Legendre duality on hypersurfaces in Kähler manifolds

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Abstract In this paper we will give a sufficient condition on real strictly Levi-convex hypersurfaces M, embedded in four-dimensional Kähler manifolds V, such that Legendre duality can be performed. We consider the contact form θ on M whose kernel is the restriction of the holomorphic tangent space of V: we will show that if there exists a Legendrian Killing vector field v, then the dual form $\beta(\cdot) := d\theta(v, \cdot)$ is a contact form on M with the same orientation than θ.

1 Introduction

In this paper we will give a sufficient condition on real strictly Levi-convex hypersurfaces, embedded in four-dimensional Kähler manifolds, such that Legendre duality can be performed, as stated by Bahri in some of his works [1], [2] for instance. Here we will consider the contact form whose kernel is the horizontal subspace, namely the restriction of the holomorphic space of the Kähler manifold.

We are going to explain what we mean by Legendre duality. We are motivated by the following fact: the standard contact form α_0 on S^3 is a pull-back from the standard contact form on $P(\mathbb{R}^3)$, that is the unit sphere cotangent bundle of S^2; therefore it is equipped with its canonical Liouville form. The Legendre duality can be completed for the Liouville form. This Legendre transform can be viewed as the data of a vector field v in $ker(\alpha_0)$, such that $\beta_0(\cdot) := d\alpha_0(v, \cdot)$ is a contact form with the same orientation than α_0.

The Legendre transform allows the transformation of a Hamiltonian problem on the cotangent sphere of S^2 into a Lagrangian problem. This duality has been extended by A.Bahri-D.Bennequin (see into the monograph [1], the

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note at pag.5) to the more general framework of contact forms α on three-dimensional compact orientable manifolds without boundary M, leading to a variational problem on a suitable space of curves. In fact, if we consider a non-vanishing tangent vector field $v \in ker(\alpha)$, let us assume that:

(D) *the non-singular dual form $\beta(\cdot) := d\alpha(v, \cdot)$ is a contact form on M with the same orientation than α;*

Condition (D) is what we call Legendre duality for α with respect to v. Now let us define the Action functional

$$A(\gamma) = \int_0^1 \alpha(\dot{\gamma})dt \tag{1}$$

on the subspace of the H^1-loops on M given by

$$C_\beta = \{\gamma \in H^1(S^1; M) \ s.t. \ \beta(\dot{\gamma}) = 0; \ \alpha(\dot{\gamma}) = \text{strictly positive constant}\}$$

If $\xi \in TM$ denotes the Reeb vector field of α, i.e.

$$\alpha(\xi) = 1, \qquad d\alpha(\xi, \cdot) = 0, \tag{2}$$

then the following result by A.Bahri-D.Bennequin holds [1]:

Theorem 1.1. *A is a C^2 functional on C_β whose critical points are periodic orbits of ξ.*

We observe that this construction is "stable under perturbation", that is the same v can be used to complete the Legendre duality for forms $u\alpha$, with $u \in C^2$ a non-vanishing function defined on M and $|u - 1|$ small.
In [12], J.Gonzalo and F.Varela introduced a family $\{\alpha_n\}_{n \in \mathbb{N}}$ of contact forms on S^3, where α_0 is the standard contact form of the sphere and for $n \geq 1$ they are all overtwisted and pairwise not contactomorphic. In [19] we considered the overtwisted contact form α_1 on S^3 and we established the existence of such a v, which is given explicitly. We then studied the cases with $n > 1$ of that family given by Gonzalo-Varela: we found that the definition of v extends, but hypothesis (D) is not satisfied anymore by this extension; another extension might work.
The existence of such a v for a given contact form on a manifold allows to compute the relative Contact Homology, by setting the problem in a suitable variational framework and by using the techniques of the theory of critical points at infinity developed by A.Bahri [3], [4], [5], [6], [7].

