orthogonal for any k and thus $\|\xi-\xi_0\|$ is forced to be 0. Hence $y = (\xi_0,\tau)$ with $\|\tau-\tau_0\| = r_0^2$. Let us seek a barrier as

$$h(x,t) = f\left((\xi_0,\tau_0)^{-1} \circ (x,t)\right),$$

for a suitable f. By left-invariance we have $L_Q h(x,t) = L_{\tilde{Q}} f((\xi_0,\tau_0)^{-1} \circ (x,t))$, where $\tilde{Q}(x,t) = Q((\xi_0,\tau_0) \circ (x,t))$. There are two cases, corresponding to two different possible choices for f.

- if the matrix $\sum_{k=1}^n(\tau^k-\tau_0^k)B^k$ is invertible, we put

$$f(x,t)=f_\alpha(x,t)=e^{-\frac{\alpha}{r_0^4}\left(\|x\|^4+\|t'\|^2+\langle t,\tau-\tau_0\rangle\right)}-e^{-\alpha},$$

  where $t'=t-\frac{1}{r_0^4}\langle t,\tau-\tau_0\rangle(\tau-\tau_0)$ is the projection of t on the orthogonal of $\tau-\tau_0$, and $\alpha>0$ has to be chosen appropriately big;

- otherwise we put P_1 the orthogonal projector on $\mathrm{Ker}(\sum_{k=1}^n(\tau^k-\tau_0^k)B^k)$, P_2 the orthogonal projector on $\left(\mathrm{Ker}(\sum_{k=1}^n(\tau^k-\tau_0^k)B^k)\right)^\perp$, and we define

$$f(x,t)=f_{\alpha,\gamma}(x,t)=e^{-\frac{\alpha}{r_0^4}\left(\|x\|^4+(\|P_1x\|^2-\gamma\|P_2x\|^2)^2+\|t'\|^2+\langle t,\tau-\tau_0\rangle\right)}-e^{-\alpha},$$

  with t' as before, and $\alpha,\gamma>0$ to be chosen.

Let us explicitly remark that the matrix $\sum_{k=1}^n(\tau^k-\tau_0^k)B^k$ cannot be the null matrix because of the linear independence for the B^k's.
We have $f(0,\tau-\tau_0)=0$ and $\nabla f(0,\tau-\tau_0)\neq 0$, since $\partial_{t_k}f(0,\tau-\tau_0)=-\frac{\alpha}{r_0^4}(\tau^k-\tau_0^k)e^{-\alpha}$. This gives $h(y)=0$ and $\nabla h(y)\neq 0$. Let us now set $U_0=\{(x,t)\in\mathbb{R}^N\,;\,\langle t,\tau-\tau_0\rangle>0\}$. We have also

$$\{(x,t)\in U_0\,;\,f(x,t)\geq 0\}\smallsetminus\{(0,\tau-\tau_0)\}\subset B^2_{r_0}(0), \tag{7}$$

and (for some choices of α,γ) there exists an open subset of U_0 containing $(0,\tau-\tau_0)$ where $L_{\tilde{Q}}f\geq 0$ (see [31, Proposition 3.3]). By translation, we have that h is a L_Q-barrier for Ω at y. □

Remark 2.8. *Here the continuity of the coefficients of the matrix Q plays no role, as in [31]. The result holds true for horizontally elliptic operators with bounded measurable coefficients, with Q uniformly positive definite.*

If we think of the example of the Heisenberg group, we are saying that the Koranyi-ball condition which is natural for the sum of squares (see Remark 2.6) is appropriate also for operators as in (4).
Let us stress that the operators involved in Theorem 2.7 have the following explicit form

$$L_Q=\sum_{i,j=1}^m q_{ij}(x,t)\left(\partial^2_{x_ix_j}+\sum_{k=1}^n(B^kx)_i\partial^2_{x_jt_k}+\frac{1}{4}\sum_{k.l=1}^n(B^kx)_i(B^lx)_j\partial^2_{t_kt_l}\right),$$

in which first order terms do not appear.

3 The presence of first-order terms

If we consider vector fields satisfying a step-2 Hörmander rank condition without any underlying Carnot group structure, what we have seen at the end of the previous section may change drastically.

Remark 3.1. *Let us consider in* $\mathbb{R}^2$ *the two vector fields* $X_1 = \partial_x$ *and* $X_2 = x\partial_t$*. If we look at the operator*

$$X_1^2 + X_2^2 = \partial^2_{xx} + x^2\partial^2_{tt},$$

we can realize that an Hopf lemma at characteristic points can be proved under the assumption of the interior homogeneous ball $\{x^4 + t^2 < r^4\}$ *(see also [24, 26]). If* $y = (0, t_0)$ *and* $\nu = \partial_t$*, we can in fact use as barrier the following* $h(x,t) = e^{-\alpha\left(\gamma x^4 - (t-t_0)\right)} - 1$*, for some positive* γ, α*. On the other hand, if we consider the operators as in* (4) *built with respect to these two vector fields, this condition is not the right one. As a matter of fact, let us pick*

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad \text{and} \quad L_Q = \partial^2_{xx} - x\partial^2_{xt} + x^2\partial^2_{tt} - \frac{1}{2}\partial_t.$$

Fix Ω *such that it is contained in the halfspace* $\{t > 0\}$*, but it is flat enough to have the interior homogeneous ball property at* $(0,0)$*. Consider the function*

$$u(x,t) = \left(t + \frac{1}{8}x^2\right)^{\alpha}$$

with $0 < \alpha - 1 < \frac{1}{26}$*. A straightforward calculation shows that this is a counterexample to Hopf for* L_Q*.*

The reason of the behavior described in Remark 3.1 is the presence of the first order term $-\frac{1}{2}\partial_t$. In order to understand what happens in presence of such terms, we will always denote in this section by Ω a bounded open set in some $\mathbb{R}^N$ such that $0 \in \partial\Omega$, the positive t-direction determines the inner unit normal and it is a characteristic direction for the operator at 0.

