Homological approach to problems with jumping non-linearity

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Abstract In this paper, we use a perturbed version of the Rabinowitz-Floer homology to find solutions to PDE's with jumping nonlinearities. As applications, we find branches for the Fucik spectrum for the Laplace equation and for systems on manifolds that fiber over S^1.

1 Introduction and main results

In this paper we will extend the Rabinowitz-Floer homology approach to semilinear problems, in particular we will deal with perturbed functionals. The unperturbed version has been studied in [20] leading to find multiple solutions to many classical problems and to the Dirac equation as in [19]; in fact one observes that the method is mainly effective for problems having a certain homogeneity: this is the reason why in this work, we will introduce a perturbation outside of the Lagrange multiplier action. This approach was first used by P. Rabinowitz [30] for hamiltonian systems in order to find periodic orbits of the Reeb vector field. This idea was then exploited in [8, 9, 10] by including the Lagrange multiplier as a variable and using a Floer type homology in a symplectic setting again, in order to exhibit periodic orbits of the Reeb vector field. This has a kind of parallel in the symplectic geometry terminology which is the leaf-wise intersection problem. A Rabinowitz-Floer homology was defined for that kind of problem in [3] and

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[4]. In our case we complete the construction for general strongly indefinite functionals with main nonlinearity having a *starshaped* potential. We use this method to deal with elliptic problems with jumping nonlinearities of the type

$$-\Delta u = \lambda_+ u_+^p - \lambda_- u_-^p \text{ on } M \tag{1}$$

or infinite dimensional systems of the form (and the related dynamical one)

$$\begin{cases} -\Delta u = \lambda_+ |v|^{q-1} v_+ - \lambda_- |v|^{q-1} v_- + g \text{ in } M \\ -\Delta v = \beta_+ |u|^{p-1} u_+ - \beta_- |u|^{p-1} u_- + f \text{ in } M \end{cases} \tag{2}$$

where we used the notation $u_+ = \max\{0, u\}$, $u_- = u_+ - u$ and here $M = S^1 \times N$ where N is a compact closed manifold: we refer the reader to Section 6 for these kind of applications.
Indeed, this method can be adapted to more general perturbations and non-linearities: here we will show explicitly the computations for the previous mentioned applications; the reader can check that in fact it can be adapted to a greater class of non-linearities.

Now we will briefly recall the notations and definitions we introduced in [20]. So, let E be a Hilbert space and let $\tilde{H} \subset E$ be a dense subspace compactly embedded in E. We consider a linear operator

$$L : \tilde{H} \longrightarrow E$$

invertible and auto-adjoint. Hence L will have a basis of eigenfunctions $\{\varphi_i\}_{i \in \mathbb{Z}}$

$$L(\varphi_i) = \lambda_i \varphi_i$$

with the convention that if $\lambda_i > 0$ then $i > 0$. This allows us to define the unbounded operator $|L|^{\frac{1}{2}}$ in the following way: if

$$u = \sum_{i \in \mathbb{Z}} a_i \varphi_i$$

then

$$L(u) = \sum_{i \in \mathbb{Z}} \lambda_i a_i \varphi_i$$

and therefore

$$|L|^{\frac{1}{2}} u = \sum_{i \in \mathbb{Z}} |\lambda_i|^{\frac{1}{2}} a_i \varphi_i$$

Now we define the space

$$H := \Big\{ u \in E \ : \ \sum_{i\in\mathbb{Z}} |\lambda_i| a_i^2 < \infty \Big\}.$$

We have $\tilde{H} \subset H \subset E$ and by denoting $\langle \cdot, \cdot \rangle$ the inner product in E, we define then the inner product of H as follows

$$\langle u, v \rangle_H = \langle |L|^{\frac{1}{2}} u, |L|^{\frac{1}{2}} v \rangle$$

We obtain the decomposition

$$H = H^+ \oplus H^-$$

where

$$H^- = \overline{span\{\varphi_i, i < 0\}}, \qquad H^+ = \overline{span\{\varphi_i, i > 0\}}$$

We will write

$$u = u^+ + u^-, \qquad \forall\, u \in H$$

according to the previous splitting. We explicitly note that

$$L(u^+ + u^-) = |L|(u^+ - u^-)$$

therefore we will write $\langle Lu, u \rangle$ in place of $\|u^+\|_H^2 + \|u^-\|_H^2$. So, now we consider the following functional defined on $\mathcal{H} = H \times \mathbb{R}$ by

$$I(u, \lambda) = \frac{1}{2} \langle Lu, u \rangle - \lambda F(u) - G(u)$$

where $F, G : E \to \mathbb{R}$ are C^2 functions with the following properties:

(F1) $|L|^{-1} \nabla F$ and $|L|^{-1} \nabla G$ are compact;

(F2) the set $S = \{u \in E \ s.t. \ F(u) = 0\}$ bounds a strictly starshaped bounded domain in E;

(G) The maps $u \to \langle \nabla G(u), u \rangle$ and $u \to G(u)$ are bounded on S.

