Multiplicity result for a nonhomogeneous Yamabe type equation involving the Kohn Laplacian

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Abstract We consider the nonhomogeneous Yamabe equation on a bounded set of the Heisenberg group $-\Delta_{\mathbb{H}}u = |u|^{q^*-2}u + f$, where f is a small perturbation in the C^0 sense. Under suitable hypotheses, we will state a multiplicity existence result for positive solutions with zero Dirichlet boundary conditions.

1 Introduction

Let Ω be a bounded domain in the Heisenberg group $\mathbb{H}^n$. In this work we are interested in finding multiple solutions of the following nonhomogeneous Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{H}}u = |u|^{q^*-2}u + f, & \text{in}\,\Omega, \\ u > 0, & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega, \\ f \in C(\bar{\Omega}),\ f \not\equiv 0\ f \geq 0 \end{cases} \tag{1}$$

Here $\Delta_{\mathbb{H}}$ denotes the sub-laplacian of the group and $q^* = (2n+2)/2$. When $f \equiv 0$, problem (1) coincides with the CR-Yamabe equation on Ω which has been intensively studied in the last years (see for instance [15], [13], [6] and the references therein). Regarding perturbation results on bounded domain, we recall the result obtained by Garagnani and Uguzzoni in [12]: they consider the homogeneous equation

$$-\Delta_{\mathbb{H}}u = |u|^{q^*-2}u + \lambda u, \quad \text{in}\,\Omega$$

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with zero Dirichlet boundary conditions; under suitable hypotheses on the boundary of Ω, they provide a multiplicity result for positive solutions involving the Lusternik-Schnirelmann category.
In [18] the authors with A.Pistoia prove the existence of concentrating solutions for the slightly sub-critical problem under a suitable assumption on $\partial\Omega$ and that the Robin's function of the domain has a non-degenerate critical point.
Here for our purpose we will also need an hypothesis on $\partial\Omega$, in particular we will require that the boundary of Ω has no characteristic points (see definition (2.1) in the next section). Then, by denoting with $H_k(\Omega)$ the k-th homology group of Ω, we will prove the following:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{H}^n$ be a bounded domain with smooth boundary and with no characteristic points. Then there exist a residual subset $D \in C^2(\bar{\Omega})$ and $\varepsilon_0 > 0$, such that for every $f \in D$ with $|f|_{C(\bar{\Omega})} < \varepsilon_0$, the problem (1) has at least $1 + \Sigma_{k=0}^{\infty} dim H_k(\Omega)$ solutions.*

The condition on Ω is needed in order to overcome some technical difficulties in proving some estimates: as consequence, if we consider $\mathbb{H}^1$ for instance, we cannot take the Heisenberg ball as our domain, since its boundary has two characteristic points; in particular any contractible domain in $\mathbb{H}^1$ with smooth boundary has characteristic points. Anyway, since the multiplicity result is due to the topology of the domain we are interested in domains with "rich" topology: for example the standard torus in $\mathbb{H}^1$ defined by $\{(R - \sqrt{x^2 + y^2})^2 + t^2 - r^2 < 0,\ R > r > 0\}$ turns out to not have any characteristic point.
We recall that the analogous problem for the standard Laplacian on bounded domains in $\mathbb{R}^n$ was solved by Hirano in [14]. Moreover we used the same technique also in [17] in which we first investigate the problem of existence and multiplicity of solutions for the non-homogeneous fourth order Yamabe type equation involving the bi-Laplacian by exhibiting a family of solutions concentrating at two points, provided the domain contains one hole and giving a multiplicity result if the domain has multiple holes (as in [7], [8]); then we prove a multiplicity result for vanishing positive solutions in a general domain.