Recently with A.Maalaoui in [18], we computed explicitly the relative Contact Homology, giving also some algebraic equivariant homology reductions, for the three-dimensional torus equipped with a family of tight contact structures: this has been possible since we were able to construct an explicit vector field v satisfying the hypothesis (D).
Finally, another application of the existence of such a vector field is that the knowing of v also allows to prove topological properties of the subspace C_β: indeed by using a particular flow along v, in [16] we showed that the injection of this subspace into the full loop space is an S^1-equivariant homotopy equivalence.
Here we will consider the case of an hypersurface M embedded in a four-dimensional Kähler manifold V, we refer the reader to the paper of Klingenberg [13] for the general notions of real hypersurfaces in Kähler manifolds; here we only recall some basic facts. First V is said to be a Kähler manifold if there exists a symplectic structure ω, a complex structure J and a Riemannian metric g such that they are compatible in the following sense (see for instance [14]):

$$\omega(X,Y) = g(X, JY) \tag{3}$$

for every pair of vector fields $X, Y \in TV$. Let us consider now a smooth compact, orientable, without boundary, embedded manifold M on V, of codimension 1, with the induced metric g. We denote by ν the inner unit normal to M, and we define the following differential form:

$$\theta(X) := g(J\nu, X) \tag{4}$$

for every vector field $X \in TM$.

Definition 1.1. *If θ is a contact form on M, we will say that M is a strictly Levi-convex hypersurface.*

We will explain in the next section the use of the previous terminology. Since V is by definition also a symplectic manifold, we note that there exists another natural differential form defined on M: since

$$rank(\omega\big|_{TM}) = 2, \qquad dim(\ker(\omega\big|_{TM})) = 1$$

let α be the form defined on TM such that $d\alpha = \omega\big|_{TM}$; in this situation if α is a contact form on M, then M is said to be of *contact type.*

Example 1.1. *(The case $\mathbb{C}^2$) If one considers an hypersurface M embedded in the usual complex space $\mathbb{C}^2(\simeq \mathbb{R}^4)$ as Kähler manifold, and if λ denotes* *the standard Liouville form in* $\mathbb{C}^2$*, then* $\lambda\big|_{TM}$ *is a contact form if* M *is the boundary of a star-shaped domain in* $\mathbb{C}^2$*. On the other hand,* θ *is a contact form if* M *is the boundary of a strictly pseudo-convex domain in* $\mathbb{C}^2$*; moreover the complexification of* $ker(\theta)$ *coincides with the restriction of the usual holomorphic tangent space of* $\mathbb{C}^2$*.*

Next we recall the definition of Killing vector fields:

Definition 1.2. *Given a Riemannian manifold* (M, g)*, a vector field* $v \in TM$ *is said a Killing vector field if:*

$$\mathcal{L}_v g = 0,$$

where $\mathcal{L}$ *denotes the Lie derivative. In particular it means that the flow of* v *generates isometry on* M*.*

We are now ready to state our result. We will prove the following

Theorem 1.2. *Let* M *be a strictly Levi-convex hypersurface. If there exists a unit non-vanishing Killing vector field* $v \in ker(\theta)$ *then hypothesis* (D) *holds.*

Remark 1.1. *We note that a similar condition already exists on contact metric manifolds, in fact if one requires that the Reeb vector field of the contact form is a Killing vector field, this gives rise to the notion of K-contact manifold (see for instance [9], [23] and [17] for definition and some applications). Here, instead, we require the existence of a Legendrian Killing vector field.*

Remark 1.2. *For the sake of simplicity and in order to have a neat statement we required that* v *is a Killing vector field: we will see from the proof that we need less. In fact condition (1.2) means that*

$$\mathcal{L}_v g(X, Y) = 0,$$

for every pair of vector fields $X, Y \in TM$*; actually we will only need the following two terms to vanish:*

$$\mathcal{L}_v g(J\nu, J\nu) = \mathcal{L}_v g(J\nu, Jv) = 0$$

Also, the requirement on v *to be unitary only simplifies the computations: the result remains true with any non-vanishing Legendrian Killing vector field.*

