Horizontal Newton operators and high-order Minkowski formula

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Abstract In this paper we study the horizontal Newton transformations, which are nonlinear operators related to the natural splitting of the second fundamental form for hypersurfaces in a complex space form. These operators allow to prove the classical Minkowski formulas in the case of real space forms: unlike the real case, the horizontal ones are not divergence-free. Here we consider the highest order of nonlinearity and we will show how a Minkowski-type formula can be obtained in this case.

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1 Introduction, definitions and statement of the results

Let M be a closed (compact, without boundary) n-dimensional hypersurface embedded in $\mathbb{R}^{n+1}$. We denote by

$$\lambda_p = g(P,\nu), \quad p \in M,$$

the supporting function of M, where we have identified the point $p \in M \subseteq \mathbb{R}^{n+1}$ with the position vector field P, ν is the outward unit normal of M at p and g stands for the standard Euclidian metric of $\mathbb{R}^{n+1}$. We also denote by ∇ the Levi-Civita connection of g and by h the second fundamental form of M

$$h(X,Y) = g(\nabla_X \nu, Y),$$

for any vector field $X, Y \in TM$, the tangent space of M. Moreover, from now on, we let $\mathcal{B} = \{X_k\}_{k=1,\dots,n}$ be any orthonormal basis for TM; therefore the coefficients of h in the basis $\mathcal{B}$ read as

$$h^i_j := h(X_i, X_j), \quad \text{for any } i,j = 1,\dots,n.$$

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We recall that the k-th order elementary symmetric function of a symmetric or Hermitian $N \times N$ matrix A is the function of the eigenvalues $\lambda_1 \dots \lambda_N$ of A defined by

$$S_k(A) = \sum_{1 \le i_1 < \dots < i_k \le N} \lambda_{i_1} \dots \lambda_{i_k} \quad \text{if } 1 \le k \le N,$$
$$S_0(A) = 1, \qquad S_k(A) = 0 \quad \text{if } k \ge N.$$

We also recall that, for $1 \le k \le N$, it holds $S_k(A) = \sum \det(A_k)$, where the sum is taken over all the principal minors A_k in A of order k. We set the normalized k-th elementary symmetric function of A:

$$\sigma_k(A) = \frac{S_k(A)}{\binom{N}{k}}.$$

Then we define the k-th elementary symmetric function of h and the normalized k-th curvature of M, respectively as

$$S_k := S_k(h), \qquad \sigma_k := \sigma_k(h), \quad \text{for } k = 0, 1, \dots, n.$$

The well-known Hsiung-Minkowski formulas read as [11, 12]:

$$\int_M \sigma_{k-1} - \lambda_p \sigma_k = 0, \qquad k = 1, \dots, n. \tag{1}$$

Integral formulas of Minkowski type are widely studied in several settings and they have a very large number of applications (see [13, 17, 18, 19, 21, 29, 30, 32, 33, 35, 39] and the references therein).
A way to prove formulas (1) is by means of the Newton operators T_k associated to h, see for instance [33, 34]. These last ones are a family of operators inductively defined as

$$T_0 = I,$$
$$T_k = S_k I - h T_{k-1}, \quad \text{for } 1 \le k \le n.$$

In the orthonormal basis $\mathcal{B}$, the coefficients of the $(k-1)$-th Newton transformation T_{k-1} associated to h read as

$$(T_{k-1})^i_j = T_{k-1}(X_i, X_j) = \frac{\partial S_k}{\partial h^j_i}, \quad \text{for any } i, j, k = 1, \dots, n.$$

Therefore one takes the divergence of the following vector fields

$$V_k = \sum_{i,j=1}^n (T_{k-1})^i_j a^j X_i,$$

where a^j are smooth functions such that $P = \sum_{j=1}^n a^j X_j + \lambda\nu$: it holds

$$\mathrm{div}(V_k) = k\binom{n}{k}\big(\sigma_{k-1} - \lambda_p\sigma_k\big), \qquad k = 1, \dots, n, \tag{2}$$

so that, by the divergence theorem, one recovers formulas (1). In order to obtain (2), one uses some structural equivalences of the Newton operators (see for instance [8, 10])

$$\sum_{i=1}^n (T_{k-1})_i^i = (n-k+1)S_{k-1}, \qquad \sum_{i,j=1}^n (T_{k-1})_j^i h_i^j = kS_k, \quad k = 1, \dots, n,$$

but most of all the following remarkable divergence-free property $\mathrm{div}(T_{k-1}) = 0$ or equivalently:

$$\sum_{i=1}^n \nabla_i^M (T_{k-1})_j^i = 0, \quad \text{for any fixed } j, k = 1, \dots, n; \tag{3}$$

here ∇_i^M denotes the covariant derivative with respect to ∇^M, the natural connection induced on M, satisfying

$$\nabla_X Y = \nabla_X^M Y - h(X,Y)\nu \quad \text{for any } X, Y \in TM.$$

Let us recall that a symmetric $(2,0)$-tensor B is Codazzi, if

$$(\nabla_X^M B)(Y,Z) = (\nabla_Y^M B)(X,Z), \quad \text{for every } X, Y, Z \in TM. \tag{4}$$

Therefore, formula (3) (sometimes called null-lagrangian property in literature) is indeed a general property satisfied by the Newton operators related to any Codazzi type tensor. Since the second fundamental form h is Codazzi for any hypersurface M embedded in a general *real* space form K_c with non-zero curvature c, the previous procedure verbatim leads to the following Minkowski formulas:

$$\int_M c_c(r)\sigma_{k-1} - \lambda_p\sigma_k = 0, \qquad k = 1, \dots, n.$$

Here r denotes the geodesic distance, $\lambda_p = g(P,\nu)$ is defined by means of the following position vector $P = s_c(r)\nabla r$ and the functions $s_c(r)$ and $c_c(r)$ are defined as follows:

$$s_c(r) = \begin{cases} \dfrac{1}{\sqrt{c}}\sin(\sqrt{c}\, r), & c > 0 \\[2ex] \dfrac{1}{\sqrt{-c}}\sinh(\sqrt{-c}\, r), & c < 0 \end{cases}, \qquad c_c(r) = \begin{cases} \cos(\sqrt{c}\, r), & c > 0 \\[2ex] \cosh(\sqrt{-c}\, r), & c < 0 \end{cases}.$$