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2 Setting of the problem

Let $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \centerdot)$ be the Heisenberg group. If we denote by $\xi = (x, y, t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ then the group law is

$$\xi_0 \centerdot \xi = (x + x_0, y + y_0, t + t_0 + 2(x \cdot y_0 - x_0 \cdot y)),\ \forall \xi, \xi_0 \in \mathbb{H}^n$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. The left translations are then given by

$$\tau_{\xi_0}(\xi) := \xi_0 \centerdot \xi$$

The dilations of the group are

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$$

for any $\lambda > 0$. We define the homogeneous norm

$$\rho(\xi) = \left((|x|^2 + |y|^2)^2 + t^2\right)^{1/4},$$

and the distance

$$d(\xi, \xi_0) = \rho(\xi_0^{-1} \centerdot \xi).$$

It holds

$$d(\delta_\lambda \xi, \delta_\lambda \xi_0) = \lambda\, d(\xi, \xi_0).$$

We will denote by $B_d(\xi, r)$ the ball with respect to the distance d, of center ξ and radius r. We have

$$B_d(\xi, r) = \tau_\xi(B_d(0, r)), \qquad B_d(0, r) = \delta_r(B_d(0, 1))$$

The canonical left-invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

The (intrinsic) gradient of the group is

$$D_H = (X_1, \dots, X_n, Y_1, \dots, Y_n)$$

The Kohn Laplacian (or sublaplacian) on $\mathbb{H}^n$ is the following second order operator invariant with respect to the left translations and homogeneous of degree two with respect to the dilations:

$$\Delta_H = \sum_{j=1}^{n} X_j^2 + Y_j^2$$

By a result in [9], the fundamental solution on $\mathbb{H}^n$ of $-\Delta_H$ with pole at the origin is

$$\Gamma(\xi) = \frac{c_q}{\rho(\xi)^{q-2}}$$

where c_q is a suitable positive constant and $q = 2n + 2$ is the homogeneous dimension of the group. The fundamental solution on $\mathbb{H}^n$ of $-\Delta_H$ with pole at the ξ will be

$$\Gamma(\xi, \eta) = \frac{c_q}{d(\xi, \eta)^{q-2}}$$

Let now

$$q^* = \frac{2q}{q-2}$$

then the following Sobolev-type inequality holds

$$\|\varphi\|_{q^*}^2 = \Big(\int_{\mathbb{H}^n} |\varphi|^{q^*} \Big)^{\frac{2}{q^*}} \leq C \int_{\mathbb{H}^n} |D_H \varphi|^2 = C \|D_H \varphi\|_2^2, \qquad \forall \varphi \in C_0^\infty(\mathbb{H}^n)$$

with C a positive constant. For every domain $\Omega \subseteq \mathbb{H}^n$, the Folland-Stein Sobolev space $S_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\| = \|D_H \varphi\|_2$$

The exponent q^* is called critical since the embedding

$$S_0^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$$

is continuous but not compact for every domain Ω. Moreover, by defining the inner product on $S_0^1(\Omega)$

$$\langle u, v \rangle = \int_\Omega \langle D_H u, D_H v \rangle$$

then there exists a natural orthogonal projection

$$P : S_0^1(\mathbb{H}^n) \longrightarrow S_0^1(\Omega)$$

Let us define the function

$$\omega(x, y, t) = \frac{c_0}{\big((1 + |x|^2 + |y|^2)^2 + t^2 \big)^{\frac{q-2}{4}}}$$

with c_0 a suitable positive constant; we recall that Jerison and Lee showed in [16] that all the positive solutions to the problem

$$-\Delta_{\mathbb{H}} u = |u|^{q^*-2} u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{2}$$

are in the form
$$\omega_{\lambda,\xi} = \lambda^{\frac{q-2}{2}} \omega \circ \delta_\lambda \circ \tau_{\xi^{-1}}$$
for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$. Using the variational framework, a positive solution of the problem (2) on Ω can be found as critical point of the following functional
$$J^\Omega : S_0^1(\Omega) \to \mathbb{R}, \qquad J^\Omega(u) = \frac{1}{2}\int_\Omega |D_H u|^2 - \frac{1}{q^*}\int_\Omega |u^+|^{q^*}$$
where $u^+ = \max\{0, u\}$ denotes the positive part of u. Moreover any variational solution is actually a classical solution ([10], [13]).
We will denote by $c = J^{\mathbb{H}}(\omega_{\lambda,\xi})$ the common critical value of the bubbles $\omega_{\lambda,\xi}$. Finally, next is the definition of characteristic points

Definition 2.1. *Let $\varphi : \mathbb{H}^n \to \mathbb{R}$ a smooth defining function for Ω, namely*
$$\Omega = \{\xi \in \mathbb{H}^n \ : \ \varphi(\xi) < 0\}, \qquad \partial\Omega = \{\xi \in \mathbb{H}^n \ : \ \varphi(\xi) = 0\}$$
A point $\xi_0 \in \partial\Omega$ is said to be characteristic if $D_{\mathbb{H}}\varphi(\xi_0) = 0$.

