A Jellett type theorem for the Levi curvature

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Abstract In this paper we prove a Jellett-type theorem for real hypersurfaces in $\mathbb{C}^2$ with respect to the Levi curvature. We provide as applications rigidity results for domains with circular symmetries.

Keywords: constant Levi curvature, Jellett theorem, sublaplacian.

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Contents

1 Introduction and state of the art

Let M be a smooth orientable hypersurface in $\mathbb{R}^{n+1}$, with $n \geq 1$. We say that M is starshaped if it is a boundary of a bounded domain which is strictly starshaped. In 1853 Jellett proved the following

Theorem (Jellett, [16]). *Any starshaped hypersurface M in $\mathbb{R}^{n+1}$ with constant mean curvature is a sphere.*

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Actually Jellett proved his theorem for two-dimensional surfaces in $\mathbb{R}^3$, however the same techniques work in any dimension.

Remark 1.1. *In the case $n = 1$, the Jellett method works without the starshapedness assumption. In fact, a closed embedded curve in $\mathbb{R}^2$ whose curvature is strictly positive bounds a convex domain, which is in particular starshaped with respect to any interior point.*

As it is very well-known, a fundamental result by Aleksandrov [1] says that the assumption of being starshaped is not needed in any dimension.
About Jellett's theorem, which dates nearly a century before Aleksandrov's, we have to mention that almost 50 years later (in 1899) Liebmann proved a weaker result that turned out to have more resonance in the future literature: he proved in [22] that the only closed convex hypersurfaces in $\mathbb{R}^3$ with constant mean curvature are the spheres. A genuine generalization of Jellett's theorem was established in 1951 by Hopf, who proved that the only compact contractible surfaces in $\mathbb{R}^3$ with constant mean curvature are the spheres (see [11]): it is worth to notice that the technique of Hopf effectively works for two-dimensional surfaces.
Just few years later, Aleksandrov proved in [1] that the spheres are the only closed hypersurfaces with constant mean curvature by introducing his celebrated moving planes technique. Later on, an alternative proof of the same result was given by Reilly in [36], where he adopted a new integral approach. We will come back again on these two different approaches.

Now, let us briefly sketch the proof of Jellett result for the reader's convenience (see also [23, Chapter 2]). Suppose that $M \subset \mathbb{R}^{n+1}$ is starshaped with respect to the origin. Let ν be the unit outward normal to M, and $p \in \mathbb{R}^{n+1}$ be the position vector. Let also $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the usual inner product and norm in $\mathbb{R}^{n+1}$, then we denote by $\psi(p) = \frac{|p|^2}{2}$ and $\lambda(p) = \langle p, \nu \rangle$ respectively the square of the distance and the support function. A straightforward computation shows that

$$\Delta_M \psi = n - nH\lambda,$$

where Δ_M and H stand respectively for the Laplace-Beltrami operator on M and for its mean curvature. In our notations nH is the trace of the second fundamental form h. Moreover, if H is constant, by the Codazzi equations we get

$$\Delta_M \lambda = nH - \|h\|^2 \lambda.$$

Here we used the notation $\|\cdot\|^2$ for the squared norm of a matrix, namely the sum of all of its squared coefficients. We recall that, for any symmetric $n \times n$ matrix Q, it holds true that

$$\|Q\|^2 \geq \frac{1}{n}(\mathrm{trace}(Q))^2, \tag{1}$$

and the equality occurs if and only if the matrix Q is a multiple of the identity. This fact and the starshapedness assumption imply

$$\Delta_M \left(H\psi - \lambda\right) = (\|h\|^2 - nH^2)\lambda \geq 0. \tag{2}$$

The function $H\psi - \lambda$ is then Δ_M-subharmonic, and thus constant being M closed (compact, without boundary). In particular, since λ is strictly positive, $\|h\|^2 = nH^2 = \frac{1}{n}(\mathrm{trace}(h))^2$. The equality case in (1) says that M is umbilical, therefore it must be a sphere.

Here we want to follow a similar strategy for the case of a threedimensional hypersurface M in $\mathbb{C}^2$ and to prove a theorem 'à la Jellett' when we consider the Levi mean curvature in place of the standard mean curvature. The Levi curvature is a sort of degenerate-elliptic analogue of the classical mean curvature: it was introduced and studied in [2, 10]. Roughly speaking, one considers the restriction of the second fundamental form to the holomorphic tangent space (see Section 2 for the precise definitions). Such restriction involves a lack of information and hence a lack of ellipticity in the relative operator. However, in the case of a suitable non-flatness, the direction of missing ellipticity is recovered through bracket commutations: the Levi operator can be thus seen as a degenerate-elliptic operator of sub-Riemannian type. This very special feature has been successfully exploited, e.g. by Citti-Lanconelli-Montanari in [9] where they were able to prove a regularity result. On the other hand, symmetry results for sub-Riemannian operators (both linear or non-linear) are extremely delicate. In particular, for rigidity results 'à la Aleksandrov', both the approaches of Aleksandrov and Reilly seem not working so far.
On one side, the moving planes method basically relies on the following main ingredients: an invariance of the curvature with respect to reflections through hyperplanes and an interior and a boundary comparison principle. In our situation, the reflections with respect to generic hyperplanes do not preserve the complex structure, and therefore the Levi curvature is not an invariant (however, in principle, other kinds of reflections might work). Regarding the comparison principles, a strong interior comparison principle

was proved in [32], whereas boundary comparison principles may fail in general. In fact, some counterexamples to the validity of the Hopf Lemma could be found (see for instance [13, Section 2] and [28, Section 5]).
Concerning the method in the Reilly's proof, it basically relies on two main ingredients: first an integral representation formula, which relates the second symmetric elementary function of the eigenvalues of the Hessian of a defining function with the mean curvature, and then the Minkowski formula. For the Levi curvatures, integral representation formulas were proved in [26], whereas an analogue of the Minkowski formula does not hold in general (see for instance [31, 41] and the examples in [29]).
Nonetheless, in literature some Aleksandrov-type results for the Levi operator have been proved under some extra-assumptions [18, 13, 33, 14, 25, 26, 27]. Most of these results are based on technical assumptions about apriori symmetries for the hypersurface, or about apriori comparisons with the classical mean curvature. To the best of our knowledge, the only Aleksandrov-type result for the Levi curvature which holds true for an explicit class of model hypersurfaces is the one in [13] (then generalized in [14]). In fact, Hounie and Lanconelli proved in [13] that the balls are the only bounded Reinhardt domains in $\mathbb{C}^2$ whose boundary has constant Levi curvature. Their proof is based on the following fact: a defining function of a Reinhardt domain in $\mathbb{C}^2$ depends on two real variables only, so one can think of the hypersurface as a graph of a function that only depends on a single real variable. In this way, they reformulated the problem in term of a (singular) ODE, and they were able to prove a uniqueness result for the given ODE.

