$\overline{Q}'$-curvature flow on Pseudo-Einstein CR manifolds

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Abstract In this paper we consider the problem of prescribing the $\overline{Q}'$-curvature on three dimensional Pseudo-Einstein CR manifolds. We study the gradient flow generated by the related functional and we will prove its convergence to a limit function under suitable assumptions.

Keywords: Pseudo-Einstein CR manifolds, $\overline{P}'$-operator

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1 Introduction and statement of the results

Let $(M, T^{1,0}M, \theta)$ be a CR three manifold, which we will always assume smooth and closed. It is known that one can construct a pair (Q, P_θ) such that under a conformal change of the contact form $\hat{\theta} = e^u\theta$, one has

$$P_\theta u + Q_\theta = Q_{\hat{\theta}} e^{2u}$$

where the Paneitz operator $P_\theta = (\Delta_b)^2 + T^2 + l.o.t.$; in particular the operator P_θ contains the space of CR pluriharmonic functions $\mathcal{P}$ in its kernel, moreover the total Q-curvature is always zero [16], hence it does not provide any extra geometric information.
Therefore, one considers another pair (P', Q'), see [3], where P' is a Paneitz type operator satisfying $P' = 4(\Delta_b)^2 + l.o.t.$ and is defined on the space of pluriharmonic functions and the Q'-curvature is defined implicitly so that

$$P'_\theta u + Q'_\theta - \frac{1}{2}P_\theta(u^2) = Q'_{\hat{\theta}} e^{2u},$$

which is equivalent to

$$P'_\theta u + Q'_\theta = Q'_{\hat{\theta}} e^{2u} \text{ mod } \mathcal{P}^\perp. \tag{1}$$

In the case of pseudo-Einstein three dimensional CR manifolds (we refer the reader to the next section for further details), in [10] the authors showed that the total Q'-curvature is not always zero and it is invariant under the conformal change of the contact structure; in

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particular it is proportional to the Burns-Epstein invariant $\mu(M)$ (see [5], [13]) and if (M,J) is the boundary of a strictly pseudo-convex domain X, then

$$\int_M Q'\theta\wedge d\theta = 16\pi^2\Big(\chi(X) - \int_X (c_2 - \frac{1}{3}c_1^2)\Big),$$

where c_1 and c_2 are the first and second Chern forms of the Kähler-Einstein metric on X obtained by solving Fefferman's equation.

Now, equation (1) has to be solved orthogonally to the infinite dimensional space $\mathcal{P}^\perp$: in order to solve this problem on a pseudo-Einstein three dimensional CR manifolds, in [9] it is introduced a new couple $(\overline{P}', \overline{Q}')$, which comes from the projection of equation (1) on to the space of L^2 CR pluriharmonic functions $\hat{\mathcal{P}}$, which is the completion of $\mathcal{P}$ under the L^2-norm. Since the P'-operator is only defined after projection on $\mathcal{P}$, we denote by $\Gamma: L^2(M)\to\hat{\mathcal{P}}$ the orthogonal projection and we let $\overline{P}' = \Gamma\circ P'$ and $\overline{Q}' = \Gamma\circ Q'$, then on a pseudo-Einstein CR manifolds, we can consider the problem of prescribing the $\overline{Q}'$-curvature, under a conformal change of the contact structure, in particular: for a given a function $f\in\hat{\mathcal{P}}$, we have the following equation

$$P'_\theta u + Q'_\theta = fe^{2u} \bmod \mathcal{P}^\perp, \tag{2}$$

that is equivalent to

$$\overline{P}'_\theta u + \overline{Q}'_\theta = \Gamma(fe^{2u}).$$

Therefore, if u solves (2), then by setting $\tilde{\theta} = e^u\theta$, one has $\overline{Q}'_{\tilde{\theta}} = f$. Let us explicitly notice the differences between the two projections. Since the space of L^2 CR pluriharmonic functions $\hat{\mathcal{P}}$ does not depend on the contact form, thus $\overline{Q}'_\theta$ is the orthogonal projection of Q'_θ on $\hat{\mathcal{P}}$ with respect to the L^2-inner product induced by θ, while $\overline{Q}'_{\tilde{\theta}}$ is the orthogonal projection of $Q'_{\tilde{\theta}}$ with respect to the L^2-inner product induced by $\tilde{\theta}$; in particular $\phi\in\hat{\mathcal{P}}_\theta$ if and only if $\phi\in\hat{\mathcal{P}}_{\tilde{\theta}}$ and $\psi\in\mathcal{P}_\theta^\perp$ if and only if $e^{-2u}\psi\in\mathcal{P}_{\tilde{\theta}}^\perp$. Therefore, by denoting Γ_u the orthogonal projection induced by $\tilde{\theta}$, one has $\Gamma_u(Q'_{\tilde{\theta}}) = f$. Let us also recall that in [10], the authors show that the non-negativity of the Paneitz operator P_θ and the positivity of the CR-Yamabe invariant imply that $\overline{P}'$ is non-negative and $\ker\overline{P}' = \mathbb{R}$. Moreover, $\int_M Q'_\theta = \int_M \overline{Q}'_\theta \le 16\pi^2$ with equality if an only if $(M, T^{1,0}M, \theta)$ is the standard sphere; in particular the previously assumptions imply that the $(M, T^{1,0}M, \theta)$ is embeddable (see [12]). Notice that unlike the Riemannian case, it remains unclear if the non-negativity of $\overline{P}'$ and $\ker\overline{P}' = \mathbb{R}$ is a sufficient condition for $\int_M Q'_\theta \le 16\pi^2$. In particular, the results presented in this paper do not fully cover the case $\overline{P}'\ge 0$ and $\ker\overline{P}' = \mathbb{R}$.

Thus, from now on we will always assume that $(M, T^{1,0}M, \theta)$ is a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and $\ker\overline{P}' = \mathbb{R}$. The problem in (2) was first studied in [10] for f constant and in the subcritical case, namely $\int_M \overline{Q}'_\theta < 16\pi^2$. Then in [21] the problem was solved for $f>0$ via a probabilistic approach again in the subcritical case; also, a solution of the problem was provided in [17] for $f>0$ and $0<\int_M \overline{Q}'_\theta < 16\pi^2$ via direct minimization.

In this paper we will study the equation (2) allowing f to change sign: our approach follows closely the methods in [1], where the authors study the analogous problem in the Riemannian setting. In particular, we will use a variational approach by defining a suitable functional on an appropriate space and then we will study the evolution problem along the negative gradient flow lines: the convergence at infinity will provide a solution to the initial problem.

Indeed, with respect to [1], new technical issues will appear, which are essentially due to our sub-Riemannian setting: in particular all the computations and the estimates regarding the convergence along the flow lines have to be done accordingly to the projection on the space of L^2 CR pluriharmonic functions, that we defined earlier. Moreover, some technical estimates on the sphere will be adapted to the CR setting as we will see in Section 5 and the Appendix. Therefore, let us define the following functional $E : H \to \mathbb{R}$, by

$$E(u) = \int_M u\overline{P}'u + 2\int_M \overline{Q}'u$$

where $H = \hat{\mathcal{P}} \cap \mathcal{S}^2(M)$ and $\mathcal{S}^2(M)$ is the Folland-Stein Sobolev space equipped with the equivalent norm (see section 2), defined by

$$\|u\|^2 = \int_M u\overline{P}'u + \int_M u^2. \tag{3}$$

We consider the following space, which will serve as a constraint

$$X = \left\{ u \in H; N(u) := \int_M \Gamma\left(fe^{2u}\right) = \int_M \overline{Q}' \right\};$$

we notice that the space is well defined since $e^u \in L^2$, see [9], Theorem 3.1.
As in the classical case, we will need the following hypotheses, depending on the sign of $\int_M \overline{Q}'$, namely:

$$\begin{cases} (i) \quad \inf_{x\in M} f(x) < 0, & \text{if } \int_M \overline{Q}' < 0 \\ (ii) \quad \sup_{x\in M} f(x) > 0,\ \inf_{x\in M} f(x) < 0 & \text{if } \int_M \overline{Q}' = 0 \\ (iii) \quad \sup_{x\in M} f(x) > 0, & \text{if } 0 < \int_M \overline{Q}' \leq 16\pi^2. \end{cases} \tag{4}$$

In the case when $\int_M \overline{Q}' = 0$, we let ℓ be the unique CR pluriharmonic function satisfying $\overline{P}'_\theta \ell + \overline{Q}' = 0$ and $\int_M \ell = 0$, see [9], Theorem 1.1. Notice that $\overline{Q}'_{e^\ell\theta} = 0$. We also recall that in the critical case $M = S^3$, there are some extra compatibility conditions of Kazdan-Warner type that f needs to satisfy in order to be the $\overline{Q}'$-curvature of a contact structure conformal to the standard one on the sphere (see Theorem 1.3. in [17]).
Now, in order to define the flow equation, we compute the first variation of E, N, and their $(\mathcal{S}^2)$ gradient, respectively:

$$\langle \nabla E(u), \phi \rangle = 2\int_M \left(\overline{P}'u + \overline{Q}'\right)\phi\ , \forall \phi \in H\ ,$$

$$\langle \nabla N(u), \phi \rangle = 2\int_M \Gamma\left(fe^{2u}\right)\phi\ , \forall \phi \in H\ ,$$

$$\nabla E(u) = 2\left(\overline{P}' + I\right)^{-1}\left(\overline{P}'u + \overline{Q}'\right)\ ,$$

$$\nabla N(u) = 2\left(\overline{P}' + I\right)^{-1}\Gamma\left(fe^{2u}\right)\ .$$

In addition, since by hypotheses (4), $\nabla N \neq 0$ on X, then X is a regular hypersurface in H and a unit normal vector field on X is given by $\nabla N/\|\nabla N\|$. Indeed, $\nabla N(u) \neq 0$ if and only if $\Gamma(e^{2u}f) \neq 0$. This last identity is clear for the hypothesis (i) and (iii). But for (ii), recall that $f \in \hat{\mathcal{P}}$, so if $\Gamma(e^{2u}f) = 0$, then $\int_M e^{2u}f^2 = 0$, leading to a contradiction. The gradient of E restricted to X is then

$$\nabla^X E = \nabla E - \left\langle \nabla E, \frac{\nabla N}{\|\nabla N\|} \right\rangle \frac{\nabla N}{\|\nabla N\|}.$$

Finally, the (negative) gradient flow equation is given by

$$\begin{cases} \partial_t u = -\nabla^X E(u) \\ \\ u(0) = u_0 \in X \end{cases} \tag{5}$$

Now we can state our main results.