Birindelli and Cutrí [5, Remark 2] noted that the Koranyi-ball condition for $\Omega \subset \mathbb{R}^3$ at 0 is enough to ensure the Hopf property also for an operator like

$$\Delta_{\mathbb{H}} + k_1(x,t)x_1\left(\partial_{x_1} - \frac{1}{2}x_2\partial_t\right) + k_2(x,t)x_2\left(\partial_{x_2} + \frac{1}{2}x_1\partial_t\right),$$

with k_j's bounded functions. Actually, we can also see that for the operators like

$$\Delta_{\mathbb{H}} + k_1(x,t)\partial_{x_1} + k_2(x,t)\partial_{x_2} + (x_1^2 + x_2^2)\gamma(x,t)\partial_t,$$

with bounded k_1, k_2, γ, we can get the Hopf property under the Koranyi-ball assumption. The barrier can be chosen as for $\Delta_{\mathbb{H}}$. This is the exact behavior Monticelli noted in [26, Lemma 4.1] for Grushin-type equations. He proved in particular a Hopf lemma in $\Omega \subset \mathbb{R}^2$ at 0 for operators like

$$\partial_{xx}^2 + x^2\partial_{tt}^2 + k(x,t)\partial_x + x^2\gamma(x,t)\partial_t,$$

with bounded k and γ, under a homogeneous interior ball condition. Having in mind these examples and Remark 3.1, we want to consider operators where the coefficient in front of ∂_t has lower degree of vanishing, or it does not vanish at all. We want thus to understand the geometry of Ω if we want an Hopf lemma to hold for model operators like

$$\Delta_{\mathbb{H}} \pm \partial_t, \qquad \partial_{xx}^2 + x^2\partial_{tt}^2 \pm \partial_t, \qquad \partial_{xx}^2 + x^2\partial_{tt}^2 \pm x\partial_t.$$

In order to do this, let us go back to the classical heat operator, seen as a degenerate elliptic operator in $\mathbb{R}^{N+1}$. Let us denote $\mathcal{H}^- = \Delta_x - \partial_t$ in $\mathbb{R}^{N+1}$. Suppose that Ω strictly contains $\{(x,t) \in \mathbb{R}^{N+1} : t + \frac{1}{2N}\|x\|^2 > 0\}$, at least locally around 0. Then $h(x,t) = t + \frac{1}{2N}\|x\|^2$ is clearly an $\mathcal{H}^-$-barrier function for Ω at 0, and thus $\mathcal{H}^-$ satisfies the Hopf lemma in Ω at 0. We stress that $t \sim -\frac{1}{2N}\|x\|^2$ is the behavior of the level set of the fundamental solution for $\mathcal{H}^-$ up to lower order terms (fundamental solution with pole at some $(0, -|t_0|)$ and passing through 0).
With the following counterexample we show that we cannot do much better than $t + \frac{1}{2N}\|x\|^2$.

Counterexample 3.2. *Suppose that Ω is contained in the region $\{(x,t) \in \mathbb{R}^{N+1} : t + \beta_0\|x\|^2 > 0\}$, for some $0 < \beta_0 < \frac{1}{2N}$. Then, we can choose $\beta_0 < \beta < \frac{1}{2N}$ and $\epsilon = \frac{(\beta-\beta_0)(1-2\beta N)}{4\beta^2} > 0$, and we can consider the function*

$$u(x,t) = (t + \beta\|x\|^2)^{1+\epsilon}.$$

This function is $C^2(\Omega) \cap C^1(\Omega \cup \{0\})$, $u(0) = 0$, $u > 0$ in Ω, and $u_t(0) = 0$. Moreover

$$\mathcal{H}^- u(x,t) = -(1-2\beta N)(1+\epsilon)(t+\beta\|x\|^2)^{-1+\epsilon}\left(t + \beta_0\|x\|^2\right) \le 0 \quad \text{in } \Omega.$$

Therefore u is a counterexample to the Hopf property for $\mathcal{H}^-$ in Ω.

The operator $\mathcal{H}^+ = \Delta_x + \partial_t$ has the same behavior than $\mathcal{H}^-$. It satisfies the Hopf lemma in the sets Ω which are "flat enough" to strictly contain the paraboloid $\{(x,t) \in \mathbb{R}^{N+1} : t > \frac{1}{2N}\|x\|^2\}$. And it does not satisfy the Hopf lemma in Ω if Ω is contained in a region delimited by a steeper paraboloid $\{(x,t) \in \mathbb{R}^{N+1} : t > \beta_0\|x\|^2\}$, for some $\beta_0 > \frac{1}{2N}$.

Regarding to the Hopf-property at 0, we can see that for the degenerate-elliptic operators

$$\Delta_{\mathbb{H}} \pm \partial_t, \qquad \partial_{xx}^2 + x^2\partial_{tt}^2 \pm \partial_t,$$

despite the non-parabolicity aspects, a similar analysis to the one for $\mathcal{H}^{\pm}$ holds true. We can summarize these facts in the next Hopf-type lemma.