In this paper we follow instead a PDE approach, which is inspired by the Jellett's proof as we just showed. We shall exploit in a crucial way the strong maximum principle for a suitable subelliptic operator on the hypersurface. For our main result we do need a technical assumption, named as condition $(\mathcal{A})$, that we will precisely state later on.

Theorem 1.1. *Let M be a starshaped hypersurface in $\mathbb{C}^2$ having constant Levi curvature and satisfying condition $(\mathcal{A})$. Then M is a sphere.*

We refer to Section 3 for the precise statement of condition $(\mathcal{A})$. Let us just mention here that it is a pointwise inequality involving the position vector and some coefficients of the Second Fundamental Form. Although such condition is technical, we do have a significant class of examples satisfying our assumptions. For instance, the condition $(\mathcal{A})$ is satisfied if the position vector p has no components along the characteristic direction of M. In particular we can prove the following

Corollary 1.1. *Let $M = \partial\Omega$ be a starshaped hypersurface in $\mathbb{C}^2$ with constant Levi curvature. Suppose Ω is a circular domain. Then Ω is a ball.*

Circular domains are meant in the sense of Carathéodory [6], who studied such domains of $\mathbb{C}^2$ for the problem of the analytic representation (see also Cartan [7]). We refer the reader to Section 4 for the definitions and further discussions. We remark here that this class of domains includes as a particular case the Reinhardt domains. At the end of Section 4 we will see how it is possible to recover from our results the Aleksandrov-type theorem by Hounie and Lanconelli.

The paper is organized as follows. In Section 2 we recall the main notions, in particular the Levi form, the Levi mean curvature, and the sublaplacian. In Section 3 we prove Theorem 1.1: the role of the Laplace-Beltrami operator in the Jellett's proof is taken here by a suitable first order perturbation of the sublaplacian. This choice will also lead to the statement of the technical condition ($\mathcal{A}$). In Section 4 we finally discuss the main applications of our theorem and we prove Corollary 1.1.

2 Levi curvature and sublaplacians

Since it requires no extra work, we are going to fix the main notions in $\mathbb{C}^{n+1}$: the comparison between the classical mean curvature and the Levi mean curvature will thus appear more evident. We will restrict later to the particular case $n = 1$.
We identify $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$, and we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the usual inner product and norm in $\mathbb{R}^{2n+2}$. For a fixed smooth connected orientable hypersurface M, boundary of a bounded domain, we put ν the unit outward normal to M and we denote by ∇ the Levi-Civita connection related to $\langle \cdot, \cdot \rangle$. Let A be the Weingarten operator, namely

$$A : TM \to TM, \quad AX := \nabla_X \nu \ ,$$

then

$$h(\cdot, \cdot) := \langle A\cdot, \cdot \rangle$$

is the Second Fundamental Form of M and $H = \frac{\mathrm{trace}(h)}{2n+1}$ is the mean curvature of M. In $\mathbb{C}^{n+1}$ we consider the standard complex structure J which is compatible with $\langle \cdot, \cdot \rangle$ and ∇ in the following sense:

$$\langle \cdot, \cdot \rangle = \langle J\cdot, J\cdot \rangle, \qquad J\nabla = \nabla J. \tag{3}$$

Thanks to J we can define the unit characteristic vector field $X_0 \in TM$ by $X_0 := -J\nu$. The horizontal distribution or Levi distribution HM is the $2n$-dimensional subspace in TM which is invariant under the action of J:

$$HM = TM \cap JTM,$$

that is a vector field $X \in TM$ belongs to HM if and only if also $JX \in HM$. Then TM splits in the orthogonal direct sum:

$$TM = HM \oplus \mathbb{R}X_0.$$

The Levi form ℓ can be defined on HM in the following way: for every $X, Y \in HM$, if $Z = \frac{1}{\sqrt{2}}(X - iJX)$ and $W = \frac{1}{\sqrt{2}}(Y - iJY)$, then

$$\ell(Z, \overline{W}) := \langle \nabla_Z \nu, \overline{W} \rangle.$$

We can compare the Levi form with the Second Fundamental Form by using the following identity (see [4], Chap.10, Theorem 2):

$$\forall X \in HM, \quad \ell(Z, \overline{Z}) = \frac{h(X,X) + h(JX,JX)}{2}. \tag{4}$$

Equivalently, we can give the definition of Levi form by using classical complex notations. We define the complex n-dimensional subspaces

$$T_{1,0}M := T^{1,0}\mathbb{C}^{n+1} \cap T^{\mathbb{C}}M, \qquad T_{0,1}M := \overline{T_{1,0}M}$$

where $T^{1,0}\mathbb{C}^{n+1}$ is the holomorphic space of $\mathbb{C}^{n+1}$, i.e. the complex space generated by the eigenvalue $+i$ of J, and $T^{\mathbb{C}}M$ is the complexified tangent space of M. Moreover

$$T_{1,0}M \oplus T_{0,1}M = H^{\mathbb{C}}M, \quad T^{\mathbb{C}}M = \mathbb{C}X_0 \oplus H^{\mathbb{C}}M.$$

The Levi form is then the hermitian operator $\ell(Z, \overline{W}) := \langle \nabla_Z \nu, \overline{W} \rangle$, for any couple of vector fields $Z, W \in T_{1,0}M$. We will say that M is Levi flat if ℓ identically vanishes; M is said strictly Levi-convex if ℓ is strictly positive definite as quadratic form.