Theorem 1.1. *Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Let us assume that $\displaystyle\int_M \overline{Q}' < 0$ and let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4). Then there exists a positive constant C_0 depending on $f^- = \max\{-f, 0\}$, M and θ, such that if*

$$e^{\tau\|u_0\|^2} \sup_{x \in M} f(x) \leq C_0$$

for a constant $\tau > 1$ depending on M and θ, then as $t \to \infty$, the flow converges in H to a solution u_∞ of (1). *Moreover, there exist constants $B, \beta > 0$ such that*

$$\|u(t) - u_\infty\| \leq B(1+t)^{-\beta},$$

for all $t \geq 0$.

Theorem 1.2. *Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Let us assume that $\displaystyle\int_M \overline{Q}' = 0$ and let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4). Then as $t \to \infty$, the flow converges in H to a function u_∞ and there exists a constant λ such that $v = u_\infty + \lambda$ satisfies*

$$\overline{P}' v + \overline{Q}' = \delta \Gamma(f e^{2v}),$$

where $\delta \in \{+1, 0, -1\}$. Moreover, there exist constants $B, \beta > 0$ such that

$$\|u(t) - u_\infty\| \leq B(1+t)^{-\beta},$$

for all $t \geq 0$. If in addition, we assume that $\int_M f e^{2\ell} \neq 0$, then $\delta \neq 0$.

Theorem 1.3. *Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Let us assume that $0 < \displaystyle\int_M \overline{Q}' < 16\pi^2$ and let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4). Then as $t \to \infty$, the flow converges in H to a solution u_∞ of* (1). *Moreover, there exist constants $B, \beta > 0$ such that*

$$\|u(t) - u_\infty\| \leq B(1+t)^{-\beta},$$

for all $t \geq 0$.

Finally, the critical case of the sphere, which is a bit different. We will consider a group G acting on S^3 preserving the CR structure. We denote by Σ the set of points fixed by G, that is

$$\Sigma = \left\{ x \in S^3;\ g \cdot x = x,\ \forall g \in G \right\}$$

and we will assume f being invariant under G, namely $f(g \cdot x) = f(x), \forall g \in G$. Then we have the following

Theorem 1.4. *Let us consider the sphere $M = S^3$ equipped with its standard contact structure and let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4) and invariant under G. Let us assume that also $u_0 \in X$ is invariant under G. If $\Sigma = \emptyset$ or*

$$\sup_{x \in \Sigma} f(x) \leq e^{-\frac{E(u_0)}{16\pi^2}},$$

then as $t \to \infty$, the flow converges in H to a solution (invariant under G) u_∞ of (1). *Moreover, there exist constants $B, \beta > 0$ such that*

$$\|u(t) - u_\infty\| \leq B(1+t)^{-\beta},$$

for all $t \geq 0$.

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2 Some definitions in pseudo-Hermitian geometry

We will follow the notations in [10]. Let M^3 be a smooth, oriented three-dimensional manifold. A CR structure on M is a one-dimensional complex sub-bundle $T^{1,0}M \subset T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ such that $T^{1,0}M \cap T^{0,1}M = \{0\}$ for $T^{0,1}M := \overline{T^{1,0}M}$. Let $\mathcal{H} = ReT^{1,0}M$ and let $J\colon \mathcal{H} \to \mathcal{H}$ be the almost complex structure defined by $J(Z + \bar{Z}) = i(Z - \bar{Z})$, for all $Z \in T^{1,0}M$. The condition that $T^{1,0}M \cap T^{0,1}M = \{0\}$ is equivalent to the existence of a contact form θ such that $\ker\theta = \mathcal{H}$. We recall that a 1-form θ is said to be a contact form if $\theta \wedge d\theta$ is a volume form on M^3. Since M is oriented, a contact form always exists, and is determined up to multiplication by a positive real-valued smooth function. We say that $(M^3, T^{1,0}M)$ is strictly pseudo-convex if the Levi form $d\theta(\cdot, J\cdot)$ on $\mathcal{H} \otimes \mathcal{H}$ is positive definite for some, and hence any, choice of contact form θ. We shall always assume that our CR manifolds are strictly pseudo-convex.

Notice that in a CR-manifold, there is no canonical choice of the contact form θ. A pseudo-Hermitian manifold is a triple $(M^3, T^{1,0}M, \theta)$ consisting of a CR manifold and a contact form. The Reeb vector field T is the vector field such that $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$. The choice of θ induces a natural L^2-dot product $\langle \cdot, \cdot \rangle$, defined by

$$\langle f, g \rangle = \int_M f(x) g(x)\ \theta \wedge d\theta.$$

A $(1,0)$-form is a section of $T^*_{\mathbb{C}}M$ which annihilates $T^{0,1}M$. An admissible coframe is a non-vanishing $(1,0)$-form θ^1 in an open set $U \subset M$ such that $\theta^1(T) = 0$. Let $\theta^{\bar{1}} := \overline{\theta^1}$ be its conjugate. Then $d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$ for some positive function $h_{1\bar{1}}$. The function $h_{1\bar{1}}$ is equivalent to the Levi form. We set $\{Z_1, Z_{\bar{1}}, T\}$ to the dual of $(\theta^1, \theta^{\bar{1}}, \theta)$. The geometric structure of a CR manifold is determined by the connection form ${\omega_1}^1$ and the torsion form $\tau_1 = A_{11}\theta^1$ defined in an admissible coframe θ^1 and is uniquely determined by

$$\begin{cases} d\theta^1 = \theta^1 \wedge {\omega_1}^1 + \theta \wedge \tau^1, \\ \omega_{1\bar{1}} + \omega_{\bar{1}1} = dh_{1\bar{1}}, \end{cases}$$

where we use $h_{1\bar{1}}$ to raise and lower indices. The connection forms determine the pseudo-Hermitian connection ∇, also called the Tanaka-Webster connection, by

$$\nabla Z_1 := {\omega_1}^1 \otimes Z_1.$$

The scalar curvature R of θ, also called the Webster curvature, is given by the expression

$$d{\omega_1}^1 = R\theta^1 \wedge \theta^{\bar{1}} \mod \theta.$$

Definition 2.1. *A real-valued function $w \in C^\infty(M)$ is CR pluriharmonic if locally $w = Ref$ for some complex-valued function $f \in C^\infty(M, \mathbb{C})$ satisfying $Z_{\bar{1}}f = 0$.*

Equivalently, [20], w is a CR pluriharmonic function if

$$\Phi w := \nabla_1\nabla_1\nabla^1 w + iA_{11}\nabla^1 w = 0$$

for $\nabla_1 := \nabla_{Z_1}$. We denote by $\mathcal{P}$ the space of all CR pluriharmonic functions and $\hat{\mathcal{P}}$ the completion of $\mathcal{P}$ in $L^2(M)$, also called the space of L^2 CR pluriharmonic functions. Let $\Gamma : L^2(M) \to \hat{\mathcal{P}}$ be the orthogonal projection on the space of L^2 pluriharmonic functions. If $S : L^2(M) \to \ker \bar{\partial}_b$ denotes the Szego kernel, then

$$\Gamma = S + \bar{S} + F, \tag{6}$$

where F is a smoothing kernel as shown in [19]. The Paneitz operator P_θ is the differential operator

$$\begin{aligned} P_\theta(w) &:= 4\mathrm{div}(\Phi w) \\ &= \Delta_b^2 w + T^2 - 4\mathrm{Im}\nabla^1\left(A_{11}\nabla^1 f\right) \end{aligned}$$

for $\Delta_b := \nabla^1\nabla_1 + \nabla^{\bar{1}}\nabla_{\bar{1}}$ the sub-Laplacian. In particular, $\mathcal{P} \subset \ker P_\theta$. Hence, $\ker P_\theta$ is infinite dimensional. For a thorough study of the analytical properties of P_θ and its kernel, we refer the reader to [19, 6, 8]. The main property of the Paneitz operator P_θ is that it is CR covariant [16]. That is, if $\hat{\theta} = e^w\theta$, then $e^{2w}P_{\hat{\theta}} = P_\theta$.

Definition 2.2. *Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Hermitian manifold. The Paneitz type operator $P'_\theta \colon \mathcal{P} \to C^\infty(M)$ is defined by*

$$\begin{aligned} P'_\theta f = {} & 4\Delta_b^2 f - 8\mathrm{Im}\left(\nabla^\alpha(A_{\alpha\beta}\nabla^\beta f)\right) - 4\mathrm{Re}\left(\nabla^\alpha(R\nabla_\alpha f)\right) \\ & + \frac{8}{3}\mathrm{Re}(\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta})\nabla^\alpha f - \frac{4}{3}f\nabla^\alpha(\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta}) \end{aligned} \tag{7}$$

for $f \in \mathcal{P}$.

The main property of the operator P'_θ is its "almost" conformal covariance as shown in [2, 10]. That is if $(M^3, T^{1,0}M, \theta)$ is a pseudo-Hermitian manifold, $w \in C^\infty(M)$, and we set $\hat{\theta} = e^w\theta$, then

$$e^{2w}P'_{\hat{\theta}}(u) = P'_\theta(u) + P_\theta(uw) \tag{8}$$

for all $u \in \mathcal{P}$. In particular, since P_θ is self-adjoint and $\mathcal{P} \subset \ker P_\theta$, we have that the operator P' is conformally covariant, mod $\mathcal{P}^\perp$.