In the following, if $(F2)$ holds, we will simply say that S is a bounded strictly starshaped surface and F is a starshaped potential. Our main result is the following

Theorem 1.1. *If F and G satisfy the hypotheses $(F1)$, $(F2)$ and (G) then the Rabinowitz-Floer homology $H_*(I)$ is well defined. Moreover*

$$H_*(I) = 0$$

If in addition I is S^1-equivariant, then we have

$$H_*^{S^1}(I) = H_*(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z}_2 & \text{if } * \text{ is even} \\ \\ 0 & \text{otherwise} \end{cases}$$

Remark 1.2. *From the proof of the result (see Section 6 also) we will show that in particular the homology we compute depends strongly on the set S in the hypothesis $(F2)$, rather than the defining function F; therefore we could use the notation $H_*(S)$ as well.*

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2 Relative index and moduli space of trajectories

First of all, we note explicitly that critical points of I satisfy the following equations:

$$\begin{cases} Lu = \lambda \nabla F(u) + \nabla G(u) \\ \\ F(u) = 0 \end{cases} \tag{3}$$

Moreover if we compute the Hessian of I at a critical point (λ, u), we get

$$Hess(I)(u,\lambda) = \begin{pmatrix} |L|^{-1}L - \lambda|L|^{-1}\nabla^2 F(u) - |L|^{-1}\nabla^2 G(u) & -|L|^{-1}\nabla F(u) \\ \\ -|L|^{-1}\nabla F(u) & 0 \end{pmatrix}$$

Hence the index and co-index of the critical point are infinite. So we need to introduce an alternative way of grading (as in [2] for instance).

Definition 2.1. *Consider two closed subspaces V and W of a Hilbert space E. We say that V is a compact perturbation of W if $P_V - P_W$ is a compact operator.*

P_V in the previous definition denote the orthogonal projection on V. Now, if V is a compact perturbation of W, we can define the relative dimension as

$$dim(V, W) = dim(V \cap W^\perp) - dim(V^\perp \cap W).$$

One can check that it is well defined and finite. Now if we have three subspaces V, W and U such that V and W are compact perturbations of U. Then V is also a compact perturbation of W and

$$dim(V, W) = dim(V, U) + dim(U, W).$$

Using this concept of relative dimension we can define a relative index as our grading.

Definition 2.2. *We denote by $V^-(u, \lambda)$ the closure of the span of the eigenfunction of the Hessian of I at a critical point (u, λ), corresponding to negative eigenvalues.*
The relative index is defined as

$$i_{rel}(u, \lambda) = dim(V^-(u, \lambda), H^- \times \mathbb{R})$$

Lemma 2.3. *If I is Morse and $(F1)$ holds then the relative index is well defined for critical points of I.*

Proof. Let $\Gamma = H^- \times \mathbb{R}$, and (u, λ) a critical point of I. The operator

$$v \mapsto Lv - \lambda \nabla^2 F(u)v - \nabla^2 G(u)v$$

has discrete spectrum since $|L|^{-1}$ is a compact operator. Then $V^-(u, \lambda)$ is well defined and it is a compact perturbation of Γ. This follows from the fact that

$$\begin{pmatrix} |L|^{-1}L & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} |L|^{-1}L - \lambda |L|^{-1}\nabla^2 F(u) - |L|^{-1}\nabla^2 G(u) & |L|^{-1}\nabla F(u) \\ |L|^{-1}\nabla F(u) & 0 \end{pmatrix},$$

is a compact operator. $\square$

We define now the moduli space of $\mathcal{H}$-gradient trajectories. Let us consider the following differential system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u^- - u^+ + \lambda |L|^{-1}\nabla F(u) + |L|^{-1}\nabla G(u) \\ \\ \dfrac{\partial \lambda}{\partial t} = F(u) \end{cases} \tag{4}$$

This system is in fact the descending gradient flow of our functional in $\mathcal{H} = H \times \mathbb{R}$ and since the right hand side is C^1, then we have local existence of the flow.
Now, given two critical points $z_0 = (u_0, \lambda_0)$ and $z_1 = (u_1, \lambda_1)$ such that

$$I(z_i) \in [a, b], \qquad \text{for} \quad i = 0, 1$$

we define the space of connecting orbits from z_0 to z_1 by

$$\mathbb{M}^{a,b}(z_0, z_1) = \left\{ z = (u, \lambda) \in C^1(\mathbb{R}, \mathcal{H}) \text{ satisfying (4) with } z(-\infty) = z_0 \, , z(+\infty) = z_1 \right\}$$

where we have denoted by

$$z(-\infty) := \lim_{t \to -\infty} z(t), \qquad z(+\infty) := \lim_{t \to +\infty} z(t)$$

The moduli space of trajectories is then defined by

$$\mathcal{M}^{a,b}(z_0, z_1) = \mathbb{M}^{a,b}(z_0, z_1)/\mathbb{R}.$$

Proposition 2.4. *Assume that $i_{rel}(z_0) > i_{rel}(z_1)$, then if I is Morse-Smale,*

$$dim\Big(\mathcal{M}^{a,b}(z_0, z_1)\Big) = i_{rel}(z_0) - i_{rel}(z_1) - 1$$

Proof. We first note that $\mathbb{M}^{a,b}(z_0, z_1) = \mathcal{F}^{-1}(0)$ where

$$\mathcal{F} : C^1(\mathbb{R}, \mathcal{H}) \mapsto \mathcal{Q}^0 = C^0(\mathbb{R}, \mathcal{H})$$

is defined by

$$\mathcal{F}(z) = \frac{dz}{dt} + \nabla I(z)$$

We will use the implicit function theorem to prove our result: we need to show that the linearized operator of $\mathcal{F}$ is Fredholm and onto. The linearized operator corresponds to

$$\partial \mathcal{F}(z) = \frac{d}{dt} + Hess(I(z))$$

and this is a linear differential equation in the Banach space $\mathcal{H}$ (see [1]). In order to show that it is Fredholm, we first notice that

$$Hess(I(z)) = \begin{pmatrix} |L|^{-1}L & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\lambda|L|^{-1}\nabla^2 F(u) - |L|^{-1}\nabla^2 G(u) & -|L|^{-1}\nabla F(u) \\ -|L|^{-1}\nabla F(u) & -1 \end{pmatrix}.$$