Lemma 3.3. *Suppose there exists an open neighborhood* U *of* 0 *such that one of the following conditions holds true:*

$$\begin{aligned}
&(i) && \left\{(x_1,x_2,t)\in U\subset\mathbb{R}^3\,:\,t\geq\frac{1}{4}(x_1^2+x_2^2)\right\}\smallsetminus\{(0,0,0)\}\subset\Omega\subset\mathbb{R}^3;\\
&(ii) && \left\{(x_1,x_2,t)\in U\subset\mathbb{R}^3\,:\,t\geq-\frac{1}{4}(x_1^2+x_2^2)\right\}\smallsetminus\{(0,0,0)\}\subset\Omega\subset\mathbb{R}^3;\\
&(iii) && \left\{(x,t)\in U\subset\mathbb{R}^2\,:\,t\geq\frac{1}{2}x^2\right\}\smallsetminus\{(0,0)\}\subset\Omega\subset\mathbb{R}^2;\\
&(iv) && \left\{(x,t)\in U\subset\mathbb{R}^2\,:\,t\geq-\frac{1}{2}x^2\right\}\smallsetminus\{(0,0)\}\subset\Omega\subset\mathbb{R}^2.
\end{aligned}\tag{8}$$

Then, respectively, we have

(i) $\Delta_{\mathbb{H}}+\partial_t$ *satisfies the Hopf lemma in* Ω *at* $(0,0,0)$*;*

(ii) $\Delta_{\mathbb{H}}-\partial_t$ *satisfies the Hopf lemma in* Ω *at* $(0,0,0)$*;*

(iii) $\partial^2_{xx}+x^2\partial^2_{tt}+\partial_t$ *satisfies the Hopf lemma in* Ω *at* $(0,0)$*;*

(iv) $\partial^2_{xx}+x^2\partial^2_{tt}-\partial_t$ *satisfies the Hopf lemma in* Ω *at* $(0,0)$*.*

Proof. The barriers can be easily taken respectively as $h(x_1,x_2,t)=t-\frac{1}{4}(x_1^2+x_2^2)$, $h(x_1,x_2,t)=t+\frac{1}{4}(x_1^2+x_2^2)$, $h(x,t)=t-\frac{1}{2}x^2$, or $h(x,t)=t+\frac{1}{2}x^2$. $\square$

On the other hand, as in Counterexample 3.2, the conditions on Ω cannot be improved too much.

Counterexample 3.4. *Suppose one of the following conditions holds true:*

$$\begin{aligned}
&(i) && \Omega\subseteq\{(x_1,x_2,t)\in\mathbb{R}^3\,:\,t>\beta_0(x_1^2+x_2^2)\}\ \textit{for some positive}\ \beta_0>\frac{1}{4};\\
&(ii) && \Omega\subseteq\{(x_1,x_2,t)\in\mathbb{R}^3\,:\,t>-\beta_0(x_1^2+x_2^2)\}\ \textit{for some positive}\ \beta_0<\frac{1}{4};\\
&(iii) && \Omega\subseteq\{(x,t)\in\mathbb{R}^2\,:\,t>\beta_0x^2\}\ \textit{for some positive}\ \beta_0>\frac{1}{2};\\
&(iv) && \Omega\subseteq\{(x,t)\in\mathbb{R}^2\,:\,t>-\beta_0x^2\}\ \textit{for some positive}\ \beta_0<\frac{1}{2}.
\end{aligned}$$

Then, respectively,

(i) $\Delta_{\mathbb{H}}+\partial_t$ *does not satisfy the Hopf lemma in* Ω *at* $(0,0,0)$*;*

(ii) $\Delta_{\mathbb{H}}-\partial_t$ *does not satisfy the Hopf lemma in* Ω *at* $(0,0,0)$*;*

(iii) $\partial^2_{xx} + x^2\partial^2_{tt} + \partial_t$ *does not satisfy the Hopf lemma in* Ω *at* $(0,0)$;

(iv) $\partial^2_{xx} + x^2\partial^2_{tt} - \partial_t$ *does not satisfy the Hopf lemma in* Ω *at* $(0,0)$.

The functions $u(x_1,x_2,t) = (t \mp \beta(x_1^2 + x_2^2))^\alpha$ *or* $u(x,t) = (t \mp \beta x^2)^\alpha$ *work as counterexamples for suitable choices of* $\alpha > 1$ *and* $\beta > 0$. *The choice of* β *has to be done respectively as* $\beta_0 > \beta > \frac{1}{4}$, $\beta_0 < \beta < \frac{1}{4}$, $\beta_0 > \beta > \frac{1}{2}$, *or* $\beta_0 < \beta < \frac{1}{2}$.

Let us stress that the case (ii) gives a counterexample to the Hopf property for $\Delta_{\mathbb{H}} - \partial_t$ in domains Ω which can satisfy the Koranyi-ball condition.

Similarly to $\mathcal{H}^\pm$, we can analyze the case of the operator

$$\partial^2_{xx} + x\partial_t, \qquad \text{in } \mathbb{R}^2.$$

This is well-studied in literature: it is the stationary part of the Kolmogorov operator, and it is an example of the so-called degenerate Ornstein-Uhlenbeck operators (see [14, 18, 7]). Suppose there exists an open neighborhood U of 0 such that

$$\left\{(x,t) \in U \subset \mathbb{R}^2 \,:\, t \geq \frac{1}{6}x^3\right\} \smallsetminus \{(0,0)\} \subset \Omega,$$

then $\partial^2_{xx} + x\partial_t$ satisfies the Hopf lemma in Ω at 0. As before, a barrier can be easily constructed as $h(x,t) = t - \frac{1}{6}x^3$.