Definition 2.3. *A pseudo-Hermitian manifold $(M^3, T^{1,0}M, \theta)$ is pseudo-Einstein if*

$$\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta} = 0.$$

Moreover, if θ induces a pseudo-Einstein structure then $e^u\theta$ is pseudo-Einstein if and only if $u \in \mathcal{P}$. The definition above was stated in [10], but it was implicitly mentioned in [16]. In particular, if $(M^3, T^{1,0}M, \theta)$ is pseudo-Einstein, then P'_θ takes a simpler form:

$$P'_\theta f = 4\Delta_b^2 f - 8\mathrm{Im}\left(\nabla^1(A_{11}\nabla^1 f)\right) - 4\mathrm{Re}\left(\nabla^1(R\nabla_1 f)\right).$$

In particular, one has

$$\int_M uP'_\theta u \geq 4\int_M |\Delta_b u|^2 - C\int_M |\nabla_b u|^2.$$

Using the interpolation inequality

$$\int_M |\nabla_b u|^2 \leq C\|u\|_{L^2}\|\Delta_b u\|_{L^2},$$

and $2ab \leq \varepsilon a^2 + \frac{1}{\epsilon}b^2$, we have the existence of $C_1 > 0$ and $C_2 > 0$, such that

$$\int_M uP'_\theta u \geq C_1\int_M |\Delta_b u|^2 - C_2\int_M u^2.$$

Hence, if P'_θ is non-negative, with trivial kernel, one has the equivalence of the Folland-Stein Sobolev norm and (3).

Definition 2.4. *Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Einstein manifold. The Q'-curvature is the scalar quantity defined by*

$$Q'_\theta = 2\Delta_b R - 4|A|^2 + R^2. \tag{9}$$

The main equation that we will be dealing with is the change of the Q'-curvature under conformal change. Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Einstein manifold, let $w \in \mathcal{P}$, and set $\hat{\theta} = e^w\theta$. Hence $\hat{\theta}$ is pseudo-Einstein. Then [2, 10]

$$e^{2w}Q'_{\hat{\theta}} = Q'_\theta + P'_\theta(w) + \frac{1}{2}P_\theta\left(w^2\right). \tag{10}$$

In particular, Q'_θ behaves as the Q-curvature for P'_θ, mod $\mathcal{P}^\perp$. Since we are working modulo $\mathcal{P}^\perp$ it is convenient to project the previously defined quantities on $\hat{\mathcal{P}}$. So we define the operator $\overline{P}'_\theta = \Gamma \circ P'_\theta$ and the $\overline{Q}'$-curvature by $\overline{Q}'_\theta = \Gamma(Q'_\theta)$. Notice that

$$\int_M Q'_\theta\, \theta \wedge d\theta = \int_M \overline{Q}'_\theta\, \theta \wedge d\theta.$$

Moreover, the operator $\overline{P}'_\theta$ has many interesting analytical properties. Indeed, $\overline{P}'_\theta : \mathcal{P} \to \hat{\mathcal{P}}$ is an elliptic pseudo-differential operator (see [9]) and if we assume that $\ker \overline{P}'_\theta = \mathbb{R}$, then its Green's function G satisfies

$$\overline{P}'_\theta G(\cdot, y) = \Gamma(\cdot, y) - \frac{1}{V},$$

where $V = \int_M \theta \wedge d\theta$ is the volume of M and $\Gamma(\cdot,\cdot)$ is the kernel of the projection operator Γ. Moreover,

$$G(x,y) = -\frac{1}{4\pi^2}\ln(|xy^{-1}|) + \mathcal{K}(x,y),$$

where $\mathcal{K}$ is a bounded kernel as proved in [7].

3 Preliminary results on the flow

First we recall one fundamental inequality that we will be using all along this paper, namely the CR version of the Beckner-Onofri inequality. This inequality was first proved in the odd dimensional spheres in [3] and then naturally extended to pseudo-Einstein 3-manifolds in [9, Theorem 3.1].

Theorem 3.1. *Assume that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Then, there exists $C > 0$ such that for all $u \in \hat{\mathcal{P}} \cap \mathcal{S}^2(M)$ with $\int_M u = 0$, we have*

$$\frac{1}{16\pi^2}\int_M u\overline{P}'u + C \geq \ln\left(⨍_M e^{2u}\right).$$

In the case of the sphere, C can be taken to be 0 and equality holds if and only if $u = J(h)$ with $h \in Aut(S^3)$ and $J(h) = det(Jac(h))$ is the determinant of the Jacobian determinant of h. The dual version of the above inequality was also investigated in [22], where the existence of extremals was investigated.
Now, we prove the global existence of solutions of (5):

Lemma 3.1. *Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4). Then for any $u_0 \in X$ there exists a solution $u \in C^\infty([0,\infty), H)$ of problem (5) such that $u(t) \in X$, for all $t \geq 0$. Moreover it holds*

$$\int_0^t \|\partial_s u(s)\|^2 ds = E(u_0) - E(u(t)),$$

for all $t \geq 0$.

Proof. Since all the functionals involved are regular, the short time existence of a solution u for (5) is ensured by the Cauchy-Lipschitz Theorem. In order to extend it to all $t \geq 0$, we notice that

$$\|\partial_t u\| = \|\nabla^X E(u)\| \leq 2\|\nabla E(u)\| \leq C_1\|u\| + C_2.$$

Thus, since

$$\partial_t \|u\|^2 = 2\langle u, \partial_t u\rangle \leq C_3\|u\|^2 + C_4,$$

by Gronwall's lemma, the solution u exists for all $t \geq 0$. In addition

$$\partial_t N(u) = \langle \nabla N(u), \partial_t u\rangle = -\left\langle \nabla N(u), \nabla^X E(u)\right\rangle = 0,$$

therefore $u(t) \in X$, for all $t \geq 0$. Finally, we have

$$\partial_t E(u) = \langle \nabla E(u), \partial_t u \rangle = -\|\partial_t u\|^2.$$

Hence, E is decreasing along the flow and the following energy identity holds

$$\int_0^t \|\partial_s u\|^2 ds = E(u_0) - E(u(t)). \tag{11}$$

$\square$

Next we prove the following lemma about the convergence

Lemma 3.2. *Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and* $\ker \overline{P}' = \mathbb{R}$. *Let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4) and let u be the solution of problem (5) obtained in the previous Lemma (3.1). If there exists a constant $C > 0$ such that $\|u(t)\| \leq C$, for all $t \geq 0$, then when $t \to \infty$, $u(t) \to u_\infty$ in H and u_∞ solves the equation*

$$\overline{P}' u + \overline{Q}' u = \lambda \Gamma(f e^{2u}),$$

for a certain $\lambda \in \mathbb{R}$. Moreover, there exist constants $B, \beta > 0$ such that

$$\|u(t) - u_\infty\| \leq B(1+t)^{-\beta},$$

for all $t \geq 0$.

Proof. Since $\|u\| \leq C$, we have that

$$|E(u)| \leq 2\|u\|^2 + C_2.$$

Therefore, by the previous energy estimate

$$\int_0^\infty \|\partial_t u\|^2 dt < \infty.$$

So there exists a sequence $t_k \to \infty$ such that

$$\|\partial_t u(t_k)\| = \|\nabla^X E(u(t_k))\| \to 0.$$

Now, from the boundedness of $\|u\|$, we also have the convergence $u(t_k) \to u_\infty$ strongly in $L^2(M)$ and weakly in $\mathcal{S}^2(M)$. From Theorem 3.1, we have that $e^{2u(t_k)} \in L^p(M)$, with $p \geq 1$, and $\|e^{2u(t_k)}\|_{L^p}$ is uniformly bounded. Thus by Egorov's lemma, we can deduce that

$$\left\| f e^{2u(t_k)} - f e^{2u_\infty} \right\|_{L^p} \to 0, \ 1 \leq p < \infty.$$

Indeed, we fix $\varepsilon > 0$. Then, there exists a set A with $Vol(A) < \varepsilon$ such that $f e^{2u(t_k)}$ converges uniformly to $f e^{2u_\infty}$ on $M \setminus A$. Therefore,

$$\begin{aligned} \left\| f e^{2u(t_k)} - f e^{2u_\infty} \right\|_{L^p} \leq & C \left\| f e^{2u(t_k)} - f e^{2u_\infty} \right\|_{L^\infty(M \setminus A)} \\ & + \left(\left\| f e^{2u(t_k)} \right\|_{L^{\tilde{p}}} + \| f e^{2u_\infty} \|_{L^{\tilde{p}}} \right) Vol(A)^{\frac{1}{p} - \frac{1}{\tilde{p}}}, \end{aligned}$$

for $p < \tilde{p} < \infty$. So the conclusion follows from the uniform boundedness of $\|e^{2u(t_k)}\|_{L^p}$, for all $1 \leq p < \infty$.
Thus $u_\infty \in X$. Now we have that

$$\nabla N(u(t_k)) = 2(\overline{P}' + I)^{-1}\Gamma\left(fe^{2u(t_k)}\right),$$

and since $fe^{2u(t_k)}$ converges strongly in $L^p(M)$ and Γ maps continuously $L^p(M)$ to $L^p(M)$ (this follows from (6) and [24]), we have by the compactness of $\overline{P}'+I$ that $\nabla N(u(t_k)$ converges strongly to $\nabla N(u_\infty)$. Also,

$$\nabla E(u(t)) = 2(\overline{P}' + I)^{-1}(\overline{P}'u + \overline{Q}') = 2u(t) + 2(\overline{P}' + I)^{-1}(\overline{Q}' - u).$$

Thus, since $\partial_t u(t_k) \to 0$ in $\mathcal{S}^2(M)$, we have that $u(t_k)$ converges strongly to u_∞ and $\nabla^X E(u_\infty) = 0$. Moreover, we have

$$(\overline{P}' + I)^{-1}(\overline{P}'u_\infty + \overline{Q}') = \lambda(u_\infty)(\overline{P}' + I)^{-1}\Gamma(fe^{2u_\infty})$$

where

$$\lambda(u_\infty) = \frac{\langle \nabla E(u_\infty), \nabla N(u_\infty)\rangle}{\|\nabla N(u_\infty)\|^2}.$$