3 Proof of Theorem (1.1)

It is known (see [19]) that a solution of the linearized problem
$$-\Delta_{\mathbb{H}} u = (q^* - 1)|\omega_{\lambda,\xi}|^{q^*-2} u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{3}$$
belongs to the following set
$$T_{\lambda,\xi} = span\Big\{\frac{\partial \omega_{\lambda,\xi}}{\partial \lambda}, \frac{\partial \omega_{\lambda,\xi}}{\partial \xi_j}, \quad j = 1, \dots, 2n+1\Big\}$$
Now we consider the eigenvalue problem
$$-\Delta_{\mathbb{H}} u = \mu g(\omega_{\lambda,\xi}) u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{4}$$
where $g(t) = (q^* - 1)|t^+|^{q^*-2}$, and let $\mu_- = (q^* - 1)^{-1}$ be the eigenvalue with eigenfunctions $\omega_{\lambda,\xi}$. Just by differentiating (2), we get that all the functions in $T_{\lambda,\xi}$ are eigenfunctions with eigenvalue $\mu_0 = 1$. We will call $E^-_{\lambda,\xi}$ the eigenspace corresponding to μ_-, $E^0_{\lambda,\xi}$ the eigenspace corresponding to μ_0, and $E^+_{\lambda,\xi} = (E^-_{\lambda,\xi} \cup E^0_{\lambda,\xi})^\perp$. Then we have that there exists $\mu_1 > 0$ such that for every $(\lambda, \xi) \in (1, \infty) \times \mathbb{H}^n$, it holds
$$\langle -\Delta_{\mathbb{H}} u - g(\omega_{\lambda,\xi}) u, u\rangle \geq \mu_1 \int_{\mathbb{H}^n} g(\omega_{\lambda,\xi}) u^2 \tag{5}$$

for all the functions $u \in E^+_{\lambda,\xi}$.
Now we need a result concerning the existence of a solution for (1). The following lemma is the analogous of the Euclidean setting: the proof is similar to that case and we will omit it (see [4], [14]).

Lemma 3.1. *Let $\Omega \subseteq \mathbb{H}^n$ be a bounded domain with smooth boundary and with no characteristic points. There exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that if $f \in C^2(\bar{\Omega})$, $f \geq 0$, $|f|_{C(\bar{\Omega})} < \varepsilon_0$, then there exists a unique solution $u_0 \in S^1_0(\Omega)$ of problem (1) with*

$$|u_0|_{C^1(\bar{\Omega})} < C_0 |f|_{C(\bar{\Omega})}$$

and

$$c_0 := \frac{1}{2}\int_\Omega |D_H u_0|^2 - \frac{1}{q^*}\int_\Omega |u_0|^{q^*} - \int_\Omega f u_0 < \frac{c}{2} \qquad .$$

Let now consider the solution u_0 obtained in Lemma (3.1) and let us define the following functional

$$J(v) = \frac{1}{2}\int_\Omega |D_H v|^2 - \frac{1}{q^*}\int_\Omega |(v+u_0)^+|^{q^*} + \frac{1}{q^*}\int_\Omega |u_0|^{q^*} + \int_\Omega |u_0|^{q^*-1} v, \ v \in S^1_0(\Omega)$$

so that for every critical point v of J, then $v + u_0$ is a critical point of the functional J^Ω_f associated to problem (1)

$$J^\Omega_f(u) = \frac{1}{2}\int_\Omega |D_H u|^2 - \frac{1}{q^*}\int_\Omega |u^+|^{q^*} - \int_\Omega f u$$