Remark 1.3. *(The case $\mathbb{C}^2$) If one considers the boundary of a star-shaped domain in $\mathbb{C}^2(\simeq \mathbb{R}^4)$ as hypersurface M, then the standard Liouville form λ restricts to a contact form on M. By direct computation one has that if in addition M is strictly convex, then there always exists a tangent vector field $v \in ker(\lambda)$ such that the Legendre duality holds.*

At the end of the paper we will show some easy examples of strictly Levi-convex hypersurfaces on which there exists a unit non-vanishing Killing vector field $v \in ker(\theta)$ such that hypothesis (D) holds.

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2 Proof of Theorem (1.2)

Here we will consider a smooth real compact orientable hypersurface without boundary M, embedded in a general four-dimensional Kähler manifold V equipped with a compatible triple (ω, J, g) in the sense of (3). We denote by ν the unit inner normal to M. The unit characteristic vector field X_0 is defined by $X_0 := J\nu$. Let A be the Weingarten operator, namely

$$A : TM \to TM, \quad AX := -\nabla_X \nu \ ,$$

where ∇ denotes the Levi-Civita connection of V; we will denote by

$$h(\cdot, \cdot) := g(A\cdot, \cdot)$$

the Second Fundamental Form of M. We also recall that both ∇ and g are compatible with the complex structure J, i.e.:

$$J\nabla = \nabla J, \qquad g(\cdot, \cdot) = g(J\cdot, J\cdot) \tag{5}$$

The complex maximal distribution or Levi distribution HM is the two-dimensional subspace in TM, invariant under the action of J:

$$HM = TM \cap JTM \tag{6}$$

i.e., a vector field $X \in TM$ belongs to HM if and only if also $JX \in HM$. Then TM splits in the g-orthogonal direct sum:

$$TM = HM \oplus \mathbb{R}X_0. \tag{7}$$

The Levi form ℓ is the hermitian operator on HM defined in the following way: for every $X, Y \in HM$, if $Z = \frac{1}{\sqrt{2}}(X - iJX)$ and $W = \frac{1}{\sqrt{2}}(Y - iJY)$, then

$$\ell(Z, \overline{W}) := g(\nabla_Z \overline{W}, \nu) \tag{8}$$

We can compare the Levi form with the Second Fundamental Form by using the following identity (see [10], Chap.10, Theorem 2):

$$\forall X \in HM, \quad 2\ell(Z, \overline{Z}) = h(X, X) + h(JX, JX) \tag{9}$$

Definition 2.1. *We will say that M is strictly Levi-convex if ℓ is strictly positive definite as quadratic form.*

We explain now some complex notation we use. Let us then define

$$T_{1,0}M := T^{1,0}V \cap T^{\mathbb{C}}M, \qquad T_{0,1}M := \overline{T_{1,0}M}$$

where $T^{1,0}V$ is the holomorphic space of V, that is the two dimensional complex space generated by the eigenvalue $+i$ of J; and $T^{\mathbb{C}}M$ is the complexified tangent space of M. Moreover

$$T_{1,0}M \oplus T_{0,1}M = H^{\mathbb{C}}M, \quad T^{\mathbb{C}}M = \mathbb{C}X_0 \oplus H^{\mathbb{C}}M$$

