NONSMOOTH VISCOSITY SOLUTIONS OF ELEMENTARY SYMMETRIC FUNCTIONS OF THE COMPLEX HESSIAN

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ABSTRACT. In this paper we prove the existence of nonsmooth viscosity solutions for Dirichlet problems involving elementary symmetric functions of the eigenvalues of the complex Hessian.

1. INTRODUCTION

Let $2 \le k \le n$. Here we will study the following equation

$$(1) \qquad \left(\sigma_k(\partial\bar\partial u)\right)^{\frac{1}{k}} = f(z, u, Du)$$

where, $\partial\bar\partial u$ is the complex Hessian of u in $\mathbb{C}^n$, Du is the Euclidean gradient of u and for every Hermitian $n \times n$ matrix M, $\sigma_k(M)$ denotes the k-th elementary symmetric function of the eigenvalues of M and f is a positive function.

The complex Hessian equation in domains of $\mathbb{C}^n$ was considered by S.-Y. Li in [11], where the author proves existence and uniqueness theorems for the Dirichlet problem for elliptic nonlinear partial differential equations that are concave symmetric functions of the eigenvalues of the complex Hessian and where the right hand side f only depends on z. The results are generalizations of those first established by L. A. Caffarelli, L. Nirenberg and J. Spruck [2] and later extended by several authors.

The complex Monge-Ampère equations have been investigated extensively over last years: we refer the reader to [8], [14] ,[13], [4], and references therein.

In this paper we will show that for any given smooth function f, under some suitable structural assumptions (see later), there always exist a small Euclidean ball B_r and a Lipschitz viscosity solution of (1) in B_r, which is not of class $C^1(B_r)$ if $k = 2$, and it is not of class $C^{1,\beta}(B_r)$, with $\beta > 1 - \frac{2}{k}$, if $k > 2$.

Let us fix some notations: we identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$, with $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j \simeq (x_j, y_j)$, for $j = 1, \ldots, n$ and we set

$$u_{z_j} = \frac{\partial u}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\Big) = \frac{1}{2}(u_{x_j} - iu_{y_j}), \qquad u_{\bar{j}} = \overline{u_j},$$

$$u_{j\bar{\ell}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_\ell} = \frac{1}{4}\Big\{\Big(\frac{\partial^2 u}{\partial x_j \partial x_l} + \frac{\partial^2 u}{\partial y_j \partial y_l}\Big) + i\Big(\frac{\partial^2 u}{\partial x_j \partial y_l} - \frac{\partial^2 u}{\partial y_j \partial x_l}\Big)\Big\} =$$

$$= \frac{1}{4}\Big\{\Big(u_{x_j x_l} + u_{y_j y_l}\Big) + i\Big(u_{x_j y_l} - u_{y_j x_l}\Big)\Big\}$$

Now we define the following $n \times 2n$ matrix

$$J := \frac{1}{2}\left(I_n, -iI_n\right)$$

and we observe that

$$\partial u = (u_{z_1}, \ldots, u_{z_n}) = J\,(u_{x_1}, \ldots, u_{x_n}, u_{y_1}, \ldots, u_{y_n}) = J\,Du$$

$$\partial\bar{\partial}u = J D^2 u\,\bar{J}^t$$

So, let Ω be a bounded open set in $\mathbb{C}^n$, we will consider the following Dirichlet problem:

(2)
$$\begin{cases} F(z, u, Du, \partial\bar{\partial}u) = 0, & \text{in } \Omega, \\ u = \phi, & \text{on } \partial\Omega, \end{cases}$$

where

(3)
$$F(z, u, Du, \partial\bar{\partial}u) := -\Big(\sigma_k\big(\partial\bar{\partial}u\big)\Big)^{\frac{1}{k}} + f(z, u, Du)$$

and ϕ is a continuous function defined on $\partial\Omega$.

We pointed out in the previous definition of F the dependence on the complex Hessian $\partial\bar{\partial}u$; anyway we see that this is equivalent to

$$F_J(z, u, Du, D^2 u) := F(z, u, Du, J D^2 u\,\bar{J}^t) = -\Big(\sigma_k\big(J D^2 u\,\bar{J}^t\big)\Big)^{\frac{1}{k}} + f(z, u, Du)$$

Since the equation (1) is not elliptic in general, we need to give the definition of some suitable cones as in [2]. First we define the open cone

$$\Gamma_k = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \sigma_j(diag(\lambda)) > 0, \text{ for every } j = 1, \ldots, k\},$$

where $diag(\lambda)$ is the $n \times n$ diagonal matrix with entries the λ_j, and we denote by $\overline{\Gamma}_k$ and $\partial\Gamma_k$ the closure and the boundary of Γ_k respectively.

We remark that F is elliptic in the cone Γ_k, i.e.

$$F(z, s, p, M) \leq F(z, s, p, N), \quad \text{for all } z \in \mathbb{C}^n, s \in \mathbb{R}, p \in \mathbb{R}^{2n},$$

and where M, N are $n \times n$ Hermitian matrices whose eigenvalues belong to the open cone Γ_k and such that $M \geq N$.

We also note that $\big(\sigma_k(\cdot)\big)^{\frac{1}{k}}$ is a concave function on the cone Γ_k.