Now let us turn our attention to the complex case. So, from now on, let M be a closed $(2n+1)$-dimensional real hypersurface embedded in $\mathbb{C}^{n+1}$. We will still denote by $\lambda_p = g(P, \nu)$, the supporting function of M, where P is the position vector and ν is the outward unit normal of M at $p \in M$. Also let g and ∇ stand for the standard Hermitian metric of $\mathbb{C}^{n+1}$ and its Levi-Civita connection respectively. The complex structure of $\mathbb{C}^{n+1}$ will be denoted by J and the following compatibility relations between g and J will be used throughout the paper:

$$\nabla J = J\nabla, \qquad g(\cdot, \cdot) = g(J\,\cdot, J\,\cdot). \tag{5}$$

The complex structure J of $\mathbb{C}^{n+1}$ induces the following orthogonal splitting of the tangent space of M

$$TM = \mathbb{R}X_0 \oplus_g HM;$$

here we have denoted by HM the horizontal (or Levi) distribution, namely the $2n$-dimensional subspace of TM invariant under the action of J, and by X_0 the characteristic (or vertical) vector field defined by $JX_0 = \nu$.

Also, we will always consider $\mathcal{B} = \{X_\alpha\}_{\alpha=0,1,\dots,2n}$ any orthonormal basis for TM; as a convention, we will use Latin letters to denote indices running from 1 to $2n$ (i.e. purely horizontal) and Greek letters to denote indices running from 0 to $2n$ (i.e. involving also the characteristic direction). We still denote by h the second fundamental form of M, and we also need to define the horizontal part of the second fundamental form

$$h^H(X,Y) = h(X,Y) \quad \text{for every } X, Y \in HM.$$

Therefore, we can define the k-th elementary symmetric function of h^H and the normalized k-th horizontal curvature of M, respectively as

$$S_k^H := S_k(h^H), \qquad \sigma_k^H := \sigma_k(h^H), \quad \text{for } k = 0, 1, \dots, 2n.$$

In the same way, we define the horizontal Newton operators T_k^H associated to h^H, in the orthonormal basis $\mathcal{B}$:

$$(T_{k-1}^H)_j^i = T_{k-1}^H(X_i, X_j) = \frac{\partial S_k(h^H)}{\partial (h^H)_i^j} = \frac{\partial S_k^H}{\partial h_i^j}, \quad \text{for any } i, j, k = 1, \dots, 2n.$$

In particular, when $k = 1$, we recognize that σ_1^H is exactly the Levi mean curvature of M; in fact we recall that the Levi form ℓ can be defined on HM in the following way: for every $X, Y \in HM$, if $Z = \frac{1}{\sqrt{2}}(X - iJX)$ and $W = \frac{1}{\sqrt{2}}(Y - iJY)$, then $\ell(Z, \overline{W}) := g(\nabla_Z \nu, \overline{W})$. We can compare the Levi form with the Second Fundamental Form by using the following identity (see [3]):

$$\forall X \in HM, \quad \ell(Z, \overline{Z}) = \frac{h(X,X) + h(JX, JX)}{2}.$$

These Levi or horizontal curvatures are a sort of degenerate-elliptic analogue of the classical curvature ([2, 6]): the restriction of the second fundamental form to the horizontal tangent space involves a lack of information and hence a lack of ellipticity

in the relative operators: for further properties and results we address the reader to [5, 23, 24, 25, 27], and the references therein.
One is then interested to Minkowski type formulas for these horizontal curvatures. A natural attempt is to mimic the Euclidean case and consider the following horizontal vector fields

$$V_k^H = \sum_{i,j=i}^{2n} (T_{k-1}^H)_j^i a^j X_i, \quad k = 1, \ldots, 2n. \tag{6}$$

Unfortunately, in general the horizontal Newton transformations T_k^H do not satisfy the divergence-free property (3), so one cannot apply the same procedure as in the real case, since reminder terms appear in computing the divergence of V_k^H; therefore one needs to find another vector field to compensate them.
Anyway, there are situations in which the reminder terms vanish. In fact, a *first* ($k = 1$) *horizontal* Minkowski formula, i.e. involving the Levi mean curvature, has been proved by Miquel [18] for compact Hopf hypersurfaces: we recall that a real hypersurface M (in a general Kähler manifold) is said to be a Hopf hypersurface if the characteristic vector field X_0 is an eigenvector for the shape operator, or equivalently $h(X_0, X_j) = 0$ for all $j = 1, \ldots, 2n$, in our notations (see [1, 4, 7, 9, 14, 16, 20, 26, 28, 31, 36, 37, 38] for Hopf hypersurfaces and their classification). If M is not Hopf, such formulas are not exact, in the sense that the reminder terms do not vanish in general; moreover (as shown in [22], where also a second ($k = 2$) horizontal Minkowski formula has been written) being Hopf is only a sufficient condition for these formulas to hold.
Quite unexpectedly, in [22] it has been proved a brand new kind of second Minkowski formula involving horizontal curvatures, that is:

$$\int_M \sigma_1^H - \lambda\widetilde{\sigma}_2 = 0,$$

where $\widetilde{\sigma}_2$ is the following calibrated combination

$$\widetilde{\sigma}_2 := \frac{(2n+1)\sigma_2\lambda - 2(2n-1)\sigma_2^H\lambda + 3(n-1)\sigma_2^\ell}{n}.$$

Here σ_2^ℓ stands for the second horizontal curvature of M taken with respect to the eigenvalues of Levi form ℓ; actually in [22] such a formula is proved for hypersurfaces embedded in a general complex space form, with an extra term depending on the curvature of the ambient manifold. Let us explicitly notice that the previous formula cannot be deduced from the classical Minkowski formulas (1), case $k = 2$ (see Remark 2.2). Its proof relies on the choice of a suitable vector field (not horizontal in general) and couple it with V_2^H, as defined in (6). Anyway, the crucial fact is that in the case $k = 2$, by the very definition of the Newton operators, T_1^H is linear as function of the coefficients of the second fundamental form. The situation changes drastically when $k > 2$ and the operators T_k^H become fully nonlinear in the h_j^i's. Here we want to study the highest order of nonlinearity, namely when $k = 2n$; we will show that a new Minkowski formula can be obtained in this case as well, in particular we will prove the following:

Theorem 1.1. *Let M be a closed real hypersurface embedded in $\mathbb{C}^{n+1}$. Then it holds*

$$\int_M \sigma^H_{2n-1} - \lambda \widetilde{\sigma}_{2n} = 0,$$

where

$$\widetilde{\sigma}_{2n} = \frac{(2n+1)\sigma_{2n} - (n+1)\sigma^H_{2n}}{n}.$$

Let us stress that for a real hypersurface embedded in $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$ the previous formula is independent of the classical Minkowski formulas (1), case $k = 2n$ (see Remark 2.2). Moreover, if the maximal order is 2, i.e. when M is embedded in $\mathbb{C}^2$, we recover the second horizontal Minkowski formula obtained in [22].
The proof of Theorem (1.1) is quite involved and it will be carried on in several steps in the next section.
Finally, in Section 3, we will show how such a formula can be obtained for hypersurfaces embedded in a general complex space form: the main computations are similar to the flat case, of course a term involving the curvature of the ambient manifold will appear.

2 Proof of Theorem 1.1

Theorem 1.1 will be proved computing the divergence of two well chosen vector fields and then by using the divergence theorem. In order to simplify the computations we will choose a suitable orthonormal frame for TM, namely we will consider a suitable orthonormal frame for TM of the form $\mathcal{E} = \{X_0, X_1, \ldots, X_{2n}\}$ where X_0 is the characteristic vector field and $X_1, \ldots, X_{2n} \in HM$ are eigenvectors for the shape operator restricted to HM, so that the horizontal second fundamental form h^H is diagonal, i.e.

$$g(\nabla_{X_i}\nu, X_j) = \delta^j_i h^j_i, \quad \text{for any } i, j = 1, \ldots, 2n.$$

With respect to the frame $\mathcal{E}$, we denote by

$$\Gamma^\gamma_{\alpha\beta} = g(\nabla_{X_\alpha} X_\beta, X_\gamma), \quad \text{for every } \alpha, \beta, \gamma = 0, \ldots, 2n$$

the coefficients of the Levi-Civita connection. Moreover we denote by B the matrix associated to the endomorphism J with respect to the basis $\mathcal{E}$

$$B^\alpha_\beta = g(X_\alpha, J(X_\beta)), \quad \alpha, \beta = 0, \ldots, 2n.$$

From the compatibility relations (5), we deduce that B is skew-symmetric and $B^\alpha_0 = -B^0_\alpha = 0$ for every $\alpha = 0, \ldots, 2n$, so that

$$J(X_i) = \sum_{j=1}^{2n} B^j_i X_j, \qquad \text{for every } i = 1, \ldots, 2n.$$

Remark 2.1. *Since $\mathcal{E}$ is an orthonormal frame, we have the identity $\Gamma^{\gamma}_{\alpha\beta} = -\Gamma^{\beta}_{\alpha\gamma}$ for any $\alpha, \beta, \gamma = 0, \ldots, 2n$. Moreover, exploiting the compatibility of the metric g with the complex structure J, we find the following relations between the coefficients of the Levi-Civita connection and the coefficients of the second fundamental form*

$$\Gamma^i_{00} = g(\nabla_{X_0} X_0, X_i) = g(\nabla_{X_0} \nu, J(X_i)) = \sum_{l=1}^{2n} B^l_i h^l_0, \quad i = 1, \ldots, 2n,$$

$$\Gamma^i_{j0} = g(\nabla_{X_j} X_0, X_i) = g(\nabla_{X_j} \nu, J(X_i)) = B^j_i h^j_j, \quad i, j = 1, \ldots, 2n.$$

In particular, since B is skew-symmetric, $\Gamma^i_{i0} = 0$.

With respect to the basis $\mathcal{E}$, the position vector reads as $P = \sum_{\alpha=0}^{2n} a^\alpha X_\alpha + \lambda\nu$, we need to compute the derivatives of the functions a^α.

Lemma 2.1. *For any $i = 1, \ldots, 2n$ we have*

$$X_0(a^0) = 1 - \lambda h^0_0 + \sum_{i,j=1}^{2n} B^i_j h^i_0 a^j,$$

$$X_i(a^0) = -\lambda h^0_i + \sum_{j=1}^{2n} B^i_j h^i_i a^j,$$

$$X_i(a^i) = 1 - \lambda h^i_i - \sum_{j=1}^{2n} \Gamma^i_{ij} a^j.$$

Proof. Keeping in mind Remark 2.1 and the fact that $\nabla_V P = V$ for any vector field V in $\mathbb{C}^{n+1}$, the thesis follows by a straightforward computation. $\square$

Now, let us consider the horizontal vector fields V^H_k defined in (6). First of all, since in our basis $\mathcal{E}$ the matrix h^H is diagonal, also the horizontal Newton operators will be diagonal as well; moreover, in order to simplify the notation, we will simply denote by $T = T_{2n-1}$ and by $F = T^H_{2n-1}$ the operators we are interested in. So we define

$$V := V^H_{2n} = \sum_{j=i}^{2n} (T^H_{2n-1})^j_j a^j X_j = \sum_{j=i}^{2n} F^j_j a^j X_j. \tag{7}$$

We will see that the divergence of V contains terms involving horizontal curvatures σ^H_{2n}, σ^H_{2n-1} and a reminder term Q_V involving mixed coefficients $h(X_0, X_i)$ (Proposition 2.1): that reminder term need to be compensated, and it turns out that the divergence of the vector field

$$W := \sum_{\alpha=0}^{2n} T^\alpha_0 a^0 X_\alpha \tag{8}$$

perfectly fits the purpose (Proposition 2.2).
Let us collect here some identities that will be useful when computing $\mathrm{div}(V)$ and $\mathrm{div}(W)$.