Definition 2.1. *The Levi mean curvature of M is defined as*

$$L = \frac{trace(\ell)}{n}.$$

By (4) we get

$$(2n+1)H = 2nL + h(X_0, X_0).$$

From now on, we fix $n = 1$. In this way, the Levi mean curvature is actually 'the' Levi curvature.
We consider an orthonormal frame for TM of the form $E := \{X_0, X_1, X_2\}$, where $X_1 \in HM$ is a unit vector field and $X_2 = JX_1$. For $j, k \in \{0, 1, 2\}$, we denote by

$$\Gamma^l_{jk} = \langle \nabla_{X_j} X_k, X_l \rangle$$

the coefficients of the Levi-Civita connection with respect to the frame E. We are going to use the relations

$$\Gamma^k_{ij} = -\Gamma^j_{ik}, \qquad \text{for any } i, j, k = 0, 1, 2. \tag{5}$$

Let us also denote by

$$h_{jk} = h(X_j, X_k), \qquad j, k = 0, 1, 2,$$

the coefficients of the Second Fundamental Form with respect to E. In such a basis we have

$$h = \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{01} & h_{11} & h_{12} \\ h_{02} & h_{12} & h_{22} \end{pmatrix}.$$

By applying the complex structure J we get

$$\begin{aligned} \Gamma^0_{01} &= \langle \nabla_{X_0} X_1, X_0 \rangle = \langle \nabla_{X_0} X_2, \nu \rangle = -h_{02} \\ \Gamma^0_{02} &= \langle \nabla_{X_0} X_2, X_0 \rangle = -\langle \nabla_{X_0} X_1, \nu \rangle = h_{01}. \end{aligned} \tag{6}$$

Let us also define the horizontal part of h

$$h_H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}.$$

We will need the following expression for the connection coefficients.

Lemma 2.1. *In our notations we have*

$$\begin{aligned}
\nabla_{X_0}X_0 &= -h_{00}\nu + h_{02}X_1 - h_{01}X_2\\
\nabla_{X_1}X_1 &= -h_{11}\nu - h_{12}X_0 + \Gamma^2_{11}X_2\\
\nabla_{X_2}X_2 &= -h_{22}\nu + h_{12}X_0 + \Gamma^1_{22}X_1\\
\nabla_{X_1}X_2 &= -h_{12}\nu + h_{11}X_0 + \Gamma^1_{12}X_1\\
\nabla_{X_1}X_0 &= -h_{01}\nu + h_{12}X_1 - h_{11}X_2\\
\nabla_{X_2}X_1 &= -h_{12}\nu - h_{22}X_0 + \Gamma^2_{21}X_2\\
\nabla_{X_2}X_0 &= -h_{02}\nu + h_{22}X_1 - h_{12}X_2\\
\nabla_{X_0}X_1 &= -h_{01}\nu - h_{02}X_0 + \Gamma^2_{01}X_2\\
\nabla_{X_0}X_2 &= -h_{02}\nu + h_{01}X_0 + \Gamma^1_{02}X_1.
\end{aligned}$$

Proof. It follows by direct computation, by exploiting (3), (5), and (6). □

Now we need to define a second order subelliptic operator in analogy to the Laplace-Beltrami operator. Firstly, for any smooth function $u : M \to \mathbb{R}$, we define the Hessian of u as follows

$$Hess(u)(X,Y) = XYu - (\nabla^M_X Y)u, \qquad \forall\, X, Y \in TM,$$

where

$$\nabla^M_X Y = \nabla_X Y + h(X,Y)\nu, \qquad \forall\, X, Y \in TM.$$

We want to consider the horizontal Hessian of u

$$Hess_H(u)(X,Y) = XYu - (\nabla^M_X Y)u, \qquad \forall\, X, Y \in HM.$$

The Laplace-Beltrami operator Δ_M acting on u can be seen either as the divergence of the gradient of u or as the trace of the Hessian of u. It is worth to notice that in this subelliptic case, the divergence of the horizontal gradient of u and the trace of the horizontal Hessian of u do not coincide in general. Here we define the horizontal laplacian of M as the trace of $Hess_H$, that is in our orthonormal frame E

$$\Delta_H u = \sum_{j=1}^{2} \left(X_j X_j u - (\nabla^M_{X_j} X_j)u \right).$$

We explicitly remark that there are not first order derivatives along X_0. In fact, by the properties of the complex structure J and the symmetries of the second fundamental form, we get

$$\langle \nabla^M_{X_1} X_1, X_0\rangle + \langle \nabla^M_{X_2} X_2, X_0\rangle = \langle \nabla_{X_1} X_2, \nu\rangle - \langle \nabla_{X_2} X_1, \nu\rangle = 0.$$

Remark 2.1. *In general Δ_H is not in divergence form. An easy way to see this fact is by considering the following identity*

$$\Delta_M = \Delta_H + X_0 X_0 - \nabla^M_{X_0} X_0.$$

Now, the characteristic vector field X_0 is always divergence free, since we have

$$\begin{aligned} divX_0 &= \langle \nabla^M_{X_0} X_0, X_0 \rangle + \langle \nabla^M_{X_1} X_0, X_1 \rangle + \langle \nabla^M_{X_2} X_0, X_2 \rangle \\ &= \langle \nabla_{X_1} \nu, X_2 \rangle - \langle \nabla_{X_2} \nu, X_1 \rangle = 0. \end{aligned}$$

Therefore the operator $\Delta_H - \nabla^M_{X_0} X_0$ is in divergence form, but the vector field $\nabla^M_{X_0} X_0$ is not divergence free in general.

As we mentioned in the Introduction, we are going to deal with a first order (horizontal) perturbation of Δ_H. For our purposes a crucial role will be played by the strong maximum principle, namely the fact that the subsolutions of the involved operator cannot have local maxima unless they are constant. This is exactly the case for operators like $\Delta_H + V_1$ (for a smooth first order vector field V_1) as long as the Levi curvature L is strictly positive in M. As a matter of fact, the Strong Maximum Principle is a local property and such operators can be written locally in the form considered in [32, Section 4] (see also [8]). We recall that the main reason is the validity of the Hörmander condition which is provided (if $L > 0$) by the relation

$$[X_1, X_2] = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 = 2L X_0 + \Gamma^1_{12} X_1 - \Gamma^2_{21} X_2.$$

Summing up, we have the following

($\mathcal{SMP}$) if $L > 0$, the operator $\Delta_H + V_1$ satisfies the strong maximum principle for every smooth first order vector field V_1 in M.