Now, the operator

$$\begin{pmatrix} |L|^{-1}L & 0 \\ 0 & 1 \end{pmatrix}$$

is time independent and hyperbolic, while the operator

$$\begin{pmatrix} -\lambda|L|^{-1}\nabla^2 F(u) - |L|^{-1}\nabla^2 G(u) & -|L|^{-1}\nabla F(u) \\ -|L|^{-1}\nabla F(u) & -1 \end{pmatrix}$$

is compact. Hence we have that $\partial\mathcal{F}$ is a Fredholm operator with index

$$ind(\partial\mathcal{F}(z)) = dim(V^-(\mathcal{F}(z_0), V^-(\mathcal{F}(z_1))$$

$$= dim(V^-(\mathcal{F}(z_0), \Gamma) + dim(\Gamma, V^-(\mathcal{F}(z_1))$$

$$= i_{rel}(z_0) - i_{rel}(z_1).$$

Moreover, from [1], we have also that $\partial\mathcal{F}(z)$ is onto if and only if the intersection is transverse.
To finish the proof now, it is enough to notice that the action of $\mathbb{R}$ is free and hence we can mod out by that action to get the desired result. $\square$

3 Compactness

3.1 Palais-Smale condition and compactification of the moduli spaces

We recall that a functional I is said to satisfies the Palais-Smale condition (PS), at the level c, if every sequence (z_k) such that

$$I(z_k) \longrightarrow c$$

and

$$\nabla I(z_k) \longrightarrow 0$$

as $k \to \infty$, has a convergent subsequence. We will say that I satisfies (PS) if the previous condition is satisfied for all $c \in \mathbb{R}$.

Proposition 3.1. *Under the assumption $(F1)$, $(F2)$ and (G), I satisfies the (PS) condition.*

Proof. Let $z_k = (u_k, \lambda_k)$ be a (PS) sequence at the level c, that is

$$\begin{cases} Lu_k - \lambda_k \nabla F(u_k) - \nabla G(u_k) = \varepsilon_k \\ \\ F(u_k) = \varepsilon_k \end{cases}$$

and

$$I(u_k, \lambda_k) = c + \varepsilon_k$$

Thus if we consider

$$\langle Lu_k - \lambda_k \nabla F(u_k) - \nabla G(u_k), u_k \rangle - 2I(u_k, \lambda_k)$$

we get

$$\lambda_k \langle \nabla F(u_k), u_k \rangle = 2c + \langle \nabla G(u_k), u_k \rangle - 2G(u_k) + \varepsilon_k(\|u_k\|_H + 2|\lambda_k|)$$

Since, $F(u_k) \to 0$ and S is strictly starshaped, there exists $a > 0$ such that

$$\langle \nabla F(u_k), u_k \rangle > a.$$

Also by (G), there exists $C > 0$ such that

$$|\langle \nabla G(u_k), u_k \rangle - 2G(u_k)| \leq C.$$

Thus

$$|\lambda_k| = C + \varepsilon_k(\|u_k\|_H + |\lambda_k|) \tag{5}$$

and we have

$$\langle Lu_k - \lambda_k \nabla F(u_k) - \nabla G(u_k), u_k^+ \rangle = \|u_k^+\|_H^2 - \lambda_k \langle \nabla F(u_k), u_k^+ \rangle - \langle \nabla G(u_k), u_k^+ \rangle$$

and

$$\langle Lu_k - \lambda_k \nabla F(u_k) - \nabla G(u_k), u_k^- \rangle = -\|u_k^-\|_H^2 - \lambda_k \langle \nabla F(u_k), u_k^- \rangle - \langle \nabla G(u_k), u_k^- \rangle$$

Hence,

$$\|u_k\|_H^2 = \lambda_k \langle \nabla F(u_k), u_k^+ - u_k^- \rangle + \langle \nabla G(u_k), u_k^+ - u_k^- \rangle + \varepsilon_k \|u_k\|_H$$

therefore

$$\|u_k\|_H^2 \leq C \left(|\lambda_k| \|\nabla F(u_k)\| + \|\nabla G(u_k)\| \right) \|u_k\| + \varepsilon_k \|u_k\|_H$$

Now, since S is a bounded hypersurface, we have that u_k is bounded in E thus

$$\|u_k\|_H \leq C|\lambda_k| + \varepsilon_k \tag{6}$$

By adding (6) and (5), one has

$$\|u_k\|_H + |\lambda_k| \leq C + C\varepsilon_k(\|u_k\|_H + |\lambda_k|) + \varepsilon_k$$

Thus both u_k and λ_k are bounded. Hence we can extract a weakly convergent subsequence of u_k and a convergent subsequence of λ_k. That is $\lambda_k \to \lambda$ and $u_k \rightharpoonup u$. Now notice since $|L|^{-1}\nabla F$ and $|L|^{-1}\nabla G$ are compact we have that

$$\|u_k^+\|_H^2 \to \lambda\langle \nabla F(u), u^+\rangle + \langle \nabla G(u), u^+\rangle$$

similarly

$$\|u_k^-\|_H^2 \to -\lambda\langle \nabla F(u), u^-\rangle - \langle \nabla G(u), u^-\rangle$$

