Contact type hypersurfaces and Legendre duality

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Abstract In this paper we study contact type hypersurfaces embedded in four-dimensional Kähler manifolds. We are interested whether the so called *Legendre duality* can be performed and we will show that this can be related to some convexity assumptions, giving a sufficient condition. As an application, in the case of convex hypersurfaces in $\mathbb{R}^4$, we will explicitly complete this duality.

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1 Introduction

Let (M, α) be an orientable and compact three dimensional contact manifold. We are interested whether the Legendre transform, as defined by A.Bahri in some of his works (see [1, 2] for instance) can be performed: let us briefly explain it. The standard contact form α_0 on S^3 is the pull-back of the standard contact form on $P(\mathbb{R}^3)$, that is the unit sphere cotangent bundle of S^2; therefore it is equipped with its canonical Liouville form. The Legendre duality can be completed for the Liouville form. This Legendre transform can be viewed as the data of a non-vanishing vector field v in $ker(\alpha_0)$, such that $\beta_0(\cdot) := d\alpha_0(v, \cdot)$ is a contact form with the same orientation of α_0.
This duality has been extended by A.Bahri-D.Bennequin (see into the monograph [1]) to the more general framework of contact forms α on three-dimensional compact orientable manifolds without boundary M. So, let us give the following:

Definition 1.1. *Let us consider a non-vanishing vector field $v \in ker(\alpha)$ and let us assume that:*

(D) the dual form $\beta(\cdot) := d\alpha(v, \cdot)$ is a contact form on M with the same orientation than α.

Condition (D) is what we call Legendre duality for α with respect to v.

We observe that the condition on v to be non-vanishing ensures to have a non-singular dual form β and a global flow defined by v, which is a crucial fact in the applications.
The previous condition appears in the following variational problem: let us define the functional

$$A(\gamma) = \int_0^1 \alpha(\dot{\gamma})dt \tag{1}$$

on the subspace of the H^1-loops on M given by

$$C_\beta = \{\gamma \in H^1(S^1; M) \ s.t. \ \beta(\dot{\gamma}) = 0; \ \alpha(\dot{\gamma}) = \text{strictly positive constant}\}.$$

If $\xi \in TM$ denotes the Reeb vector field of α, i.e. $\alpha(\xi) = 1, \quad d\alpha(\xi, \cdot) = 0$, then the following result by A.Bahri-D.Bennequin holds [1]:

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Theorem 1.1. *A is a C^2 functional on C_β whose critical points are periodic orbits of ξ.*

The existence of such a v for a given contact form α allows to compute the Contact Homology relative to α, by using the techniques of the theory of critical points at infinity developed by A.Bahri [3, 4, 5, 6, 7, 8].
Recently in [18], we explicitly computed the relative Contact Homology for the three-dimensional torus equipped with a family of tight contact structures: this has been possible since we were able to construct an explicit vector field v satisfying the hypothesis (D).
Moreover, another application of the existence of such a vector field v allowed us to prove a topological property of Smale's type: indeed in [15], by using a suitable flow along v, we showed that the injection of the subspace C_β into the full loop space is an S^1-equivariant homotopy equivalence.
Therefore, we are motivated by these reasons to understand on which manifolds one can perform this Legendre transform and most of all whether an explicit vector field v can be found in order to carry out precise computations.
In [11], J.Gonzalo and F.Varela introduced a family $\{\alpha_n\}_{n\in\mathbb{N}}$ of contact forms on S^3, where α_0 is the standard contact form of the sphere and for $n \geq 1$ they are all overtwisted and pairwise not contactomorphic. In [20] it is considered the overtwisted contact form α_1 on S^3 and it is established the existence of such a v, which is given explicitly (see [7] for the homology computation related to that v); let us note that this is the first case in which such a v is given for an overtwisted contact structure on a compact manifold.
In [21] it is given a sufficient condition on real strictly Levi-convex hypersurfaces M, embedded in four-dimensional Kähler manifolds V, with the contact form on M whose kernel is the restriction of the holomorphic tangent space of V: it is showed that if there exists a Legendrian Killing vector field, the Legendre duality can be performed (see also [9, 16, 19, 26], for further details and applications of Killing vector fields on contact manifolds).