Obviously, we also have that F_J is elliptic and concave in the set of $2n \times 2n$ real symmetric matrices M such that eigenvalues of $J M \bar{J}^t$ belong to the open cone Γ_k.

Therefore, we give the following

Definition 1.1. *Let $z_0 \in \mathbb{C}^n$ and let φ be a C^2 function in a neighborhood of z_0. We will say that φ is strictly k-plurisubharmonic, in brief strictly k-p.s.h. (respectively k-p.s.h.) at z_0, if the vector $\lambda = (\lambda_1, \ldots \lambda_n)$ of the eigenvalues of $\partial\bar{\partial}\varphi(z_0)$ belongs to the open cone Γ_k (respectively $\overline{\Gamma}_k$). We remark that the cone Γ_k is invariant with respect to the permutation of λ_j.*
We will say that φ is strictly k-p.s.h. (respectively k-p.s.h.) in $\Omega \subset \mathbb{C}^n$ if φ is strictly k-p.s.h. (respectively k-p.s.h.) at z_0 for every $z_0 \in \Omega$.
Moreover if $\rho : \mathbb{C}^n \to \mathbb{R}$ is a smooth defining function for a bounded open set $\Omega \subset \mathbb{C}^n$, that is

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}, \qquad \partial\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$$

then we will say that the domain Ω is strictly k-p.s.h. if ρ is strictly k-p.s.h.

In the sequel we will work essentially in the ball $B_r \subseteq \mathbb{C}^n$: by the previous definition, we see that for any $r > 0$ and for any $k = 1, \ldots, n$, the ball B_r is strictly k-p.s.h., since the defining function $\rho(z) = |z|^2 - r^2$ is strictly k-p.s.h.
We refer to [10], [3] for a full detailed exposition on the theory of viscosity solutions: we will give the basic definitions of sub- and super-solution.

Definition 1.2. *Let us consider the equation*

$$(4) \qquad\qquad F(z, u, Du, \partial\bar{\partial}u) = 0, \qquad in\ \Omega,$$

We say that an upper semicontinuous function u (in brief $u \in USC(\Omega)$) is a viscosity sub-solution for (4) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $z_0 \in \Omega$ is a local maximum for the function $u - \varphi$, then φ is k-p.s.h. at z_0 and

$$(5) \qquad\qquad F\big(z_0, u(z_0), D\varphi(z_0), \partial\bar{\partial}\varphi(z_0)\big) \leq 0.$$

We say that a lower semicontinuous function u (in brief $u \in LSC(\Omega)$) is a viscosity super-solution for (4) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $z_0 \in \Omega$ is a local minimum for the function $u - \varphi$, then either φ is k-p.s.h. at z_0 and

$$(6) \qquad\qquad F\big(z_0, u(z_0), D\varphi(z_0), \partial\bar{\partial}\varphi(z_0)\big) \geq 0$$

or φ is not k-p.s.h. at z_0.
A continuous function u is a viscosity solution for (4) if it is either a viscosity sub-solution and a viscosity super-solution for (4).
We say that a function $u \in USC(\overline{\Omega})$ is a viscosity sub-solution for (2) if u is a viscosity sub-solution for (4) and in addition $u \leq \phi$ on $\partial\Omega$.
We say that a function $u \in LSC(\overline{\Omega})$ is a viscosity super-solution for (2) if u is a viscosity super-solution for (4) and in addition $u \geq \phi$ on $\partial\Omega$.
A viscosity solution for (2) is either a viscosity sub-solution and a viscosity super-solution for (2).

We will also need a comparison principle for F in the set of k-p.s.h. functions to ensure the uniqueness of the viscosity solution of the Dirichlet problem (2). By following the analysis on comparison principle in [3], we see that if the function f is continuous, positive and increasing with respect to u, then F is proper in the set of k-p.s.h. functions, according to the definition in [3], and by using the fact that in the set of the Hermitian matrices M such that $\sigma_j(M) \geq 0$ for all $j \leq k$, the function $-(\sigma_k(M))^{1/k}$ is monotone decreasing, convex and homogeneous of degree one, i.e. $\sigma_k^{1/k}(\lambda M) = \lambda \sigma_k^{1/k}(M)$ for all $\lambda \in \mathbb{R}^+$, then F satisfies the hypotheses in [3]. So, we have

Proposition 1.1. *If the function f is continuous, positive and increasing with respect to u, then if $\underline{u}$ and $\overline{u}$ are respectively viscosity sub-and super-solution of (2) in B_r, such that $\underline{u} \leq \overline{u}$ on ∂B_r, then $\underline{u} \leq \overline{u}$ in $\overline{B}_r$.*

The paper is organized as follows. In Section 2 we prove the existence of a Lipschitz viscosity solution of (2) in B_r, as limit of a sequence $\{u^\varepsilon\}$ of smooth solutions of a regularized elliptic problem, whose gradient is bounded independently on ε. In Section 3 we prove our main result:

Theorem 1.1. *Let us denote by $|\partial_z f|, |f_u|, |f_p|$ the derivative of f with respect to its arguments. Suppose $2 \leq k \leq n$, and $f \in C^\infty(B_1 \times \mathbb{R} \times \mathbb{R}^{2n})$ is a positive function, monotone increasing with respect to u and there exists a constant $C > 0$ such that $|\partial_z f|, |f_u|, |f_p| \leq C$. Then there exist $R \in (0, 1)$ and a k-p.s.h. viscosity solution u to the equation*

$$(7) \qquad\qquad F(z, u, Du, \partial\bar{\partial}u) = 0 \quad in \quad B_R,$$

such that $u \in Lip(\bar{B}_R)$.
Moreover, if $k > 2$ then $u \notin C^{1,\beta}(\bar{B}_R)$ for any $\beta > 1 - \frac{2}{k}$, if $k = 2$ then $u \notin C^1(\bar{B}_R)$.

