Some integral formulas for the characteristic curvature.

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Abstract We show some integral formulas involving the characteristic curvature for closed real hypersurfaces in complex spaces.

Mathematics Subject Classification 2010. Primary 32V40; Secondary 35J70, 53C21.

Keywords: Hypersurfaces in complex spaces, Null lagrangian, Minkowski type formula.

1 Introduction

Let M denote a closed embedded hypersurface in $\mathbb{R}^{n+1}$. In a series of well known papers [23], [22], R.C.Reilly first proved some integral formulas involving the symmetric elementary functions of the eigenvalues of the Hessian of a given defining function for M and the curvatures related to the second fundamental form of M; then he used these formulas to prove some isoperimetric inequalities involving the curvatures of M and finally with the help of the Minkowski formula he gave an new elegant proof of the Alexandrov theorem [1].

Many generalizations of the previous results have been investigated in different contexts by several authors, see for instance [5], [24], [20], [26] and the references therein. In general, one is interested in some special subspaces of the tangent space: usually the so called *horizontal space*. In particular, in [13], we focused on the Levi curvatures for a real hypersurface M in $\mathbb{C}^{n+1}$: these are elementary symmetric functions of the eigenvalues of the Levi Form, which is defined on a (horizontal) subspace HM of codimension one in the full tangent space of M; there we proved an integral formula that led to an isoperimetric estimate and a Alexandrov type result (see also [21], [4] or [16], [17] for other symmetry results on Levi curvatures).
In this short note we will focus on the curvature related to the missing direction. As we will see in the sequel, this *characteristic curvature* is highly degenerate as differential operator: nevertheless we will prove that some integral formulas still hold with respect to this characteristic curvature.
Here for the sake of clarity, we will present the case of real hypersurfaces in $\mathbb{C}^{n+1}$ with its canonical hermitian metric. However generalizations of these formulas can be easily obtained in the case of real hypersurfaces in the other two standard models of complex space forms: the complex projective space $\mathbb{C}P^{n+1}$ with the Fubini-Study metric, and the complex hyperbolic space $\mathbb{C}H^{n+1}$ with the Bergman metric (see Remark 2.2).

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So, let M be a closed and embedded smooth real hypersurface in $\mathbb{C}^{n+1}$ and let us identify $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$, with

$$z = (z_1, \dots, z_{n+1}) \simeq (x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}), z_k = x + iy, \quad k = 1, \dots, n+1.$$

Therefore M can be seen as a CR-manifold of CR-codimension equal to one, with the standard CR structure induced by the holomorphic structure of $\mathbb{C}^{n+1}$. Thus for every $p \in M$ the tangent space T_pM splits in two subspaces: the $2n$-dimensional horizontal subspace H_pM (the largest subspace in T_pM invariant under the action of the standard complex structure J of $\mathbb{C}^{n+1}$) and the vertical one-dimensional subspace generated by the characteristic direction $T_p := JN_p$, where N_p is the unit inward normal at p. Moreover, if g is the standard metric on $\mathbb{C}^{n+1}$ (that we induce on M as well), then it holds

$$T_pM = H_pM \oplus \mathbb{R}T_p \ ,$$

and the sum is g-orthogonal. Let us denote by h the Second Fundamental Form on TM

$$h(U, V) = g(\nabla_U V, N), \quad U, V \in TM \ ,$$

where ∇ is the Levi-Civita connection for g.

Definition 1.1. *We will call the characteristic curvature of M, the following*

$$\mathcal{C} := h(T, T) = g(\nabla_T T, N) \ .$$

A defining function for M is a smooth function $f : \mathbb{C}^{n+1} \to \mathbb{R}$ such that

$$\Omega = \{z \in \mathbb{C}^{n+1} : f(z) < 0\}, \quad M = \partial\Omega = \{z \in \mathbb{C}^{n+1} : f(z) = 0\},$$

with $Df \neq 0$ on $\partial\Omega$. Hence $N = -Df/|Df|$ is the inner unit normal and the characteristic direction $T \in TM$ is basically the normalized Hamiltonian vector field related to the Hamiltonian f (see [11] or [9], for instance).
A direct computation shows that for any $z \in M$, we have an explicit formula for $\mathcal{C}$ that involves a defining function f:

$$\mathcal{T}f(z) := \mathcal{C}(z) = \frac{1}{|Df(z)|^3} tra\Big(A(Df(z))D^2 f(z)\Big) \ , \tag{1}$$

where A is the following $(2n+2) \times (2n+2)$ symmetric matrix:

$$A(Df(z)) = \begin{pmatrix} f_y \otimes f_y & -f_y \otimes f_x \\ -f_x \otimes f_y & f_x \otimes f_x \end{pmatrix} \ .$$