3 Proof of Theorem 1.1

Suppose for the moment that M does not pass through the origin. Thus we can always write, at least locally on M, the position vector $p \in M$ as

$$p = \sum_{j=0}^{2} a_j X_j + \lambda \nu$$

for some smooth functions a_j, $j = 0, 1, 2$. We keep on denoting by $\psi(p) = \frac{|p|^2}{2}$, for $p \in M$. We need the derivatives of the functions a_j along X_1 and X_2. These are given by

$$\begin{aligned}
X_1(a_1) &= 1 + \langle p, \nabla_{X_1} X_1 \rangle = 1 - h_{12} a_0 + \Gamma_{11}^2 a_2 - h_{11}\lambda \\
X_1(a_0) &= \langle p, \nabla_{X_1} X_0 \rangle = h_{12} a_1 - h_{11} a_2 - h_{01}\lambda \\
X_1(a_2) &= \langle p, \nabla_{X_1} X_2 \rangle = h_{11} a_0 + \Gamma_{12}^1 a_1 - h_{12}\lambda \\
X_2(a_2) &= 1 + \langle p, \nabla_{X_2} X_2 \rangle = 1 + h_{12} a_0 + \Gamma_{22}^1 a_1 - h_{22}\lambda \\
X_2(a_0) &= \langle p, \nabla_{X_2} X_0 \rangle = h_{22} a_1 - h_{12} a_2 - h_{02}\lambda \\
X_2(a_1) &= \langle p, \nabla_{X_2} X_1 \rangle = -h_{22} a_0 + \Gamma_{21}^2 a_2 - h_{12}\lambda.
\end{aligned} \tag{7}$$

We have the following

Lemma 3.1. *In our notations we have*

$$\Delta_H \psi = 2 - 2L\lambda.$$

Proof. First of all we have

$$X_j \psi = \langle X_j, p \rangle = a_j, \qquad j = 0, 1, 2.$$

Hence we get

$$\begin{aligned}
&(X_1)^2 \psi - (\nabla_{X_1}^M X_1)\psi = X_1(\langle X_1, p \rangle) - \langle \nabla_{X_1}^M X_1, p \rangle = \\
&= 1 + \langle \nabla_{X_1} X_1, p \rangle - \langle \nabla_{X_1} X_1 + h_{11}\nu, p \rangle = 1 - h_{11}\lambda.
\end{aligned}$$

and

$$\begin{aligned}
&(X_2)^2 \psi - (\nabla_{X_2}^M X_2)\psi = X_2(\langle X_2, p \rangle) - \langle \nabla_{X_2}^M X_2, p \rangle = \\
&= 1 + \langle \nabla_{X_2} X_2, p \rangle - \langle \nabla_{X_2} X_2 + h_{22}\nu, p \rangle = 1 - h_{22}\lambda.
\end{aligned}$$

The statement readily follows. ☐

Now let us define the quantities

$$A_1 = h_{01} h_{12} - h_{11} h_{02}, \qquad A_2 = h_{22} h_{01} - h_{12} h_{02},$$

and we recall that in our notations we have put

$$\|h_H\|^2 = (h_{11})^2 + (h_{22})^2 + 2(h_{12})^2.$$

In the following lemma we differentiate the function λ by means of the Codazzi equations. This is the first time that the constant Levi-curvature condition appears.

Lemma 3.2. *If L is constant we have*

$$\Delta_H \lambda = 2L - \|h_H\|^2 \lambda - \left((h_{01})^2 + (h_{02})^2\right)\lambda + 2a_1A_1 + 2a_2A_2.$$

Proof. First of all we have

$$\begin{aligned} X_1\lambda &= X_1(\langle p,\nu\rangle) = \langle p, \nabla_{X_1}\nu\rangle = a_0h_{01} + a_1h_{11} + a_2h_{21} \\ X_2\lambda &= X_2(\langle p,\nu\rangle) = \langle p, \nabla_{X_2}\nu\rangle = a_0h_{02} + a_1h_{12} + a_2h_{22}. \end{aligned}$$

Therefore we get

$$\begin{aligned} &(\nabla^M_{X_1}X_1)\lambda + (\nabla^M_{X_2}X_2)\lambda = \Gamma^2_{11}X_2\lambda + \Gamma^1_{22}X_1\lambda = \\ &= \Gamma^2_{11}(a_0h_{02} + a_1h_{12} + a_2h_{22}) + \Gamma^1_{22}(a_0h_{01} + a_1h_{11} + a_2h_{21}). \end{aligned}$$

With the aid of (7) let us compute the last term we need

$$\begin{aligned} &(X_1)^2\lambda + (X_2)^2\lambda = \\ &= X_1(a_0h_{01} + a_1h_{11} + a_2h_{21}) + X_2(a_0h_{02} + a_1h_{12} + a_2h_{22}) = \\ &= h_{01}(h_{12}a_1 - h_{11}a_2 - h_{01}\lambda) + h_{11}(1 - h_{12}a_0 + \Gamma^2_{11}a_2 - h_{11}\lambda) + \\ &+ h_{21}(h_{11}a_0 + \Gamma^1_{12}a_1 - h_{12}\lambda) + h_{02}(h_{22}a_1 - h_{12}a_2 - h_{02}\lambda) + \\ &+ h_{12}(-h_{22}a_0 + \Gamma^2_{21}a_2 - h_{12}\lambda) + h_{22}(1 + h_{12}a_0 + \Gamma^1_{22}a_1 - h_{22}\lambda) + \\ &+ a_0[X_1(h_{01}) + X_2(h_{02})] + a_1[X_1(h_{11}) + X_2(h_{12})] + a_2[X_1(h_{21}) + X_2(h_{22})] = \\ &= h_{11} + h_{22} - ((h_{11})^2 + (h_{22})^2 + 2(h_{12})^2 + (h_{01})^2 + (h_{02})^2))\lambda + \\ &+ h_{01}(a_1h_{12} - a_2h_{11}) + h_{11}a_2\Gamma^2_{11} + h_{21}a_1\Gamma^1_{12} + \\ &+ h_{02}(a_1h_{22} - a_2h_{12}) + h_{12}a_2\Gamma^2_{21} + h_{22}a_1\Gamma^1_{22} + \\ &+ a_0[X_1(h_{01}) + X_2(h_{02})] + a_1[X_1(h_{11}) + X_2(h_{12})] + a_2[X_1(h_{21}) + X_2(h_{22})]. \end{aligned}$$