From now on we will consider a unit vector field $X_1 \in HM$, and $X_2 := JX_1$, such that $\{X_0, X_1, X_2\}$ is an orthonormal frame for TM. In the same way, if $Z = \frac{1}{\sqrt{2}}(X_1 - iX_2)$ then $Z \in T_{1,0}M$ and $\{X_0, Z, \overline{Z}\}$ is an orthonormal frame for $T^{\mathbb{C}}M$. Let us define the following differential form:

$$\theta : TM \to \mathbb{R}, \quad \theta(\cdot) := g(X_0, \cdot)$$

It is clear that $\ker(\theta) = HM$. If M is strictly Levi-convex then θ is a contact form with contact structure HM. In fact, by using (5), the following relation holds:

$$d\theta(X_1, X_2) = -\theta([X_1, X_2]) = -g(X_0, [X_1, X_2]) = -g(X_0, \nabla_{X_1}X_2 - \nabla_{X_2}X_1) =$$

$$= -g(\nu, \nabla_{X_1}X_1 + \nabla_{X_2}X_2) = -2g(\nu, \nabla_Z \overline{Z}) = -2\ell(Z, \overline{Z})$$

Therefore if M is strictly Levi-convex then

$$\theta \wedge d\theta(X_0, X_1, X_2) = -2\ell(Z, \overline{Z}) < 0$$

Remark 2.1. *We note that we are considering a "normalized" contact form θ, that is we are taking into account the characteristic direction X_0 related to the unit normal ν; one could use any other characteristic direction giving rise to a new contact form of the type $u\theta$, with u a never vanishing function defined on M: in this situation the kernel will be of course the same, but the main result could be not true in general, since the dynamics of the Legendrian vector field v (as well as the Reeb vector field) is totally nonlinear.*

We will need some formulas for some commutators and for the covariant derivatives along the vector fields of the orthonormal basis $E := \{X_0, X_1, X_2\}$. We denote by

$$\Gamma^l_{jk} := g(\nabla_{X_j} X_k, X_l), \qquad j, k, l = 0, 1, 2$$

the coefficients of the Levi-Civita connection with respect to the basis E; also we denote by

$$h_{jk} := h(X_j, X_k) = g(\nabla_{X_j} X_k, \nu), \qquad j, k = 0, 1, 2$$

the coefficients of the Second Fundamental Form with respect to the basis E. We have then

Lemma 2.1. *Let $E := \{X_0, X_1, X_2\}$ be an orthonormal basis for TM with $X_2 := JX_1$, and let $Z = \frac{1}{\sqrt{2}}(X_1 - iX_2)$. It holds:*

$$\begin{aligned}
\nabla_{X_0} X_0 &= h_{00}\nu + h_{02} X_1 - h_{01} X_2 \\
\nabla_{X_1} X_1 &= h_{11}\nu - h_{12} X_0 + \Gamma^2_{11} X_2 \\
\nabla_{X_2} X_2 &= h_{22}\nu + h_{12} X_0 + \Gamma^2_{21} X_1 \\
\nabla_{X_1} X_2 &= h_{12}\nu + h_{11} X_0 + \Gamma^1_{12} X_1 \\
\nabla_{X_1} X_0 &= h_{01}\nu + h_{12} X_1 - h_{11} X_2 \\
\nabla_{X_2} X_1 &= h_{12}\nu - h_{22} X_0 + \Gamma^2_{21} X_2 \\
\nabla_{X_2} X_0 &= h_{02}\nu + h_{22} X_1 - h_{12} X_2 \\
\nabla_{X_0} X_1 &= h_{01}\nu - h_{02} X_0 + \Gamma^2_{01} X_2 \\
\nabla_{X_0} X_2 &= h_{02}\nu + h_{01} X_0 + \Gamma^1_{02} X_1
\end{aligned} \tag{10}$$

Moreover:

$$
\begin{aligned}
&[X_0, X_1] = -h_{02}X_0 - h_{12}X_1 + (h_{11} + \Gamma^2_{01})X_2 \\
&[X_0, X_2] = h_{01}X_0 - (h_{22} + \Gamma^2_{01})X_1 + h_{12}X_2 \\
&[X_1, X_2] = 2\ell(Z, \overline{Z})X_0 + \Gamma^1_{12}X_1 - \Gamma^2_{21}X_2
\end{aligned}
\tag{11}
$$