Arguing as in [3] we have that there exists $\varepsilon_1 > 0$ such that for every $f \in C(\bar{\Omega})$, $f \geq 0$, $f \not\equiv 0$, $|f|_{C(\bar{\Omega})} < \varepsilon_1$ and for every $v \in S^1_0(\Omega)$, $v^+ \not\equiv 0$, there exists a unique positive number t_v such that $J(t_v v)$ is increasing on $[t_1, t_v)$ for some $t_1 > 0$, decreasing on $(t_v, +\infty)$ and $J(t_v v) = \max\{J(tv) : \ t > 0\}$.
Now let us define the following set

$$S = \left\{ u \in S^1_0(\Omega) \setminus \{0\} \ s.t. \ \int_\Omega |D_{\mathbb{H}} u|^2 = \int_\Omega |u^+|^{q^*} \right\}$$

and the Nehari type manifold

$$\mathcal{S} = \left\{ t_v v : \ v \in S^1_0(\Omega) \setminus \{0\} \right\}$$

Then one has that $J(v) > 0$ on S and every non zero critical point of J is contained in $\mathcal{S}$. Moreover, by the concentration compactness principle in our subelliptic setting and the representation theorem for Palais-Smale

sequences proved in [5], we get that J satisfies the Palais-Smale condition on the interval $(0, c)$.
We introduce now the functions $W_{\lambda,\xi} := P\omega_{\lambda,\xi}$, namely the $S_0^1(\Omega)$ projections of $\omega_{\lambda,\xi}$, defined by

$$W_{\lambda,\xi} = \omega_{\lambda,\xi} - h_{\lambda,\xi}$$

where

$$\begin{cases} -\Delta_{\mathbb{H}} W_{\lambda,\xi} = -\Delta_{\mathbb{H}} \omega_{\lambda,\xi} = \omega_{\lambda,\xi}^{q^*-1}, & \text{in } \Omega, \\ W_{\lambda,\xi} = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta_{\mathbb{H}} h_{\lambda,\xi} = 0, & \text{in } \Omega, \\ h_{\lambda,\xi} = \omega_{\lambda,\xi}, & \text{on } \partial\Omega, \end{cases}$$

In the next lemma we provide some estimates on the functions $W_{\lambda,\xi}$ and we explicitly remark that in [14], the author does not use any projection: indeed he considers the bubbles themselves times a cut-off function.
Let us first define the following sets, for every $\rho > 0$:

$$\Pi(\rho) : \big\{(\lambda, \xi) \in (1, \infty) \times \Omega : \ d(\xi)\lambda = \rho\big\}$$

$$\overline{\Pi}(\rho) : \big\{(\lambda, \xi) \in (1, \infty) \times \Omega : \ d(\xi)\lambda \geq \rho\big\}$$

where $d(\xi) = \min\{d(\partial\Omega, \xi), d_0\}$ and d_0 is a small positive number. Then we have the following estimates:

Lemma 3.2. *Let $\Omega \subseteq \mathbb{H}^n$ be a bounded domain with smooth boundary and with no characteristic points. Let $\rho_0 > 2$, then for $(\lambda, \xi) \in \overline{\Pi}(\rho_0)$ we have:*

$$\|W_{\lambda,\xi}\|^2 \leq qc + O(d(\xi)\lambda)^{-(q-2)} \tag{6}$$

$$\|W_{\lambda,\xi}\|^{q^*}_{L^{q^*}(\Omega)} \geq qc - O(d(\xi)\lambda)^{-(q-2)} \tag{7}$$

$$\int_\Omega u_0 W_{\lambda,\xi}^{q^*-1} \geq O(d(\xi)\lambda^{-\frac{(q-2)}{2}}) \tag{8}$$

$$\int_\Omega u_0^{q^*/2} W_{\lambda,\xi}^{q^*/2} \leq O(d(\xi)^{q^*/2}\lambda^{-q^*/2}|\log\lambda|) \tag{9}$$