Now by integration we have that

$$\int_M \overline{Q}' = \lambda(u_\infty)\int_M \Gamma\left(fe^{2u_\infty}\right),$$

and since $u_\infty \in X$, we have if $\displaystyle\int_M \overline{Q}' \neq 0$, that $\lambda(u_\infty) = 1$ and hence u_∞ solves the desired equation. On the other hand, if $\displaystyle\int_M \overline{Q}' = 0$, either $\lambda(u_\infty) = 0$ and thus

$$\overline{P}'u_\infty + \overline{Q}' = 0,$$

or $\lambda(u_\infty) > 0$, thus setting $v = u_\infty + \frac{1}{2}\ln(\lambda(u_\infty))$ we have

$$\overline{P}'v + \overline{Q}' = \Gamma(fe^{2v}),$$

and similarly if $\lambda(u_\infty) < 0$, we have a function $v = u_\infty + \frac{1}{2}\ln(-\lambda(u_\infty))$ such that

$$\overline{P}'v + \overline{Q}' = -\Gamma(fe^{2v}).$$

In particular, if we assume that $\int_M fe^{2\ell} \neq 0$ in the case $\lambda(u_\infty) = 0$, we have $u_\infty - \ell$ is constant. Hence,

$$0 = \int_M fe^{2u_\infty} = e^{2(u_\infty - \ell)}\int_M fe^{2\ell} \neq 0,$$

which is a contradiction.
The polynomial convergence of the flow can be deduced from the Lojasiewicz-Simon inequality following Theorem 3 in [25] and Lemma 3.2 in [1]. Let $\eta : H \to T_{u_\infty}X$ be the natural

projection, where $T_{u_\infty}X$ denotes the tangent space of the manifold X at the point u_∞. We have, for $v \in T_{u_\infty}X$

$$\left(\nabla^X\right)^2 E(u_\infty)v = \eta\left(\nabla^2 E(u_\infty)v - \frac{\langle \nabla E(u_\infty), \nabla N(u_\infty)\rangle}{\|\nabla N(u)\|^2}\nabla^2 N(u_\infty)v + R^\perp v\right)$$

where $R^\perp v$ is the component along $\nabla N(u_\infty)$. Thus, since $\eta\left(R^\perp v\right) = 0$, we have that

$$\left(\nabla^X\right)^2 E(u_\infty)v = 2\left(I - \eta(\overline{P}' + 1)\right)v - 4\frac{\langle \nabla E(u_\infty), \nabla N(u_\infty)\rangle}{\|\nabla N(u)\|^2}(\overline{P}' + 1)^{-1}\Gamma(fe^{2u_\infty}v).$$

It can be checked that $(\nabla^X)^2 E(u_\infty) : T_{u_\infty}X \to T_{u_\infty}X$ is a Fredholm operator, then there exists a constant $\delta > 0$ and $0 < \kappa < \frac{1}{2}$ such that if $\|u(t) - u_\infty\| < \delta$, it holds

$$\|\nabla^X E(u)\| \geq (E(u(t)) - E(u_\infty))^{1-\kappa}.$$

We note that if $E(u(t_0)) = E(u_\infty)$ for some $t_0 \geq 0$, then the flow is stationary and the estimate is trivially satisfied. So we can assume that $E(u(t)) - E(u_\infty) > 0$, for all $t \geq 0$. Since $\lim_{n\to\infty}\|u(t_n) - u_\infty\| = 0$, for a given $\varepsilon > 0$, there exists $n_0 > 0$ such that for $n \geq n_0$ we have,

$$\|u(t_n) - u_\infty\| < \frac{\varepsilon}{2}$$

and

$$\frac{1}{\kappa}(E(u(t_n)) - E(u_\infty))^\kappa < \frac{\varepsilon}{2}.$$

We set $\varepsilon = \frac{\delta}{2}$ and

$$T := \sup\left\{t \geq t_{n_0}; \|u(s) - u_\infty\| < \delta; s \in [t_{n_0}, t]\right\},$$

and we assume for the sake of contradiction that $T < \infty$. Now we have

$$-\partial_t[E(u(t)) - E(u_\infty)]^\kappa = -\kappa\partial_t E(u(t))[E(u(t)) - E(u_\infty)]^{\kappa-1},$$

but

$$-\partial_t E(u(t)) = -\langle E(u), \partial_t u\rangle = \|\nabla^X E(u)\|\|\partial_t u\|.$$

Thus, for $t \in [t_{n_0}, T]$ we have

$$-\partial_t[E(u(t)) - E(u_\infty)]^\kappa \geq \kappa\|\partial_t u\|,$$

and since E is non-increasing along the flow, we have after integration in the interval $[t_{n_0}, T]$

$$\|u(T) - u(t_{n_0})\| \leq \int_{t_{n_0}}^T \|\partial_s u\| ds \leq \frac{1}{\kappa}[E(u(t_{n_0})) - E(u_\infty)]^\kappa < \frac{\varepsilon}{2}.$$

Hence,

$$\|u(T) - u_\infty\| \leq \|u(T) - u(t_{n_0})\| + \|u(t_{n_0}) - u_\infty\| < \varepsilon = \frac{\delta}{2}$$

which is a contradiction and so $T = +\infty$. We set now $g(t) = E(u(t)) - E(u_\infty)$, for $t \in [t_{n_0}, +\infty)$. Then we have

$$g'(t) = -\|\nabla^X E(u)\|^2 \geq g^{2\kappa-1}(t).$$

By integration we obtain

$$g^{2\kappa-1}(t) \geq g^{2\kappa-1}(t_{n_0}) + (1-2\kappa)(t-t_{n_0}).$$

Since $2\kappa - 1 < 0$, then

$$g(t) \leq [g^{2\kappa-1}(t_{n_0}) + (1-2\kappa)(t-t_{n_0})]^{\frac{1}{2\kappa-1}} \leq Ct^{\frac{1}{2\kappa-1}}.$$

Now, by taking $t' > t$, we have

$$\|u(t) - u(t')\| \leq \int_t^{t'} \|\partial_s u\| ds \leq \frac{1}{\theta}[E(u(t)) - E(u_\infty)]^\kappa \leq \frac{1}{\kappa} g^\kappa(t) \leq Ct^{\frac{\kappa}{2\kappa-1}}.$$

For $t' = t_n$, letting $n \to \infty$ and setting $\beta = \frac{\kappa}{1-2\kappa}$, we get that for $t > t_{n_0}$

$$\|u(t) - u_\infty\| \leq Ct^{-\beta}$$

Therefore, since $\|u(t) - u_\infty\|$ is bounded for $t > t_{n_0}$, we have the existence of $B > 0$ such that for all $t \geq 0$

$$\|u(t) - u_\infty\| \leq B(1+t)^{-\beta}.$$

□

Corollary 3.1. *Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\overline{P}'$ is non-negative and* $\ker \overline{P}' = \mathbb{R}$. *Let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4) and let u be the solution of problem (5) obtained in the Lemma (3.1). If $\bar{u} = \frac{1}{V}\int_M u$ is uniformly bounded then the flow converges. Here $V = \int_M \theta \wedge d\theta$ is the volume of M.*

Proof. From the energy identity (11) we have that

$$\int_M u\overline{P}'u + 2\int_M \overline{Q}'u \leq E(u_0),$$

but we also know from the Poincaré-type inequality (or the non-negativity of the operator $\overline{P}'$), that

$$\int_M u\overline{P}'u \geq \lambda_1 \int_M (u-\bar{u})^2.$$

Here λ_1 is the first non-zero eigenvalue of the operator $\overline{P}'$. In particular, from Young's inequality, we obtain that

$$\int_M u\overline{P}'u \leq E(u_0) + \varepsilon \int_M (u-\bar{u})^2 + C(\varepsilon)\|\overline{Q}'\|_{L^2}^2 - 2\bar{u}\int_M \overline{Q}'.$$

Hence, for ε small enough, we get

$$\int_M u\overline{P}'u \leq C,$$

since $\bar{u}$ is uniformly bounded, then the uniform boundedness of $\|u\|$ and the conclusion follows from Lemma 3.2. □

Therefore, in the rest of the paper, we will show the uniform boundedness of $\bar{u}$ along the flow, in order to have convergence at infinity.

4 The sub-critical case

Along all this section we will assume that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Next we consider the three separate cases in which $\int_M \overline{Q}' < 16\pi^2$. Also we let $V = \int_M \theta \wedge d\theta$ be the volume of M.

4.1 Case $\int_M \overline{Q}' < 0$ and proof of Theorem 1.1

Lemma 4.1. *There exists a positive constant $C > 0$ depending on M and θ such that for any measurable subset $K \subset M$ with $Vol(K) > 0$, we have*

$$\int_M u \leq |E(u_0)| + \frac{C}{Vol(K)} + \frac{4V}{Vol(K)} \max\left(\int_K u, 0\right)$$

Proof. Without loss of generality we can assume that $\int_M u > 0$ otherwise the inequality is trivially satisfied. First

$$\int u\overline{P}'u \leq E(u_0) - 2\int_M \overline{Q}'u$$

and

$$\|u - \bar{u}\|^2_{L^2} \leq \frac{1}{\lambda_1}\int_M u\overline{P}'u.$$

Hence,

$$\int_M u^2 \leq \frac{1}{\lambda_1}E(u_0) - \frac{2}{\lambda_1}\int_M \overline{Q}'u + \frac{1}{V}\left(\int_M u\right)^2$$

Now if $\int_K u \leq 0$, then we have

$$\left(\int_M u\right)^2 \leq \left(\int_{K^c} u\right)^2 \leq Vol(K^c)\int_M u^2,$$

hence

$$\frac{Vol(K)}{V}\int_M u^2 \leq \frac{1}{\lambda_1}E(u_0) - \frac{2}{\lambda_1}\int_M \overline{Q}'u.$$

