On the characteristic direction of real hypersurfaces in $\mathbb{C}^{n+1}$ and a symmetry result.

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Abstract In this paper we show the following property of a non Levi flat real hypersurface in $\mathbb{C}^{n+1}$: if the unit characteristic direction T is a geodesic, then it is an eigenvector of the second fundamental form and the relative eigenvalue is constant. As an application we prove a symmetry result, of Alexandrov type, for compact hypersurfaces in $\mathbb{C}^{n+1}$ with positive constant Levi mean curvature.

1 Introduction

By using Codazzi equations and Chow Theorem, we show a characterization result for non Levi flat real smooth hypersurfaces in $\mathbb{C}^{n+1}$, whose unit characteristic direction T is a geodesic. By denoting with h the second fundamental form of M and with $h_{TT} := h(T,T)$, the main result of our work is:

Theorem 1.1. *Let M be a non Levi flat real hypersurface in $\mathbb{C}^{n+1}$. If the characteristic direction T is a geodesic for M, then h_{TT} is constant.*

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Theorem 1.1 cannot be inverted. Indeed, in Section 4 we will show a non Levi flat hypersurface whose characteristic direction is not a geodesic, but h_{TT} is constant.
As an application of Theorem 1.1 we get a result of characterization of spheres, of Alexandrov type:

Corollary 1.2. *Let M be a compact real hypersurface in $\mathbb{C}^{n+1}$ with positive constant Levi mean curvature. If the characteristic direction T is a geodesic for M, then M is a sphere.*

The problem of characterizing compact hypersurfaces with positive constant Levi mean curvature has recently received attention from many mathematicians. Klingenberg in [4] showed that if the characteristic direction of a compact hypesurface is a geodesic and the Levi form is diagonal and positive definite, then M is a sphere. Later on Hounie and Lanconelli proved that the boundary of a compact Reinhardt domain in $\mathbb{C}^2$ with constant Levi curvature is a sphere. Monti and Morbidelli in [8] proved that every Levi umbilical hypersurface for $n \geq 2$, is contained either in a sphere or in the boundary of a tube domain with spherical section.

Our paper is organized as follows. In Section 2 we introduce notations and we prove that the characteristic direction T is a geodesic iff it is a curvature line. In Section 3 we recall the celebrated Codazzi equations for the Levi-Civita connection. In Section 4 we prove Theorem 1.1 by using Chow Theorem and we use the classical Alexandrov Theorem to show Corollary 1.2.

2 Curvature lines and geodesics

We recall some elementary facts in order to fix the notations. Let M be a hypersurface in $\mathbb{C}^{n+1}$ and let TM be the tangent space to M. We denote by N the inner unit normal, and we define the characteristic direction $T \in TM$ as:

$$T = J(N) \tag{1}$$

where J is the standard complex structure in $\mathbb{C}^{n+1}$ (corresponding to the multiplication by $\pm i$). The complex maximal distribution or Levi distribution HM is the largest subspace in TM invariant under the action of J

$$HM = TM \cap J(TM) \tag{2}$$

i.e., a vector field $X \in TM$ belongs to HM if and only if also $J(X) \in HM$. Moreover, if g is the standard metric on M induced by $\mathbb{C}^{n+1}$, then every element in TM can be written as a direct sum of an element of HM and one of the space generated by T, in formulas

$$TM = HM \oplus \mathbb{R}T \tag{3}$$

where $dim(HM) = 2n$ and the sum is g-orthogonal:

$$\forall X \in HM \quad g(T, X) = 0 \tag{4}$$

In the sequel we shall use the following notation: we will use a tilde for all the objects in $\mathbb{C}^{n+1}$ that induce on M the relative induced objects. As an example, with $\tilde{g}$ we refer to the metric on $\mathbb{C}^{n+1}$ and with g we refer to the metric on M induced by $\tilde{g}$.