Here, starting by the definition of being of "contact type" in symplectic manifolds (see below), we recognize a sort of *convexity* of the manifold along v: therefore we will consider hypersurfaces in Kähler manifolds, where a canonical metric exists, and we relate our condition (D) to the standard convexity defined by the Second Fundamental form, giving a sufficient condition (Proposition 3.1). Finally, in the case of convex hypersurfaces in $\mathbb{R}^4$, we will explicitly exhibit a vector field v (Theorem 3.1).

2 Preliminaries

We will recall some basic definitions in symplectic and Kählerian geometry, see for instance [12, 14] for further details.
Let (V, ω) be a symplectic four dimensional manifold and let M be an orientable embedded hypersurface. Since $dimTM = 3$, then the restriction of ω to TM must be degenerate and of rank 2. Its kernel is therefore one dimensional and we define the one dimensional subspace:

$$Q_M := \Big\{\eta \in TM \quad s.t. \quad \omega(\eta, v) = 0, \qquad \forall v \in TM\Big\}.$$

Now, we give the following:

Definition 2.1. *Let (V, ω) be a symplectic four dimensional manifold and let M be an orientable embedded compact hypersurface. We say that (M, α) is of contact type if there exists a one-form α on M such that:*

$$(i) \quad \alpha(\eta) \neq 0, \quad \textit{for all } \eta \in Q_M, \quad \eta \neq 0;$$

$$(ii) \quad d\alpha = \omega, \quad \textit{on } TM.$$

Indeed, it holds that $\alpha \wedge d\alpha \neq 0$ on M, that motivates the terminology; moreover the one-form α can be extended to a neighborhood of M in V. Thus, let (M, α) be a contact type hypersurface in (V, ω) and let us assume that there exists a non-vanishing vector field $v \in ker(\alpha)$: we define on M the dual one-form $\beta(\cdot) := d\alpha(v, \cdot)$ and we let $\xi \in Q_M$ be the Reeb vector field of α. We know that $[\xi, v] \in ker(\alpha)$, so assuming for a moment that $\{\xi, v, [\xi, v]\}$ is a basis for TM, a direct computation shows that

$$\beta \wedge d\beta(\xi, v, [\xi, v]) = \beta([\xi, v])d\beta(\xi, v) = -\Big(d\alpha(v, [\xi, v])\Big)^2.$$

Since

$$\alpha \wedge d\alpha(\xi, v, [\xi, v]) = d\alpha(v, [\xi, v]) \neq 0,$$

then condition (D) reads as $(\beta \wedge d\beta/\alpha \wedge d\alpha > 0)$

$$d\alpha(v, [\xi, v]) < 0. \tag{2}$$

For what we will do in the sequel, let us notice that if we use in the computations a vector field $\eta \in Q_M$ with $\alpha(\eta) > 0$, instead of the Reeb vector field ξ, the condition on formula (2) remains the same.

The last equation (2) can be thought as a convexity condition on M along ξ and v: our aim is to show that, when a metric exists, then this is exactly the case.

Therefore, we are going to consider as ambient space, a four-dimensional Kähler manifold V, where a suitable metric exists: we refer to [10, 13, 14], for the notion of Kähler manifold and real hypersurfaces in it; here we only recall some basic facts.

First $V = V(\omega, g, J)$ is said to be a Kähler manifold if there exist a symplectic structure ω, a complex structure J and a Riemannian metric g such that they are compatible in the following sense:

$$\omega(X, Y) = g(X, JY) \tag{3}$$

for every pair of vector fields $X, Y \in TV$.

We consider now a smooth compact, orientable, embedded manifold M on V, of codimension 1, with the induced metric g, and we assume that there exists a one-form α such that (M, α) is of contact type. Using the symplectic structure, by the rank condition on ω, there exists a non-vanishing vector field ν transverse to M such that $d\alpha(\eta, \nu) \neq 0$, for any non-zero $\eta \in Q_M$. Now, we can use the complex structure J to make these choices unique; in fact, let us denote by ν the inner unit normal to M, then we define $T = J\nu$ and by (3) we get that $T \in Q_M$ and $|T|_g = 1$. We will need in the sequel also the following one-form on M:

$$\theta(\cdot) := g(T, \cdot) \tag{4}$$

In general θ is a not a contact form on M: when this happens M is said a strictly Levi-convex hypersurface. However the kernel of θ defines a two-dimensional subspace of TM which is called *horizontal*: in particular the complexification of $ker(\theta)$ coincides with the restriction of the usual holomorphic tangent space of V; see for instance [17, 22, 23, 24, 25], for some properties and applications regarding Levi-convex hypersurfaces. We observe that in general $ker(\alpha)$ and $ker(\theta)$ intersect transversally (they could coincide as well), but the vector field T is always transverse to both: in particular, by the very definition, T is g-orthogonal to $ker(\theta)$. Let now A be the Weingarten operator, namely

$$A : TM \to TM, \quad AX := -\nabla_X \nu\ ,$$

where ∇ denotes the Levi-Civita connection of V; we will denote by

$$h(\cdot, \cdot) := g(A\cdot, \cdot)$$

the Second Fundamental Form of M: we recall that a hypersurface is said to be (strictly) convex if the Second Fundamental Form is positive definite as quadratic form. We also recall that ∇ is compatible with the complex structure J, i.e.:

$$J\nabla = \nabla J \tag{5}$$

Since $T \in Q_M$, up to a change of sign, we can assume that $\alpha(T) > 0$; then let us write the equation (2) in this situation:

$$d\alpha(v, [T, v]) = \omega(v, [T, v]) = g(v, J[T, v]) = g(v, J(\nabla_T v - \nabla_v T)) =$$
$$= g(v, \nabla_T Jv + \nabla_v \nu)) = g(v, \nabla_T Jv) - h(v, v),$$

where $v \in ker(\alpha)$ is a non-vanishing vector field. Now let us suppose that we have a non-zero vector field $X \in ker(\theta)$, then we can define $v \in ker(\alpha)$ as $v = X + \rho T$, where $\rho : M \to \mathbb{R}$ is a function that solves the following equation:

$$0 = \alpha(v) = \alpha(X + \rho T) = \alpha(X) + \rho\alpha(T), \qquad \alpha(T) > 0 \tag{6}$$

Remark 2.1. *Let us also explicitly observe that since the vector fields X and T are both non-zero and orthogonal, in particular $|T|_g = 1$, we have that v is non-vanishing, in fact:*

$$|v|_g^2 = g(v, v) = g(X + \rho T, X + \rho T) = |X|_g^2 + \rho^2 > 0.$$

Hence, since if $X \in ker(\theta)$ then also $Y := JX \in ker(\theta)$, we get:

$$d\alpha(v, [T, v]) = g(v, \nabla_T Jv) - h(v, v) =$$
$$= g(X + \rho T, \nabla_T Y - \rho\nu) - h(X + \rho T, X + \rho T) =$$
$$= g(X, \nabla_T Y) - h(X, X). \tag{7}$$

Now, if M is convex, then $h(X, X) > 0$ for any non-zero vector field $X \in TM$; therefore we have showed the following:

Proposition 2.1. *Let M be a convex real hypersurface in V of contact type. If there exists a non-vanishing vector field $X \in ker(\theta)$, such that $g(X, \nabla_T Y) \leq 0$, then the condition (D) holds.*

At this point, the fact that the manifold M is of contact type, convex and the existence of such a vector field X depend on the ambient space V and on the manifold M itself: as an application, in the next section we will examine the explicit case of convex hypersurfaces in $\mathbb{C}^2$.

3 Convex hypersurfaces

Here we consider convex real hypersurfaces in $\mathbb{C}^2$ with its canonical hermitian metric. We will use the same notation of the previous section and we observe that all the computations we will make can be repeated in the case of real hypersurfaces in the other two standard models of complex space form: the complex projective space $\mathbb{C}P^{n+1}$ with the Fubini-Study metric, and the complex hyperbolic space $\mathbb{C}H^{n+1}$ with the Bergman metric. These three prototypes differ in the sign of the holomorphic sectional curvature (respectively zero, positive, and negative), but what one really needs is the fact that the connection coefficients are constant with respect to suitable coordinates. So, let us make the identification $\mathbb{C}^2 \simeq \mathbb{R}^4$, with

$$(z_1, z_2) \in \mathbb{C}^2, \quad z_1 = x_1 + iy_1, \ z_2 = x_2 + iy_2, \quad (x_1, y_1, x_2, y_2) \in \mathbb{R}^4$$

and let us consider $(\mathbb{R}^4, \omega)$, where $\omega = d\lambda$ and λ is the standard Liouville one-form:

$$\lambda = \frac{1}{2}(y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2)$$