In order to deal with the three terms into the square brackets, we are going to exploit the Codazzi equations (see, e.g., [19]) which we can write as follows: for all $X, Y \in TM$

$$(\nabla^M_X h)(Y,\cdot) = (\nabla^M_Y h)(X,\cdot)$$

where we have denoted the Bortolotti derivative by

$$(\nabla^M_X h)(\cdot,\cdot) = X(h(\cdot,\cdot)) - h(\nabla^M_X \cdot,\cdot) - h(\cdot,\nabla^M_X \cdot).$$

With respect to the the first term in the brackets, we have the following relation

$$\begin{aligned} &X_1(h_{01}) + X_2(h_{02}) = \\ &= X_0(h_{11}) + h(\nabla^M_{X_1}X_1, X_0) + h(X_1, \nabla^M_{X_1}X_0) - 2h(\nabla^M_{X_0}X_1, X_1) + \\ &+ X_0(h_{22}) + h(\nabla^M_{X_2}X_2, X_0) + h(X_2, \nabla^M_{X_2}X_0) - 2h(\nabla^M_{X_0}X_2, X_2) = \\ &= X_0(2L) - h_{12}h_{00} + \Gamma^2_{11}h_{02} + 2h_{01}h_{02} - 2\Gamma^2_{01}h_{12} + h_{12}h_{00} + \\ &+ \Gamma^1_{22}h_{01} - 2h_{01}h_{02} - 2\Gamma^1_{02}h_{12} = \Gamma^2_{11}h_{02} + \Gamma^1_{22}h_{01}. \end{aligned} \tag{8}$$

The second term gives

$$\begin{aligned}
&X_1(h_{11}) + X_2(h_{12}) = -X_1(h_{22}) + X_2(h_{12}) = \\
&= -2h(\nabla^M_{X_1} X_2, X_2) + h(\nabla^M_{X_2} X_1, X_2) + h(X_1, \nabla^M_{X_2} X_2) = \\
&= -2h_{11}h_{02} - 2\Gamma^1_{12}h_{12} - h_{22}h_{02} + \Gamma^2_{21}h_{22} + h_{12}h_{01} + \Gamma^1_{22}h_{11}.
\end{aligned}$$

About the third one, we have

$$\begin{aligned}
&X_1(h_{21}) + X_2(h_{22}) = X_1(h_{21}) - X_2(h_{11}) = \\
&= h(\nabla^M_{X_1} X_2, X_1) + h(X_2, \nabla^M_{X_1} X_1) - 2h(\nabla^M_{X_2} X_1, X_1) = \\
&= h_{11}h_{01} + \Gamma^1_{12}h_{11} - h_{12}h_{02} + \Gamma^2_{11}h_{22} + 2h_{22}h_{01} - 2\Gamma^2_{21}h_{12}.
\end{aligned}$$

In all the three terms, besides (5), we used in a crucial way that $2L = h_{11} + h_{22}$ is constant throughout M. Putting all together, we finally get

$$\begin{aligned}
&\Delta_H \lambda = (X_1)^2\lambda + (X_2)^2\lambda - (\nabla^M_{X_1} X_1)\lambda - (\nabla^M_{X_2} X_2)\lambda = \\
&= h_{11} + h_{22} - ((h_{11})^2 + (h_{22})^2 + 2(h_{12})^2 + (h_{01})^2 + (h_{02})^2))\lambda + \\
&+h_{01}(a_1h_{12} - a_2h_{11}) + h_{11}a_2\Gamma^2_{11} + h_{21}a_1\Gamma^1_{12} + h_{02}(a_1h_{22} - a_2h_{12}) + \\
&+h_{12}a_2\Gamma^2_{21} + h_{22}a_1\Gamma^1_{22} + +a_0[\Gamma^2_{11}h_{02} + \Gamma^1_{22}h_{01}] + \\
&+a_1[-2h_{11}h_{02} - 2\Gamma^1_{12}h_{12} - h_{22}h_{02} + \Gamma^2_{21}h_{22} + h_{12}h_{01} + \Gamma^1_{22}h_{11}] + \\
&+a_2[h_{11}h_{01} + \Gamma^1_{12}h_{11} - h_{12}h_{02} + \Gamma^2_{11}h_{22} + 2h_{22}h_{01} - 2\Gamma^2_{21}h_{12}] + \\
&-\Gamma^2_{11}(a_0h_{02} + a_1h_{12} + a_2h_{22}) - \Gamma^1_{22}(a_0h_{01} + a_1h_{11} + a_2h_{21}) = \\
&= h_{11} + h_{22} - ((h_{11})^2 + (h_{22})^2 + 2(h_{12})^2 + (h_{01})^2 + (h_{02})^2))\lambda + \\
&+2a_1(h_{01}h_{12} - h_{11}h_{02}) + 2a_2(h_{22}h_{01} - h_{12}h_{02}).
\end{aligned}$$

□

Let us consider the function $u : M \longrightarrow \mathbb{R}$ defined as

$$u = L\psi - \lambda.$$

From the previous lemmas we deduce that

$$\Delta_H u = (\|h_H\|^2 - 2L^2)\lambda + ((h_{01})^2 + (h_{02})^2))\lambda - 2(a_1A_1 + a_2A_2)$$

if L is constant. The term $(\|h_H\|^2 - 2L^2)\lambda$ is the exact analogous of the Euclidean case (2). The inequality (1) applied to the matrix h_H and the starshapedness assumption imply the nonnegativity of such term. The problem relies on the remaining term, for which we have no clue about its sign. Even more importantly, we don't have a significant class of examples for

which we can guarantee that $((h_{01})^2+(h_{02})^2))\lambda - 2(a_1A_1+a_2A_2) \geq 0$. This is the reason why we introduce a first order perturbation of the horizontal laplacian Δ_H. Such a choice is motivated by the applications we will present in the next section.
Thus, let us consider the following horizontal vector field

$$V = \nabla^M_{X_0} X_0 = h_{02}X_1 - h_{01}X_2 \in HM.$$

The subelliptic operator we want to deal with is the following

$$\mathcal{L} = \Delta_H + 4V.$$

Remark 3.1. *We saw in Remark 2.1 that $\Delta_H - V$ is in divergence form: it is in fact the divergence of the horizontal gradient. Hence, our operator $\mathcal{L}$ is (in general) neither the trace of the horizontal Hessian nor the divergence of the horizontal gradient.*

Let us also define the following quantity

$$R := 2L(a_1h_{02} - a_2h_{01}) + (a_1A_1 + a_2A_2) - \left((h_{01})^2 + (h_{02})^2\right)\lambda.$$

The following lemma says that we do have an average information on R under the constant Levi-curvature assumption.