But again,

$$\langle Lu_k - \lambda_k \nabla F(u_k) - \nabla G(u_k), u^+\rangle \to \|u^+\|_H^2 - \langle \lambda \nabla F(u) - \nabla G(u), u^+\rangle$$

and

$$\langle Lu_k - \lambda_k \nabla F(u_k) - \nabla G(u_k), u^-\rangle \to -\|u^-\|_H^2 - \langle \lambda \nabla F(u) - \nabla G(u), u^-\rangle$$

Combining both we get

$$\|u\|_H^2 = \langle \lambda \nabla F(u) - \nabla G(u), u^+ - u^-\rangle$$

Hence we have the convergence in norm of u_k and thus (PS) holds.

□

In fact following the previous proof, we have proved the following

Lemma 3.2. *If*

$$\|\nabla I(u,\lambda)\| \leq \varepsilon$$

and

$$|I(u,\lambda)| \leq M$$

for some $\varepsilon, M > 0$, *then there exists* $C = C(M) > 0$ *such that*

$$|\lambda| \leq C$$

and

$$\|u\|_H \leq C.$$

Next we state a uniform boundedness on the flow lines as in [8], for instance.

Proposition 3.3. *Under hypotheses* $(F1)$, $(F2)$ *and* (G), *the flow lines between critical points (or more precisely the moduli space) are uniformly bounded.*

The proof of the previous Proposition is similar to the one in [20] (Proposition 3.3.), since the presence of the perturbation G does not play a crucial role: for this reason we will omit it.

Proposition 3.4. *There exists a space* $\hat{H}$ *that embeds compactly in* H *such that any gradient flow-line is bounded in*

$$\hat{\mathcal{H}} = \hat{H} \times \mathbb{R}$$

Proof. We recall again that the space H is characterized by

$$\|u\|_H^2 = \sum_{i\in\mathbb{Z}} |a_i|^2 |\lambda_i| < \infty$$

for any $u \in E$. In a similar way, one can define $\hat{H}$ to be the set of vectors $u \in E$ such that

$$\|u\|_{\hat{H}}^2 = \sum_{i\in\mathbb{Z}} |a_i|^2 |\lambda_i|^2 < \infty$$

Now let $\tilde{G}$ be the fundamental solution of the operator

$$\frac{d}{dt} + P_+ - P_-$$

where $P_\pm$ is the projection on $H^\pm$, then from the equation of the flow, we have that

$$u(t) = \int_{-\infty}^{+\infty} \tilde{G}(t-s)\left(\lambda(s)|L|^{-1}\nabla F(u) + |L|^{-1}\nabla G(u)\right) ds$$

and since ∇F and ∇G map H to E, we have that

$$|L|^{-1}\nabla F \in \hat{H}, \qquad |L|^{-1}\nabla G \in \hat{H}$$

and therefore $u \in \hat{H}$. $\square$

From the previous proposition we have that the moduli spaces are modeled on the affine spaces

$$\mathcal{Q}^1(z_0, z_1) = \tilde{z} + C_0^1(\mathbb{R}, \hat{\mathcal{H}})$$

where $\tilde{z}$ is a flow line between z_0 and z_1. We will consider the map

$$ev : \mathcal{M}(z_0, z_1) \longmapsto \hat{\mathcal{H}}$$

defined by $ev(z) = z(0)$. This map is onto and hence the set $\mathcal{M}(z_0, z_1)$ is precompact.

Now the arguments for the compactification by broken trajectories follow the same construction in [20] (Subsection 3.1.1), therefore we refer the reader to [20] for the details.

4 Construction of the homology

In this section we will define the different chain complexes and their homologies. We will give an explicit computation later on, under specific assumptions.

Let F and G be functions satisfying $(F1)$, $(F2)$ and (G), and let I denote the related energy functional. For $a < b$ we define the critical sets

$$Crit_k^{[a,b]}(I)$$

as the set of critical points of I with energy in the interval $[a, b]$ and relative index k.

We notice that if I is Morse and satisfies (PS) (which we can always assume as we will see later on), then $Crit_k^{[a,b]}(I)$ is always finite. Now we define the chain complex $C_k^{[a,b]}(I)$ as the vector space over $\mathbb{Z}_2$ generated by $Crit_k^{[a,b]}(I)$, for every $k \in \mathbb{Z}$. That is

$$C_k^{[a,b]}(I) = Crit_k^{[a,b]}(I) \otimes \mathbb{Z}_2$$

The boundary operator ∂ is defined for any $z \in Crit_k^{[a,b]}(I)$ by

$$\partial z = \sum_{y \in Crit_{k-1}^{[a,b]}(I)} \Big(\sharp \mathcal{M}(z, y) mod[2] \Big) y$$

Using the compactness results of the previous subsections we have that

$$\partial^2 = 0$$

and therefore $(C_*^{[a,b]}(I), \partial)$ is indeed a chain complex and we will denote it by

$$H_*^{[a,b]}(I) = H_*(C_*^{[a,b]}(I), \partial)$$

its homology.