Proof. It follows by direct computation, by using (5) and the symmetries of the coefficients of the Levi-Civita connection. □

Now let us fix a generic unit vector field $X_1 \in HM$ and let us define the following dual differential form:

$$\beta(\cdot) := d\theta(X_1, \cdot)$$

In this notation the Reeb vector field of θ is given by

$$\xi = \frac{\zeta}{2\ell(Z, \overline{Z})}, \qquad \zeta := 2\ell(Z, \overline{Z})X_0 - h_{01}X_1 - h_{02}X_2$$

where as usual $X_2 := JX_1$ and $Z = \frac{1}{\sqrt{2}}(X_1 - iX_2)$; we recall that $\ell(Z, \overline{Z})$ never vanishes since M is strictly Levi-convex. Moreover:

$$\beta(X_2) = -2\ell(Z, \overline{Z}), \qquad \ker(\beta) = span\{\xi, X_1\}$$

Lemma 2.2. *It holds:*

$$\beta \wedge d\beta(X_0, X_1, X_2) = \tag{12}$$

$$= 2\ell(Z, \overline{Z})\Big\{h_{02}X_1\big(\log \ell(Z, \overline{Z})\big) - X_1\big(h_{02}\big) + h_{02}\Gamma^2_{21} - 2(h_{11} + \Gamma^2_{01})\ell(Z, \overline{Z})\Big\}$$

Proof. We will use (5), (10) and (11). First we have

$$\beta(X_0) = d\theta(X_1, X_0) = -\theta\big([X_1, X_0]\big) = -h_{02}$$

Then we get:

$$\beta \wedge d\beta(X_0, X_1, X_2) = \beta(X_0)d\beta(X_1, X_2) - \beta(X_2)d\beta(X_1, X_0) =$$

$$= -h_{02}\Big\{X_1\big(\beta(X_2)\big) - \beta\big([X_1, X_2]\big)\Big\} + 2\ell(Z, \overline{Z})\Big\{X_1\big(\beta(X_0)\big) - \beta\big([X_1, X_0]\big)\Big\} =$$

$$= -h_{02}\Big\{ - 2X_1\big(\ell(Z, \overline{Z})\big) - \beta\big(2\ell(Z, \overline{Z})X_0 + \Gamma^1_{12}X_1 - \Gamma^2_{21}X_2\big)\Big\} +$$

$$+2\ell(Z,\overline{Z})\Big\{-X_1\big(h_{02}\big)-\beta\big(h_{02}X_0+h_{12}X_1-(h_{11}+\Gamma^2_{01})X_2\big)\Big\}=$$

$$=-h_{02}\Big\{-2X_1\big(\ell(Z,\overline{Z})\big)+2h_{02}\ell(Z,\overline{Z})-2\Gamma^2_{21}\ell(Z,\overline{Z})\Big\}+$$

$$+2\ell(Z,\overline{Z})\Big\{-X_1\big(h_{02}\big)+(h_{02})^2-2(h_{11}+\Gamma^2_{01})\ell(Z,\overline{Z})\Big\}=$$

$$=2h_{02}X_1\big(\ell(Z,\overline{Z})\big)-2\ell(Z,\overline{Z})X_1\big(h_{02}\big)+2\ell(Z,\overline{Z})h_{02}\Gamma^2_{21}-4(h_{11}+\Gamma^2_{01})\big(\ell(Z,\overline{Z})\big)^2$$