The proof of this theorem uses Pogorelov's counterexamples (see [6, Section 5.5]) and its extensions developed in [16] , [7], [12].

2. Existence of Lipschitz continuous viscosity solutions

Here we want to prove the existence of a Lipschitz viscosity solution of (2) in B_r, as limit of a sequence $\{u^\varepsilon\}$ of smooth solutions of a regularized problem, whose gradient is bounded independently on ε.
For any given $\varepsilon > 0$, we define

$$F^\varepsilon(z, u, Du, \partial\bar{\partial}u) := -\left(\sigma_k(\partial\bar{\partial}u + \varepsilon \, trace(\partial\bar{\partial}u)I_n)\right)^{\frac{1}{k}} + f(z, u, Du)$$

It turns out that $-F^\varepsilon$ is uniformly elliptic with ellipticity constants depending on ε, in particular we have:

Lemma 2.1. *Let $k = 1, \dots, n$. There exist constants $0 < \lambda_\varepsilon < \Lambda_\varepsilon$, depending on ε, such that*

$$\lambda_\varepsilon \, trace(N) \leq -F^\varepsilon(z, r, p, M + N) + F^\varepsilon(z, r, p, M) \leq \Lambda_\varepsilon \, trace(N)$$

for every $n \times n$ Hermitian and positive definite matrix N, for every $z \in \Omega, r \in \mathbb{R}, p \in \mathbb{R}^{2n}$, and for every $n \times n$ Hermitian matrix M such that the eigenvalues of $M + \varepsilon \ trace(M) \ I_n$ are in the cone Γ_k.

Proof. It's a straightforward computation, by taking into account that the functions $(\sigma_k)^{\frac{1}{k}}$ are homogeneous of degree one, monotone increasing and concave. We have

$$- F^\varepsilon(z, r, p, M + N) + F^\varepsilon(z, r, p, M)$$

$$= \left(\sigma_k \left(M + N + \varepsilon \ trace(M + N)I_n\right)\right)^{\frac{1}{k}} - \left(\sigma_k \left(M + \varepsilon \ trace(M)I_n\right)\right)^{\frac{1}{k}}$$

$$\geq \left(\sigma_k \left(N + \varepsilon \ trace(N)I_n\right)\right)^{\frac{1}{k}} \geq \left(\sigma_k \left(\varepsilon \ trace(N)I_n\right)\right)^{\frac{1}{k}}$$

$$\geq \varepsilon \ trace(N)\left(\sigma_k \left(I_n\right)\right)^{\frac{1}{k}} = \lambda_\varepsilon \ trace(N)$$

Moreover, by the monotonicity, the homogeneity and the concavity of $(\sigma_k)^{\frac{1}{k}}$ and by Lagrange theorem there is $\theta \in]0, 1[$ such that

$$- F^\varepsilon(z, r, p, M + N) + F^\varepsilon(z, r, p, M)$$

$$= \left(\sigma_k \left(M + N + \varepsilon \ trace(M + N)I_n\right)\right)^{\frac{1}{k}} - \left(\sigma_k \left(M + \varepsilon \ trace(M)I_n\right)\right)^{\frac{1}{k}}$$

$$\leq \left(\sigma_k \left(M + \varepsilon \ trace(M)I_n + (1 + \varepsilon) \ trace(N)I_n\right)\right)^{\frac{1}{k}} - \left(\sigma_k \left(M + \varepsilon \ trace(M)I_n\right)\right)^{\frac{1}{k}}$$

$$= \left(\partial_{r_{jj}}(\sigma_k)^{\frac{1}{k}} \left(M + \varepsilon \ trace(M)I_n + \theta(1 + \varepsilon) \ trace(N)I_n\right)\right) \cdot (1 + \varepsilon) \ trace(N)$$

$$\leq \left(\partial_{r_{jj}}(\sigma_k)^{\frac{1}{k}} \left(\theta(1 + \varepsilon) \ trace(N)I_n\right)\right) \cdot (1 + \varepsilon) \ trace(N)$$

$$= \left(\partial_{r_{jj}}(\sigma_k)^{\frac{1}{k}} \left(I_n\right)\right) \cdot (1 + \varepsilon) \ trace(N) = \Lambda_\varepsilon \ trace (N)$$

where $\partial_{r_{jj}}\left((\sigma_k(r))^{\frac{1}{k}}\right)$ denotes the sum in j of partial derivatives of $(\sigma_k)^{\frac{1}{k}}$ with respect to r_{jj}, which is homogeneous of degree zero. $\qquad \square$