We shall denote by $\tilde{\nabla}$ the Levi-Civita connection in $\mathbb{C}^{n+1}$. We recall that both $\tilde{\nabla}$ and $\tilde{g}$ are compatible with the complex structure J, i.e.:

$$J\tilde{\nabla} = \tilde{\nabla}J, \quad \tilde{g}(\cdot, \cdot) = \tilde{g}(J(\cdot), J(\cdot)) \tag{5}$$

The second fundamental form h is defined as:

$$h(V, W) = \tilde{g}(\tilde{\nabla}_V W, N) = g(A(V), W), \ \forall V, W \in TM \tag{6}$$

where A is the Weingarten operator, defined by

$$A(V) = -\tilde{\nabla}_V N, \ \forall\ V \in TM \tag{7}$$

The Levi form l is the hermitian operator on HM defined in the following way:

$\forall X_1, X_2 \in HM$, if $Z_1 = X_1 - iJ(X_1)$ and $Z_2 = X_2 - iJ(X_2)$, then

$$l(X_1, X_2) = \tilde{g}(\tilde{\nabla}_{Z_1} \bar{Z}_2, N) \tag{8}$$

We compare the Levi form with the second fundamental form by using the identity (see [2], Chap.10, Theorem 2):

$$\forall X \in HM, \quad l(X,X) = h(X,X) + h(J(X),J(X)) \tag{9}$$

We recall that M is non Levi flat if in every point of M the Levi form is not identically zero.

The classical mean curvature H and the Levi mean curvature L are respectively:

$$H = \frac{1}{2n+1}tr(h), \quad L = \frac{1}{n}tr(l) \tag{10}$$

where tr is the canonical trace operator. A direct calculation lead to the relation between H and L [7]:

$$H = \frac{1}{2n+1}(2nL + h_{TT}) \tag{11}$$

Definition 2.1. *Let $V \in TM$. V is a eigenvector for A (or for h) if there exists a function (eigenvalue) $\lambda : M \to \mathbb{R}$ such that $A(V) = \lambda V$ on M.*
Let γ be the integral curve of V, i.e. $\gamma \subseteq M$ is a line such that $\dot{\gamma} = V$. If V is a eigenvector for A then we refer to γ as a curvature line. Moreover, if V is unitary, then the value of λ is $\lambda = h(V,V)$ because

$$h(V,V) = g(A(V),V) = g(\lambda V,V) = \lambda g(V,V) = \lambda$$

Definition 2.2. *Let $V \in TM$. The integral curve of V is a geodesic if $\nabla_V V = 0$ or equivalently: if $\tilde{\nabla}_V V \in \mathbb{R}N$, i.e. if the field $\tilde{\nabla}_V V$ is normal to M.*
It is well known that this definition of geodesic coincides with that one of minimizing curve for the distance functional $d_{p,q}(\gamma)$, induced by the metric $\tilde{g}$ of $\mathbb{C}^{n+1}$, i.e. if $p,q \in M$, for all curves $\gamma : [t_1,t_2] \to M$ such that $\gamma(t_1) = p$ and $\gamma(t_2) = q$

$$d_{p,q}(\gamma) = \int_{t_1}^{t_2} \sqrt{g(\dot{\gamma},\dot{\gamma})}dt$$

and the geodesic is the curve that realizes $min\big(d_{p,q}(\gamma)\big)$

With an abuse of language, we will also refer to the vector field V as a curvature line or a geodesic if the corresponding integral curve is a curvature line or a geodesic respectively.

Lemma 2.3. *Let T be the characteristic direction of M. T is a curvature line if and only if it is a geodesic.*

Proof. If T is a curvature line, one has

$$A(T) = \lambda T, \quad \lambda = h_{TT} \tag{12}$$

For all $X \in HM$, by using (4), one realizes that

$$g(\tilde{\nabla}_T N, X) = g(-A(T), X) = -h_{TT} g(T, X) = 0 \tag{13}$$

Then for the compatibility of the complex structure J with the connection $\tilde{\nabla}$ and with the metric $\tilde{g}$, for all $X \in HM$ we have

$$0 = \tilde{g}(\tilde{\nabla}_T N, X) = \tilde{g}(J(\tilde{\nabla}_T N), J(X)) = \tilde{g}(\tilde{\nabla}_T T, J(X)) \tag{14}$$