Lemma 3.3. *Suppose that L is constant on M. Then it holds*

$$\int_M R\, d\sigma = 0.$$

Proof. The first step is to prove that the vector field

$$W = JV = h_{01}X_1 + h_{02}X_2$$

is divergence free. In fact we have

$$\begin{aligned} \mathrm{div} W &= \langle \nabla^M_{X_0} W, X_0\rangle + \langle \nabla^M_{X_1} W, X_1\rangle + \langle \nabla^M_{X_2} W, X_2\rangle = \\ &= h_{01}\Gamma^0_{01} + h_{02}\Gamma^0_{02} + X_1(h_{01}) + h_{02}\Gamma^1_{12} + h_{01}\Gamma^2_{21} + X_2(h_{02}) = \\ &= h_{01}\Gamma^0_{01} + h_{02}\Gamma^0_{02} = -h_{01}h_{02} + h_{02}h_{01} = 0. \end{aligned}$$

Here we used (5), (6), and the Codazzi equation (8) (for which the fact that L is constant is needed).
Moreover, if we differentiate the function a_0 along W we obtain by (7)

$$\begin{aligned} W(a_0) &= h_{01}X_1(a_0) + h_{02}X_2(a_0) = \\ &= h_{01}(a_1h_{12} - a_2h_{11} - \lambda h_{01}) + h_{02}(a_1h_{22} - a_2h_{12} - \lambda h_{02}) = \\ &= a_1A_1 + a_2A_2 + 2L(a_1h_{02} - a_2h_{01}) - \left((h_{01})^2 + (h_{02})^2\right)\lambda = \\ &= R. \end{aligned} \tag{9}$$

Now, since M is closed, by the divergence theorem we have

$$\int_M R\, d\sigma = \int_M W(a_0)\, d\sigma = -\int_M a_0 \, \mathrm{div} W\, d\sigma = 0$$

and the lemma is proved. □

This means in particular that the function R has to change its sign, unless it vanishes identically on M. Our key assumption, which we have already mentioned in the Introduction, is that we require a pointwise information for R. Precisely, in Theorem 1.1 we assume the following

$$(\mathcal{A}) \qquad 3((h_{01})^2 + (h_{02})^2))\lambda + 2R \geq 0 \quad \forall\, p \in M.$$

From the last lemma we know that the condition $(\mathcal{A})$ is surely satisfied in average, under the starshapedness hypothesis.
Let us note that, if $h_{01} = h_{02} = 0$, then $R = 0$ and condition $(\mathcal{A})$ is satisfied. The vanishing at every point of h_{01} and h_{02} means that the characteristic vector field X_0 is an eigenvector for the Weingarten operator, which is exactly the definition for M to be of Hopf type (about this subject, see [5, 20, 30, 31]). We recall, for instance, the rigidity result in [18] where it is assumed a parallelism condition for the horizontal distribution which implies, in particular, to be of Hopf type. Moreover, Hopf hypersurfaces are classified in general complex space forms, and symmetry results can also be found in [35, 38, 39, 40, 17, 34, 3, 25].
Before discussing more in depth how we can guarantee the validity of condition $(\mathcal{A})$, we now complete the proof of our theorem.

Proof of Theorem 1.1. Without loss of generality we can assume that M is starshaped with respect to the origin, namely $\lambda(p) > 0$ for all $p \in M$. In particular $0 \notin M$. We consider the function

$$u = L\psi - \lambda,$$

and we compute $\mathcal{L}u$. Lemmas 3.1 and 3.2 tell us how to compute $\Delta_H u$, since L is constant on M. We are left with Vu. We get

$$\begin{aligned} Vu &= h_{02}X_1 u - h_{01}X_2 u = \\ &= h_{02}(La_1 - a_0 h_{01} - a_1 h_{11} - a_2 h_{21}) + \\ &- h_{01}(La_2 - a_0 h_{02} - a_1 h_{12} - a_2 h_{22}) = \\ &= L(a_1 h_{02} - a_2 h_{01}) + a_1 A_1 + a_2 A_2. \end{aligned}$$

Hence we have

$$\begin{aligned}\mathcal{L}u &= (\|h_H\|^2 - 2L^2)\lambda + ((h_{01})^2 + (h_{02})^2))\lambda + \\ &+ 4L(a_1 h_{02} - a_2 h_{01}) + 2(a_1 A_1 + a_2 A_2) = \\ &= (\|h_H\|^2 - 2L^2)\lambda + 3((h_{01})^2 + (h_{02})^2))\lambda + 2R.\end{aligned}$$

The condition $(\mathcal{A})$, the starhapedness, and the algebraic inequality (1) applied to the matrix h_H imply that

$$\mathcal{L}u \geq 0 \quad \text{in } M,$$

i.e. u is a smooth subsolution for $\mathcal{L}$. On the other hand, since M is compact, u must have a maximum point. From the strong maximum principle for $\mathcal{L}$ $(\mathcal{SMP})$, u is forced to be constant. Let us remark explicitly that we need $L > 0$ to ensure the $(\mathcal{SMP})$-property. This is exactly the case in our situation: the compactness of M says that L has to be positive somewhere and therefore everywhere since L is constant.
The fact that u is constant is saying that $\mathcal{L}u \equiv 0$. Since $3((h_{01})^2+(h_{02})^2))\lambda+2R \geq 0$, we get $(\|h_H\|^2 - 2L^2)\lambda = 0$ and thus

$$\|h_H\|^2 = 2L^2$$

having $\lambda > 0$. This is the equality case in (1) which implies that h_H is a multiple of the identity matrix. This is a horizontal umbilicality condition for M, with no information on the coefficients h_{01}, h_{02}, and h_{00}. Still, this is sufficient to conclude that M is a sphere thanks to our result in [27]. The horizontal umbilicality says that $h_{11} = h_{22}$ and $h_{12} = 0$. The hypothesis (H) in [27, Theorem 1.1] (case $c = 0$, $n = 1$) is thus satisfied, and the theorem applies. Therefore M must be a sphere. □

4 Applications

In this section we want to go back to condition $(\mathcal{A})$, and discuss the main application of Theorem 1.1. We want to point out that condition $(\mathcal{A})$ is easily satisfied under the starshaped assumption if $a_0 \equiv 0$. Recalling that $a_0 = \langle X_0, p\rangle = \langle \nu, Jp\rangle$, the vanishing of a_0 is equivalent to say that p has no components along the characteristic direction or that Jp is tangent.