Remark 2.1. *Since*

$$trace(\partial\bar\partial u) = trace(J \ D^2u \ \bar{J}^t) = \frac{1}{4}trace(D^2u) = \frac{1}{4}\Delta u$$

the previous lemma states the uniform ellipticity for any fixed $\varepsilon > 0$, with respect to the classical Euclidian Hessian, of the functional

$$F_J^\varepsilon(z, u, Du, D^2u) := -\left(\sigma_k(J \ D^2u \ \bar{J}^t + \frac{\varepsilon}{4} \ \Delta u \ I_n)\right)^{\frac{1}{k}} + f(z, u, Du)$$

in the set of the $2n \times 2n$ symmetric matrices M such that the eigenvalues of

$$J \ M \ \bar{J}^t + \frac{\varepsilon}{4} \ trace(M) \ I_n$$

are in the cone Γ_k.

We will need some structural assumptions on f, in order to ensure the existence of a smooth solution. Indeed, we assume that f is a smooth function and we require on $B_1 \times \mathbb{R} \times \mathbb{R}^{2n}$ the following hypotheses:

(H1) $f_u \geq 0$;
(H2) there exists a constant C such that $|\partial_z f|, |f_u|, |f_p| \leq C$.

We notice that hypothesis $(H1)$ will give us the uniqueness of the solution, by the comparison principle; on the other hand $(H2)$ will imply bounds for the second derivatives and for their Hölder seminorms as in [15], [9], since F^ε is uniformly elliptic in the sense of [15]. Estimates for higher derivatives follow from the linear uniformly elliptic theory [5, Lemma 17.16]. These estimates allow us to apply the method of continuity [5, Theorem 17.8]. Therefore, under hypotheses $(H1), (H2)$, for every $\phi \in C^{2,\alpha}(\bar{B}_r)$, $r \leq 1$, there exists a (unique) classical solution $u^\varepsilon \in C^{2,\alpha}(B_r)$ of the Dirichlet problem related to $F^\varepsilon = 0$ in B_r (from further regularity results, u^ε is actually C^∞); moreover u^ε is strictly k-p.s.h.

We will prove a gradient bound for u^ε, uniform in ε. Thus, by taking the uniform limit as ε goes to zero we will find a Lipschitz continuous viscosity solution u of the Dirichlet problem related to $F = 0$; in this last limit process we can use the stability property of the viscosity solutions with respect to the uniform convergence, since the sets of k-p.s.h. functions satisfy a crucial property of inclusion as ε decreases (see for instance [16]).

We will make use of particular sub- and super-solutions that we will build with the help of a suitable convex function ϕ.

We have, for $2 \leq k \leq n$:

Proposition 2.1. *Let f be a positive function satisfying $(H1), (H2)$ and let $\phi \in C^{2,\alpha}(\bar{B}_1) \cap Lip(\bar{B}_1)$ be a convex function such that $\sigma_k(\partial\bar{\partial}\phi) = 0$ in B_1. Then there exists $r_0 \leq 1$ such that for any $0 < r < r_0$, the problem (2) in B_r has a viscosity solution $u \in Lip(\bar{B}_r)$ satisfying*

$$(8) \qquad \|u\|_{L^\infty(\bar{B}_r)} + \|u\|_{Lip(\bar{B}_r)} \leq C,$$

where C only depends on r, $\|\phi\|_{L^\infty(\bar{B}_1)}$, $\|D\phi\|_{L^\infty(\bar{B}_1)}$

We will prove the Proposition 2.1 in some steps. First of all, since $-F^\varepsilon$ is uniformly elliptic, by the hypotheses on f, there exists a smooth solution u^ε of the problem

$$(9) \qquad \begin{cases} F^\varepsilon(z, u, Du, \partial\bar{\partial}u) = 0, & \text{in } B_r, \\ u = \phi, & \text{on } \partial B_r, \end{cases}$$

Moreover, applying the technique of [1, Lemma 1.4], we get

$$(10) \qquad \sup_{B_r} |Du^\varepsilon| \leq \sup_{\partial B_r} |Du^\varepsilon| + c$$

where c is a positive constant independent of ε and only depending on the constant C in $(H2)$. Now, we are going to find explicitly global sub- and super-solutions.

Lemma 2.2. *Let $\phi \in C^2(\overline{B}_1) \cap Lip(\overline{B}_1)$ be a convex function such that $\sigma_k(\partial\bar{\partial}\phi) = 0$ in B_1. Then there exists $\varepsilon_0 \leq 1$ such that for every $0 < \varepsilon < \varepsilon_0$ we have*

(11)
$$F^\varepsilon(z, \phi, D\phi, \partial\bar{\partial}\phi) > 0$$

Proof. Let us define
$$\inf_{B_1 \times \mathbb{R} \times \mathbb{R}^{2n}} f = c_0 > 0$$

Therefore
$$F^\varepsilon(z, \phi, D\phi, \partial\bar{\partial}\phi) = -\sigma_k\left(\partial\bar{\partial}\phi + \varepsilon \ trace(\partial\bar{\partial}\phi)I_n\right)^{1/k} + f(z, \phi, D\phi) \geq$$

$$\geq -\sigma_k\left(\partial\bar{\partial}\phi + \varepsilon \ trace(\partial\bar{\partial}\phi)I_n\right)^{1/k} + c_0$$