Counterexample 3.5. *If we want to construct counterexamples analogue to the previous ones, let us define the following function*

$$f_{\beta_0^\pm}(x) = \begin{cases} \beta_0^+ x^3 & \text{if } x > 0 \\ \beta_0^- x^3 & \text{if } x < 0. \end{cases}$$

Suppose that

$$\Omega \subseteq \left\{(x,t) \in \mathbb{R}^2 \,:\, t > f_{\beta_0^\pm}(x)\right\}$$

for some $\beta_0^+ > \frac{1}{6}$ *and* $0 < \beta_0^- < \frac{1}{6}$. *Then, we can consider the function*

$$u(x,t) = (t - f_{\beta^\pm})^\alpha,$$

with $\beta_0^+ > \beta^+ > \frac{1}{6}$, $\beta_0^- < \beta^- < \frac{1}{6}$, *and* $\alpha > 1$. *The function* $f_{\beta^\pm}$ *is smooth enough to ensure that* $u \in C^2(\Omega) \cap C^1(\Omega \cup \{0\})$. *Moreover* u *is positive in* Ω, *and* $u(0) = u_t(0) = 0$. *Suitable choices of* β^+, β^-, α *give* $\left(\partial^2_{xx} + x\partial_t\right) u \leq 0$ *in* Ω *and* u *is thus a counterexample to the Hopf lemma in* Ω.

Let us now turn our attention to the degenerate-elliptic operator

$$\partial^2_{xx} + x^2\partial^2_{tt} + x\partial_t.$$

Also for this one, assuming that there exists an open neighborhood U of 0 such that

$$\left\{(x,t)\in U\subset\mathbb{R}^2\,:\,t\geq\frac{1}{6}x^3\right\}\smallsetminus\{(0,0)\}\subset\Omega, \tag{9}$$

$(\partial^2_{xx}+x^2\partial^2_{tt}+x\partial_t)$ satisfies the Hopf property in Ω at 0: $h(x,t)=t-\frac{1}{6}x^3$ is in fact still a barrier. However we cannot be as precise as in Counterexample 3.5. We are able to find a counterexample just assuming that

$$\Omega\subseteq\left\{(x,t)\in\mathbb{R}^2\,:\,t>\beta_0x^2\right\}$$

for some positive β_0. In this case, by taking $0<\beta<\beta_0$, the function $u(x,t)=(t-\beta x^2)^\alpha$ works as counterexample to Hopf in Ω for some $\alpha>1$. We can slightly improve the condition (9) by asking the following: there exists some positive γ such that

$$\left\{(x,t)\in U\subset\mathbb{R}^2\,:\,t\geq\frac{1}{6}x^3+\gamma x^4\right\}\smallsetminus\{(0,0)\}\subset\Omega.$$

The $(\partial^2_{xx}+x^2\partial^2_{tt}+x\partial_t)$-barrier in Ω can be then built as $h(x,t)=e^{-\alpha\left(\frac{1}{6}x^3+\gamma x^4-t\right)}-1$ for some big α. A similar improvement could also be done with the conditions (8) by adding a term like γx^4 (we are going to exploit this fact in the proof of Proposition 4.1 below).

Remark 3.6. *Let us just mention that, for any $k\in\mathbb{N}$, the operator*

$$\partial^2_{xx}+x^k\partial_t,\qquad \textit{in } \mathbb{R}^2,$$

behaves regarding the Hopf property similar to $\mathcal{H}^+=\partial^2_{xx}+\partial_t$ (in the case of k even) or to $\partial^2_{xx}+x\partial_t$ (for k odd), with the natural adjustments.

The behavior observed in the previous specific degenerate-elliptic examples occurs also in different situations. Let us consider in $\mathbb{R}^3$ the two vector fields

$$X_1=\partial_{x_1}+b^1(x)\partial_t,\qquad X_2=\partial_{x_2}+b^2(x)\partial_t, \tag{10}$$

where $b=(b^1,b^2):U_0\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is smooth, defined in an open neighborhood U_0 of $(0,0)$, and such that $b(0,0)=(0,0)$. Suppose that

$$[X_1,X_2](0)=b^2_{x_1}(0,0)-b^1_{x_2}(0,0)\neq 0. \tag{11}$$

We want to investigate the operator

$$X_1^2+X_2^2=\Delta_x+2b^1(x)\partial^2_{x_1t}+2b^2(x)\partial^2_{x_2t}+\left((b^1(x))^2+(b^2(x))^2\right)\partial^2_{tt}+\operatorname{div}(b)\partial_t.$$

Since $b(0)=0$, the positive t-direction is characteristic at 0. Let us define the following polynomial of degree 3

$$F_b(x)=\frac{1}{2}\left\langle\mathcal{J}^sb(0)x,x\right\rangle+\frac{1}{6}\left(b^1_{x_1,x_1}(0)x_1^3+3b^1_{x_1,x_2}(0)x_1^2x_2+3b^2_{x_1,x_2}(0)x_1x_2^2+b^2_{x_2,x_2}(0)x_2^3\right),$$

where $\mathcal{J}^sb(0)$ is the symmetric part of the Jacobian matrix of b at $(0,0)$.

Theorem 3.7. *Let X_1, X_2 the two vector fields in $\mathbb{R}^3$ defined in* (10)*, satisfying* (11)*. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded open set with $0 \in \partial\Omega$, and with $(0,0,1)$ as inner unit normal at 0. Suppose also that there exist an open neighborhood U of $(0,0,0)$ and a positive constant γ such that*

$$\Big\{(x,t) \in U \subset \mathbb{R}^3 \,:\, t \geq F_b(x) + \gamma \|x\|^4\Big\} \smallsetminus \{(0,0,0)\} \subset \Omega.$$

Then the operator $\mathcal{L} = X_1^2 + X_2^2$ satisfies the Hopf lemma in Ω at 0.