Proof. First we note that since the boundary of our domain has no characteristic points, then the intrinsic normal is always defined, in particular we obtain by Lemma (3.1) and the Hopf Lemma (see [2]) that there exists a constant $\ell > 0$ such that

$$\ell d(\eta) \leq u_0(\eta), \qquad \forall\, \eta \in \Omega \tag{10}$$

We recall now that the Green's function G and its regular part H are defined by

$$G(\xi,\eta) = \Gamma(\xi,\eta) - H(\xi,\eta)$$

and

$$\begin{cases} -\Delta_{\mathbb{H}} H(\xi,\cdot) = 0, & \text{in}\,\Omega, \\ H(\xi,\cdot) = \Gamma(\xi,\cdot), & \text{on}\,\partial\Omega, \end{cases}$$

where $\Gamma(\xi,\cdot)$ is the fundamental solution of $-\Delta_H$ with pole at ξ. By using the maximum principle for $\Delta_{\mathbb{H}}$ we have a control on the L^∞ norm of $h_{\lambda,\xi}$, in particular

$$h_{\lambda,\xi}(\eta) = \frac{H(\xi,\eta)}{\lambda^{\frac{q-2}{2}}} \int_{\mathbb{H}^n} \omega^{q^*-1} + o\Big(\frac{1}{\lambda^{\frac{q-2}{2}}}\Big)$$

Other useful estimates on the Green's function and the projections can be found in [11]; we recall also some similar estimates in the Appendix of [20]: the only technical assumption that we will add is that $\partial\Omega$ is without characteristic points.
Moreover, we explicitly note that at best of our knowledge, we don't know any explicit formula for the Green's function for any bounded domain in the Heisenberg group.
The first estimate (6) is essentially contained in [6] (Proposition 5.1.): we only need to rewrite it, taking into account also the distance $d(\xi)$. For the second one we can argue in the same way. In fact for some $r > 0$, we consider a ball $B_d(\xi, r)$ contained in Ω centered at ξ. We get

$$\int_\Omega W_{\lambda,\xi}^{q^*} \geq \int_{B_d(\xi,r)} W_{\lambda,\xi}^{q^*} = \int_{B_d(\xi,r)} (\omega_{\lambda,\xi} - h_{\lambda,\xi})^{q^*} \geq$$

$$\geq \int_{B_d(\xi,r)} \omega_{\lambda,\xi}^{q^*} - \int_{B_d(\xi,r)} q^* h_{\lambda,\xi}\omega_{\lambda,\xi}^{q^*-1} =$$

$$= \int_{\mathbb{H}} \omega_{\lambda,\xi}^{q^*} - \int_{\mathbb{H}\setminus B_d(\xi,r)} \omega_{\lambda,\xi}^{q^*} - \int_{B_d(\xi,r)} q^* h_{\lambda,\xi}\omega_{\lambda,\xi}^{q^*-1}$$

By rescaling the last two integrals after a change of variables, and then by direct computation, we get (7). Now by (10) we have that for every $\eta \in \Omega$ it holds

$$\frac{\ell}{2} d(\xi) \leq u_0(\eta), \qquad \forall\, \eta \; s.t. \; d(\xi) \leq 2d(\eta)$$

Moreover $d(\xi)\lambda > 2$ since $\rho_0 > 2$. Then

$$\int_\Omega u_0 W_{\lambda,\xi}^{q^*-1} \geq \frac{\ell}{2} d(\xi) \int_{\Omega\cap\{d(\xi)\leq 2d(\eta)\}} W_{\lambda,\xi}^{q^*-1} \geq \frac{\ell}{2} d(\xi) \int_{B_d(\xi,1/\lambda)} W_{\lambda,\xi}^{q^*-1} \geq$$

$$\geq \frac{\ell}{2}d(\xi)\int_{B_d(\xi,1/\lambda)}\omega_{\lambda,\xi}^{q^*-1}-\frac{\ell}{2}d(\xi)(q^*-1)\int_{B_d(\xi,1/\lambda)}h_{\lambda,\xi}\omega_{\lambda,\xi}^{q^*-2}=$$

$$=\frac{\ell}{2}d(\xi)\int_{\mathbb{H}}\omega_{\lambda,\xi}^{q^*-1}-\frac{\ell}{2}d(\xi)\int_{\mathbb{H}\setminus B_d(\xi,1/\lambda)}\omega_{\lambda,\xi}^{q^*-1}-\frac{\ell}{2}d(\xi)(q^*-1)\int_{B_d(\xi,1/\lambda)}h_{\lambda,\xi}\omega_{\lambda,\xi}^{q^*-2}$$

and again after a rescaling we get (8). The last estimate (9) can be obtained in a similar way. □