Again using Young's inequality we obtain

$$\int_M u^2 \leq \frac{2V}{\lambda_1 Vol(K)}E(u_0) + \frac{4\|\overline{Q}'\|^2_{L^2}V^2}{\lambda_1^2 Vol(K)^2},$$

but

$$\left(\int_M u\right)^2 \leq V\int_M u^2 \leq \frac{2V^2}{\lambda_1 Vol(K)}E(u_0) + \frac{4\|\overline{Q}'\|^2_{L^2}V^3}{\lambda_1^2 Vol(K)^2}$$
$$\leq |E(u_0)|^2 + \frac{V^4}{\lambda_1^2 Vol(K)^2} + \frac{4\|\overline{Q}'\|^2_{L^2}V^3}{\lambda_1^2 Vol(K)^2},$$

which yields

$$\int_M u \leq |E(u_0)| + \frac{C}{VolK}.$$

We assume now that $\displaystyle\int_K u > 0$. Then one has

$$\int_M u^2 \leq \frac{1}{\lambda_1} E(u_0) - \frac{2}{\lambda_1}\int_M \overline{Q}' u + \frac{1}{V}\left(\left(\int_K u\right)^2 + \left(\int_{K^c} u\right)^2 + 2\int_{K^c} u \int_K u\right),$$

and

$$\frac{2}{V}\int_{K^c} u \int_K u \leq \frac{2Vol(K^c)}{Vol(K)V}\left(\int_K u\right)^2 + \frac{Vol(K)}{2V}\int_M u^2.$$

Hence,

$$\frac{Vol(K)}{2V}\int_M u^2 \leq \frac{1}{\lambda_1}E(u_0) - \frac{2}{\lambda_1}\int_M \overline{Q}' u + \frac{3}{Vol(K)}\left(\int_K u\right)^2.$$

By using that

$$\left|\frac{2}{\lambda_1}\int_M \overline{Q}' u\right| \leq \frac{Vol(K)}{4V}\int_M u^2 + \frac{4V\|\overline{Q}'\|^2_{L^2}}{\lambda_1^2 Vol(K)},$$

we have,

$$\int_M u^2 \leq \frac{4V}{\lambda_1 Vol(K)}|E(u_0)| + \frac{16V^2\|\overline{Q}'\|^2_{L^2}}{\lambda_1^2 Vol(K)^2} + \frac{12V}{Vol(K)^2}\left(\int_M u\right)^2.$$

Hence,

$$\left(\int_M u\right)^2 \leq \frac{4V^2}{\lambda_1 Vol(K)}|E(u_0)| + \frac{16V^3\|\overline{Q}'\|^2_{L^2}}{\lambda_1^2 Vol(K)^2} + \frac{12V^2}{Vol(K)^2}\left(\int_M u\right)^2,$$

and therefore

$$\int_M u \leq |E(u_0)| + \frac{C}{Vol(K)} + \frac{4V}{Vol(K)}\int_K u.$$

□

Lemma 4.2. *Let K be a measurable subset of M such that $Vol(K) > 0$. Then there exists a constant $\alpha > 1$ depending on M and θ and a constant $C_K > 1$ depending on $Vol(K)$ such that*

$$\int_M e^{2u} \leq C_K e^{\alpha\|u_0\|^2} \max\left(\left(\int_K e^{2u}\right)^\alpha, 1\right).$$

Proof. Recall that from Theorem 3.1 one has the existence of $C > 0$ such that

$$\int_M e^{2u} \leq C\exp\left(\frac{1}{16\pi^2}\int_M u\overline{P}' u + \frac{2}{V}\int_M u\right).$$

Again, by the energy identity (11) and Young's inequality, we have

$$\begin{aligned}\int_M u\overline{P}' u &\leq E(u_0) - 2\int_M \overline{Q}'(u-\bar{u}) - 2\bar{u}\int_M \overline{Q}' \\ &\leq E(u_0) - 2\bar{u}\int_M \overline{Q}' + \frac{1}{\varepsilon}\|\overline{Q}'\|^2_{L^2} + \frac{\varepsilon}{\lambda_1}\int_M u\overline{P}' u.\end{aligned}$$

Thus, for $\varepsilon = \frac{\lambda_1}{2}$,

$$\frac{1}{2}\int_M u\overline{P}'u \le E(u_0) - 2\bar{u}\int_M \overline{Q}' + \frac{2}{\lambda_1}\|\overline{Q}'\|_{L^2}^2.$$

Therefore

$$\int_M e^{2u} \le C\exp\left(\frac{1}{8\pi^2}E(u_0) + \frac{\|\overline{Q}'\|_{L^2}^2}{4\lambda_1\pi^2} + \left(2 - \frac{1}{4\pi^2}\int_M \overline{Q}'\right)\bar{u}\right).$$

Now we notice that $E(u_0) \le \|u_0\|^2 + \|\overline{Q}'\|_{L^2}^2$, hence there exist constants C_1 and C_2 such that

$$\int_M e^{2u} \le C_1\exp\left(\frac{1}{8\pi^2}\|u_0\|^2 + C_2\int_M u\right).$$

By using Lemma 4.1, we get

$$\int_M e^{2u} \le \bar{C}_K\exp\left(A_1\|u_0\|^2 + \frac{A_2}{Vol(K)}\max\left(\int_K u, 0\right)\right),$$

where $\bar{C}_K$ depends on $Vol(K)$. Now, we set $\alpha = \max\left(A_1, \frac{A_2}{2}, 2\right) > 1$, and we get

$$\int_M e^{2u} \le \bar{C}_K\exp\left(\alpha\|u_0\|^2 + \frac{\alpha}{Vol(K)}\max\left(\int_K 2u, 0\right)\right).$$

But Jensen's inequality yields

$$\exp\left(\frac{1}{Vol(K)}\int_K 2u\right) \le \frac{1}{Vol(K)}\int_K e^{2u},$$

in particular

$$\exp\left(\frac{\alpha}{Vol(K)}\max\left(\int_K u, 0\right)\right) \le \max\left(\left(\frac{1}{Vol(K)}\int_K e^{2u}\right)^{\alpha}, 1\right).$$

Therefore, by adjusting the constant eventually

$$\int_M e^{2u} \le C_K e^{\alpha\|u_0\|^2}\max\left(\left(\int_K e^{2u}\right)^{\alpha}, 1\right)$$

which completes the proof. □

Next we move to the proof of Theorem 1.1. We set

$$K = \left\{x \in M; f(x) \le \frac{1}{2}\inf_{x\in M} f(x)\right\}.$$

From the compatibility condition (i) in (4), we have that $Vol(K) > 0$, and since

$$\int_M \overline{Q}' = \int_M fe^{2u_0},$$

we obtain

$$\frac{\int_M \overline{Q}'}{\inf\limits_{x\in M} f(x)} \le \int_M e^{2u_0}.$$

Thus, there exists $C > 0$ (we will assume $C > 1$ actually) such that

$$\int_M e^{2u_0} \le C \exp\left[C\left(\int_M u_0 \overline{P}' u_0 + \int_M u_0^2\right)\right] = Ce^{C\|u_0\|^2}.$$

Hence,

$$\frac{\int_M \overline{Q}'}{\inf\limits_{x\in M} f(x)} \le Ce^{C\|u_0\|^2}. \tag{12}$$

Next we will prove the following

Lemma 4.3. *Let C_K and α be the constants found in Lemma 4.2. Let*

$$r = C_K(8C)^\alpha e^{(C+1)\alpha\|u_0\|^2},$$

and let us assume that

$$e^{\tau\|u_0\|^2} \sup_{x\in M} f(x) \le C_0,$$

where $\tau = \alpha(C+1) - C$ and

$$C_0 = -\frac{\inf\limits_{x\in M} f(x)}{8^\alpha C_K C^{\alpha-1}}.$$

Then for all $t \ge 0$, it holds

$$\int_M e^{2u} \le 2r.$$

Proof. Let

$$T = \sup\left\{s \ge 0; \int_M e^{2u} \le 2r \text{ in } [0, s]\right\}$$

and let us assume for the sake of contradiction that $T < \infty$. We notice that by continuity, we obtain that

$$\int_M e^{2u(T)} = 2r.$$

We assume first that

$$\int_M f^+ e^{2u(T)} \le \frac{1}{2}\int_M f^- e^{2u(T)},$$

where $f^+ := \max\{f, 0\}$ and $f^- = f^+ - f$ denote the positive and negative part of f respectively. Then we get

$$\int_M f^- e^{2u(T)} \le -2\int_M f e^{2u(T)} = -2\int_M \overline{Q}' \le -4\int_M \overline{Q}'.$$

Since in K we have $f^-(x) \ge -\frac{1}{2}\inf\limits_{x\in M} f(x)$, we have

$$\int_K e^{2u(T)} \le \frac{8\int_M \overline{Q}'}{\inf\limits_{x\in M} f(x)}$$

which combined with (12) gives

$$\int_K e^{2u(T)} \le 8Ce^{C\|u_0\|^2}.$$

But from Lemma 4.2, we have

$$\int_M e^{2u(T)} \leq C_K e^{\alpha\|u_0\|^2} \max\left(\left(\int_K e^{2u}\right)^\alpha, 1\right).$$

Thus

$$\int_M e^{2u(T)} \leq C_K e^{\alpha\|u_0\|^2} \left(8Ce^{C\|u_0\|^2}\right)^\alpha = r,$$

which is a contradiction.
So we move to the next case, where

$$\int_M f^+ e^{2u(T)} > \frac{1}{2}\int_M f^- e^{2u(T)}.$$

Then we have

$$-\frac{1}{2}\inf_{x\in M} f(x)\int_K e^{2u(T)} \leq \int_M f^- e^{2u(T)} < 2\int_M f^+ e^{2u(T)} \leq 4r \sup_{x\in M} f(x).$$

Hence,

$$\int_K e^{2u(T)} \leq -\frac{8r\sup\limits_{x\in M} f(x)}{\inf\limits_{x\in M} f(x)}.$$

By using our assumption, we obtain that

$$\int_K e^{2u(T)} \leq -\frac{8re^{-\tau\|u_0\|^2}C_0}{\inf\limits_{x\in M} f(x)},$$

and by Lemma 4.2, we have

$$\int_M e^{2u(T)} \leq C_K e^{\alpha\|u_0\|^2}\left(\frac{8re^{-\tau\|u_0\|^2}C_0}{-\inf\limits_{x\in M} f(x)}\right)^\alpha \leq r,$$

leading again to a contradiction. Hence $T = +\infty$ and $\int_M e^{2u}$ is uniformly bounded. $\square$

Now, by Jensen's inequality we have

$$\exp\left(\frac{1}{V}\int_M 2u\right) \leq \frac{1}{V}\int_M e^{2u} \leq \frac{2r}{V},$$

thus $\bar{u}$ is bounded from above. Now again using the energy identity (11), we get

$$\int_M u\overline{P}'u + 2\int_M \overline{Q}'(u-\bar{u}) + 2\bar{u}\int_M \overline{Q}' \leq E(u_0),$$

and

$$\int_M u\overline{P}'u + 2\int_M \overline{Q}'(u-\bar{u}) \geq \frac{1}{2}\int_M u\overline{P}'u - \frac{2\|\overline{Q}'\|_{L^2}^2}{\lambda_1} \geq -C_3.$$

Therefore

$$2\bar{u}\int_M \overline{Q}' \leq E(u_0) + C_3,$$

and since $\int_M \overline{Q}' < 0$ we have that $\bar{u}$ is uniformly bounded from below which finishes the proof of Theorem 1.1.