Due to the structure of the matrix A, the characteristic curvature is highly degenerate as differential operator acting on functions; to see this, let us consider the hypersurface M (locally) as the graph of some function $u : \mathbb{R}^{2n+1} \supseteq \Omega \to \mathbb{R}$ such that $(\xi, u(\xi)) \in M$ for all $\xi \in \Omega$. So we set

$$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n), \quad x_{n+1} = t, \quad y_{n+1} = s, \quad \xi = (x, y, t) \ ,$$

and we take as defining function

$$f(z) = f(x, y, t, s) = u(x, y, t) - s = u(\xi) - s, \qquad |Df|^2 = 1 + |Du|^2 \ .$$

Then we have

$$\mathcal{T}u := \frac{1}{(1+|Du|^2)^{\frac{3}{2}}} tra\Big(A(Du)\, D^2u\Big) ,$$

where A is the following symmetric matrix:

$$A(Du) = \begin{pmatrix} u_y \otimes u_y & -u_y \otimes u_x & -u_y \\ -u_x \otimes u_y & u_x \otimes u_x & u_x \\ -u_y & u_x & 1 \end{pmatrix} . \tag{2}$$

The characteristic curvature operator $\mathcal{T}$ is a second order quasilinear degenerate elliptic operator on $\mathbb{R}^{2n+1}$: in fact, by (2) we see that the following $2n$ independent vector fields

$$\partial_{x_k} + u_{y_k}\partial_t \quad , \qquad \partial_{y_k} - u_{x_k}\partial_t \quad , \qquad k = 1, \ldots, n ,$$

are eigenvectors for A with eigenvalue identically equals to zero; instead the vector field

$$-u_{y_1}\partial_{x_1} - u_{y_n}\partial_{x_n} + u_{x_1}\partial_{y_1} + u_{x_n}\partial_{y_n} + \partial_t ,$$

is an eigenvector for A with eigenvalue equals to $(1+|u_x|^2+|u_y|^2)$. Anyway, in [11] we prove for this operator existence and uniqueness of viscosity solutions for the Dirichlet Problem with prescribed curvature and we get also the Lipschitz regularity of solutions by using a Bernstein method (see [25], [14], [12] for the same results on the Levi curvatures); on the other hand, we show with two counterexamples, that neither the Strong Comparison Principle nor the Hopf Lemma hold for this operator $\mathcal{T}$. This is a substantial difference between the highly degenerate Characteristic Curvature operator and the Levi Curvature operators, for which Lanconelli and Montanari in [8] proved the Strong Comparison Principle: indeed the principal part of Levi Curvature operators are degenerate only with respect to one direction and when computed on strictly pseudoconvex functions, the $2n$ vector fields, respect to which the operator is strictly elliptic, satisfy the Hörmander rank condition. Moreover in [15] we also proved some properties of the characteristic curvature by using the Codazzi equations.

Here we show that regarding some integral formulas instead, the horizontal and the characteristic curvature have the same behavior. We have the following representation formula:

Theorem 1.1. *Let Ω be a bounded domain of $\mathbb{C}^{n+1}$ with boundary a real smooth hypersurface M. For every defining function f of M, we have*

$$\int_\Omega \big\{\sigma_2(D^2f(z)) - 16\sigma_2(\partial\bar\partial f(z))\big\}dz = \frac{1}{2}\int_M \mathcal{C}(z)|Df(z)|^2 d\sigma(z), \tag{3}$$

where $\sigma_2(D^2f)$ and $\sigma_2(\partial\bar\partial f)$ denote the second elementary symmetric functions of the eigenvalues of the real Hessian D^2f and the Hermitian complex Hessian $\partial\bar\partial f$, respectively.