Proof. We want to prove that

$$h(x,t) = e^{-\alpha\left(\gamma\|x\|^4 + F_b(x) - t\right)} - 1$$

is an $\mathcal{L}$-barrier function in Ω at 0. To this aim we need some preliminaries. First of all we have

$$\begin{aligned}\Delta F_b(x) &= b^1_{x_1}(0) + b^2_{x_2}(0) + b^1_{x_1,x_1}(0)x_1 + b^1_{x_1,x_2}(0)x_2 + b^2_{x_1,x_2}(0)x_1 + b^2_{x_2,x_2}(0)x_2 \\ &= \operatorname{div}(b)(x) + \|x\|^2 k(x)\end{aligned}$$

where k is bounded in a neighborhood of $(0,0)$. Moreover we get

$$\begin{aligned}\nabla F_b(x) - b(x) &= \begin{pmatrix} b^1_{x_1}(0)x_1 + b^1_{x_2}(0)x_2 + \frac{1}{2}[X_1,X_2](0)x_2 + O(\|x\|^2) - b^1(x) \\ b^2_{x_1}(0)x_1 + b^2_{x_2}(0)x_2 - \frac{1}{2}[X_1,X_2](0)x_1 + O(\|x\|^2) - b^2(x) \end{pmatrix} \\ &= \frac{1}{2}[X_1,X_2](0)\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} + \|x\|^2 \begin{pmatrix} k_1(x) \\ k_2(x) \end{pmatrix},\end{aligned}$$

with k_1 and k_2 bounded functions in a neighborhood of $(0,0)$. Denoting by $f(x,t) = \gamma\|x\|^4 + F_b(x) - t$ and by

$$\sigma(x) = \begin{pmatrix} 1 & 0 & b^1(x) \\ 0 & 1 & b^2(x) \end{pmatrix},$$

we can write

$$\begin{aligned}\mathcal{L}h(x,t) &= \operatorname{Tr}\left(\sigma^T(x)\sigma(x)\mathcal{H}h(x,t)\right) + \operatorname{div}(b)(x)\partial_t h(x,t) \\ &= \alpha e^{-\alpha f(x,t)}\left(\operatorname{div}(b)(x) - \Delta F_b(x) - 16\gamma\|x\|^2 + \alpha\|\sigma(x)\nabla f(x,t)\|^2\right) \\ &= \alpha e^{-\alpha f(x,t)}\left(-\|x\|^2 k(x) - 16\gamma\|x\|^2 + \alpha\left\|4\gamma\|x\|^2 x + \nabla F_b(x) - b(x)\right\|^2\right) \\ &= \alpha\|x\|^2 e^{-\alpha f(x,t)}\left(-k(x) - 16\gamma + \alpha\left(\frac{1}{4}\left([X_1,X_2](0)\right)^2 + o(1)\right)\right).\end{aligned}$$

Thus, in a small neighborhood of 0, for α big enough we get $\mathcal{L}h \geq 0$ because of the assumption (11). This proves that h is an $\mathcal{L}$-barrier since $h(0) = 0$ and $h_t(0) = \alpha$. $\square$

The comparison with the degree-3 polynomial F_b is suggested by the examples in the first part of this section (and looking at the first order term $\operatorname{div}(b)\partial_t$). In the case of the Heisenberg vector fields, where $b(x) = \frac{1}{2}(-x_2, x_1)$, we have $F_b \equiv 0$ and Theorem 3.7 gives back the flatness condition of Birindelli and Cutrí.

4 A remark on the global behavior

Throughout the paper we have considered conditions on the behavior of $\partial\Omega$ around a characteristic point. Here we would like to exploit such analysis in order to construct bounded open sets Ω in which our operators satisfy the Hopf lemma at every boundary point. It is not difficult to see that this is not possible for operators like ∂^2_{xx} in $\mathbb{R}^2$ or $\Delta_x \pm \partial_t$ in $\mathbb{R}^{N+1}$. Nonetheless, it is possible for the family of operators L_Q as in (4) which we have treated in Section 2 and in Section 3, and it is possible in a uniform way with respect to uniformly positive definite matrices Q.

By looking at the proof of Theorem 2.7 we may recognize that we have already proved that in the sets

$$B^2_{r_0}(\xi_0,\tau_0) = (\xi_0,\tau_0) \circ \Big\{(x,t) \in \mathbb{R}^m \times \mathbb{R}^n \,:\, \|x\|^4 + \|t\|^2 < r^4\Big\} \subset \mathbb{R}^N$$

every operator $L_Q = \sum_{i,j=1}^m q_{i,j}(x,t)X_iX_j$ (with $X_i = \partial_{x_i} + \frac{1}{2}\sum_{k=1}^n (B^k x)_i \partial_{t_k}$ as in (5)) satisfies the Hopf lemma at every boundary point of $B^2_{r_0}(\xi_0,\tau_0)$. This holds true for any symmetric uniformly positive definite matrix $Q(x,t)$. As a matter of fact, we have seen with (6) that the characteristic points of $\partial B^2_{r_0}(\xi_0,\tau_0)$ are just the ones of the form (ξ_0,τ) (with $\|\tau - \tau_0\| = r_0^2$) and we have showed that the barrier functions at those points are actual L_Q-barrier functions in $B^2_{r_0}(\xi_0,\tau_0)$ (see (7)).

On the other hand, we have seen in Remark 3.1 that, for the vector fields $X_1 = \partial_x$ and $X_2 = x\partial_t$ in $\mathbb{R}^2$, the operators L_Q may not satisfy the Hopf lemma in the homogeneous ball $\{(x,t) \in \mathbb{R}^2 \,:\, x^4 + t^2 < 1\}$. We have to change this set accordingly to what we have showed in Section 3. To this aim, let us fix $\Lambda \geq \lambda > 0$ and define the following bounded open set

$$B_{\frac{\Lambda}{\lambda}} = \left\{(x,t) \in \mathbb{R}^2 \,:\, x^4 - \frac{1}{2}\left(\frac{\Lambda}{\lambda} - 1\right)x^2 + t^2 < 1\right\} \subset \mathbb{R}^2.$$

Proposition 4.1. *For any 2×2 symmetric matrix $Q(x,t)$ such that $\lambda\mathbb{I}_2 \leq Q \leq \Lambda\mathbb{I}_2$, the operator $L_Q = \sum_{i,j=1}^2 q_{i,j}(x,t)X_iX_j$ satisfies the Hopf lemma in $B_{\frac{\Lambda}{\lambda}}$ at any point of its boundary.*

Proof. By putting

$$\sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix},$$

we can write

$$L_Q u(x,t) = \mathrm{Tr}\Big(\sigma(x)Q(x,t)\sigma(x)\mathcal{H}u(x,t)\Big) + q_{1,2}(x,t)\partial_t u(x,t).$$

It is easy to see that the only characteristic points are the ones on the line $x = 0$, i.e. $(0,\pm 1)$. Let us just consider the point $(0,-1)$, the other case will follow analogously.