4.2 Case $\int_M \overline{Q}' = 0$ and proof of Theorem 1.2

Since $\int_M \overline{Q}' = 0$, we have that

$$\langle \nabla E(u), 1\rangle = 2\int_M \overline{P}' u = 0$$

and

$$\langle \nabla N(u), 1\rangle = 2\int_M \Gamma\left(fe^{2u}\right) = 2\int_M \overline{Q}' = 0.$$

Hence,

$$0 = \int_M \partial_t u = \partial_t \int_M u,$$

which means that the average value of u is preserved. Therefore $\bar{u} = \bar{u}_0$ and by Corollary 3.1, we have the convergence of the flow. This completes the proof of Theorem 1.2.

4.3 Case $0 < \int_M \overline{Q}' < 16\pi^2$ and proof of Theorem 1.3

First, we have again from the energy identity (11)

$$\int_M u\overline{P}' u + 2\int_M \overline{Q}'(u-\bar{u}) + 2\bar{u}\int_M \overline{Q}' \leq E(u_0). \tag{13}$$

Hence

$$2\bar{u}\int_M \overline{Q}' \leq E(u_0) - \frac{1}{2}\int_M u\overline{P}' u + \frac{2}{\lambda_1}\|\overline{Q}'\|_{L^2}^2$$

and then $\bar{u}$ is bounded from above; we will need a bound from below. Since $u \in X$, we get

$$\int_M \overline{Q}' = \int_M fe^{2u} \leq \|f\|_\infty \int_M e^{2u},$$

and therefore

$$\ln\left(\frac{\int_M \overline{Q}'}{\|f\|_\infty}\right) \leq \ln\left(\int_M e^{2u}\right).$$

Now again from Theorem 3.1 we have

$$\ln\left(\frac{\int_M \overline{Q}'}{\|f\|_\infty}\right) \leq C + \frac{1}{16\pi^2}\int_M u\overline{P}' u + \frac{2}{V}\int_M u. \tag{14}$$

Let $\delta > 0$ to be determined later, we sum equation (13) and $-\delta$ times equation (14), obtaining

$$\ln\left(\frac{\int_M \overline{Q}'}{\|f\|_\infty}\right) - \delta E(u_0) \leq C + \left(\frac{1}{16\pi^2} - \delta\right)\int_M u\overline{P}' u + 2\left(1 - \delta\int_M \overline{Q}'\right)\bar{u} - 2\delta\int_M \overline{Q}'(u-\bar{u}).$$

Since $\int_M \overline{Q}' < 16\pi^2$, we choose δ such that $\int_M \overline{Q}' < \frac{1}{\delta} < 16\pi^2$, and we set

$$c_1 = 2\left(1 - \delta \int_M \overline{Q}'\right), \qquad c_2 = \delta - \frac{1}{16\pi^2}.$$

We have

$$\ln\left(\frac{\int_M \overline{Q}'}{\|f\|_\infty}\right) - \delta E(u_0) - C + c_2 \int_M u\overline{P}'u + 2\delta \int_M \overline{Q}'(u - \bar{u}) \leq c_1 \bar{u}.$$

Now we notice that

$$c_2 \int_M u\overline{P}'u + 2\delta \int_M \overline{Q}'(u - \bar{u}) \geq (c_2\lambda_1 - \delta\varepsilon)\|u - \bar{u}\|^2_{L^2} - \frac{\delta}{\varepsilon}\|\overline{Q}'\|^2_{L^2},$$

therefore for ε small enough we have that

$$c_2 \int_M u\overline{P}'u + 2\delta \int_M \overline{Q}'(u - \bar{u}) \geq -c_3.$$

It follows that $\bar{u}$ is bounded from below and therefore from Corollary 3.1 this finishes the proof.

5 The critical case and proof of Theorem 1.4

Here we will study the case $\int_M \overline{Q}' = 16\pi^2$, where $M = S^3$ is the sphere equipped with its standard contact structure. We will see S^3 as a subset of $\mathbb{C}^2$ with coordinates (ζ_1, ζ_2) such that

$$S^3 = \left\{(\zeta_1, \zeta_2) \in \mathbb{C}^2\ ;\ |\zeta_1|^2 + |\zeta_2|^2 = 1\right\}.$$

We recall, following the notations in [3, page 15], that every C^4 conformal mapping of $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$ comes from the action of $SU(2,1)$ and it can be written as the composition of the following four transformations:

- left translations: $(z,t) \to (z',t') * (z,t)$, where here $*$ denotes the group operation on $\mathbb{H}^1$,
- dilations: $(z,t) \to (\delta z, \delta^2 t)$ for $\delta > 0$,
- rotations: $(z,t) \to (az, t)$, where $a \in S^1 \subset \mathbb{C}$,
- inversion: $(z,t) \to \left(-\frac{z}{|z|^2 + it}, \frac{t}{t^2 + |z|^4}\right)$.

The group of conformal transformation of the Heisenberg group $\mathbb{H}^1$, also called the group of CR automorphisms, will be denoted by $Aut(\mathbb{H}^1)$. Using the Cayley transform $C : \mathbb{H}^1 \to S^3 \setminus \{(0, -i)\}$ one has a clear description of the set $Aut(S^3)$:

$$Aut(S^3) = \{C \circ h \circ C^{-1}; h \in Aut(\mathbb{H}^1)\}.$$

For $p \in S^3$ and $r \geq 1$, we will write $h_{p,r}$ the element of $Aut(S^3)$ corresponding to a Cayley transform centered at p and a dilation of size r. That is, if $h \in Aut(\mathbb{H}^1)$ is a dilation with $\delta = r$ and $C_p : \mathbb{H}^1 \to S^3 \setminus \{-p\}$ is the Cayley transform sending zero to p (instead of $(0,i)$), then $h_{p,r} = C_p \circ h \circ C_p^{-1}$. Now, for $u \in X$ we set

$$v_{p,r} = u \circ h_{p,r} + \frac{1}{2}\ln(J(h_{p,r})),$$

where we denoted $J(h) = det(Jac(h))$, the Jacobian determinant of h. We have

$$E(v_{p,r}) = E(u) \leq E(u_0),$$

and since $u \in X$

$$\int_{S^3} f \circ h_{p,r} e^{2v_{p,r}} = \int_{S^3} f e^{2u},$$

hence

$$\int_{S^3} e^{2v_{p,r}} \geq \frac{16\pi^2}{\sup\limits_{x \in S^3} f(x)}.$$

From [3, page 38], we know that for all $t \geq 1$ there exists $r(t) \geq 1$ and $p(t) \in S^3$ such that

$$\int_{S^3} \xi_i e^{2v_{p(t),r(t)}} = 0,\ i = 1,2.$$

So we let $v(t) = v_{p(t),r(t)}$ and $h(t) = h_{p(t),r(t)}$. Then using Corollary A.2 in the Appendix, one has the existence of $a < \dfrac{1}{16\pi^2}$ and a constant C_1 such that

$$a\int_{S^3} v(t)\overline{P}'v(t) + 2\int_{S^3} v(t) - \ln\left(\int_{S^3} e^{2v(t)}\right) + C_1 \geq 0.$$

Since $E(v(t)) \leq E(u_0)$, we find that

$$\int_{S^3} v(t)\overline{P}'v(t) \leq C,$$

and

$$\left|\int_{S^3} v(t)\right| \leq C.$$

In particular we have that for all $p \geq 1$

$$\int_{S^3} e^{2|pv(t)|} \leq C_p,$$

and hence

$$\int_{S^3} v^2(t) \leq C$$

leading to the boundedness of $v(t)$ in H. We need the following concentration-compactness lemma in order to prove uniform boundedness.