We have also the following integral formula of Minkowski type

Theorem 1.2. *Let M be a smooth, closed and embedded hypersurface in $\mathbb{C}^{n+1}$. We have*

$$\int_M 1 + \lambda(z)\mathcal{C}(z) + Q(z)d\sigma(z) = 0 , \tag{4}$$

where $\lambda(z) = g(z, N)$, for all $z \in M$.

Here Q depends on some coefficients of the Second fundamental Form and will be explicitly written in the proof: in particular, when M is of Hopf type then Q vanishes identically.
We recall that there are many generalizations of the various Minkowski formulas, in particular in [20] and [26], it is proved the analogous of the classical Minkowski formula for the horizontal mean curvature for Hopf hypersurfaces in complex space forms: we can recover those formulas by (4), differentiating only along the characteristic vector field T, without any differentiation along horizontal vector fields, getting an easier proof; moreover we can relax the assumption on being Hopf (see Remarks 2.1 and 2.3).

2 Proofs of the results

Proof. (of Theorem 1.1)
The proof of the representation formula is straightforward: as in [23] and [13], we will use the null lagrangian property for elementary symmetric functions in the eigenvalues of both the Hessians and then the divergence theorem.
Let M be a smooth boundary of a bounded domain Ω in $\mathbb{C}^{n+1}$ with defining function f. For every $k, l = 1, \ldots, n$ we have

$$\frac{\partial f}{\partial z_k} = \frac{1}{2}\Big(\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\Big), \qquad |\partial f| = \frac{1}{2}|Df|\ ,$$

and

$$\frac{\partial^2 f}{\partial z_k \partial \bar{z}_\ell} = \frac{1}{4}\Big(\frac{\partial^2 f}{\partial x_k \partial x_l} + \frac{\partial^2 f}{\partial y_k \partial y_l}\Big) + \frac{i}{4}\Big(\frac{\partial^2 f}{\partial x_k \partial y_l} - \frac{\partial^2 f}{\partial y_k \partial x_l}\Big)\ .$$

Therefore we have:

$$\sum_{l=1}^{n+1} \frac{\partial}{\partial z_l}\left(\frac{\partial \sigma_2(\partial\bar{\partial} f)}{\partial f_{z_l \bar{z}_k}}\right) = 0, \qquad \forall k = 1, \ldots, n+1$$

and

$$\sum_{l=1}^{2n+2} \frac{\partial}{\partial \xi_l}\left(\frac{\partial \sigma_2(D^2 f)}{\partial f_{\xi_l \xi_k}}\right) = 0, \qquad \forall k = 1, \ldots, 2n+2,$$

where $\xi_k = x_k$ for $k = 1, \ldots, n+1$ and $\xi_k = y_k$ for $k = n+2, \ldots, 2n+2$. The null lagrangian properties read as:

$$\sum_{l,k=1}^{n+1} \frac{\partial \sigma_2(\partial\bar{\partial} f)}{\partial f_{z_l \bar{z}_k}} \frac{\partial f}{\partial z_l} \frac{\partial f}{\partial \bar{z}_k} = -\sum_{1 \le i_1 < i_2 \le n+1} \Delta_{(z_{i_1}, z_{i_2})}(f)$$

and

$$\sum_{l,k=1}^{2n+2} \frac{\partial \sigma_2(D^2 f)}{\partial f_{\xi_l \xi_k}} \frac{\partial f}{\partial \xi_l} \frac{\partial f}{\partial \xi_k} = -\sum_{1 \le i_1 < i_2 \le 2n+2} \Delta_{(\xi_{i_1}, \xi_{i_2})}(f)\ ,$$

where

$$\Delta_{(z_{i_1}, z_{i_2})}(f) = \det\begin{pmatrix} 0 & f_{\bar{z}_{i_1}} & f_{\bar{z}_{i_2}} \\ f_{z_{i_1}} & f_{z_{i_1}, \bar{z}_{i_1}} & f_{z_{i_1}, \bar{z}_{i_2}} \\ f_{z_{i_2}} & f_{z_{i_2}, \bar{z}_{i_1}} & f_{z_{i_2}, \bar{z}_{i_2}} \end{pmatrix}\ ,$$

$$\Delta_{(\xi_{i_1},\xi_{i_2})}(f) = \det \begin{pmatrix} 0 & f_{\xi_{i_1}} & f_{\xi_{i_2}} \\ f_{\xi_{i_1}} & f_{\xi_{i_1},\xi_{i_1}} & f_{\xi_{i_1},\xi_{i_2}} \\ f_{\xi_{i_2}} & f_{\xi_{i_2},\xi_{i_1}} & f_{\xi_{i_2},\xi_{i_2}} \end{pmatrix} .$$