Let us define now the following sets:

$$\mathcal{M}=\{W_{\lambda,\xi}\ :\ (\lambda,\xi)\in(1,\infty)\times\Omega\}$$

and

$$\mathcal{N}=\{sW_{\lambda,\xi}\ :\ (\lambda,\xi)\in(1,\infty)\times\Omega,\ s\in(1/2,2)\}$$

We will also denote by

$$F^-_{\lambda,\xi}=\{sW_{\lambda,\xi}\ :\ s\in\mathbb{R}\},\qquad F^+_{\lambda,\xi}=(T_{\lambda,\xi}\mathcal{N})^\perp,\qquad F_{\lambda,\xi}=F^-_{\lambda,\xi}\oplus F^+_{\lambda,\xi}$$

and for every function $v=v_-+v_+\in F_{\lambda,\xi}$ we will denote $Kv=v_--v_+$. The following two results are the same as in [14]: we refer to it for the proof. First we have a sort of "convexity" property:

Lemma 3.3. *There exist positive numbers $r_1,\rho_1,C_1,\varepsilon_2$, with $\rho_1>\rho_0$ and such that if $|f|_{C(\bar\Omega)}<\varepsilon_2$, $(\lambda,\xi)\in\overline{\Pi}(\rho_1)$ and $w\in B_{r_1}(W_{\lambda,\xi})$ then*

$$\langle-\Delta_{\mathbb{H}}v-g(w+u_0)v,Kv\rangle\geq C_1\|v\|^2$$

for every $v\in F_{\lambda,\xi}$.

Next we have the existence of a suitable function:

Lemma 3.4. *There exist positive numbers ρ_2,C_2, such that if $|f|_{C(\bar\Omega)}<\varepsilon_2$, $(\lambda,\xi)\in\overline{\Pi}(\rho_2)$ then there exists $w_{\lambda,\xi}\in S\cap B_{\frac{r_1}{2}}(W_{\lambda,\xi})$ with*

$$\|w_{\lambda,\xi}-W_{\lambda,\xi}\|\leq C_2\|\nabla J(W_{\lambda,\xi})\|$$

and

$$J(w_{\lambda,\xi})=\min_{v\in F^+_{\lambda,\xi}\cap B_{\frac{r_1}{2}}(0)}\ \max_{w\in F^-_{\lambda,\xi}\cap B_{\frac{r_1}{2}}(0)}J(w_{\lambda,\xi}+v+w)=$$

$$=\max_{w\in F^-_{\lambda,\xi}\cap B_{\frac{r_1}{2}}(0)}J(w_{\lambda,\xi}+w)=\min_{v\in F^+_{\lambda,\xi}\cap B_{\frac{r_1}{2}}(0)}J(w_{\lambda,\xi}+v)$$

We need now a transversality result: here we state it, in the Appendix we will prove it as byproduct of a more general statement.

Lemma 3.5. *Let $\Omega \subseteq \mathbb{H}^n$ be a bounded domain with smooth boundary and with no characteristic points. There exists a residual subset $D \subseteq C^2(\bar{\Omega})$ such that for $f \in D$, every solution u of the problem (1) is a nondegenerate critical point for the functional J_f^Ω.*