Lemma 5.1. *Either*

(i) $\|u(t)\| \leq C$, *for some constant* C*;*
or
(ii) *there exists a sequence* $t_n \to \infty$ *and a point* $p_0 \in S^3$ *such that for all* $r > 0$

$$\lim_{n\to\infty} \int_{B_r(p_0)} f e^{2u(t_n)} = 16\pi^2.$$

Moreover, for any $\tilde{x} \in S^3 \setminus \{p_0\}$*, and any* $r < d(\tilde{x}, p_0)$*, we have*

$$\lim_{n\to\infty} \int_{B_r(\tilde{x})} f e^{2u(t_n)} = 0.$$

Proof. We assume first that $r(t)$ is bounded. Then we get

$$0 < C_1 \leq J(h_{p(t),r(t)}) \leq C_2.$$

Thus, from the uniform boundedness of $v(t)$ we have

$$\int_{S^3} |u(t)| \leq C.$$

Therefore, from Lemma 3.1, it follows that $\|u(t)\|$ is uniformly bounded.
So now we assume that $r(t)$ is not bounded, then there exists a sequence $t_n \to \infty$ such that $r(t_n) \to \infty$ and without loss of generality, by compactness of S^3 we can assume that $p(t_n) \to p_0$. From the uniform boundedness of $v(t)$, we can also assume that $v(t_n) \to v_\infty$ strongly in $L^2(S^3)$ and weakly in H. We let then $r > 0$ and set $K_n = h(t_n)^{-1}(B_r(p_0))$. Then we have

$$\left| \int_{S^3} f \circ h(t_n) e^{2v(t_n)} - \int_{K_n} f \circ h(t_n) e^{2v(t_n)} \right| \leq \left(\sup_{x \in S^3} f(x) \right) \left(Vol(K_n^c) \int_{S^3} e^{4|v(t_n)|} \right)^{\frac{1}{2}}.$$

Since $h(t_n)(x) \to p_0$ a.e. then $\lim_{n\to\infty} Vol(K_n) = V$, and thus

$$\int_{B_r(p_0)} f e^{2u(t_n)} = \int_{K_n} f \circ h(t_n) e^{2v(t_n)} = \int_{S^3} f \circ h(t_n) e^{2v(t_n)} + o(1).$$

We have also

$$\int_{S^3} f \circ h(t_n) e^{2v(t_n)} = 16\pi^2,$$

and then

$$\lim_{n\to\infty} \int_{B_r(p_0)} f e^{2u(t_n)} = 16\pi^2.$$

Now if we consider $\tilde{x} \in S^3 \setminus \{p_0\}$ and $r < d(p_0, \tilde{x})$ we have that $h(t_n)(x) \notin B_r(\tilde{x})$ for n big enough, since $\lim_{n\to\infty} h(t_n)(x) = p_0$ a.e.; in particular

$$\lim_{n\to\infty} \chi_{h(t_n)^{-1}(B_r(\tilde{x}))} = 0,$$

where χ is the characteristic function. Therefore

$$\lim_{n\to\infty} \int_{B_r(\tilde{x})} f e^{2u(t_n)} = \lim_{n\to\infty} \int_{h(t_n)^{-1}(B_r(\tilde{x}))} f \circ h(t_n) e^{2v(t_n)} = 0.$$

□

Let us assume now that $\Sigma = \emptyset$. By using the previous lemma, if $\|u(t)\|$ is not uniformly bounded, then there exists $p_0 \in S^3$ such that

$$\lim_{n\to\infty} \int_{B_r(p_0)} f e^{2u(t_n)} = 16\pi^2,$$

and if $p_1 \neq p_0$ and $r < d(p_0, p_1)$, then

$$\lim_{n\to\infty} \int_{B_r(p_1)} f e^{2u(t)} = 0.$$

Since $\Sigma = \emptyset$, then there exists $g \in G$ such that $p_1 = g \cdot p_0 \neq p_0$. But

$$16\pi^2 = \lim_{n\to\infty} \int_{B_r(p_0)} f e^{2u(t)} = \lim_{n\to\infty} \int_{B_r(g\cdot p_0)} f e^{2u(t)} = \lim_{n\to\infty} \int_{B_r(p_1)} f e^{2u(t)} = 0$$

which is a contradiction. Hence $\|u(t)\|$ is uniformly bounded.
Now we assume that $\Sigma \neq \emptyset$ and that $\|u(t)\|$ is not uniformly bounded. We have that the concentration point $p_0 \in \Sigma$, otherwise we reach a contradiction arguing as in the previous case. So we obtain

$$\int_{B_r(p_0)} f e^{2u(t_n)} \leq \sup_{x\in B_r(p_0)} f(x) \int_{B_r(p_0)} e^{2u(t_n)} \leq \max\left(\sup_{x\in B_r(p_0)} f(x), 0 \right) \int_{S^3} e^{2u(t_n)}.$$

By using the the sphere version of Theorem 3.1, proved in [3], we have that

$$\frac{1}{V} \int_{S^3} e^{2u(t_n)} \leq e^{\frac{E(u(t_n))}{V}}.$$

Thus

$$\int_{B_r(p_0)} f e^{2u(t_n)} < \max\left(\sup_{x\in B_r(p_0)} f(x), 0 \right) V e^{\frac{E(u_0)}{V}}.$$

Now we first let $n \to \infty$, then $r \to \infty$ and we get

$$16\pi^2 < V \max(f(p_0), 0) e^{\frac{E(u_0)}{V}}.$$

Therefore $f(p_0) > 0$ and

$$1 < f(p_0) e^{\frac{E(u_0)}{16\pi^2}},$$

hence

$$f(p_0) > e^{-\frac{E(u_0)}{16\pi^2}},$$

which leads to a contradiction of the assumption in Theorem 1.4. Therefore we get the uniform boundedness of $\|u\|$ also in this case, which yields the convergence of the flow and it ends the proof.

A Appendix: Improved Moser-Trudinger Inequality

In what follows we will consider S^3 as a subset of $\mathbb{C}^2$ with coordinates (ζ_1, ζ_2) such that $|\zeta_1|^2 + |\zeta_2|^2 = 1$. We recall here the Improved Moser-Trudinger inequality introduced in [3] in order to prove the existence of a minimizer:

Proposition A.1. *([3], Proposition 3.4) Given $\frac{1}{2} < a < 1$, there exist constants $C_1(a)$, $C_2(a)$ such that for $u \in H$ with $\int_{S^3} \zeta_i e^{2u} = 0$, $i = 1, 2$, it holds:*

$$\frac{a}{16\pi^2} \int_{S^3} uP'u + 2\int_{S^3} u - \ln\left(\int_{S^3} e^{2u}\right) + C_1(a)\|(-\Delta_b)^{\frac{3}{4}}u\|_2^2 + C_2(a) \geq 0$$

This improved estimate will not be useful to us in our setting since it contains the term $C_1(a)\|(-\Delta_b)^{\frac{3}{4}}u\|_2^2$ that we cannot bound along the flow. Notice that in [3], the authors exploit Ekeland's principle to exhibit a good minimizing Palais-Smale sequence that allows the control of this extra term. In our setting, we will prove a result that can be seen as intermediate between Proposition A.1 and the usual Moser-Trudinger inequality in Theorem 3.1. In fact in [3] the authors gave hints on how to prove this result, knowing that this method only works in dimension 3 and 5. We will follow a technique used in [11], since it is simpler and it allows even more improved estimates.
We set

$$P_k := \Big\{\text{polynomials of } \mathbb{C}^2 \text{ with degree at most } k\Big\}$$

and

$$P_{k,0} := \Big\{f \in P_k; \int_{S^3} f = 0\Big\}.$$

For a given $m \in \mathbb{N}$ we let

$$\mathcal{N}_m := \left\{ \begin{array}{l} N \in \mathbb{N}; \exists x_1, \cdots, x_N \in S^3, \nu_1, \cdots, \nu_N \in \mathbb{R}^+ \text{ with } \sum_{k=1}^N \nu_k = 1 \\ \text{and for any } f \in P_{m,0}; \sum_{k=1}^N \nu_k f(x_k) = 0 \end{array} \right\}.$$

We let then $N_m = \min \mathcal{N}_m$. As it was shown in [11], one has $N_1 = 2$ and $N_2 = 4$. We recall from [3] that one has the following inequality on the standard sphere:
There exists a constant $A_2 > 0$ such that

$$\int_{S^3} \exp\left[A_2 \frac{|u - \bar{u}|^2}{\|\Delta_b u\|_{L^2}^2}\right] \leq C_0.$$

In fact the sharp constant A_2 was explicitly computed in [3] and it has the value $A_2 = 32$. With this result we can easily deduce that if $u \in \mathcal{S}^2(S^3)$ then $e^{2u} \in L^p(S^3)$ for all $1 \leq p < \infty$.

Lemma A.1. *Consider a sequence of functions $u_k \in \mathcal{S}^2(S^3)$ such that*

$$\bar{u}_k = 0, \quad \|\Delta_b u_k\|_{L^2} \leq 1$$

and suppose that $u_k \rightharpoonup u$ weakly in $\mathcal{S}^2(S^3)$ and

$$|\Delta_b u_k|^2 \rightharpoonup |\Delta_b u|^2 + \sigma \text{ in measure },$$

where σ is a measure on S^3. Let $K \subset S^3$ be a compact set with $\sigma(K) < 1$, then for all $1 \leq p < \frac{1}{\sigma(K)}$ we have

$$\sup_k \int_K \exp\left[pA_2 u_k^2\right] < \infty.$$

Proof. Let φ be a fixed smooth compactly supported function on S^3. We set $v_k = u_k - u$. Then $v_k \to 0$ strongly in L^2 and weakly in $\mathcal{S}^2(S^3)$. Now we compute

$$\begin{aligned}\int_{S^3} |\Delta_b(\varphi v_k)|^2 &= \int_{S^3} (\varphi \Delta_b v_k + v_k \Delta_b \varphi + 2\nabla_H \varphi \nabla_H v_k)^2 \\ &= \int_{S^3} \varphi^2 (\Delta_b v_k)^2 + v_k^2 (\Delta_b \varphi)^2 + 4|\nabla_H v_k \nabla_H \varphi|^2 + 2\varphi v_k \Delta_b \varphi \Delta_b v_k + \qquad (15) \\ &\quad + 4\varphi(\nabla_H \varphi \nabla_H v_k)\Delta_b v_k + 4 v_k (\nabla_H v_k \nabla_H \varphi)\Delta_b \varphi.\end{aligned}$$

Hence,

$$\int_{S^3} |\Delta_b(\varphi v_k)|^2 \to \int_{S^3} \varphi^2 d\sigma.$$

Assume that $1 \le p_1 < \frac{1}{\sigma(K)}$ and take φ so that $\varphi_{|K} = 1$, and $\int_{S^3} \varphi^2 d\sigma < \frac{1}{p_1}$. Then we have for k large,

$$\|\Delta_b(\varphi v_k)\|_{L^2}^2 < \frac{1}{p_1}.$$

Therefore,

$$\int_K \exp\left[p_1 A_2 (v_k - \overline{\varphi v_k})^2\right] \le \int_{S^3} \exp\left[p_1 A_2 (\varphi v_k - \overline{\varphi v_k})^2\right] \le \int_{S^3} \exp\left[A_2 \frac{(\varphi v_k - \overline{\varphi v_k})^2}{\|\Delta_b \varphi v_k\|_{L^2}^2}\right] \le C_0.$$