4.0.1 The S^1-equivariant case

Here we will define the equivariant homology for a particular group action, namely the S^1 action. Hence in this case, we assume F and G to be S^1-invariant, that is

$$F(e^{i\theta}u) = F(u), \qquad G(e^{i\theta}u) = G(u) \qquad \theta \in \mathbb{R}$$

We define the critical groups by

$$C_k^{[a,b],S^1}(I) = \frac{Crit_k^{[a,b]}(I)}{S^1} \otimes \mathbb{Z}_2$$

We notice that this definition makes sense since here the critical points are in fact critical circles, since I is equivariant. Now, by breaking the symmetry perturbing the functional, each

$$z_k \in \frac{Crit_k^{[a,b]}(I)}{S^1}$$

splits into a max and a min respectively

$$z_k^+ \in Crit_{k+1}^{[a,b]}(\tilde{I}), \qquad z_k^- \in Crit_k^{[a,b]}(\tilde{I})$$

where $\tilde{I}$ is the perturbed functional. Hence we define for any

$$z_{k+1} \in \frac{Crit_{k+1}^{[a,b]}(I)}{S^1}$$

the boundary operator

$$\partial_{S^1} z_{k+1} = \sum_{z_k \in \frac{Crit_k^{[a,b]}(I)}{S^1}} \sharp\Big(\mathcal{M}(z_{k+1}^+, z_k^+) mod[2]\Big) z_k$$

We see that ∂_{S^1} is well defined; now we need to show that indeed it is a boundary operator.

Lemma 4.1. *We have*

$$\partial_{S^1}^2 = 0$$

Proof. First we define the following chain complex

$$\overline{C}_k = \bigoplus_{z_k \in \frac{Crit_k^{[a,b]}(F_B)}{S^1}} (z_k^+, z_k^-) \otimes \mathbb{Z}_2$$

with the following boundary operator

$$\overline{\partial}(z_{k+1}^{+}, z_{k+1}^{-}) = \sum_{z_k \in \frac{Crit_k^{[a,b]}(F_B)}{S^1}} (< z_{k+1}^{+}, z_k^{+} > z_k^{+}, < z_{k+1}^{-}, z_k^{-} > z_k^{-})$$

where we put

$$< x, y >= \sharp\Big(\mathcal{M}(x,y) mod[2]\Big)$$

We claim that $\overline{\partial}^2 = 0$, so that $(\overline{C}_*, \overline{\partial})$ is a chain complex. Indeed this claim is the same of the one in [20] (Lemma 4.1.), so we will omit its proof.
Next, we consider the map

$$f_* : \overline{C}_* \longrightarrow C_*^{[a,b],S^1}(I)$$

defined by

$$f_*((z_k^{+}, z_k^{-}) = z_k$$

We notice that f is well defined and it is an isomorphism. By the S^1 action we have that

$$< z_{k+1}^{+}, z_k^{+} >=< z_{k+1}^{-}, z_k^{-} >$$

and we obtain that

$$\partial_{S^1} = f^{-1} \circ \overline{\partial} \circ f$$

Which completes the proof of the lemma. □

5 Stability and transversality

5.1 Stability

In this section we will consider, for a fixed function F, two functions G_1 and G_2 satisfying (G) and we will show that under suitable conditions

$$H_*(I_1) = H_*(I_2)$$

where we called I_i the energy functional related to G_i, with $i = 1, 2$. The proof will be done in the general case and there is absolutely no difference in the equivariant case since all the perturbations can be taken to be equivariant. So, let η be a smooth function on $\mathbb{R}$ such that

$$\begin{cases} \eta(t) = 1, & t \geq 1 \\ \eta(t) = 0, & t \leq 0 \end{cases}$$

$$G_t = (1 - \eta(t))G_1 + \eta(t)G_2$$

and for a fixed $t \in \mathbb{R}$, we will denote by I_t the energy functional related to G_t. Now we define the non-autonomous gradient flow by

$$z'(t) = -\nabla I_t(z(t)),$$

where ∇I_t is the gradient with respect to z for a fixed t. Given z_1 a critical point of I_1 and z_2 a critical point of I_2, we let $z(t)$ be the flow line from z_1 to z_2.

Lemma 5.1. *Assume that $G_1 - G_2$ is bounded then $z(t)$ is uniformly bounded by a constant depending only on z_1 and z_2.*

Proof. Here again one needs to worry about the boundedness of λ along the flow. First we notice that

$$\frac{\partial I_t(z(t))}{\partial t} = -\|z'(t)\|^2 + \eta'(t)\,(G_1 - G_2)$$

Therefore, we have

$$I_t(z(t)) \leq I_1(z_1) + C\int_0^t |\eta'(s)|ds$$

and

$$\int_{-\infty}^{+\infty} \|z'(t)\|^2 dt \leq I_2(z_2) - I_1(z_1) + C.$$

So we consider the following function:

$$\tau(s) = \inf\{t \geq 0; ||\nabla I_t(z(t+s)|| \geq \varepsilon\} \tag{7}$$

where ε is as in Lemma (3.2), where I has to be replaced by I_t. We need a bound for this last one. We have

$$\int_{-\infty}^{+\infty} \|\nabla I_t(z)(t)\|^2 dt \leq I_2(z_2) - I_1(z_1) + C$$

hence

$$\int_s^{s+\tau(s)} \|\nabla I_t(z)(t)\|^2 dt \leq I_2(z_2) - I_1(z_1) + C$$

Thus

$$\varepsilon^2 \tau(s) \leq I_2(z_2) - I_1(z_1) + C$$

Now

$$\lambda(s) = \lambda(s + \tau(s)) - \int_s^{s+\tau(s)} \lambda'(t)dt$$

and

$$|\lambda(s)| \le C + \sqrt{\tau(s)}\Big(I_2(z_2) - I_1(z_1) + C\Big)^{\frac{1}{2}}$$