Now, by the homogeneity of the function σ_2 and using the divergence theorem, a direct computation shows that:

$$\int_\Omega \big\{\sigma_2(D^2f) - 16\sigma_2(\partial\bar{\partial}f)\big\}dz =$$

$$= \frac{1}{2}\int_\Omega \sum_{l,k=1}^{2n+2} \frac{\partial}{\partial \xi_l}\left(\frac{\partial\sigma_2(D^2f)}{\partial f_{\xi_l\xi_k}}\frac{\partial f}{\partial \xi_k}\right) - 16\sum_{l,k=1}^{n+1}\frac{\partial}{\partial z_l}\left(\frac{\partial\sigma_2(\partial\bar{\partial}f)}{\partial f_{z_l\bar{z}_k}}\frac{\partial f}{\partial \bar{z}_k}\right)dz =$$

$$= \frac{1}{2}\int_M \frac{1}{|Df|}\sum_{l,k=1}^{2n+2}\left(\frac{\partial\sigma_2(D^2f)}{\partial f_{\xi_l\xi_k}}\frac{\partial f}{\partial \xi_k}\frac{\partial f}{\partial \xi_l}\right) - \frac{16}{|Df|}\sum_{l,k=1}^{n+1}\left(\frac{\partial\sigma_2(\partial\bar{\partial}f)}{\partial f_{z_l\bar{z}_k}}\frac{\partial f}{\partial \bar{z}_k}\frac{\partial f}{\partial z_l}\right)d\sigma(z) =$$

$$= \frac{1}{2}\int_M -\frac{1}{|Df|}\sum_{1\le i_1<i_2\le 2n+2}\Delta_{(\xi_{i_1},\xi_{i_2})}(f) + \frac{16}{|Df|}\sum_{1\le i_1<i_2\le n+1}\Delta_{(z_{i_1},z_{i_2})}(f)d\sigma(z) =$$

$$= \frac{1}{2}\int_M \frac{1}{|Df|}tra\Big(A(Df(z))D^2f(z)\Big)d\sigma(z) = \frac{1}{2}\int_M \mathcal{C}(z)|Df|^2 d\sigma(z).$$

□

Now we are going to prove the Theorem 1.2.

Proof. (of Theorem 1.2)
We will denote by $z \in M$ the position vector (with respect to the origin) and we will consider the squared distance function:

$$\varphi : M \to \mathbb{R}, \qquad \varphi(z) = \frac{|z|^2}{2} = \frac{g(z,z)}{2} .$$

For any tangent vector field $V \in TM$, the derivative of φ along V is

$$V(\varphi(z)) = \frac{1}{2}V(g(z,z)) = g(z,V) .$$

So, if $\{X_1,\dots,X_n,Y_1,\dots,Y_n\}$ is any orthonormal basis of the horizontal space HM, with $Y_k = JX_k$ and $k = 1,\dots,n$, then we have

$$z = \lambda N + T(\varphi)T + \sum_{k=1}^{n}\big\{X_k(\varphi)X_k + Y_k(\varphi)Y_k\big\} ,$$

where we have set

$$\lambda(z) = g(z, N_z) .$$

Now we observe that the characteristic vector field T is divergence free, in fact:

$$div(T) = g(\nabla_T T, T) + \sum_{k=1}^{n}\big\{g(\nabla_{X_k}T, X_k) + g(\nabla_{Y_k}T, Y_k)\big\} =$$

$$= \sum_{k=1}^{n}\big\{g(-\nabla_{X_k}N, Y_k) + g(\nabla_{Y_k}N, X_k)\big\} = \sum_{k=1}^{n}\big\{h(X_k,Y_k) - h(Y_k,X_k)\big\} = 0 .$$