Lemma 2.2. *We have the following equalities on the coefficients of the $(2n-1)$-th Newton transformation T:*

$$\begin{aligned} T_0^0 &= S_{2n-1}^H; \\ T_0^l &= -h_0^l \frac{\partial S_{2n-1}^H}{\partial h_l^l}, \quad \textit{for any fixed } l = 1, \ldots, 2n; \\ S_{2n} &= S_{2n}^H + \sum_{\alpha=0}^{2n} T_0^\alpha h_\alpha^0. \end{aligned}$$

Proof. Since S_{2n} is the sum of all the $2n$-by-$2n$ principal minors of h we can write

$$S_{2n} = \frac{1}{2n!} \sum_{\{i_1,\ldots,i_{2n}\} \subseteq \{0,\ldots,2n\}} \delta \begin{pmatrix} i_1 & \ldots & i_{2n} \\ j_1 & \ldots & j_{2n} \end{pmatrix} h_{j_1}^{i_1} \ldots h_{j_{2n}}^{i_{2n}}.$$

Here the Kronecker symbol $\delta \begin{pmatrix} i_1 & \ldots & i_{2n} \\ j_1 & \ldots & j_{2n} \end{pmatrix}$ is zero if $i_k = i_l$ or $i_k = j_l$ for some $k \neq l$ or if $\{i_1, \ldots i_{2n}\} \neq \{j_1, \ldots j_{2n}\}$ as sets, otherwise it is the sign of the permutation $(i_1, \ldots i_{2n}) \to (j_1, \ldots j_{2n})$. Since h^H is diagonal, if $i_l \in \{1, \ldots, 2n\}$ for some $l = 1, \ldots, 2n$ we just need to consider $h_{j_l}^{i_l}$ with $j_l = i_l$. Using this notation, the expression for the coefficients T_0^0 and T_0^l of the $(2n-1)$-th Newton transformation read as

$$\begin{aligned} T_0^0 = \frac{\partial S_{2n}}{\partial h_0^0} &= \frac{1}{(2n-1)!} \sum_{\{i_1,\ldots,i_{2n-1}\} \subseteq \{1,\ldots,2n\}} \delta \begin{pmatrix} i_1 & \ldots & i_{2n-1} & 0 \\ i_1 & \ldots & i_{2n-1} & 0 \end{pmatrix} h_{i_1}^{i_1} \ldots h_{i_{2n-1}}^{i_{2n-1}} \\ &= S_{2n-1}^H \end{aligned}$$

and

$$\begin{aligned} T_0^l = \frac{\partial S_{2n}}{\partial h_l^0} &= \frac{1}{(2n-2)!} \sum_{\{i_1,\ldots,i_{2n-2}\} \subseteq \{1,\ldots,2n\}} \delta \begin{pmatrix} i_1 & \ldots & i_{2n-2} & 0 & l \\ i_1 & \ldots & i_{2n-2} & l & 0 \end{pmatrix} h_{i_1}^{i_1} \ldots h_{i_{2n-2}}^{i_{2n-2}} h_0^l \\ &= -h_0^l \frac{\partial S_{2n-1}^H}{\partial h_l^l} \quad \text{for any } l = 1, \ldots, 2n. \end{aligned}$$

Finally, it is easy to see that

$$S_{2n} - \sum_{\alpha=0}^{2n} T_0^\alpha h_\alpha^0 = S_{2n} - T_0^0 h_0^0 - \sum_{l=1}^{2n} T_0^l h_l^0 = S_{2n}^H.$$

$\square$

Lemma 2.3. *The following identities hold true for any fixed $i, j \in \{1, \dots, 2n\}$ with $i \neq j$.*

$$i) \qquad \frac{\partial^2 S_k^H}{\partial h_i^i \partial h_j^j} = \frac{\partial^2 S_k^H}{\partial h_j^j \partial h_i^i}, \qquad \textit{for any } k = 1, \dots, 2n;$$

$$ii) \qquad \frac{\partial S_{2n-1}^H}{\partial h_i^i} h_i^i = S_{2n-1}^H - F_i^i;$$

$$iii) \quad \frac{\partial^2 S_{2n-1}^H}{\partial h_i^i \partial h_j^j} h_i^i = \frac{\partial S_{2n-1}^H}{\partial h_j^j} - \frac{\partial F_i^i}{\partial h_j^j}.$$

Proof. The first identity is trivial. To prove *ii*) recall that S_{2n-1}^H is the sum of all the $(2n-1)$-by-$(2n-1)$ principal minors of h^H i.e.

$$S_{2n-1}^H = \sum_{1 \leq i_i < \dots < i_{2n-1} \leq 2n} h_{i_1}^{i_1} \dots h_{i_{2n-1}}^{i_{2n-1}}$$

and compute

$$\begin{aligned} \frac{\partial S_{2n-1}^H}{\partial h_i^i} h_i^i &= \sum_{\substack{1 \leq i_i < \dots < i_{2n-2} \leq 2n \\ i_1, \dots, i_{2n-2} \neq i}} h_{i_1}^{i_1} \dots h_{i_{2n-2}}^{i_{2n-2}} h_i^i \\ &= \sum_{1 \leq i_i < \dots < i_{2n-1} \leq 2n} h_{i_1}^{i_1} \dots h_{i_{2n-1}}^{i_{2n-1}} - \sum_{\substack{1 \leq i_1 < \dots < i_{2n-1} \leq 2n \\ i_1, \dots, i_{2n-1} \neq i}} h_{i_1}^{i_1} \dots h_{i_{2n-1}}^{i_{2n-1}} \\ &= S_{2n-1}^H - F_i^i. \end{aligned}$$

Finally, *iii*) follows from *ii*), simply differentiating with respect to h_j^j. □

Next we compute the divergence of V.