Thus, if we fix $\varepsilon > 0$, we can write

$$\begin{aligned}u_k^2 &= (v_k - \overline{\varphi v_k} + u + \overline{\varphi v_k})^2 \\ &= (v_k - \overline{\varphi v_k})^2 + 2(v_k - \overline{\varphi v_k})(u + \overline{\varphi v_k}) + (u + \overline{\varphi v_k})^2 \\ &\le (1+\varepsilon)(v_k - \overline{\varphi v_k})^2 + 2(1 + \frac{1}{\varepsilon})u^2 + 2(1 + \frac{1}{\varepsilon})^2 \overline{\varphi v_k}^2.\end{aligned}$$

Hence, given $p < \frac{1}{\sigma(K)}$ we can take $p_1 \in (p, \frac{1}{\sigma(K)})$ such that

$$\int_K e^{A_2 p_1 u_k^2} < C_0,$$

which finishes the proof. $\square$

Corollary A.1. *We consider the same assumptions as in Lemma A.1 and we let* $\ell = \max_{x \in S^3} \sigma(\{x\}) \le 1$. *Then the following hold*

- *If* $\ell < 1$, *then for any* $1 \le p < \frac{1}{\ell}$, $e^{A_2 u_k^2}$ *is bounded in* $L^p(S^3)$. *In particular* $e^{A_2 u_k^2} \to e^{A_2 u^2}$ *in* L^1.
- *If* $\ell = 1$, *then there exists* $x_0 \in S^3$ *such that* $\sigma = \delta_{x_0}$, $u = 0$ *and after passing to a subsequence if necessary, we have*

$$e^{A_2 u_k^2} \rightharpoonup 1 + c_0 \delta_{x_0},$$

  for some $c_0 \ge 0$.

Proof. Assume that $\ell < 1$ and let $1 \le p < \frac{1}{\ell}$. Then for all $x \in S^3$, $\sigma(\{x\}) < \frac{1}{p}$. By continuity, there exists $r_x > 0$ such that $\sigma(\overline{B_{r_x}(x)}) < \frac{1}{p}$. Since S^3 is compact we can find a finite collection of balls of the form $B_{r_i}(x_i)$ such that

$$S^3 = \bigcup_{i=1}^{N} \overline{B_{r_i}(x_i)}.$$

So using Lemma A.1, we have

$$\sup_k \int_{\overline{B_{r_i}(x_i)}} \exp\left[pA_2 u_k^2\right] < \infty.$$

Thus,

$$\sup_k \int_{S^3} \exp\left[pA_2 u_k^2\right] < \infty.$$

We assume now that $\ell = 1$. Since $\|\Delta_b u_k\|^2 \le 1$ we have that $\|\Delta_b u\|^2 + \sigma(S^3) \le 1$. Therefore, we have $u = 0$ and there exists $x_0 \in S^3$ such that $\sigma = \delta_{x_0}$. Hence, for r small, we have that

$$\sup_k \int_{S^3 \setminus B_r(x_0)} \exp\left[qA_2 u_k^2\right] < \infty,$$

for all $q \ge 1$. Therefore, $e^{A_2 u_k^2} \to 1$ in $L^1(S^3 \setminus B_r(x_0))$ for every $r > 0$ and small. Hence, after passing to a subsequence if necessary we have that $e^{A_2 u_k^2} \rightharpoonup 1 + c_0 \delta_{x_0}$ in measure. $\square$

Proposition A.2. *Let $\alpha > 0$ and consider a sequence $m_k \to \infty$ and $u_k \in \mathcal{S}^2(S^3)$ such that $\overline{u_k} = 0$ and $\|\Delta_b u_k\|_{L^2} = 1$ such that $u_k \rightharpoonup u$ weakly in $\mathcal{S}^2(S^3)$ and $(\Delta_b u_k)^2 \rightharpoonup (\Delta_b u)^2 + \sigma$ in measure. We assume moreover that*

$$\ln\left(\int_{S^3} e^{2m_k u_k}\right) \ge \alpha m_k,$$

and

$$\frac{e^{2m_k u_k}}{\int_{S^3} e^{2m_k u_k}} \rightharpoonup \nu \text{ in measure.}$$

We set $R = \left\{x \in S^3; \sigma(\{x\}) \ge A_2\alpha\right\} = \{x_1, \cdots, x_N\}$. Then $\nu = \sum_{i=1}^N \nu_i \delta_{x_i}$ with $\nu_i \ge 0$ and $\sum_i \nu_i = 1$.

Proof. Let $K \subset S^3$ such that $\sigma(K) < A_2\alpha$. By continuity, we can find a compact set K_1 such that $K \subset int(K_1)$ and $\sigma(K_1) < A_2\alpha$. Now given $\frac{1}{A_2\alpha} < p < \frac{1}{\sigma(K_1)}$, we have

$$\sup_k \int_{K_1} e^{pA_2 u_k^2} \le C_0.$$

Since $2m_k u_k \le pA_2 u_k^2 + \frac{m_k^2}{pA_2}$, we have

$$\int_{K_1} e^{2m_k u_k} \le Ce^{\frac{m_k^2}{A_2 p}}.$$

Therefore,

$$\frac{\displaystyle\int_{K_1} e^{2m_k u_k}}{\displaystyle\int_{S^3} e^{2m_k u_k}} \leq Ce^{\left(\frac{1}{A_2 p}-\alpha\right)m_k^2}.$$

So $\nu(K) \leq \nu(K_1) = 0$ and $\nu(K) = 0$. Thus, if $\sigma(\{x\}) < A_2\alpha$, then there exists $r_x > 0$ small enough so that $\sigma(\overline{B_{r_x}(x)}) < A_2\alpha$. Hence, $\nu(\overline{B_{r_x}(x)}) = 0$. We deduce then that $\nu(S^3 \setminus R) = 0$. Therefore

$$\nu = \sum_{k=1}^{N} \nu_k \delta_{x_k},$$

with $\nu_k \geq 0$ and $\sum_{k=1}^{N} \nu_k = 1$. □

Let $f_1, \cdots, f_\ell \in C(S^3)$. We define

$$\mathcal{S}_f = \left\{ u \in \mathcal{S}^2(S^3); \overline{u} = 0; \int_{S^3} f_k e^{2u} = 0; k = 1, \cdots, \ell \right\}.$$

Proposition A.3. *If $f_j \in P_{m,0}$ for $j = 0, \cdots, \ell$ and $\alpha > \frac{1}{A_2 N_m}$, then there exists $C \in \mathbb{R}$ such that*

$$\ln\left(\int_{S^3} e^{2u}\right) \leq \alpha \|\Delta_b u\|_2^2 + C, \forall u \in \mathcal{S}_f.$$

Proof. We assume that the inequality

$$\ln\left(\int_{S^3} e^{2u}\right) \leq \alpha \|\Delta_b u\|_2^2 + C$$

does not hold for $u \in \mathcal{S}_f$. Then there exists a sequence $u_k \in \mathcal{S}_f$ such that

$$\ln\left(\int_{S^3} e^{2u_k}\right) - \alpha \|\Delta u_k\|_{L^2}^2 \to \infty.$$

Therefore, it follows that $\int_{S^3} e^{2u_k} \to \infty$ and $\|\Delta_b u_k\|_{L^2} \to \infty$. So we let $m_k = \|\Delta_b u_k\|_{L^2}$ and $v_k = \frac{u_k}{m_k}$. Then $m_k \to \infty$, $\|\Delta_b v_k\|_{L^2}^2 = 1$. Hence, after passing to a subsequence, we have

$$\begin{cases} v_k \rightharpoonup v \text{ weakly in } \mathcal{S}^2(S^3), \\ |\Delta_b v_k|^2 \rightharpoonup |\Delta_b v|^2 + \sigma \text{ in measure}, \\ \dfrac{e^{2m_k v_k}}{\displaystyle\int_{S^3} e^{2m_k v_k}} \rightharpoonup \nu \text{ in measure}. \end{cases}$$

So we let $R = \{x \in S^3; \sigma(\{x\}) \geq A_2\alpha\} = \{x_1, \cdots, x_N\}$. It follows from Proposition A.2 that $\nu = \sum_{j=1}^{N} \nu_j \delta_{x_j}$, with $\sum_{j=1}^{N} \nu_j = 1$ and $\nu_j \geq 0$.
But since $u_k \in \mathcal{S}_f$, we have

$$\int_{S^3} f_j d\nu = 0.$$

Therefore,

$$\sum_{i=1}^{N} \nu_i f_j(x_i) = 0, \text{ for all } 1 \leq j \leq \ell.$$

On the other hand, $A_2 \alpha N \leq 1$. In particular, if $f_j \in P_{m,0}$, we have that $N \in \mathcal{N}_m$. Therefore,

$$\alpha \leq \frac{1}{A_2 N} \leq \frac{1}{A_2 N_m}.$$

Hence, if $\alpha = \frac{1}{A_2 N_m} + \varepsilon$ we get a contradiction and the result holds. □

Therefore, if we define

$$\mathcal{S}_0 = \left\{ u \in \mathcal{S}^2(S^3); \overline{u} = 0; \int_{S^3} f e^{2u} = 0 \text{ for all } f \in P_{1,0} \right\},$$

the following corollary holds

Corollary A.2. *There exist $a < \frac{1}{16\pi^2}$ and $C > 0$ such that for all $u \in \mathcal{P} \cap \mathcal{S}_0$, we have*

$$a \int_{S^3} u \overline{P}' u + 2 \int_M u - \ln \left(\int_M e^{2u} \right) \geq -C.$$

Indeed, this corollary follows from the fact that

$$\int_{S^3} u \overline{P}' u \geq \int_{S^3} |2\Delta_b u|^2$$

for all $u \in \hat{\mathcal{P}}$ and $8A_2 > 16\pi^2$.

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