We know there exist positive constants $\{c_1, \dots, c_n\}$ such that
$$0 \leq c_n\left(\sigma_n\left(\partial\bar{\partial}\phi\right)\right)^{1/n} \leq \dots \leq c_k\left(\sigma_k\left(\partial\bar{\partial}\phi\right)\right)^{1/k} \leq \dots \leq c_1 \ trace(\partial\bar{\partial}\phi)$$

Since $\sigma_k(\partial\bar{\partial}\phi) = 0$ in B_1, we have that, if
$$\sup_{z \in B_1} \ trace(\partial\bar{\partial}\phi(z)) = 0$$

then $\partial\bar{\partial}\phi = 0$ in B_1, therefore for any positive ε
$$F^\varepsilon(z, \phi, D\phi, \partial\bar{\partial}\phi) = f(z, \phi, \partial\phi) \geq c_0 > 0$$

in B_1. On the other hand, let
$$\sup_{z \in B_1} \ trace(\partial\bar{\partial}\phi(z)) = M > 0$$

then we have, for $0 < \varepsilon \leq 1$,
$$\sigma_k\left(\partial\bar{\partial}\phi + \varepsilon \ trace(\partial\bar{\partial}\phi)I_n\right) = \sigma_k\left(\partial\bar{\partial}\phi\right) + (\varepsilon \ trace(\partial\bar{\partial}\phi))\sigma_{k-1}\left(\partial\bar{\partial}\phi\right) + \dots$$

$$+(\varepsilon \ trace(\partial\bar{\partial}\phi))^{k-1} \ trace(\partial\bar{\partial}\phi) + (\varepsilon \ trace(\partial\bar{\partial}\phi))^k \leq$$

$$\leq (\varepsilon \ trace(\partial\bar{\partial}\phi))\frac{c_1}{c_{k-1}}(\ trace(\partial\bar{\partial}\phi))^{k-1} + \dots$$

$$+(\varepsilon \ trace(\partial\bar{\partial}\phi))^{k-1} \ trace(\partial\bar{\partial}\phi) + (\varepsilon \ trace(\partial\bar{\partial}\phi))^k \leq$$

$$\leq kc^*\varepsilon(\ trace(\partial\bar{\partial}\phi))^k \leq kc^*\varepsilon M^k$$

where
$$c^* = \max\left\{\frac{c_1}{c_{k-1}}, \dots, \frac{c_1}{c_2}\right\}$$

Therefore, if we let $\varepsilon_0 = \min\{1, \frac{c_0^k}{kc^*M^k}\}$, we get
$$F^\varepsilon(z, \phi, D\phi, \partial\bar{\partial}\phi) = -\sigma_k\left(\partial\bar{\partial}\phi + \varepsilon \ trace(\partial\bar{\partial}\phi)I_n\right)^{1/k} + f(z, \phi, D\phi) \geq$$

$$\geq -(kc^*\varepsilon M^k)^{1/k} + c_0 > 0$$

for any $0 < \varepsilon < \varepsilon_0$. $\qquad\square$

Lemma 2.3. *Let $\phi \in C^2(\overline{B}_1) \cap Lip(\overline{B}_1)$ be a convex function. For positive λ and r, we define*

$$u_\lambda(z) := \phi(z) + \lambda\rho(z),$$

where $\rho(z) = |z|^2 - r^2$. Then, there exists $\lambda^* \geq 1$, depending on $\sup_{B_1} |\phi|$ and $\sup_{B_1} |D\phi|$, such that

(12) $$F^\varepsilon(z, u_{\lambda^*}, Du_{\lambda^*}, \partial\bar{\partial}u_{\lambda^*}) < 0 \text{ in } B_r,$$

for any $r < 1/\lambda^*$.

Proof. Since ϕ is a plurisubharmonic function,

$$\partial\bar{\partial}u_\lambda = \partial\bar{\partial}\phi + \lambda I_n \geq \lambda I_n$$

In particular u_λ is k-p.s.h. for every $\lambda > 0$ and

$$\partial\bar{\partial}u_\lambda + \varepsilon \, trace(\partial\bar{\partial}u_\lambda) \, I_n \geq \partial\bar{\partial}u_\lambda$$

Moreover, as a consequence of the monotonicity of the function $\sigma_k^{1/k}(\cdot)$ and of its homogeneity, i.e. $\sigma_k^{1/k}(\lambda\cdot) = \lambda\sigma_k^{1/k}(\cdot)$ for every $\lambda > 0$, we get

$$F^\varepsilon(z, u_\lambda, Du_\lambda, \partial\bar{\partial}u_\lambda) \leq F(z, u_\lambda, Du_\lambda, \partial\bar{\partial}u_\lambda) =$$

$$= -\sigma_k\left(\partial\bar{\partial}u_\lambda\right)^{1/k} + f(z, u_\lambda, Du_\lambda) \leq -\lambda + f(z, \phi(z) + \lambda\rho(z), D\phi(z) + 2\lambda z)$$