Proposition 2.1. *We have*

$$\operatorname{div}(V) = 2n\sigma_{2n-1}^H - 2n\lambda\sigma_{2n}^H + Q_V,$$

where the reminder term Q_V is:

$$Q_V = -2 \sum_{i,j=1}^{2n} F_i^i B_i^j h_0^i a^j.$$

Proof. Since h^H is diagonal and S_{2n}^H is the determinant of h^H we have the following useful identities

$$\frac{\partial S^H_{2n}}{\partial h^i_i} h^i_i = S^H_{2n}, \quad \text{for every } i = 1, \dots, 2n; \tag{9}$$

$$\frac{\partial^2 S^H_{2n}}{\partial h^i_i \partial h^j_j} h^i_i = F^j_j, \quad \text{for every } i = 1, \dots, 2n \text{ with } i \neq j; \tag{10}$$

$$\sum_{j=i}^{2n} F^j_j = S^H_{2n-1}. \tag{11}$$

Let us start with the computation of $\operatorname{div}(V)$.

$$\begin{aligned}
\operatorname{div}(V) &= \sum_{\alpha=0}^{2n} \sum_{j=1}^{2n} g\left(\nabla_{X_\alpha} F^j_j a^j X_j, X_\alpha\right) \\
&= \sum_{j=1}^{2n} X_j\left(F^j_j\right) a^j + \sum_{j=1}^{2n} F^j_j X_j(a^j) + \sum_{\alpha=0}^{2n} \sum_{j=1}^{2n} \Gamma^\alpha_{\alpha j} F^j_j a^j.
\end{aligned}$$

By Remark 2.1 we get:

$$\operatorname{div}(V) = \sum_{j=1}^{2n} X_j\left(F^j_j\right) a^j + \sum_{j=1}^{2n} F^j_j X_j(a^j) + \sum_{i,j=1}^{2n} \Gamma^i_{ij} F^j_j a^j - \sum_{i,j=1}^{2n} B^i_j h^i_0 F^j_j a^j. \tag{12}$$

Now we use the Codazzi equations (4) and then the symmetries of the coefficients of the Levi-Civita connection to rewrite the first summation in the right hand side of expression above. The idea is to get rid of the derivatives of the coefficients of the second fundamental form exploiting the fact that h^H is diagonal, so that for $i \neq j$, $X_j(h^j_i) = 0$, while in case $i = j$ the term $\frac{\partial^2 S^H_{2n}}{\partial h^j_j \partial h^i_i} = 0$. We have

$$\begin{aligned}
\sum_{j=1}^{2n} X_j\left(F^j_j\right) a^j &= \sum_{i,j=1}^{2n} \frac{\partial^2 S^H_{2n}}{\partial h^i_i \partial h^j_j} X_j(h^i_i) a^j \\
&= \sum_{i,j=1}^{2n} \frac{\partial^2 S^H_{2n}}{\partial h^i_i \partial h^j_j} \left(X_i(h^i_j) - \Gamma^\alpha_{ij} h^i_\alpha - \Gamma^\alpha_{ii} h^\alpha_j + 2\Gamma^\alpha_{ji} h^i_\alpha\right) a^j \\
&= \sum_{i,j=1}^{2n} \frac{\partial^2 S^H_{2n}}{\partial h^i_i \partial h^j_j} \left(B^i_j h^i_i h^i_0 - \Gamma^i_{ij} h^i_i - \Gamma^j_{ii} h^j_j - 2B^j_i h^j_j h^i_0\right) a^j.
\end{aligned}$$

Then, by (10)

$$\sum_{j=1}^{2n} X_j\left(F^j_j\right) a^j = \sum_{i,j=1}^{2n} \left(B^i_j h^i_0 F^j_j - \Gamma^i_{ij} F^j_j - \Gamma^j_{ii} F^i_i - 2B^j_i h^i_0 F^i_i\right) a^j. \tag{13}$$

Now we consider the second summation in the right hand side of (12). By Lemma 2.1 and identities (9),(11)

$$\sum_{j=1}^{2n} F_j^j X_j(a^j) = S_{2n-1}^H - 2n\lambda S_{2n}^H + \sum_{i,j=1}^{2n} F_i^i \Gamma_{ii}^j a^j. \tag{14}$$

Substituting (13) and (14) in (12) we get

$$\operatorname{div}(V) = S_{2n-1}^H - 2n\lambda S_{2n}^H - \sum_{i,j=1}^{2n} 2B_i^j h_0^i F_i^i a^j$$

and we just need to normalize S_{2n-1}^H and S_{2n} to finally get the thesis. □

We notice that the terms in Q_V depend linearly on the mixed terms h_0^i, so they vanish identically if the hypersurface is of Hopf type: therefore from the divergence of V above, we will get a formula of Miquel type [18] for Hopf hypersurfaces embedded in general complex space forms (see Remark 3.2).
The following Lemma will help the computation of $\operatorname{div}(W)$ instead.

Lemma 2.4. *It holds:*

$$\sum_{\alpha=0}^{2n} X_\alpha(T_0^\alpha) + \sum_{\alpha,\beta=0}^{2n} T_0^\alpha \Gamma_{\beta\alpha}^\beta = 0.$$

Proof. We have:

$$\begin{aligned}
&\sum_{\alpha=0}^{2n} X_\alpha(T_0^\alpha) + \sum_{\alpha,\beta=0}^{2n} T_0^\alpha \Gamma_{\beta\alpha}^\beta \\
&= X_0(T_0^0) + \sum_{i=1}^{2n} \big(X_i(T_0^i) + T_0^i \Gamma_{0i}^0\big) + \sum_{j=1}^{2n} T_0^0 \Gamma_{j0}^j + \sum_{i,j=0}^{2n} T_0^i \Gamma_{ji}^j \\
&= X_0(T_0^0) + \sum_{i=1}^{2n} \big(X_i(T_0^i) + T_0^i \Gamma_{0i}^0\big) + \sum_{i,j=0}^{2n} T_0^i \Gamma_{ji}^j \\
&= \sum_{i=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} X_0(h_i^i) - \sum_{i=1}^{2n} X_i(h_0^i) \frac{\partial S_{2n-1}^H}{\partial h_i^i} - \sum_{i,k=1}^{2n} h_0^i \frac{\partial^2 S_{2n-1}^H}{\partial h_j^j \partial h_i^i} X_i(h_j^j) \\
&\quad - \sum_{i=1}^{2n} h_0^i \frac{\partial S_{2n-1}^H}{\partial h_i^i} \Gamma_{0i}^0 - \sum_{i,j=1}^{2n} h_0^i \frac{\partial S_{2n-1}^H}{\partial h_i^i} \Gamma_{ji}^j.
\end{aligned}$$