This leads to

$$|\lambda(s)| \le C + \frac{I_2(z_2) - I_1(z_1) + C}{\varepsilon}$$

Therefore we have the uniform bound for λ.
Similarly $\|u\|_H$ is uniformly bounded. In fact,

$$\|u(s)\|_H \le \|u(s + \tau(s))\|_H + \frac{I_2(z_2) - I_1(z_1) + C}{\varepsilon}$$

and since $\|u(s+\tau(s))\|_H \le C$ and λ is bounded, we have the desired result.

□

Now, as in the autonomous case, this uniform boundedness implies pre-compactness, therefore we can define the moduli space of trajectories of the non-autonomous gradient flow,

$$\mathbb{M}(z_1, z_2)$$

and we omit the similar gluing construction that can be done to compactify it. In fact, we can show that it is a finite dimensional manifold with

$$dim\Big(\mathbb{M}(z_1, z_2)\Big) = i_{rel}(z_1) - i_{rel}(z_2)$$

Moreover, if

$$i_{rel}(z_1) - i_{rel}(z_2) = 1$$

we have that

$$\partial\mathbb{M}(z_1, z_2) = \bigcup_{x \in Crit_{i_{rel}(z_2)}(I_1)} \mathbb{M}(z_1, x) \times \mathbb{M}(x, z_2)$$

$$\bigcup_{y \in Crit_{i_{rel}(z_1)}(I_2)} \mathbb{M}(z_1, y) \times \mathbb{M}(y, z_2)$$

With this in mind we can construct the continuation isomorphism

$$\Phi_{12} : C_*(I_1) \longrightarrow C_*(I_2)$$

defined at the chain level by

$$\Phi_{12}(z) = \sum_{x \in Crit_{i_{rel}(z)}(I_2)} (\sharp \mathbb{M}(z,x) mod[2])x$$

By the previous remark on the boundary of the moduli space in the non-autonomous case, one sees that

$$\partial_1 \Phi_{12} + \Phi_{12} \partial_2 = 0$$

this shows that it is a chain homomorphism, hence it descends at the homology level. The last thing to check is that it is an isomorphism, by taking a homotopy of homotopies (see for instance Schwarz [33]). Finally we have the following result

Corollary 5.2. *Assume that F, G_1 and G_2 satisfy the assumptions $(F1)$, $(F2)$ and (G), such that $G_1 - G_2$ is bounded. Then*

$$H_*(I_1) = H_*(I_2)$$

As we said before, the same stability results hold for the equivariant cases.

5.2 Transversality

In this section we will show that up to a small and smooth perturbation of F we can always assume that I is Morse. Then, it can be approximated by a Morse-Smale functional with the same critical points and the same connections.

Lemma 5.3. *Consider two functions F and G satisfying $(F1)$, $(F2)$ and (G), then for a generic perturbation K in $C_0^3(E)$, the energy functional $\tilde{I}$ related to $G + K$ is Morse.*

Proof. We consider the functional

$$\psi : \mathcal{H} \times C_0^3(E) \longrightarrow \mathcal{H}$$

defined by

$$\psi(z, K) = \nabla \tilde{I}(z)$$

Let us notice first that the inverse image of zero corresponds to critical points of the functional related to $G+K$. Also, for $(z, K) \in \psi^{-1}(0)$ we have

$$\partial_z \psi(z, K)v = Hess(\tilde{I}(z))v,$$

which is a perturbation of a compact operator, and hence it is a Fredholm operator of index zero. Now it remains to show that $\nabla\psi(z,K)$ is surjective. So, let us compute the differential with respect to K:

$$\partial_K\psi(z,K)(H,0) = \begin{pmatrix} -|L|^{-1}\nabla H(u) \\ 0 \end{pmatrix}$$

Therefore, for the functional component of the space, we see that by taking

$$H(u) = \langle f, u\rangle$$

then we have that the range of the first component is dense since f can be any function of E and the operator $|L|^{-1}$ maps E to a dense subspace. Now for the scalar component, we consider the differential with respect to u for $(z,K) \in \psi^{-1}(0)$:

$$\partial_\lambda\psi(z,K)(0,\alpha) = \begin{pmatrix} 0 \\ \alpha|L|^{-1}\nabla F(u) \end{pmatrix}$$

with $\alpha \in \mathbb{R}$. Since S is starshaped, we have that $\nabla F \neq 0$ for $u \in S$. Thus we have the surjectivity. Therefore by the transversality theorem, 0 is a regular point of $\psi(\cdot, K)$ for a generic K and this is equivalent to say that $\tilde{I}$ is Morse. □

Notice also that the perturbation K can be chosen to be S^1-equivariant if F and G are so.

Lemma 5.4. *Assume that I is Morse and satisfies (PS) in $[a,b]$, then for every $\varepsilon > 0$ there exists a functional I^ε such that*

(i) $\|I - I^\varepsilon\|_{C^2} < \varepsilon$

(ii) *I^ε satisfies (PS) in $[a-\varepsilon, b+\varepsilon]$*

(iii) *I^ε has the same critical points than I with the same connections (number of connecting orbits).*

The proof of this result is similar to the one in [2] for that it will be omitted.