Now, we differentiate the function φ twice along T to get

$$T^2(\varphi) = T(g(z, T_z)) = g(T,T) + g(z, \nabla_T T) =$$

$$= 1 + g(\lambda N + T(\varphi)T + \sum_{k=1}^{n} \{X_k(\varphi)X_k + Y_k(\varphi)Y_k\}, \nabla_T T) =$$

$$= 1 + \lambda \mathcal{C} + \sum_{k=1}^{n} \{X_k(\varphi)h(T, Y_k) - Y_k(\varphi)h(T, X_k)\} = 1 + \lambda\mathcal{C} + Q \ ,$$

where we have called

$$Q := \sum_{k=1}^{n} \{X_k(\varphi)h(T, Y_k) - Y_k(\varphi)h(T, X_k)\} \ .$$

Therefore we have that

$$0 = \int_M T^2(\varphi(z))d\sigma(z) = \int_M 1 + \lambda(z)\mathcal{C}(z) + Q(z)d\sigma(z) \ .$$

□

Remark 2.1. *We can recover the formula for the horizontal subspace, in fact we have on* M *that*

$$\Delta_M = \Delta_H + T^2 \ ,$$

where we have denoted by Δ_H *the sublaplacian related to* HM*. Now, let* $\mathcal{H}$ *and* $\mathcal{L}$ *be the classical mean curvature and the horizontal (Levi) mean curvature, respectively. Since it holds* $(2n+1)\mathcal{H} = 2n\mathcal{L} + \mathcal{C}$*, then by the classical Minkowski formula, we have*

$$0 = \int_M \Delta_H(\varphi(z))d\sigma(z) = \int_M 2n + 2n\lambda(z)\mathcal{L}(z) - Q(z)d\sigma(z) \ .$$

Remark 2.2. *Let us explicitly observe, that the same computations can be repeated in the case of real hypersurfaces in the other two standard models of complex space form other than* $\mathbb{C}^{n+1}$*: the complex projective space* $\mathbb{C}P^{n+1}$ *with the Fubini-Study metric, and the complex hyperbolic space* $\mathbb{C}H^{n+1}$ *with the Bergman metric. These three prototypes differ in the sign of the holomorphic (constant) sectional curvature* K*, respectively zero, positive, and negative. Denoting by* r *the geodesic distance function of* $z \in M$ *from a given point* o*, in the case of* $\mathbb{C}H^{n+1}$ *the function to consider is* $\varphi(z) = \log\cosh r$*; instead in the case of* $\mathbb{C}P^{n+1}$ *the function to differentiate is* $\varphi(z) = \log\cos r$*: the only assumption to add in the case of* $\mathbb{C}P^{n+1}$ *is that the hypersurface* M *must be contained in the geodesic ball of center* o *and radius smaller that* $\pi/\sqrt{-K}$ *in order to avoid conjugate points and to ensure that the function* φ *is smooth.*

Remark 2.3. *The previous formulas simplify in the case* M *is a Hopf hypersurface, namely whether the characteristic vector field is an eigenvector for the Second Fundamental form*

$$\nabla_T T = \mathcal{C}N, \qquad h(T, X_k) = h(T, Y_k) = 0 \ .$$

We address the reader, for instance to [3], [19], or [7] and the references therein, for further details on Hopf hypersurfaces. However on these manifolds we have that Q *identically vanishes, so the Minkowski formula holds for both the horizontal (see [20] and [26]) and the*

characteristic subspaces.
However, in a paper with G.Tralli [18], using the Codazzi equations we found that there exists a special vector field $V \in HM$ such that

$$\int_M Q(z)d\sigma(z) = \int_M \varphi(z)div(V)d\sigma(z)\ .$$

Therefore, we can relax the assumption on being Hopf in order to obtain the Minkowski formula: in particular, we can exhibit explicit examples of hypersurfaces that are not Hopf but for which the Minkowski formula still holds.

Finally let us recall that in the case of the Reinhardt domains (i.e. toric domains), it holds identically

$$T(\varphi(z)) = g(z,T) = 0, \qquad \forall z \in M\ .$$

In [10] we proved with this technique a symmetry result of Alexandrov type with respect to the characteristic curvature $\mathcal{C}$, even though the Minkowski formula does not hold.

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