The second equality follows from Remark 2.1, while the last one follows replacing T_0^0 and T_0^i with the expressions obtained in Lemma 2.2 and recalling that T_0^α is function of the coefficients of the second fundamental form h^i_0 and h^i_i. Then, by Remark 2.1 we find

$$\begin{aligned}\sum_{\alpha=0}^{2n} X_\alpha(T^\alpha_0) + \sum_{\alpha,\beta=0}^{2n} T^\alpha_0 \Gamma^\beta_{\beta\alpha} = {} & \sum_{i=1}^{2n} \frac{\partial S^H_{2n-1}}{\partial h^i_i} X_0(h^i_i) - \sum_{i=1}^{2n} X_i(h^i_0) \frac{\partial S^H_{2n-1}}{\partial h^i_i} \\ & - \sum_{i,k=1}^{2n} h^i_0 \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} X_i(h^j_j) - \sum_{i,j=1}^{2n} \frac{\partial S^H_{2n-1}}{\partial h^i_i} B^j_i h^i_0 h^j_0 \\ & - \sum_{i,j=1}^{2n} h^i_0 \frac{\partial S^H_{2n-1}}{\partial h^i_i} \Gamma^j_{ji}.\end{aligned} \tag{15}$$

We use Codazzi equations (4) and then Remark 2.1 to simplify the third term in the right hand side of (15). As before the idea is to get rid of the derivatives of the coefficients of the second fundamental form exploiting the fact that h^H is diagonal, so that for $i = j$ the term $\frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} = 0$.

$$\begin{aligned}& \sum_{i,j=1}^{2n} h^i_0 \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} X_i(h^j_j) \\ & = \sum_{i,j=1}^{2n} \left(h^i_0 \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} X_j(h^j_i) + \sum_{\alpha=0}^{2n} h^i_0 \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} \left(-\Gamma^\alpha_{ji} h^j_\alpha - \Gamma^\alpha_{jj} h^\alpha_i + 2\Gamma^\alpha_{ij} h^j_\alpha \right) \right) \\ & = \sum_{i,j=1}^{2n} h^i_0 \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} \left(-\Gamma^0_{ji} h^j_0 - \Gamma^j_{ji} h^j_j - \Gamma^i_{jj} h^i_i + 2\Gamma^0_{ij} h^j_0 \right) \\ & = \sum_{i,j=1}^{2n} \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} \left(B^j_i h^j_j h^i_0 h^j_0 - \Gamma^j_{ji} h^j_j h^i_0 + \Gamma^j_{ji} h^i_i h^i_0 - 2B^i_j h^i_i h^i_0 h^j_0 \right).\end{aligned}$$

With the aid of Lemma 2.3, it is possible to rewrite equality above as

$$\begin{aligned}& \sum_{i,j=1}^{2n} h^i_0 \frac{\partial^2 S^H_{2n-1}}{\partial h^j_j \partial h^i_i} X_i(h^j_j) \\ & = \sum_{i,j=1}^{2n} \left(\frac{\partial S^H_{2n-1}}{\partial h^i_i} - \frac{\partial F^j_j}{\partial h^i_i} \right) \left(B^j_i h^i_0 h^j_0 - \Gamma^j_{ji} h^i_0 \right) \\ & + \sum_{i,j=1}^{2n} \left(\frac{\partial S^H_{2n-1}}{\partial h^j_j} - \frac{\partial F^j_j}{\partial h^i_i} \right) \left(\Gamma^j_{ji} h^i_0 - 2B^i_j h^i_0 h^j_0 \right) \\ & = \sum_{i,j=1}^{2n} \frac{\partial S^H_{2n-1}}{\partial h^i_i} \left(B^j_i h^i_0 h^j_0 - \Gamma^j_{ji} h^i_0 \right) + \sum_{i,j=1}^{2n} \frac{\partial S^H_{2n-1}}{\partial h^j_j} \left(\Gamma^j_{ji} h^i_0 - 2B^i_j h^i_0 h^j_0 \right) \\ & - 3 \sum_{i,j=1}^{2n} \frac{\partial F^j_j}{\partial h^i_i} B^j_i h^j_0 h^i_0.\end{aligned}$$

Now, let us focus the attention on the sum

$$\sum_{i,j=1}^{2n} \frac{\partial F_j^j}{\partial h_i^i} B_i^j h_0^j h_0^i = \sum_{i,j=1}^{2n} \frac{\partial^2 S_{2n}^H}{\partial h_j^j \partial h_i^i} B_i^j h_0^j h_0^i.$$

Since B is skew-symmetric, we can split the summation over indices $i < j$ and $i > j$ to see that they cancel each other out. Hence the considered summation equals zero and

$$\begin{aligned} &\sum_{i,j=1}^{2n} h_0^i \frac{\partial^2 S_{2n-1}^H}{\partial h_j^j \partial h_i^i} X_i(h_j^j) \\ &= \sum_{i,j=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} \left(B_i^j h_0^i h_0^j - \Gamma_{ji}^j h_0^i \right) + \sum_{i,j=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_j^j} \left(\Gamma_{ji}^j h_0^i - 2B_j^i h_0^i h_0^j \right). \end{aligned} \tag{16}$$

Similarly we use Codazzi equations (4) and then Remark 2.1 to simplify the following expression

$$\begin{aligned} \sum_{i=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} X_0(h_i^i) &= \sum_{i=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} \left(X_i(h_0^i) + \sum_{\alpha=0}^{2n} \left(-\Gamma_{i0}^\alpha h_\alpha^i + \Gamma_{i\alpha}^i h_0^\alpha + 2\Gamma_{0i}^\alpha h_\alpha^i \right) \right) \\ &= \sum_{i=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} X_i(h_0^i) + \sum_{i,j=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} \left(\Gamma_{ij}^i h_0^j + 2\Gamma_{0i}^0 h_0^i \right) \\ &= \sum_{i=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} X_i(h_0^i) + \sum_{i,j=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} \left(\Gamma_{ij}^i h_0^j - 2B_i^j h_0^j h_0^i \right). \end{aligned} \tag{17}$$