Now, let

$$L := \sup_{z\in B_1} |D\phi(z)|, \qquad E := \sup_{(z,p)\in B_1\times B_{L+2}} f(z, \sup_{B_1}\phi, p), \qquad \lambda^* := \max\{1, E\}$$

so, if $r < 1/\lambda^*$ and $z \in B_r$, it holds

$$|Du_{\lambda^*}(z)| = |D\phi(z) + 2\lambda^* z| \leq \sup_{B_r}|D\phi| + 2\lambda^* r \leq \sup_{B_1}|D\phi| + 2\lambda^* r \leq L + 2$$

Then, since ρ is negative in B_r and f is increasing with respect to u, we have for any $r < 1/\lambda^*$

$$f(z, \phi(z) + \lambda^*\rho(z), D\phi(z) + 2\lambda^* z) \leq f(z, \sup_{B_1}\phi, D\phi(z) + 2\lambda^* z) \leq$$

$$\leq \sup_{(z,p)\in B_1\times B_{L+2}} f(z, \sup_{B_1}\phi, p) \leq \lambda^*$$

and this ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 2.1 . Let $u_{\lambda^*} = \phi + \lambda^*\rho$ be the function given by the λ^* as defined in the previous lemma. So, if $r < r_0 := 1/\lambda^*$, then $u_{\lambda^*} \in C^2(\overline{B}_r)$ and it is a classical sub-solution to $F^\varepsilon = 0$ in B_r. Moreover, $u_{\lambda^*} = \phi$ on ∂B_r.
On the other hand, if $\varepsilon < \varepsilon_0$ then ϕ is a classical super-solution to $F^\varepsilon = 0$ in B_r.
So, by the comparison principle we have

$$u_{\lambda^*} \leq u^\varepsilon \leq \phi \text{ in } \overline{B}_r, \quad r < r_0, \quad \varepsilon < \varepsilon_0$$

Since

$$|u_{\lambda^*}(z)| = |\phi(z) + \lambda^* \rho(z)| \leq \sup_{B_r} |\phi| + \lambda^* r^2 \leq \|\phi\|_{L^\infty(\bar{B}_1)} + r$$

and

$$|Du_{\lambda^*}(z)| = |D\phi(z) + 2\lambda^* z| \leq \sup_{B_r} |D\phi| + 2\lambda^* r \leq \|D\phi\|_{L^\infty(\bar{B}_1)} + 2$$

then, if u denotes the uniform limit of u^ε as ε goes to zero, we can conclude that $u \in \mathrm{Lip}\,(\bar{B}_r)$ independently on ε and

(13)
$$\|u\|_{L^\infty(\bar{B}_r)} + \|u\|_{\mathrm{Lip}\,(\bar{B}_r)} \leq C,$$

with C depending on r, $\|\phi\|_{L^\infty(\bar{B}_1)}$, $\|D\phi\|_{L^\infty(\bar{B}_1)}$ □

3. Existence of nonsmooth solutions

Throughout this section we denote by

$$z = (z_1, z', z''), \quad z' = (z_2, \ldots, z_k), \quad z'' = (z_{k+1}, \ldots, z_n)$$

For $0 \leq \varepsilon < 1$ and $0 < r < r_0$ such that Proposition 2.1 holds true, we define

(14)
$$w_\varepsilon(z) = w_\varepsilon(z) := (r^2 + |z_1|^2)(\varepsilon^2 + |z'|^2)^\alpha, \quad \alpha = 1 - \frac{1}{k},$$

and

$$\psi_\varepsilon(z) := M w_\varepsilon(z), \quad \phi_\varepsilon(z) = \phi_\varepsilon(z) := 2M(\varepsilon^2 + |z'|^2)^\alpha,$$

with M a positive constant to be determined.

Lemma 3.1. *There exists $M = M(r)$ such that*

(15)
$$F(z, \psi_\varepsilon, D\psi_\varepsilon, \partial\bar{\partial}\psi_\varepsilon) < 0 \quad in \ B_r, \quad \forall\, \varepsilon \in]0, r[.$$

Proof. First, $w_\varepsilon(z)$ is independent on z'' therefore $\partial\bar{\partial}w_\varepsilon(z)$ has $n - k$ null rows by construction and so

(16)
$$\sigma_k(\partial\bar{\partial}w_\varepsilon(z)) = \det \partial\bar{\partial}_{(z_1, z')} w_\varepsilon(z).$$

Direct computations show that

(17)
$$\det \partial\bar{\partial}_{(z_1, z')} w_\varepsilon(z) = f_\varepsilon(z)$$

with

$$f_\varepsilon(z) = \alpha^k (r^2 + |z_1|^2)^{k-2} \frac{r^2(\alpha^{-1}\varepsilon^2 + |z'|^2) + \alpha^{-1}\varepsilon^2|z'|^2}{(\varepsilon^2 + |z'|^2)}.$$

Indeed, we have

$$\partial\bar{\partial}_{(z_1, z')} w_\varepsilon(z) = (\varepsilon^2 + |z'|^2)^{\alpha-1} \begin{pmatrix} \varepsilon^2 + |z'|^2 & \alpha\bar{z}_1\, z' \\ \alpha z_1\, (\bar{z}')^T & (r^2 + |z_1|^2)\left(\alpha I_{k-1} + \alpha(\alpha-1)\frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2}\right) \end{pmatrix}$$