We will denote by g, J, ∇, the canonical metric, the complex structure, and the Levi-Civita connection respectively. Now, let M be a convex and compact hypersurface in $\mathbb{R}^4$; we have that in particular M bounds a starshaped domain, let us say with respect to the origin, so the restriction

$$\alpha := \lambda_{|M}, \qquad \alpha(\cdot) = \frac{1}{2} g(p, J\cdot), \quad p \in M$$

is a contact form on M: therefore M is in particular a contact type hypersurface. Now, let us recall a known sufficient condition (see [1], Proposition 1, pag. 2): if $v \in ker(\alpha)$ defines a Hopf fibration, then the dual form $\beta(\cdot) = d\alpha(v, \cdot)$ is a contact form. Of course if v generates a Hopf type fibration, then it is non-vanishing and all its orbits are closed; anyway in general one does not know such a v explicitly. Here we will prove the following result:

Theorem 3.1. *Let M be a compact and convex embedded hypersurface in $\mathbb{R}^4$. Then there exists a non-vanishing vector field $X^d \in ker(\theta)$, which is given explicitly, such that the Legendre duality holds with respect to $v^d = X^d + \rho T$.*

We recall that $\theta(\cdot) := g(T, \cdot)$, $T = J\nu$ where ν is the unit (inner) normal to M and ρ is a function such that $v^d \in ker(\alpha)$.
First, we will prove the following lemma showing that in any embedded real hypersurface in $\mathbb{C}^2$ we can find a non-vanishing vector field having a special property. We have:

Lemma 3.1. *Let M be a compact embedded and orientable real hypersurface in $\mathbb{C}^2$. Then there exists a non-vanishing vector field $X^d \in ker(\theta)$, such that*

$$g(X^d, \nabla_T Y^d) = -h(T, T),$$

where $Y^d := JX^d$.

Proof. Let $f : \mathbb{C}^2 \to \mathbb{R}$ be a smooth defining function for M, namely

$$M = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ f(z_1, z_2) = 0\}.$$

Denoting by $\partial f = (\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_1}) = (f_1, f_2)$ the complex gradient of f and by $|\cdot|^2 = g(\cdot, \cdot)$, then the unit inner normal is given by:

$$\nu = -\frac{1}{|\partial f|}(f_{\bar{1}} \partial_{z_1} + f_{\bar{2}} \partial_{z_2} + f_1 \partial_{\bar{z}_1} + f_2 \partial_{\bar{z}_2})$$

Since

$$\partial_{z_k} = \frac{1}{2}(\partial_{x_k} - i\partial_{y_k}), \quad \partial_{\bar{z}_k} = \frac{1}{2}(\partial_{x_k} + i\partial_{y_k}), \ k = 1, 2$$

and

$$J(\partial_{x_k}) = \partial_{y_k}, \quad J(\partial_{y_k}) = -\partial_{x_k}, \ k = 1, 2,$$

we observe that the $\mathbb{C}$-linear extension of the complex structure satisfies

$$J(\partial_{z_k}) = i\partial_{z_k}, \quad J^{\mathbb{C}}(\partial_{\bar{z}_k}) = -i\partial_{\bar{z}_k}, \ k = 1, 2.$$

Therefore

$$T = J(\nu) = -\frac{i}{|\partial f|}(f_{\bar{1}} \partial_{z_1} - f_1 \partial_{\bar{z}_1} + f_{\bar{2}} \partial_{z_2} - f_2 \partial_{\bar{z}_2}).$$