Corollary 4.1. *Let M be a starshaped hypersurface in $\mathbb{C}^2$ with constant Levi curvature. Suppose that Jp is a tangent vector field. Then M must be a sphere.*

Proof. In (9) we saw that $R = W(a_0)$. Thus, under our assumption, R is not just vanishing in average but we actually have $R \equiv 0$. The remaining quantity involved in condition $(\mathcal{A})$ is then $3((h_{01})^2 + (h_{02})^2))\lambda$ which is of course nonnegative if M is starshaped. Therefore, Theorem 1.1 applies. $\square$

The condition $a_0 \equiv 0$ is more geometrical than technical. Under non-flatness, it is equivalent to say that the tangent part of Jp is a Killing vector field. Indeed, the fact that $Jp - a_0\nu$ is Killing is equivalent to have

$$\langle \nabla_U (Jp - a_0\nu), V\rangle + \langle \nabla_V (Jp - a_0\nu), U\rangle = 0 \quad \forall U, V \in TM.$$

On the other hand, by (3), we have

$$\begin{aligned} & \langle \nabla_U (Jp - a_0\nu), V\rangle + \langle \nabla_V (Jp - a_0\nu), U\rangle = \\ = \; & \langle \nabla_U p, JV\rangle + \langle \nabla_V p, JU\rangle - a_0 \left(\langle \nabla_U \nu, V\rangle + \langle \nabla_V \nu, U\rangle\right) = \\ = \; & \langle U, JV\rangle + \langle V, JU\rangle - 2a_0 h(U,V) \\ = \; & -2a_0 h(U,V). \end{aligned}$$

However, under the assumptions of our result, the second fundamental form cannot be the null form since L has to be a positive constant. Thus, the Killing property is equivalent to the vanishing of a_0.

Let us specify the meaning of the property $a_0 \equiv 0$ explicitly in terms of a defining function for M. So, let

$$M = \partial\Omega = \{z \in \mathbb{C}^2 : f(z) = 0\} \quad \text{and} \quad \Omega = \{z \in \mathbb{C}^2 : f(z) < 0\}$$

for some smooth $f : \mathbb{C}^2 \to \mathbb{R}$ (with $\nabla f \neq 0$ on M). We identify $\mathbb{C}^2 \simeq \mathbb{R}^4$, with $z_j = x_j + iy_j$ (for $j = 1, 2$), and we write $f(z_1, z_2) = f(x_1, y_1, x_2, y_2)$. Then we have

$$\nu = \frac{1}{|\nabla f|}(f_{x_1}\partial_{x_1} + f_{y_1}\partial_{y_1} + f_{x_2}\partial_{x_2} + f_{y_2}\partial_{y_2}),$$

$$X_0 = -J\nu = \frac{1}{|\nabla f|}(f_{y_1}\partial_{x_1} - f_{x_1}\partial_{y_1} + f_{y_2}\partial_{x_2} - f_{x_2}\partial_{y_2})$$

(for us $J\partial_{x_j} = \partial_{y_j}$). Therefore

$$a_0 = \langle p, X_0\rangle = \frac{1}{|\nabla f|}\left(x_1 f_{y_1} - y_1 f_{x_1} + x_2 f_{y_2} - y_2 f_{x_2}\right). \tag{10}$$

Definition 4.1. *An open set $\Omega \subset \mathbb{C}^2$ is said to be circular (with center at the origin) if*

$$(z_1, z_2) \in \Omega \quad \Longrightarrow \quad (e^{i\theta} z_1, e^{i\theta} z_2) \in \Omega$$

for all $\theta \in \mathbb{R}$.

Circular domains were investigated by Carathéodory and Cartan (respectively in [6] and in [7]) for the problems of the analytic representation. Moreover, Cartan studied the role of the circular starshaped domains (*cerclé étoilé* in [7]) and he proved that holomorphic functions in a circular domain extend holomorphically to the smallest circular starshaped domain containing the initial domain. It is due to Cartan also the fact that holomorphic functions in a circular domain can be expanded in series of homogeneous polynomials, and the fact that the only automorphisms of a bounded circular domain preserving the origin are linear (see also [21]).
For our purposes it is important to notice that a defining function of a circular domain Ω can be written actually as a function of three real independent variables. In fact (see also [6]), if we write $z_j = |z_j|\, e^{i\theta_j}$, then $(z_1, z_2) \in \Omega$ iff $(e^{i(\theta_1 - \theta_2)} |z_1|, |z_2|) \in \Omega$. Thus a defining function f for Ω can be written

$$\begin{aligned} f(z_1, z_2) &= h\Big(|z_1|^2, |z_2|^2, |z_1|\,|z_2| \cos(\theta_1 - \theta_2), |z_1|\,|z_2| \sin(\theta_1 - \theta_2)\Big) \\ &= h\left(x_1^2 + y_1^2, x_2^2 + y_2^2, x_1 x_2 + y_1 y_2, y_1 x_2 - x_1 y_2\right). \end{aligned} \tag{11}$$

Hence we can directly check that $a_0 \equiv 0$ for circular domains, and prove Corollary 1.1.