Substituting (16) and (17) in (15) we conclude:

$$\sum_{\alpha=0}^{2n} X_\alpha(T_0^\alpha) + \sum_{\beta=0}^{2n} T_0^\alpha \Gamma_{\beta\alpha}^\beta =$$

$$= \sum_{i,j=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_i^i} \left(\Gamma_{ij}^i h_0^j - 2B_i^j h_0^j h_0^i \right) - \sum_{i,j=1}^{2n} \frac{\partial S_{2n-1}^H}{\partial h_j^j} \left(\Gamma_{ji}^j h_0^i - 2B_j^i h_0^i h_0^j \right) = 0.$$

□

Therefore we get the following

Proposition 2.2. *The divergence of the vector field* $W = \sum_{\alpha=0}^{2n} T_0^\alpha a^0 X_\alpha$ *is*

$$\operatorname{div}(W) = 2n\sigma_{2n-1}^H - \lambda \left((2n+1)\sigma_{2n} - \sigma_{2n}^H \right) + Q_W,$$

where Q_W *is the following reminder term:*

$$Q_W = -\sum_{i,j=1}^{2n} F_i^i B_i^j h_0^i a^j.$$

Proof. We compute

$$\begin{aligned}\operatorname{div}(W) &= \sum_{\alpha,\beta=0}^{2n} g\left(\nabla_{X_\beta} T_0^\alpha a^0 X_\alpha, X_\beta\right) \\ &= \sum_{\alpha=0}^{2n} X_\alpha\left(T_0^\alpha\right) a_0 + \sum_{\alpha=0}^{2n} T_0^\alpha X_\alpha\left(a_0\right) + \sum_{\alpha,\beta=0}^{2n} T_0^\alpha a_0 \Gamma_{\beta\alpha}^\beta\end{aligned}$$

and we use Lemmas 2.1, 2.2 and 2.4 to get

$$\operatorname{div}(W) = S_{2n-1}^H - \lambda(S_{2n} - S_{2n}^H) + Q_W$$

where the reminder term Q_W has the form

$$Q_W = \sum_{i,j=1}^{2n} a^j\left(T_0^0 B_j^i h_0^i + T_0^i B_j^i h_i^i\right).$$

Finally, in order to write Q_W as in the statement, we simply substitute T_0^0 and T_0^i by their equivalent expression obtained in Lemma 2.2 and recall the identity *ii)* proved in Lemma 2.3. The thesis follows normalizing S_{2n-1}^H, S_{2n} and S_{2n}^H. □

Now we have all the necessary tools to easily prove our main theorem.

Proof of Theorem 1.1. If we look at the statements of Propositions 2.1 and 2.2 where we compute the divergence of W and V respectively, we see that the reminder term Q_W is exactly half of Q_V, therefore we consider the vector field $U := 2W - V$, and we integrate over M

$$\int_M \operatorname{div}(U) = \int_M 2n\sigma_{2n-1}^H - 2(2n+1)\lambda\sigma_{2n} + 2(n+1)\lambda\sigma_{2n}^H. \tag{18}$$

On the other hand, since M is closed, by the divergence theorem, (18) must be equal to 0, therefore dividing by $2n$, we get the desired formula. □

Remark 2.2. *Let us show that our formula is independent of the classical Minkowski formula (case $k = 2n$) and cannot be deduced from it. In fact, in our notation, the standard Minkowski formula is obtained by taking the divergence of the following vector field*

$$V_{2n} = \sum_{\alpha,\beta=0}^{2n} T_\beta^\alpha a^\beta X_\alpha.$$

Let us expand it, taking into account the horizontal indices

$$V_{2n} = \sum_{i,j=1}^{2n} T_j^i a^j X_i + \sum_{j=1}^{2n} T_j^0 a^j X_0 + \sum_{\alpha=0}^{2n} T_0^\alpha a^0 X_\alpha,$$

and the third term in the right hand side is exactly the vector field W. Now we recall from Lemma 2.2 that

$$S_{2n} = S_{2n}^H + \sum_{\alpha=0}^{2n} T_0^\alpha h_\alpha^0,$$

so that

$$T_j^i = \frac{\partial S_{2n}}{\partial h_i^j} = \frac{\partial S_{2n}^H}{\partial h_i^j} + \sum_{\alpha=0}^{2n} \frac{\partial T_0^\alpha}{\partial h_i^j} h_\alpha^0, \quad i, j = 1, \ldots, 2n,$$

and we get

$$V_{2n} = V_{2n}^H + W + Z = V + W + Z,$$

where we have set

$$Z = \sum_{i,j=1}^{2n} \sum_{\alpha=0}^{2n} \frac{\partial T_0^\alpha}{\partial h_i^j} h_\alpha^0 a^j X_i + \sum_{j=1}^{2n} T_j^0 a^j X_0.$$

Therefore V_{2n} is the sum of three vector fields: by a straightforward computation, the divergence of each of them contains different reminder terms which cancel out when summing up. We found a new combination of only two of them, $U = 2W - V$, such that the reminder terms disappear.

3 The case of complex space forms

Here we will show how to get the formula in Theorem 1.1, in the case of hypersurfaces embedded in a general complex space form: an additional term involving the curvature of the ambient manifold will appear, however the main computations are basically the same as in the flat case.
So, let M be a closed, real hypersurface embedded in a complex space form K_c, i.e. a Kähler manifold of real dimension $2n+2$ with constant holomorphic sectional curvature $4c \neq 0$. It is known that, according to the sign of the sectional curvature, positive or negative, the models for such manifolds are respectively the complex projective space $\mathbb{C}P^{n+1}$ endowed with the Fubini Study metric and the complex hyperbolic space $\mathbb{C}H^{n+1}$ with the Bergman metric (see [15, 31]). We arbitrarily fix a point as origin in K_c and denote by $r = r(p)$ the geodesic distance of any given point $p \in K_c$ from the origin. The following potential function will allow us to distinguish the sign of the curvature c, handling simultaneously the different cases:

$$\psi(r) = \begin{cases} -\dfrac{1}{c} \log\left(\cos(\sqrt{c}\, r)\right), & c > 0 \\[2ex] -\dfrac{1}{c} \log\left(\cosh(\sqrt{-c}\, r)\right), & c < 0. \end{cases}$$

Remark 3.1. *In the case of $\mathbb{C}P^{n+1}$, i.e. when c is positive, we will always require M to be contained in the geodesic ball of center the origin and radius smaller than $\pi/\sqrt{4c}$: under this assumption conjugate points are avoided and the function ψ turns out to be smooth.*