We claim that there exists a positive γ such that

$$\left\{(x,t)\in\mathbb{R}^2\,:\,0>t\geq -1-\frac{1}{4}\left(\frac{\Lambda}{\lambda}-1\right)x^2+\gamma x^4,\,|x|\leq\rho\right\}\smallsetminus\{(0,-1)\}\subset B_{\frac{\Lambda}{\lambda}},$$

for a suitable positive ρ small enough. In fact, if $(x,t)\neq(0,-1)$ belongs to the set in the left-hand side, we have

$$x^4-\frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)x^2+t^2\leq 1+x^4\left(1+\frac{1}{16}\left(\frac{\Lambda}{\lambda}-1\right)^2-2\gamma-x^2\gamma\frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)+x^4\gamma^2\right)$$

which is strictly less than 1 if $2\gamma>1+\frac{1}{16}\left(\frac{\Lambda}{\lambda}-1\right)^2$ and $|x|$ is small. With such a choice of γ, we want to consider the function

$$h(x,t)=e^{-\alpha\left(\gamma x^4-\frac{1}{4}\left(\frac{\Lambda}{\lambda}-1\right)x^2-t\right)}-e^{-\alpha},$$

for some positive α to be chosen. Of course, $h(0,-1)=0$ and $h_t(0,-1)=\alpha e^{-\alpha}\neq 0$. Moreover, by denoting $v(x,t)=\gamma x^4-\frac{1}{4}\left(\frac{\Lambda}{\lambda}-1\right)x^2-t$, we get

$$\begin{aligned}L_Qh(x,t)\; &= \;\alpha e^{-\alpha v(x,t)}\left(q_{1,2}(x,t)+q_{1,1}(x,t)\left(\frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)-12\gamma x^2\right)+\right.\\ &+\;\left.\alpha\left\langle Q(x,t)\sigma(x)\nabla v(x,t),\sigma(x)\nabla v(x,t)\right\rangle\right).\end{aligned}$$

The bounds on the eigenvalues of Q give that $q_{1,2}(x,t)+\frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)q_{1,1}(x,t)\geq 0$. Hence we have

$$\begin{aligned}L_Qh(x,t)\; &\geq\; \alpha e^{-\alpha v(x,t)}\left(-12\Lambda\gamma x^2+\alpha\lambda\left\|\sigma(x)\nabla v(x,t)\right\|^2\right)\\ &=\; \alpha x^2e^{-\alpha v(x,t)}\left(-12\Lambda\gamma+\alpha\lambda\left(1+\left(4\gamma x^2-\frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)\right)^2\right)\right)\\ &\geq\; \alpha x^2e^{-\alpha v(x,t)}\left(\alpha\lambda-12\Lambda\gamma\right)\end{aligned}$$

which is nonnegative if α is big enough. This proves that h is an L_Q-barrier function in $B_{\frac{\Lambda}{\lambda}}$ at $(0,-1)$. And it concludes the proof. $\square$

We would like to stress that, in the case $\Lambda=\lambda$, L_Q is forced to be $\lambda(\partial^2_{xx}+x^2\partial^2_{tt})$: the set $B_{\frac{\Lambda}{\lambda}}$ coincides with the homogeneous ball $\{x^4+t^2<1\}$ and we recover the condition in [24, 26].

5 A nonlinear example: the Levi operator

Now we want to consider boundary comparison principles of Hopf-type for some non-linear second order degenerate-elliptic operators. Just to fix some notations, let us put $\mathfrak{L}u = F(x, u, \nabla u, \mathcal{H}u)$ for some function F smooth with respect to its entries. We will always assume that $\mathfrak{L}$ satisfies the interior comparison principle and $\mathfrak{L}$ is degenerate-elliptic, i.e.

$$F(x, u, p, M) \geq F(x, u, p, N) \quad \text{whether} \quad M \geq N.$$

Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth boundary, $y \in \partial\Omega$, and ν the inner unit normal to $\partial\Omega$ at y. We say that $\mathfrak{L}$ satisfies the Hopf lemma in Ω at $y \in \partial\Omega$ if, for every couple of functions $u, v \in C^2\left((\Omega \cap W) \cup \{y\}\right)$, we have

$$\begin{cases} \mathfrak{L}u \leq \mathfrak{L}v \text{ in } \Omega \cap W, \\ u > v \text{ in } \Omega \cap W, \\ u(y) = v(y), \end{cases} \quad \Rightarrow \quad \frac{\partial u}{\partial \nu}(y) > \frac{\partial v}{\partial \nu}(y), \tag{12}$$

where W is an open neighborhood of y.