6 Computation of the homology and applications

Here we point out that in fact the most important assumption on F, is the hypothesis (F2). Indeed, F needs to have the geometry of a local minimum around 0. In fact, the homology that we compute is intrinsically related to the surface S since the critical points will be located on S itself: given F, if G_1, G_2 are bounded functions on S and if we use the notation $H_*(S)$ instead of $H_*(I)$, then $H_*(S) = H_*(I_1) = H_*(I_2)$. This can be done using a cut off function η: if the hypersurface S is located in the ball $B_A(0)$ we choose η such that

$$\eta(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq \sqrt{A} \\ 0 \text{ for } t > 2\sqrt{A} \end{cases}$$

Thus we consider the modified perturbation

$$G_\eta = \eta(\|u\|^2) G_2(u) + (1 - \eta(\|u\|^2)) G_1(u).$$

Then G_η is bounded. Hence if we are interested in critical points in a specific surface S, we can disregard the behavior of the perturbation outside a bounded set and make it equal to a reference functional that we know how to compute its homology and this is the main trick in order to compute the homology.

Using this fact, it is enough to know the value of the homology $H_*(I_0)$, where I_0 corresponds to the unperturbed functional (that is $G_0 \equiv 0$), as computed in [20] to obtain the main result.

Moreover, if the functional has some invariance, the homology is richer and has infinitely many generators which yield infinitely many solutions to the desired equations. On the other hand, in the case of vanishing of the homology, one needs to consider the local version of it, that is one has to restrict the energy on an interval $[a, b]$. By following the variation of the energy along the perturbation one can exhibit the existence of at least one solution, see for instance [19].

Now we present some examples of PDEs and systems for which one can apply our previous results to get existence and multiplicity of solutions.We observe that as in [20] here we will consider only examples in which the relevant operator L has unbounded spectrum from above and below, which is the interesting case for our methods; however our results apply to operator such as laplacian, bilaplacian or sublaplacian as well, giving rise to the usual Morse homology: for instance, we address the reader to the papers [17], [15], [16], [24], [22], [31], [21], [26], [23], [25] and the reference therein, for other

kind of methods to obtain different type of existence and multiplicity results, even with perturbations.
In the following two examples we will consider a manifold $M = S^1 \times N$ where N is a compact closed manifold. Thus, we can define the S^1 action on functions by translation on the first component. That is,

$$e^{it} \cdot u(e^{is}, y) = u(e^{i(s+t)}, y)$$

6.1 Non-linear Fucik spectrum

We start here by taking a simple differential operator. We consider the problem

$$-\Delta u = \lambda_+ u_+^p - \lambda_- u_-^p, \text{ on } M \tag{8}$$

for $1 \leq p < \frac{n+2}{n-2}$. This type of problem was heavily investigated since the nonlinearity has different behavior at $+\infty$ and $-\infty$ (see for instance [13] and the references therein). We notice that in our case, we take p to be subcritical to insure compactness of the embedding $H^1(M) \subset L^{p+1}(M)$; the critical case, $p = \frac{n+2}{n-2}$ presents other phenomena such as bubbling and concentration: however under the assumption of the symmetry of the domain (for instance when there is a group action for which also the operator is invariant), one could try to get the compactness on some subspace (as in [23, 25], for instance) and then extend these techniques even to the critical case. Regarding (8) we will take the operator $L = -\Delta$,

$$F(u) = \frac{1}{p+1} \int_M (|u|^{p+1} - 1) dx \tag{9}$$

and

$$G(u) = \frac{\alpha}{p+1} \int_M |u_+|^{p+1} dx \tag{10}$$

with any non-zero real parameter α. Then we have the following byproduct of the main Theorem 1.1:

Corollary 6.1. *Let $M = S^1 \times N$ where N is a compact closed manifold and let us consider the problem (8) with F and G given by (9) and (10) respectively. Therefore we have the existence of an infinite sequence of solutions $(\lambda_k, u_k)_{k \geq 0}$ such that*

$$-\Delta u_k = \lambda_{+,k} |u_k|^{p-1} u_{+,k} - \lambda_{-,k} |u_k|^{p-1} u_{-,k}$$

where

$$\lambda_{+,k} = \lambda_k + \alpha, \qquad \lambda_{-,k} = \lambda_k$$

In fact, we first notice that the assumptions $(F1), (F2), (G)$ are fulfilled; in particular the choice of the exponent p ensures the compactness hypothesis. Then, since the manifold M is a product $S^1 \times N$, it is easy to see that the functional I is invariant under the S^1 action on u. Therefore the thesis follows by the considerations in the previous sections.
We also notice that in the case $p = 1$. we are dealing with the classical Fucik spectrum (see [32, 27, 29]), namely we find a sequence of solutions to the problem

$$-\Delta u = \lambda_+ u_+ - \lambda_- u_-$$

One can also consider problems with different kinds of non-linearities, for instance different exponents on p and q for F and G as long as $1 \leq q \leq p < \frac{n+2}{n-2}$, in order to ensure the compactness of the Sobolev embedding as stated above and the boundedness of G.