Proof of Corollary 1.1. Suppose Ω is a circular starshaped domain whose boundary has constant Levi curvature. If we write a defining function for $M = \partial\Omega$ as in (11), by (10) we get

$$\begin{aligned} a_0 &= \frac{1}{|\nabla f|}\left(x_1(2y_1 h_1 + y_2 h_3 + x_2 h_4) - y_1(2x_1 h_1 + x_2 h_3 - y_2 h_4) + \right. \\ &\qquad \left. + x_2(2y_2 h_2 + y_1 h_3 - x_1 h_4) - y_2(2x_2 h_2 + x_1 h_3 + y_1 h_4)\right) = \\ &= \frac{1}{|\nabla f|}\left(2(x_1 y_1 - y_1 x_1) h_1 + 2(x_2 y_2 - y_2 x_2) h_2 + \right. \\ &\qquad + (x_1 y_2 - y_1 x_2 + x_2 y_1 - y_2 x_1) h_3 + \\ &\qquad \left. + (x_1 x_2 + y_1 y_2 - x_2 x_1 - y_2 y_1) h_4\right) = \\ &= 0. \end{aligned}$$

Hence $a_0 \equiv 0$ in M, and Corollary 4.1 applies. $\square$

Reinhardt domains form a special class of circular domains. We want to show that for such domains we can improve our result. Let us recall the definition.
A Reinhardt domain Ω (with center at the origin) is an open subset of $\mathbb{C}^2$ such that

$$\text{if} \quad (z_1, z_2) \in \Omega \quad \text{then} \quad (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in \Omega \tag{12}$$

for all the real numbers θ_1, θ_2. These domains were introduced by Reinhardt in [37], and they naturally arise in the theory of several complex variables as the logarithmically convex Reinhardt domains are the domains of convergence of power series (see, e.g., [21, 12, 15]). In comparison to circular domains, they enjoy one more symmetry since one has $(z_1, z_2) \in \Omega$ iff $(|z_1|, |z_2|) \in \Omega$. Thus, a Reinhardt domain Ω can be naturally described with a domain in $\mathbb{R}^2$, and $\partial\Omega$ as a curve in $\mathbb{R}^2$. The fact of having constant Levi curvature gives rise to an ODE for the function parametrizing such curve. This is what Hounie and Lanconelli considered in [13]. To be more precise, by writing locally the defining function $f(z_1, z_2)$ as $F(|z_2|^2) - |z_1|^2$, they showed that F has to be solution of the ODE

$$sFF'' = sF'^2 - L(F + sF'^2)^{3/2} - FF'.$$

They proved a uniqueness result for the solutions to this degenerate second order ODE starting at $s = 0$, which lead to their Aleksandrov-type theorem. Their technique has then been used in [14] to prove an Aleksandrov theorem for bounded Reinhardt domains in $\mathbb{C}^{n+1}$ with an additional rotational symmetry in two complementary sets of variables. Still for Reinhardt domains in $\mathbb{C}^{n+1}$, in [24] the first named author proved a similar result of symmetry considering the characteristic curvature $h(X_0, X_0)$ rather than the Levi one. Here we want to show that from our Corollary 1.1 we can recover Hounie-Lanconelli result in [13], namely that

the only bounded Reinhardt domains in $\mathbb{C}^2$ whose boundary has constant Levi curvature are the balls.

What we have to show is that, for the case of Reinhardt domains, the starshapedness assumption is not needed. In fact it happens something similar to what we pointed out in Remark 1.1 for the case $n = 1$ in the classical Jellett theorem.

Lemma 4.1. *Let M be the boundary of a bounded Reinhardt domain Ω in $\mathbb{C}^2$. If the Levi curvature of M is strictly positive for any $p \in M$, then Ω is starshaped with respect to the origin.*

Proof. Let us consider a defining function $f : \mathbb{C}^2 \to \mathbb{R}$ depending only on the radii in the following way: let $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(z_1, z_2) = g(s_1, s_2) = g(s), \qquad s_k = \frac{z_k \bar{z}_k}{2} = \frac{x_k^2 + y_k^2}{2}, \quad k = 1, 2.$$

We first compute the function λ in this case:

$$\lambda = \langle p, \nu \rangle = \langle (x_1, y_1, x_2, y_2), \nu \rangle = \frac{2(s_1 g_1 + s_2 g_2)}{|\nabla f|} = \frac{\widetilde{\lambda}}{|\nabla f|}$$

with $\widetilde{\lambda} = 2(s_1 g_1 + s_2 g_2)$. We have to prove that the function $\widetilde{\lambda}$ is strictly positive, for any $p \in M$. We use the explicit formula (see [32] for instance) to compute the Levi curvature of M in term of a defining function. When applied to this case, it becomes

$$L = \frac{\widetilde{\lambda} g_1 g_2 + 2 s_1 s_2 D_2}{|\nabla f|^3}, \tag{13}$$

where we have denoted by

$$D_2 = g_2^2 g_{11} - 2 g_1 g_2 g_{12} + g_1^2 g_{22}$$

the term involving the second derivatives of g. We consider the following set in $\mathbb{R}^2$ defined by the function g

$$M_1 = \{s \in \mathbb{R}^2 : g(s) = 0,\ s_1 \geq 0,\ s_2 \geq 0\}.$$

M_1 is a curve in the first quadrant: it is smooth and well defined up to the axes $s_1 = 0$ and $s_2 = 0$. Let us consider the following function

$$\eta = \langle s, \nu(M_1) \rangle_{\mathbb{R}^2},$$

where $\nu(M_1)$ denotes the outward unit normal to M_1. Basically η measures the starshapedness of M_1 with respect to the origin in $\mathbb{R}^2$. Now let us suppose that η vanishes at some point $\bar{s} \in M_1$; therefore the curvature $K(M_1)$ of M_1 would be non-positive at the same point $\bar{s} \in M_1$. We can compute

$$K(M_1) = \frac{g_2^2 g_{11} - 2 g_1 g_2 g_{12} + g_1^2 g_{22}}{|\nabla g|^3} = \frac{D_2}{|\nabla g|^3}$$

and

$$\eta = \langle s, \nu(M_1) \rangle_{\mathbb{R}^2} = \frac{s_1 g_1 + s_2 g_2}{|\nabla g|} = \frac{1}{2} \frac{\widetilde{\lambda}}{|\nabla g|}.$$

Since both $|\nabla g|$ and $|\nabla f|$ never vanish for $s \in M_1$ or $p \in M$, assuming that there exists a point $\bar{p} \in M$ such that $\widetilde{\lambda}(\bar{p}) = 0$ is equivalent of saying that there exists a point $\bar{s} \in M_1$ such that $\eta(\bar{s}) = 0$. Then $D_2(\bar{s}) = D_2(\bar{p}) \leq 0$, and thus by the formula (13) we would have $L \leq 0$ at $\bar{p}$ which is a contradiction.
This proves that $\widetilde{\lambda}$ cannot vanish. Since it must be positive somewhere, it has to be positive everywhere. $\square$

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