Since $(\alpha - 1)k = -1$, we get

$$\det \partial \bar{\partial}_{(z_1,z')} w_\varepsilon(z) =$$

$$= (\varepsilon^2 + |z'|^2)^{-1} \det \begin{pmatrix} \varepsilon^2 + |z'|^2 & \alpha \bar{z}_1 \, z' \\ \alpha z_1 \, (\bar{z}')^T & (r^2 + |z_1|^2)\left(\alpha I_{k-1} + \alpha(\alpha - 1)\frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2} \right) \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & \alpha(\varepsilon^2 + |x'|^2)^{-1/2} \, \bar{z}_1 \, z' \\ \alpha(\varepsilon^2 + |x'|^2)^{-1/2} \, z_1 \, (\bar{z}')^T & (r^2 + |z_1|^2)\left(\alpha I_{k-1} + \alpha(\alpha - 1)\frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2} \right) \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 \\ \alpha(\varepsilon^2 + |x'|^2)^{-1/2} \, z_1 \, (\bar{z}')^T & (r^2 + |z_1|^2)\left(\alpha I_{k-1} + \alpha(\alpha - 1)\frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2} \right) - \alpha^2 |z_1|^2 \frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2} \end{pmatrix}$$

$$= \det \left((r^2 + |z_1|^2)\left(\alpha I_{k-1} + \alpha(\alpha - 1)\frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2} \right) - \alpha^2 |z_1|^2 \frac{z' \otimes \bar{z}'}{\varepsilon^2 + |z'|^2} \right)$$

$$:= \det \Gamma,$$

where Γ is a $(k-1) \times (k-1)$ symmetric matrix. It is easy to see that $\lambda_1 = \alpha(r^2 + |z_1|^2)$ is an eigenvalue of Γ with multiplicity $k - 2$. Now, trace $\Gamma = (k - 2)\lambda_1 + \lambda_2$ with

$$\lambda_2 = (r^2 + |z_1|^2)\left(\alpha + \alpha(\alpha - 1)\frac{|z'|^2}{\varepsilon^2 + |z'|^2} \right) - \alpha^2 |z_1|^2 \frac{|z'|^2}{\varepsilon^2 + |z'|^2}$$

$$= \alpha^2 \frac{r^2 \left(\frac{\varepsilon^2}{\alpha} + |z'|^2 \right) + \frac{\varepsilon^2}{\alpha}|z'|^2}{\varepsilon^2 + |z'|^2}$$

Thus

$$\det \Gamma = \lambda_1^{k-2} \lambda_2 = \alpha^k (r^2 + |z_1|^2)^{k-2} \frac{r^2 \left(\frac{\varepsilon^2}{\alpha} + |z'|^2 \right) + \frac{\varepsilon^2}{\alpha}|z'|^2}{\varepsilon^2 + |z'|^2} = f_\varepsilon$$

which completes the proof of (17). In particular, if $\varepsilon < r$

$$f_\varepsilon \geq \alpha^k r^{2(k-1)} \geq \frac{\alpha^{k-1} r^{2(k-1)}}{2}$$

Since $\psi_\varepsilon = Mw_\varepsilon$, by (16), we can choose r small, such that

$$\sigma_k \left(\partial \bar{\partial} \psi_\varepsilon \right)^{1/k} > \alpha^\alpha r^{2\alpha} M.$$

On the other side, direct computations show that

$$|Dw_\varepsilon|^2 = 4\left(|z_1|^2 (\varepsilon^2 + |z'|^2)^{2\alpha} + \alpha^2 |z'|^2 (r^2 + |z_1|^2)^2 (\varepsilon^2 + |z'|^2)^{2(\alpha - 1)} \right)$$

and for every $\varepsilon \in \,]0, r[$,

$$(18) \qquad\qquad |Dw_\varepsilon|^2 \leq 2^{2\alpha + 3} r^{4\alpha + 2} \quad \text{in } B_r.$$

From (18), we obtain

$$(19) \qquad\qquad |D\psi_\varepsilon| \leq 2^{\alpha + 3/2} M r^{2\alpha + 1} \quad \text{in } B_r.$$

Choosing $M = 2^{-\alpha-(3/2)}r^{-2\alpha-1}$, the right hand side of (19) equals 1, and $\psi_\varepsilon \leq 2^{-1/2}r < 1$. The strategy now is to take r such that

$$\sup_{(z,p)\in B_1\times B_1} f(z,1,p) < \alpha^\alpha 2^{-\alpha-(3/2)}r^{-1}.$$

Then, by the increasing monotonicity of $s \to f(\cdot, s, \cdot)$, in B_r we obtain

$$F(z, \psi_\varepsilon, D\psi_\varepsilon, \partial\bar\partial\psi_\varepsilon) < -\alpha^\alpha 2^{-\alpha-(3/2)}r^{-1} + f(z, \psi_\varepsilon, D\psi_\varepsilon)$$
$$< -\alpha^\alpha 2^{-\alpha-(3/2)}r^{-1} + f(z, 1, D\psi_\varepsilon) < 0.$$

$\square$

Proof of Theorem 1.1. We have

$$\psi_0 \leq \psi_\varepsilon \leq \phi_\varepsilon, \quad \text{in } B_1.$$

Since $k \geq 2$, the exponent $\alpha \geq \frac{1}{2}$ and so ϕ_ε is convex for $\varepsilon \geq 0$.