Remark 5.1. *If $\mathfrak{L}$ is strictly elliptic, i.e. the derivative of F with respect to the matrix entrance is strictly positive definite, then it is well-known that $\mathfrak{L}$ satisfies the Hopf lemma in Ω at $y \in \partial\Omega$, at every boundary point y of every smooth Ω. Let us briefly sketch the proof, having in mind the barrier-type argument in Remark 2.1. By definition, for any C^2-function v, we have that the linearized operator $\mathcal{L}_v$ is an elliptic linear operator defined by*

$$\mathfrak{L}(v + \epsilon h) = \mathfrak{L}v + \epsilon \mathcal{L}_v(h) + o(\epsilon), \qquad \text{as } \epsilon \to 0, \text{ for any } h \in C^2.$$

Since smooth sets Ω have the interior ball property, we can consider a function $h = h_\alpha$ as in Remark 2.2. By the ellipticity of $\mathcal{L}_v$, we can find a neighborhood U of y where $\mathcal{L}_v h \geq c_v > 0$ (see (2)). We can always assume $U \subset W$. We have to compare the functions u and $v + \epsilon h$ on the set $V = \{p \in U : h(p) > 0\} \subset \Omega \cap W$. For small positive ϵ we get

$$\begin{cases} \mathfrak{L}u \leq \mathfrak{L}v + \epsilon(\mathcal{L}_v(h) + o(1)) = \mathfrak{L}(v + \epsilon h) \text{ in } V, \\ u \geq v + \epsilon h \text{ on } \partial V. \end{cases}$$

By the interior comparison principle we deduce that $u \geq v + \epsilon h$ in V, and thus

$$\frac{\partial u}{\partial \nu}(y) - \frac{\partial v}{\partial \nu}(y) \geq \epsilon \frac{\partial h}{\partial \nu}(y) > 0.$$

On the other hand, if $\mathfrak{L}$ is not strictly elliptic, we have to deal with linearized operators with non-trivial kernels. These last ones may depend on the functions on which we are linearizing. For this reason we need a suitable definition of characteristic points.

Definition 5.2. *Let $y \in \partial\Omega$, and W be an open neighborhood of y. Let also $v \in C^2((\Omega \cap W) \cup \{y\})$. We say that y is characteristic for $(\mathfrak{L}, v, \Omega)$ if*

$$y \text{ is characteristic for } (\mathcal{L}_v, \Omega)$$

according to Definition 1.1.

With this definition, we have readily the following

Proposition 5.3. *Let $y \in \partial\Omega$, and W be an open neighborhood of y. Suppose we have $u, v \in C^2((\Omega \cap W) \cup \{y\})$. Assume also that y is not characteristic for $(\mathfrak{L}, v, \Omega)$ or for $(\mathfrak{L}, u, \Omega)$ (for at least one of the two). Then* (12) *holds true.*

Proof. If y is not characteristic for $(\mathfrak{L}, v, \Omega)$, we follow the proof in Remark 5.1. On the other hand, if y is not characteristic for $(\mathfrak{L}, u, \Omega)$, we can compare the functions $u - \epsilon h$ and v with the same type of choice for the barrier h. The key point is that, in both cases, the functions h as in Remark 2.2 satisfy $\mathcal{L}_v h \geq c_v > 0$ or $\mathcal{L}_u h \geq c_u > 0$ in a suitable neighborhood of y. This is enough to ensure that $\mathfrak{L}u \leq \mathfrak{L}(v + \epsilon h)$ or $\mathfrak{L}(u - \epsilon h) \leq \mathfrak{L}v$ in the desired set and to conclude the proof. □

At the points y which are characteristic for both $(\mathfrak{L}, v, \Omega)$ and $(\mathfrak{L}, u, \Omega)$, the Hopf property may not hold true. As an example, let us introduce the nonlinear operator describing the Levi curvature for a real hypersurface in $\mathbb{C}^2$ (see e.g. [8, 23], and the references therein, for a complete exposition about this operator). Let us fix some notations.

For a smooth function u defined in $\mathbb{R}^3$, let us put

$$A(\nabla u) = \begin{pmatrix} 1 + u_t^2 & 0 & u_y - u_x u_t \\ 0 & 1 + u_t^2 & -u_x - u_y u_t \\ u_y - u_x u_t & -u_x - u_y u_t & u_x^2 + u_y^2 \end{pmatrix}.$$

This symmetric matrix is nonnegative definite: it has eigenvalues $1 + u_t^2$, $1 + |\nabla u|^2$, and 0. The eigenvector of $A(\nabla u)$ related to the eigenvalue 0 is

$$(-u_y + u_x u_t, u_x + u_y u_t, 1 + u_t^2).$$

We want to consider the following operator

$$\mathfrak{L}u = \frac{1}{(1 + |\nabla u|^2)^{3/2}} \mathrm{Tr}\Big(A(\nabla u)\ \mathcal{H}u\Big).$$

We say that u is strictly pseudoconvex at a point if $\mathfrak{L}u > 0$ at that point. This convexity property is equivalent to a Hörmander-type condition. In fact, by defining the non-linear vector fields

$$X_u = \partial_x + \frac{u_y - u_x u_t}{1 + u_t^2}\partial_t, \qquad Y_u = \partial_y - \frac{u_x + u_y u_t}{1 + u_t^2}\partial_t,$$

we can write the operator as the following sum of squares

$$\mathfrak{L}u = \frac{(1+u_t^2)^2}{(1+|\nabla u|^2)^{3/2}}\left(X_u^2 u + Y_u^2 u\right),$$

and we have

$$[X_u, Y_u] = -\frac{(1+|\nabla u|^2)^{3/2}}{(1+u_t^2)^2}\mathfrak{L}u\,\partial_t.$$