Now, we define the unit vector field

$$Z^d = \frac{\sqrt{2}}{|\partial f|}(f_2\partial_{z_1} - f_1\partial_{z_2}),$$

where "d" stands for the defining function. For a compact notation, let us set

$$\nu^k = \frac{f_k}{|\partial f|}, \qquad \mu^1 = \nu^2 \ , \ \mu^2 = -\nu^1,$$

and let us use the summation convention, so that:

$$\nu = -(\bar{\nu}^k\partial_{z_k} + \nu^k\partial_{\bar{z}_k}),$$

$$T = -i(\bar{\nu}^k\partial_{z_k} - \nu^k\partial_{\bar{z}_k}),$$

$$Z^d = \sqrt{2}(\nu^2\partial_{z_1} - \nu^1\partial_{z_2}) = \sqrt{2}\mu^k\partial_{z_k}.$$

We have:

$$g(\nabla_T T, N) = ig(\nabla_T \overline{Z}^d, Z^d). \tag{8}$$

In fact, first of all, we notice that

$$W(\bar{\nu}^k)\nu_k = W(\bar{\mu}^k)\mu_k, \qquad \forall\, W \in TM.$$

Then

$$g(\nabla_T \overline{Z}^d, Z^d) = g(\nabla_T \sqrt{2}\bar{\mu}^k\partial_{\bar{z}_k}, \sqrt{2}\mu^j\partial_{z_j}) =$$
$$= T(\bar{\mu}^k)\mu_k = T(\bar{\nu}^k)\nu_k,$$

and

$$g(\nabla_T T, N) = ig(\nabla_T(\bar{\nu}^k\partial_{z_k} - \nu^k\partial_{\bar{z}_k}), \bar{\nu}^j\partial_{z_j} + \nu^j\partial_{\bar{z}_j}) =$$
$$= \frac{i}{2}(T(\bar{\nu}^k)\nu_k - T(\nu^k)\bar{\nu}_k) = iT(\bar{\nu}^k)\nu_k.$$

Now, we denote by X^d and Y^d the unitary real and the imaginary part of Z^d, i.e.:

$$Z^d = \frac{1}{\sqrt{2}}(X^d - iY^d), \quad Y^d = J(X^d), \quad |X^d| = |Y^d| = 1.$$

We have that:

$$0 = g(T, Z^d) = \frac{1}{\sqrt{2}}\Big(g(T, X^d) - ig(T, Y^d)\Big),$$

therefore $X^d, Y^d \in ker(\theta)$ and $\{X^d, Y^d, T\}$ is an orthonormal basis for TM. Moreover

$$g(\nabla_T \overline{Z}^d, Z^d) = g(\nabla_T \frac{1}{\sqrt{2}}(X^d + iY^d), \frac{1}{\sqrt{2}}(X^d - iY^d)) =$$

$$= \frac{1}{2}\{g(\nabla_T X^d, X^d) + g(\nabla_T Y^d, Y^d) + i(g(\nabla_T Y^d, X^d) - g(\nabla_T X^d, Y^d))\} = ig(\nabla_T Y^d, X^d).$$

Hence, by (8) we get that

$$g(X^d, \nabla_T Y^d) = -g(\nabla_T T, N) = -h(T, T).$$

$\square$

Remark 3.1. *Actually, once a defining function is given, since* $|X^d| = |Y^d| = 1$*, one has that any linear combination of* X^d *and* Y^d *with non-zero constant* $a, b \in \mathbb{R}$ *has the following property: if* $\widetilde{X} = aX^d + bY^d$*, then* $\widetilde{X}$ *is a non-vanishing vector field in* $ker(\theta)$ *such that*

$$g(\widetilde{X}, \nabla_T J\widetilde{X}) = -(a^2 + b^2)h(T, T).$$

Now the proof of the main result is an easy corollary:

Proof of Theorem 3.1. If M is convex , then M is starshaped and we can assume it is with respect to the origin. Therefore, since ν is the inward unit normal, we get

$$\alpha(T) = \frac{1}{2}g(p, JT) = -\frac{1}{2}g(p, \nu) > 0.$$

So, if we consider the vector field X^d given by the previous lemma, we get by (6) that

$$\rho = -\frac{\alpha(X^d)}{\alpha(T)} = \frac{g(p, Y^d)}{g(p, \nu)}$$

Finally, since M is convex, by the previous lemma and (7) we have

$$d\alpha(v^d, [T, v^d]) = g(X^d, \nabla_T Y^d) - h(X^d, X^d) = -\Big(h(T, T) + h(X^d, X^d)\Big) < 0.$$

□

Let us give next some examples in which we compute everything explicitly, showing also different dynamics of the vector fields X^d and T.