Moreover T is unitary ($g(T,T) = 1$), and by differentiating along T one has

$$\tilde{g}(\tilde{\nabla}_T T, T) = 0 \tag{15}$$

Therefore, by using (14) and (15) it is proved that

$$\tilde{\nabla}_T T \in \mathbb{R}N, \quad \nabla_T T = 0 \tag{16}$$

To prove the converse we can argue by inverting the previous procedure. □

3 A Codazzi equation

In this section we write a Codazzi equation (see [5]) with the notations of Section 2. The celebrated Codazzi equations assert that: for all $V, W, Z \in T(M)$

$$(\nabla_V h)(W, Z) = (\nabla_W h)(V, Z) \tag{17}$$

where

$$(\nabla_V h)(W, Z) = V(h(W, Z)) - h(\nabla_V W, Z) - h(W, \nabla_V Z) \tag{18}$$

By writing equation (17) with $V = X$ e $W = Z = T$, where T is the characteristic direction, we get

$$(\nabla_X h)(T,T) = (\nabla_T h)(X,T) \tag{19}$$

Let

$$B = \{T, X_1, \ldots, X_n, J(X_1), \ldots, J(X_n)\} = \{T, X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\}$$

be an orthonormal basis of TM. For $k = 1, \ldots, 2n$, we denote

$$\Gamma^k_{XT} = g(\nabla_X T, X_k), \quad \Gamma^k_{TX} = g(\nabla_T X, X_k), \quad \Gamma^T_{TX} = g(\nabla_T X, T)$$

In particular, by using (4) and (5) one has

$$\begin{aligned}\Gamma^T_{TX} &= \tilde{g}(\tilde{\nabla}_T X, T) = -\tilde{g}(\tilde{\nabla}_T T, X) = \tilde{g}(\tilde{\nabla} N, J(X)) \\ &= -g(A(T), J(X)) = -h(T, J(X)).\end{aligned}$$

Therefore, with the usual convention to sum up and low equal indices, (19) becomes:

$$X(h_{TT}) - 2h(\nabla_X T, T) = T(h(T,X)) - h(\nabla_T X, T) - h(X, \nabla_T T)$$

$$X(h_{TT}) - 2h(\Gamma^k_{XT} X_k, T) = T(h(T,X)) - h(\Gamma^k_{TX} X_k + \Gamma^T_{TX} T, T) - h(X, \nabla_T T)$$

$$X(h_{TT}) = T(h(T,X)) + \Big(2\Gamma^k_{XT} - \Gamma^k_{TX}\Big) h(X_k, T) - h_{TT} h(T, J(X)) - h(X, \nabla_T T) \tag{20}$$

4 An Alexandrov type result

In this section we first prove Theorem 1.1 by using (20). Then, by using the classical Alexandrov Theorem for compact hypersurfaces with constant mean curvature, we prove our symmetry result, Corollary 1.2. Let us start with a lemma

Lemma 4.1. *If M is non Levi flat, then M has the following H-connectivity property: for every couple of points $p, q \in M$ there exists a curve $\gamma : [0,1] \to M$, such that $\gamma(0) = p$, $\gamma(1) = q$ and $\dot{\gamma}(t) \in HM$ for all $t \in [0,1]$.*

Proof. It has been proved in [6, Corollary 3.1 and Remark 3.1] that if M is not Levi flat then there is a basis $\{X_j, j = 1, \dots, 2n\}$ of HM such that the Hörmander's rank condition holds:

$$\dim\Big(span\big\{X_j, [X_\ell, X_k], \quad j,k,\ell = 1,\dots,2n\big\}\Big) = 2n+1 \tag{21}$$

With the notations of the present paper, an easier proof of (21) can be obtained. Indeed, if M is non Levi flat then at every point of M then there exists at least a vector field $X \in HM$ such that $l(X,X) \neq 0$. For $Y = J(X)$ and $Z = X - iY$, one has

$$l(X,X) = \tilde{g}(\tilde{\nabla}_Z \bar{Z}, N) = \tilde{g}(\tilde{\nabla}_{X-iY} X + iY, N) = \tilde{g}(\tilde{\nabla}_X X + \tilde{\nabla}_Y Y, N) =$$

$$= \tilde{g}(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, T) = \tilde{g}([X,Y],T) \neq 0$$