Therefore, X_u and Y_u satisfy the Hörmander condition iff u is strictly pseudoconvex. For a fixed function u, the linearized operator $\mathcal{L}_u$ is given by the following

$$\begin{aligned}
\mathcal{L}_u h &= \frac{1}{(1+|\nabla u|^2)^{3/2}}\Bigg(\mathrm{Tr}\Big(A(\nabla u)\ \mathcal{H}h\Big)+\\
&+ \left(2u_x u_{tt} - 2u_t u_{xt} - 2u_{yt} - 3\frac{\mathrm{Tr}(A(\nabla u)\mathcal{H}u)}{1+|\nabla u|^2}u_x\right)h_x +\\
&+ \left(2u_y u_{tt} + 2u_{xt} - 2u_t u_{yt} - 3\frac{\mathrm{Tr}(A(\nabla u)\mathcal{H}u)}{1+|\nabla u|^2}u_y\right)h_y +\\
&+ \left(2u_t u_{xx} + 2u_t u_{yy} - 2u_x u_{xt} - 2u_y u_{yt} - 3\frac{\mathrm{Tr}(A(\nabla u)\mathcal{H}u)}{1+|\nabla u|^2}u_t\right)h_t\Bigg).
\end{aligned}$$

From now on, let us assume Ω to be a bounded open set in $\mathbb{R}^3$, with $0 \in \partial\Omega$, and $\nu = (0,0,1)$ as inner unit normal at 0. The fact that 0 is characteristic for $(\mathfrak{L}, u, \Omega)$ is thus equivalent to

$$u_x^2(0) + u_y^2(0) = \langle A(\nabla u(0))\nu, \nu\rangle = 0.$$

We want to construct two functions for which the Hopf property does not hold true.

Counterexample 5.4. *Let us consider Ω described by $\{(x,y,t) \in \mathbb{R}^3\,;\, t > x^2+y^2\}$ at least locally around 0, and let*

$$v(x,y,t) = x^2+y^2, \qquad u(x,y,t) = x^2+y^2 - \frac{(x^2+y^2)^2}{2} + \frac{t^2}{2}.$$

We have

$$\nabla v(x,y,t) = (2x, 2y, 0), \qquad \nabla u(x,y,t) = (2x(1-x^2-y^2), 2y(1-x^2-y^2), t),$$

and

$$\begin{aligned}
\mathfrak{L}v(x,y,t) &= \frac{4}{\Big(1+4(x^2+y^2)\Big)^{\frac{3}{2}}},\\
\mathfrak{L}u(x,y,t) &= \frac{4(1+t^2)\left(1+(x^2+y^2)\Big(\dfrac{(1-x^2-y^2)^2}{1+t^2}-2\Big)\right)}{\Big(1+4(x^2+y^2)(1-x^2-y^2)^2+t^2\Big)^{\frac{3}{2}}}.
\end{aligned}$$

In particular we note that $\mathfrak{L}u(0) = \mathfrak{L}v(0) = 4$. *Let us put*

$$f(x, y, t) = \mathfrak{L}v(x, y, t) - \mathfrak{L}u(x, y, t).$$

A straightforward calculation shows that

$$\nabla f(0) = (0, 0, 0), \qquad \mathcal{H}f(0) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

and therefore the existence of a local minimum for f *at* 0. *Then, we can take* W *such that* $f(x, y, t) \geq f(0, 0, 0)$ *in* W, *i.e.*

$$\mathfrak{L}v(x, y, t) \geq \mathfrak{L}u(x, y, t), \qquad \forall (x, y, t) \in W.$$

Thus the following is true

$$\begin{cases} \mathfrak{L}u \leq \mathfrak{L}v & \text{in } \Omega \cap W \\ u > v & \text{in } \Omega \cap W \\ u(0) = v(0) & \end{cases}$$

but

$$\frac{\partial u}{\partial \nu}(0) = \frac{\partial v}{\partial \nu}(0) = 0.$$

We notice that a counterexample to the Hopf lemma for a nonlinear operator related to $\mathfrak{L}$ was shown in [21, Example 7.2]. For the operator treated in [21] a Hörmander-type condition is not available and also strong comparison principles may fail. On the other hand, for the Levi operator $\mathfrak{L}$ strong comparison theorems hold true under pseudoconvexity conditions [23], but we have just seen that boundary comparison principles of Hopf type are not true in general even assuming a Hörmander condition/strict pseudoconvexity.

Finally, with the previous sections in mind, we want to find a criterion on Ω in order to have the Hopf-type property for $\mathfrak{L}$ for any given couple of functions such that 0 is characteristic for both of them.

Proposition 5.5. *Let* $u, v \in C^2\left((\Omega \cap W) \cup \{y\}\right)$. *Assume* 0 *is characteristic for* $(\mathfrak{L}, u, \Omega)$ *and* $(\mathfrak{L}, v, \Omega)$. *Put*

$$\beta_0 = \max\left\{-\mathfrak{L}u(0)\frac{u_t(0)}{\sqrt{1 + u_t^2(0)}}, -\mathfrak{L}v(0)\frac{v_t(0)}{\sqrt{1 + v_t^2(0)}}\right\}.$$

Suppose there exists $\beta < \beta_0$ *such that*

$$\left\{(x, y, t) \in U \,:\, t \geq \frac{\beta}{4}(x^2 + y^2)\right\} \setminus \{(0, 0, 0)\} \subset \Omega$$

for some neighborhood U *of* 0, *then* (12) *is satisfied.*

Proof. First of all, by assumption, we have $u_x(0) = u_y(0) = v_x(0) = v_y(0) = 0$. Let us consider
$$h(x, y, t) = t - \frac{\beta}{4}(x^2 + y^2).$$
Suppose $\beta_0 = -\mathfrak{L}v(0)\frac{v_t(0)}{\sqrt{1+v_t^2(0)}}$: as in Remark 5.1, we are going to compare u and $v + \epsilon h$. Otherwise we can work with $u - \epsilon h$ and v.
A simple calculation shows that
$$\mathcal{L}_v h(0) = \frac{1}{1 + v_t^2(0)}\left(-\beta\sqrt{1 + v_t^2(0)} - \mathfrak{L}v(0)v_t(0)\right) > 0.$$
By continuity $\mathcal{L}_v h$ is strictly positive in a neighborhood of 0. We can then conclude as in Remark 5.1. □

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