Example 3.1. *(The ellipsoid* E_1 *in* $\mathbb{C}^2$*)*
Let us consider $\mathbb{C}^2$ *with coordinates* $z_k = x_k + iy_k,\ k = 1, 2$ *and*

$$E_1 = \{(z_1, z_2) \in \mathbb{C}^2 \ s.t. \ f(z_1, z_2) := a|z_1|^2 + c|z_2|^2 - 1 = 0,\ a, c > 0\}.$$

Then (E_1, α) *is a convex and contact type hypersurface. Now, we have*

$$Z^d = \frac{\sqrt{2}}{|\partial f|}(f_2\partial_{z_1} - f_1\partial_{z_2}) = \frac{\sqrt{2}}{|\partial f|}(c\bar{z}_2\partial_{z_1} - a\bar{z}_1\partial_{z_2}),$$

and thus

$$X^d = \frac{1}{|\partial f|}(2cx_2\partial_{x_1} - 2cy_2\partial_{y_1} - 2ax_1\partial_{x_2} + 2ay_1\partial_{y_2}),$$

$$Y^d = \frac{1}{|\partial f|}(2cy_2\partial_{x_1} + 2cx_2\partial_{y_1} - 2ay_1\partial_{x_2} - 2ax_1\partial_{y_2}),$$

and

$$\nu = -\frac{1}{|\partial f|}(2ax_1\partial_{x_1} + 2ay_1\partial_{y_1} + 2cx_2\partial_{x_2} + 2cy_2\partial_{y_2}),$$

$$T = \frac{1}{|\partial f|}(2ay_1\partial_{x_1} - 2ax_1\partial_{y_1} + 2cy_2\partial_{x_2} - 2cx_2\partial_{y_2}).$$

Moreover, since $\alpha(T) = 1/|\partial f|$ *we get:*

$$\rho = -\frac{\alpha(X^d)}{\alpha(T)} = \frac{g(p, Y^d)}{g(p, \nu)} = (a - c)(x_1y_2 + y_1x_2).$$

Example 3.2. *(The ellipsoid E_2 in $\mathbb{R}^4$)*
Let us consider $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) and

$$E_2 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \ s.t. \ f(x_1, y_1, x_2, y_2) := ax_1^2 + by_1^2 + cx_2^2 + dy_2^2 - 1 = 0\},$$

with $a, b, c, d > 0$. Then (E_2, α) is a convex and contact type hypersurface. Now, we have

$$X^d = \frac{1}{|\partial f|}(2cx_2\partial_{x_1} - 2dy_2\partial_{y_1} - 2ax_1\partial_{x_2} + 2by_1\partial_{y_2}),$$

$$Y^d = \frac{1}{|\partial f|}(2dy_2\partial_{x_1} + 2cx_2\partial_{y_1} - 2by_1\partial_{x_2} - 2ax_1\partial_{y_2}),$$

and

$$\nu = -\frac{1}{|\partial f|}(2ax_1\partial_{x_1} + 2by_1\partial_{y_1} + 2cx_2\partial_{x_2} + 2dy_2\partial_{y_2}),$$

$$T = \frac{1}{|\partial f|}(2by_1\partial_{x_1} - 2ax_1\partial_{y_1} + 2dy_2\partial_{x_2} - 2cx_2\partial_{y_2}).$$

Also:

$$\rho = -\frac{\alpha(X^d)}{\alpha(T)} = \frac{g(p, Y^d)}{g(p, \nu)} = (a - d)x_1y_2 + (b - c)y_1x_2.$$

We notice that, of course if $a = b$ and $c = d$, then E_2 coincides with E_1. The difference between the two ellipsoids is in the structure of the vector fields X^d and T. In fact, in E_1, the vector field X^d has always closed orbits, therefore it defines a Hopf fibration; the vector field T in general does not have closed orbits, unless the ratio b/a is rational: when this happens both X^d and T generate a Hopf fibration. On the other hand, for the second ellipsoid E_2, depending on the choice of a, b, c, d, both the vector fields X^d and T do not have in general closed orbits.

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