This means that for every basis $\{X_j, j = 1, \dots, 2n\}$ of HM the Hörmander's rank condition (21) holds. By Chow's theorem we then get the H-connectivity property. □

Proof of Theorem 1.1. By Lemma 2.3 the characteristic direction T is a curvature line. If T is a curvature line for M, then for all $V \in HM$

$$h(T,V) = g(A(T),V) = h_{TT}\, g(T,V) = 0$$

Moreover, since T is a geodesic, then $\nabla_T T = 0$ on M. Let $X \in HM$, the equation (20) becomes

$$\begin{aligned} X(h_{TT}) =& T(h(T,X)) + \Big(2\Gamma^k_{XT} - \Gamma^k_{TX}\Big) h(X_k,T) + \\ & - h_{TT} h(T,J(X)) - h(X, \nabla_T T) = 0 \end{aligned} \tag{22}$$

and h_{TT} is constant on HM.

Since M is not Levi flat, by Lemma 4.1 for every couple of points $p, q \in M$ there exists a curve $\gamma : [0,1] \to M$, such that $\gamma(0) = p$, $\gamma(1) = q$ and $\dot{\gamma}(t) \in HM$ for all $t \in [0,1]$. Therefore, by using an arbitrary basis $\{X_1, \dots, X_{2n}\}$ of HM, one obtains:

$$\dot{\gamma}(h_{TT}) = \alpha^k X_k(h_{TT}) = 0$$

Then h_{TT} is constant along γ and therefore on M □

In general the converse of Theorem 1.1 does not hold, i.e. if the coefficient of the second fundamental form h_{TT} is constant, one cannot conclude that the characteristic direction T is a geodesic (or a curvature line), as the following example shows

Example 4.2. *In $\mathbb{C}^2$ with coordinates $z_k = x_k + iy_k$, $k = 1, 2$, we consider the domain*

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : f(x_1, y_1, x_2, y_2) = x_1^2 + (ay_1 + bx_2)^2 - 1 < 0\}$$

with a, b constants such that $a^2 + b^2 = 1$. Let M be the real hypersurface defined by $M := \partial\Omega$. We claim that if $0 < a < 1$ then M is non Levi flat and $h_{TT} = a^2$, but T is not a geodesic. Indeed, let $r = ay_1 + bx_2$, then on M one has

$$Df = 2(x_1, ar, br, 0), \quad |Df| = 2$$

where D is the Euclidean gradient in $\mathbb{R}^4$. Therefore, by identifying vector fields with first order partial differential operators, we get

$$N = -(x_1\partial_{x_1} + ar\partial_{y_1} + br\partial_{x_2}), \quad T = J(N) = ar\partial_{x_1} - x_1\partial_{y_1} - br\partial_{y_2}$$

Then, by using $T(r) = -ax_1$, one has

$$h_{TT} = h(T, T) = \tilde{g}(\tilde{\nabla}_T T, N) = -T(ar)x_1 - T(x_1)ar = a^2x_1^2 + a^2r^2 = a^2$$

We notice that M is isometric to the cylinder $S^1 \times \mathbb{R}^2$ whose three principal curvatures are 1,0,0; therefore the classical mean curvature of M is $H = \dfrac{1}{3}$. From (11) it follows that $2L = b^2$, and since $b \neq 0$ then M is non Levi flat. Moreover, since

$$\tilde{\nabla}_T T = T(ar)\partial_{x_1} - T(x_1)\partial_{y_1} - T(br)\partial_{y_2} \notin \mathbb{R}N$$

then T is not a geodesic.

As a consequence of Theorem 1.1 we get the proof of Corollary 1.2.

Proof of Corollary 1.2. If M has constant positive Levi mean curvature, then M is non Levi flat, and since T is a curvature line, one has that h_{TT} is constant on M. By using the compactness of M, by (11) and by the classical Alexandrov's theorem [1] we get that M is a sphere. □

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