Therefore we obtain (12). □

Proof. of Theorem 1.2
By hypothesis there exists a Legendrian unit Killing vector field v. By definition (1.2) and by using the conditions on the Levi-Civita connection, we see that

$$\big(\mathcal{L}_v g\big)(X,Y)=v\big(g(X,Y)\big)-g(\mathcal{L}_v X,Y)-g(X,\mathcal{L}_v Y)=$$

$$=g(\nabla_X v,Y)+g(X,\nabla_Y v)=0 \tag{13}$$

for every pair of vector fields $X,Y\in TM$. Now let us rename $X_1:=v$, and $X_2:=JX_1$, $Z=\frac{1}{\sqrt{2}}(X_1-iX_2)$. By using the formulas (13) and (10), we can find conditions on some of the coefficients of the Second Fundamental Form and of the connection, with respect to the orthonormal basis $\{X_0,X_1,X_2\}$. In particular we have:

$$0=g(\nabla_{X_0}X_1,X_2)+g(X_0,\nabla_{X_2}X_1)=\Gamma^2_{01}-h_{22} \tag{14}$$

$$0=g(\nabla_{X_0}X_1,X_0)+g(X_0,\nabla_{X_0}X_1)=-2h_{02} \tag{15}$$

Moreover, by property (9) we have

$$2\ell(Z,\overline{Z})=h_{11}+h_{22} \tag{16}$$

By putting (14), (15) and (16) in the equation (12) we get

$$\beta\wedge d\beta(X_0,X_1,X_2)=-8\big(\ell(Z,\overline{Z})\big)^3 \tag{17}$$

Since M is strictly Levi-convex, $\ell(Z,\overline{Z})$ never vanishes, therefore β is a contact form on M. Finally

$$\frac{\beta\wedge d\beta(X_0,X_1,X_2)}{\theta\wedge d\theta(X_0,X_1,X_2)}=\frac{-8\big(\ell(Z,\overline{Z})\big)^3}{-2\ell(Z,\overline{Z})}=4\big(\ell(Z,\overline{Z})\big)^2>0$$

Therefore β is a contact form on M with the same orientation than θ. This means that the Legendre duality can be completed and it ends the proof. □

Example 2.1. *(The standard sphere S^3 in $\mathbb{C}^2$)*
Let us consider $\mathbb{C}^2$ with coordinates $z_k = x_k + iy_k$, $k = 1, 2$ and

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \ s.t. \ |z_1|^2 + |z_2|^2 = 1\}$$

Then (S^3, θ) is a strictly Levi-convex hypersurface, where

$$\theta = y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2$$

Now, we define the (unit) tangent vector field

$$v = x_2 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2}$$

so that $v \in ker(\theta)$. By a direct computation, one finds that v generates isometries and therefore it is a Legendrian Killing vector field. Finally, if $\beta(\cdot) = d\theta(v, \cdot)$, then

$$\frac{\beta \wedge d\beta}{\theta \wedge d\theta} = 4$$

Hence condition (D) holds.

Example 2.2. *(A cylinder $S^1 \times \mathbb{R}^2$ in $\mathbb{C}^2$)*
Let us consider $\mathbb{C}^2$ with coordinates $z_k = x_k + iy_k$, $k = 1, 2$ and

$$M = \{(z_1, z_2) \in \mathbb{C}^2 \ s.t. \ x_1^2 + x_2^2 = 1\}$$

Then (M, θ) is a strictly Levi-convex hypersurface, where

$$\theta = -x_1 dy_1 - x_2 dy_2$$

Now, we define the (unit) tangent vector field

$$v = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

so that $v \in ker(\theta)$. By a direct computation, one finds that v generates isometries and therefore it is a Legendrian Killing vector field. Finally, if $\beta(\cdot) = d\theta(v, \cdot)$, then

$$\theta \wedge d\theta = \beta \wedge d\beta$$

Hence condition (D) holds.

Remark 2.2. *Both the previous examples are hypersurfaces in $\mathbb{C}^2$ of Hopf type, that is the characteristic vector field X_0 is an eigenvector for the Weingarten operator (equivalently X_0 is a geodesic vector field, see for instance [20]). In order to construct other examples, one might also try to consider Hopf hypersurfaces in more general Kähler manifolds, see [8], [11], [15], [21], [22] for the related literature.*

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