Unless otherwise stated, we use the same notation as in the previous sections; therefore let $\mathcal{E}$ be an orthonormal frame for TM of the form $\mathcal{E} := \{X_0, X_1, \ldots, X_{2n}\}$, where X_0 is the characteristic vector field and $X_1, \ldots, X_{2n} \in HM$ are eigenvectors for the shape operator restricted to HM. With respect to the basis $\mathcal{E}$, the position vector field, defined as

$$P = \nabla\psi = \psi'\nabla r, \tag{19}$$

at a point $p \in M$, reads as

$$P = \sum_{\alpha=0}^{2n} a^\alpha X_\alpha + \lambda\nu.$$

Here a^α are again suitable smooth functions, and ∇r denotes the gradient of r. We will also need the tangential and the horizontal part of the position vector P

$$P^\tau = P - \lambda\nu, \quad P^H = P - a^0 X_0 - \lambda\nu.$$

Finally, let us define the terms that will appear as effect of the curvature c of the ambient manifold:

$$\Phi = \frac{1}{2n}\left(T\left(P^\tau, a^0 X_0\right) - T\left((JP)^\tau, -\lambda X_0\right)\right),$$
$$\Phi^H = \frac{1}{2n}\left(F\left(P^H, P^H\right) - F\left(JP^H, JP^H\right)\right).$$

Now, from the definition of geodesic distance we have

$$g(\nabla r, \nabla r) = 1, \qquad \nabla_{\nabla r}\nabla r = 0,$$

and

$$\nabla_{J\nabla r}\nabla r = \frac{1 - c(\psi')^2}{\psi'}\nabla r.$$

Moreover, for any vector field $V \in TK_c$ such that $g(V, \nabla r) = g(V, J\nabla r) = 0$, it holds

$$\nabla_V \nabla r = \frac{1}{\psi'}V.$$

Using the last three identities and (19), it is possible to write the covariant derivative of the position vector field P along any vector field $V \in TK_c$ as:

$$\nabla_V P = V + cg(V, P)P - cg(V, JP)JP. \tag{20}$$

With this notation, we have the following expression for the derivatives of the coefficients of the position vector field.

Lemma 3.1. *For any $i = 1, \dots, 2n$ we have*

$$X_0(a^0) = 1 - \lambda h_0^0 + \sum_{i,j=1}^{2n} B_j^i h_0^i a^j + c\left((a^0)^2 - \lambda^2\right),$$

$$X_i(a^0) = -\lambda h_i^0 + \sum_{j=1}^{2n} B_j^i h_i^i a^j + c a^i a^0 + c\lambda \sum_{j=1}^{2n} a^j B_j^i,$$

$$X_i(a^i) = 1 - \lambda h_i^i - \sum_{j=1}^{2n} \Gamma_{ij}^i a^j + c(a_i)^2 - c \sum_{j,k=1}^{2n} a^j a^k B_j^i B_k^i.$$

Proof. It is similar to the proof of Lemma 2.1, taking into account formula (20). □

We explicitly recall that for hypersurfaces in complex space forms K_c, with non-zero curvature, the second fundamental form h is not a Codazzi tensor (see formula (4)), in particular the Codazzi equations for h read as follow (see [15, 31]): for all $X, Y, Z \in TM$ we have

$$(\nabla_X^M h)(Y,Z) - (\nabla_Y^M h)(X,Z) = \tag{21}$$

$$= c\Big(g(Y,X_0)g(\varphi X, Z) - g(X,X_0)g(\varphi Y, Z) - 2g(X,\varphi Y)g(X_0,Z)\Big),$$

where φ is the endomorphism defined by $JX = \varphi X + g(X,X_0)\nu$, for any $X \in TM$. As in Section 2, we obtain the $2n$-th horizontal Minkowski formula computing the divergence of the vector field $U = 2W - V$, with W and V as defined in (8) and (7) respectively. We choose again an orthonormal frame for TM of the form $\mathcal{E} = \{X_0, X_1, \dots, X_{2n}\}$ where X_0 is the characteristic vector field and $X_1, \dots, X_{2n} \in HM$ are eigenvectors for the shape operator restricted to HM. Then, the computations proceed exactly as in Proposition 2.1, Lemma 2.4 and Proposition 2.2, using Lemma 3.1 instead of Lemma 2.1: here, the crucial fact is that, in the case $X, Y, Z \in HM$, or $X = X_0$ and $Y = Z$ belongs to HM, the right hand side of (21) is zero. By a straightforward computation, then we get the following

Proposition 3.1. *It holds:*

$$\operatorname{div}(V) = 2n\sigma_{2n-1}^H - 2n\lambda\sigma_{2n}^H + Q_V + 2nc\Phi^H \tag{22}$$

$$\operatorname{div}(W) = 2n\sigma_{2n-1}^H - \lambda\left((2n+1)\sigma_{2n} - \sigma_{2n}^H\right) + Q_W + 2nc\Phi$$

with $Q_V = 2Q_W = -2\sum_{i,j=1}^{2n} F_i^i B_i^j h_0^i a^j$.

Remark 3.2. *From the divergence of V, formula (22), we get a formula of Miquel type [18] for Hopf hypersurfaces, involving only purely horizontal curvatures: in fact the terms in Q_V depend linearly on the mixed coefficients h_0^i, so they vanish identically if the hypersurface is of Hopf type. Therefore, if M is a closed real Hopf hypersurface in a complex space form K_c, then it holds*

$$\int_M \sigma_{2n-1}^H - \lambda\sigma_{2n}^H = c\int_M \Phi^H.$$

Finally, by means of the divergence theorem, $\int_M \operatorname{div}(2W - V) = 0$ and we get the desired formula.

Corollary 3.1. *Let M be a closed real hypersurface embedded in K_c. Then it holds:*

$$\int_M \sigma_{2n-1}^H - \lambda \widetilde{\sigma}_{2n} = c \int_M \Phi^H - 2\Phi,$$

where

$$\widetilde{\sigma}_{2n} = \frac{(2n+1)\sigma_{2n} - (n+1)\sigma_{2n}^H}{n}.$$

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