6.2 Systems of elliptic equations with jumping nonlinearities

Let us consider again $M = S^1 \times N$ as before and let us define the following elliptic system:

$$\left\{ \begin{array}{l} -\Delta u = \lambda_+ |v|^{q-1} v_+ - \lambda_- |v|^{q-1} v_-, \text{ on } M \\ -\Delta v = \beta_+ |u|^{p-1} u_+ - \beta_- |u|^{p-1} u_-, \text{ on } M \end{array} \right. \tag{11}$$

We take

$$L = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix}$$

so this operator have a discrete unbounded spectrum from above and below. Next we define

$$F(u,v) = \int_M \left(\frac{1}{p+1} |u|^{p+1} + \frac{1}{q+1} |v|^{q+1} - 1 \right) dx \tag{12}$$

and

$$G(u,v) = \int_M \frac{\alpha_1}{p+1} |u_+|^{p+1} + \frac{\alpha_2}{q+1} |v_+|^{q+1} dx \tag{13}$$

with $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$ and non-zero real parameters α_1, α_2 (see [18, 5, 14]). We have

Corollary 6.2. *Let $M = S^1 \times N$ where N is a compact closed manifold and let us consider the problem (11) with F and G given by (12) and (13) respectively. Therefore this system admits infinitely many solutions with energy going to infinity.*

Indeed, even in this case, the level set $\{F = 0\}$ bounds a *spherical* domain so in particular it is strictly starshaped and thus (F2) holds; moreover from the restriction on p and q we have that assumptions $(F1)$ and (G) hold as well, so that the results follows again.

We observe that one can also consider the non-homogeneous case by taking

$$G(u,v) = \int_M \frac{\alpha_1}{p+1}|u_+|^{p+1} + \frac{\alpha_2}{q+1}|v_+|^{q+1} + fu + gvdx$$

for some suitable functions f, g (for instance in L^∞). This will give a sequence of solutions to the problem

$$\begin{cases} -\Delta u = \lambda_+|v|^{q-1}v_+ - \lambda_-|v|^{q-1}v_- + g, \text{ on } M \\ -\Delta v = \beta_+|u|^{p-1}u_+ - \beta_-|u|^{p-1}u_- + f, \text{ on } M \end{cases} \tag{14}$$

Also, by taking $p = q = 1$ and $\alpha_1 = 0$ then one has a sequence of solutions to the Fucik spectrum of the bi-Laplacian (see [28] for a similar result on a fourth order problem): that is

$$\Delta^2 v = \lambda_+ v_+ - \lambda_- v_-$$

6.3 An infinite dimensional dynamical system with jumping nonlinearities

We consider here any compact closed manifold N as domain for the space variables (on which we apply the laplacian) and we propose to find periodic solutions (in time) to the following infinite dimensional dynamical system:

$$\begin{cases} \frac{\partial}{\partial t}u - \Delta u = \lambda_+|v|^{q-1}v_+ - \lambda_-|v|^{q-1}v_- + g, \text{ on } S^1 \times N \\ -\frac{\partial}{\partial t}v - \Delta v = \beta_+|u|^{p-1}u_+ - \beta_-|u|^{p-1}u_- + f, \text{ on } S^1 \times N \end{cases} \tag{15}$$

for

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{n}{n+2}.$$

We define then

$$F(u,v) = \int_{S^1 \times N} \left(\frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1} - 1 \right) dt\, dx \tag{16}$$

and

$$G(u,v) = \int_{S^1 \times N} \frac{\alpha_1}{p+1}|u_+|^{p+1} + \frac{\alpha_2}{q+1}|v_+|^{q+1} + fu + gvdt\, dx \tag{17}$$

with non-zero real parameters α_1, α_2 and for functions $f, g \in L^\infty(S^1 \times N)$. We have

Corollary 6.3. *Let us consider the problem (15) with F and G given by (16) and (17) respectively. Therefore there is a sequence of periodic solutions with energy going to infinity.*

This type of problems has been deeply investigated in the case of power type non-linearity, we cite for example the works [11], [12], [6], [7] and the references therein: however, to the best of our knowledge, we do not know any references solving this type of problems with jumping non-linearity. In order to prove our statement, we have in this case to define the operator

$$L = \begin{pmatrix} 0 & \frac{\partial}{\partial t} - \Delta \\ -\frac{\partial}{\partial t} - \Delta & 0 \end{pmatrix}.$$

This is an unbounded operator on $L^2(S^1 \times N)$ and it is auto-adjoint with spectrum

$$\sigma(L) = \{\pm\sqrt{j^2 + \lambda_k^2}; j \in \mathbb{Z}, \lambda_k \in \sigma(-\Delta), k \in \mathbb{N}\}$$

The corresponding eigenfunctions, in complex notations, are of the form

$$\psi_{j,k} = e^{ijt}\varphi_k(x)$$

where the φ_k are eigenfunctions of the Laplace operator on N. The natural space of functions to consider is then

$$H = \{u = \sum_{k,j} u_{k,j}\psi_{k,j} \in L^2(S^1 \times N); \sum_{k,j} |j^2 + \lambda_k^2|^{\frac{1}{4}} u_{k,j}\psi_{k,j} \in L^2(S^1 \times N)\}$$

Then we again notice that the assumptions $(F1), (F2), (G)$ are fulfilled; in particular the assumption on p, q ensures the compactness hypothesis. Then, since in this case the group action of S^1 is explicit, the thesis follows again by the considerations in the previous sections.

Finally, we notice that in particular, this allows us to find periodic solutions to the beam equation with jumping non-linearity by taking $p = 1$ and $f = 0$.

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