Proof. of Theorem (1.1)
Let us consider a function $f \in C(\bar{\Omega})$, $f \not\equiv 0$ $f \geq 0$ and $f \in D$, the residual set of the Lemma (3.5). Following [14], by estimates in Lemma (3.2) and Lemma (3.3) and (3.4) we have that there exist $\rho_3 > \rho_0$, $0 < \varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$ and a subset $\Psi \subseteq \overline{\Pi}(\rho_3)$ with $\Psi, \overline{\Pi}(\rho_3)$ and Ω homological equivalent and such that if $|f|_{C(\bar{\Omega})} < \varepsilon_3$ then

$$J(w_{\lambda,\xi}) < c, \qquad \forall\ (\lambda, \xi) \in \Psi$$

where $w_{\lambda,\xi}$ is the function obtained in Lemma (3.4). Moreover

$$J(w_{\lambda,\xi}) > c, \qquad \forall\ (\lambda, \xi) \in \partial\overline{\Pi}(\rho_3)$$

We also define a functional

$$\bar{J} : \overline{\Pi}(\rho_3) \to \mathbb{R}, \qquad \bar{J}(\lambda, \xi) = J(w_{\lambda,\xi})$$

By results in [1] and [21] we have that if $(\lambda, \xi) \in \mathrm{int}\overline{\Pi}(\rho_3)$ is a critical point for $\bar{J}$, then $w_{\lambda,\xi}$ is a critical point for J; moreover $\bar{J}$ satisfies the Palais-Smale condition on $(0, c)$. We set, for every non zero class $[\alpha] \in H_*(\Psi)$, the value

$$c_\alpha = \min_{\alpha \in [\alpha]} \max_{(\lambda,\xi)\in\alpha} J(w_{\lambda,\xi})$$

It follows that there exists a critical point $(\lambda, \xi) \in \mathrm{int}\overline{\Pi}(\rho_3)$ for $\bar{J}$, with critical value c_α. This implies that there exists a critical point $w_{\lambda,\xi}$ for J with $J(w_{\lambda,\xi}) = c_\alpha$. Since $f \in D$, by the transversality theorem, every critical point of J is nondegenerate. Therefore, by using the previous minmax argument, the number of critical points of J, and therefore of J_f^Ω, is at least $1 + \Sigma_{k=0}^{\infty} dim H_k(\Omega)$. □

A Appendix

Here we state a well known transversality theorem from which we derive the Lemma (3.5).

Theorem A.1. *Let X, Y, Z be separable Banach spaces and $\Phi : X \times Y \to Z$ a C^1 map. Suppose that*

(*i*) $\forall\ (x, y) \in \Phi^{-1}(z),\ D_x\Phi(x, y) : X \to Z$ *is Fredholm of index 0*

(*ii*) $\forall\ (x, y) \in \Phi^{-1}(z),\ D_x\Phi(x, y) : X \to Z$ *is surjective*

Then the set of $y \in Y$ such that $z \in Z$ is a regular value of $\Phi(\cdot, y)$ is residual in Y.

Proof. of Lemma (3.5)
We are going to apply the previous theorem to DJ_f^{Ω}. Let

$$X = S^2(\Omega) \cap S_0^1(\Omega), \quad Y = C^2(\bar{\Omega}), \quad Z = L^2(\Omega)$$

$$\Phi(u, f) = \Delta_{\mathbb{H}} u + |u|^{q^*-2} u + f$$

For every $u \in X$, the map

$$D_u\Phi(u, f)v = \Delta_{\mathbb{H}} v + g(u)v$$

is Fredholm of index zero. Let $z = 0$ and $(u, f) \in \Phi^{-1}(0)$, namely

$$-\Delta_{\mathbb{H}} u = |u|^{q^*-2} u + f$$

We have that $u \in C(\bar{\Omega})$ and the kernel of $\Delta_{\mathbb{H}} + g(u)$ is a finite dimensional subspace of $C^2(\bar{\Omega})$. Now we want to prove that there exist $(v, \bar{f}) \in X \times Y$ such that, for every $h \in Z$

$$D\Phi(u, f)v = D_u\Phi(u, f)v + D_f\Phi(u, f) = \Delta_{\mathbb{H}} v + g(u)v + \bar{f} = h$$

Now $\bar{P}h \in C^2(\bar{\Omega})$, where $\bar{P}$ is the projection from X to the kernel. Then if we set $\bar{f} = \bar{P}h$ then it follows that such a v exists. □

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