Moreover, ϕ_ε is smooth for $\varepsilon > 0$, and independent of z_1 and z'', therefore $\partial\bar\partial\phi_\varepsilon$ has $n - k + 1$ null eigenvalues. Therefore:

(20) $\qquad F(z, \phi_\varepsilon, D\phi_\varepsilon, \partial\bar\partial\phi_\varepsilon) = f(z, \phi_\varepsilon, D\phi_\varepsilon) > 0 \quad \text{in } B_1, \quad \forall\, \varepsilon \in\,]0, 1[.$

Thus, applying Proposition 2.1, there exists $0 < r < r_0$ such that the Dirichlet problem

$$F = 0 \quad \text{in } B_r, \qquad u = \phi_\varepsilon \quad \text{on } \partial B_r,$$

with $\varepsilon \in\,]0, 1[$, has a viscosity solution u_ε such that

$$\|u_\varepsilon\|_{L^\infty(\overline{B}_r)} + \|u_\varepsilon\|_{Lip(\overline{B}_r)} \leq C(r, \varepsilon, M)$$

with $C(r, \varepsilon, M)$ depending on ε only through

$$C(\phi_\varepsilon) := \|\phi_\varepsilon\|_{L^\infty(\overline{B}_r)} + \|D\phi_\varepsilon\|_{L^\infty(\overline{B}_r)}.$$

On the other hand, an elementary computation shows that $C(\phi_\varepsilon) \leq 8M$. Then, we can choose $C(r, \varepsilon, M)$ independent of ε, and so

(21) $\qquad\qquad\qquad \|u_\varepsilon\|_{L^\infty(\overline{B}_r)} + \|u_\varepsilon\|_{Lip(\overline{B}_r)} \leq C(r, M).$

Now we can use the Comparison Principle to compare u_ε with ψ_ε and ϕ_ε. Indeed, if $\varepsilon < r$, by (20) and Lemma 3.1, ϕ_ε and ψ_ε are, respectively, classical super-solution and sub-solution to $F = 0$ in B_r. On the other hand $\psi_\varepsilon \leq \phi_\varepsilon$ in B_1, in particular, $\psi_\varepsilon \leq \phi_\varepsilon$ on ∂B_r. Thus, by the Comparison Principle,

(22) $\qquad\qquad\qquad \psi_\varepsilon \leq u_\varepsilon \leq \phi_\varepsilon \quad \text{in } B_r, \quad \forall \varepsilon \in\,]0, r[.$

The uniform estimate (21) implies the existence of a sequence $\varepsilon_j \searrow 0$ such that $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ uniformly converges to a viscosity solution $u \in Lip(\overline{B}_r)$ to the Dirichlet problem

$$F = 0 \quad \text{in } B_r, \quad u = \phi_0 \text{ on } \partial B_r.$$

Moreover, from the comparison principle, we get

(23) $\qquad\qquad\qquad \psi_0 \leq u \leq \phi_0 \quad \text{in } B_r.$

So

(24)
$$Mr^2|z_1|^{2\alpha} \le u(z_1, 0, \ldots, 0) \le 2M|z_1|^{2\alpha}.$$

and in particular

(25)
$$Mr^2|x_1|^{2\alpha} \le u((x_1, 0), 0, \ldots, 0) \le 2M|x_1|^{2\alpha}.$$

As in the proof of [7, Theorem 1] inequalities in (25) imply, if $k > 2$:

$$\partial_{x_1} u \notin C^\beta, \text{ for every } \beta > 2\alpha - 1 = 1 - \frac{2}{k} \quad \text{if } 2\alpha > 1 \text{ (i.e. } k > 2).$$

Indeed, if $2\alpha > 1$, then $\partial_{x_1} u(0) = 0 = u(0)$ so that, if u was $C^{1,\beta}$, with $\beta > 2\alpha - 1$, we would have $u((x_1, 0), 0, \ldots, 0) \le C|x_1|^{1+\beta}$ for a suitable $C > 0$ and for every x_1 sufficiently small. Hence, by the first inequality in (25), we would have $\beta \le 2\alpha - 1$, a contradiction.

If $k = 2$, in the same way, we see that $\partial_{x_1} u$ is not continuous and this ends the proof. $\qquad\square$

References

[1] L. Caffarelli, J.J. Kohn, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations.* Comm. Pure Appl. Math. 38 (1985), no. 2, 209–252.

[2] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian,* Acta Math. 155 (1985), 261–301.

[3] M.G. Crandall, H.Ishii and P.L. Lions, *User's guide to viscosity solutions of second order Partial differential equations,* Bull. Amer. Soc. 27, (1992), 1–67.

[4] S. Dinew; S. Kolodziej, *A priori estimates for complex Hessian equations.* Anal. PDE 7 (2014), no. 1, 227–244.

[5] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order,* second edition, Springer-Verlag, 1983.

[6] C. E. Gutiérrez, *T*he Monge-Ampère Equation. Progress in Nonlinear Differential Equations and Their Applications, Birkhaüser 2001.

[7] C. E. Gutiérrez, E. Lanconelli, A. Montanari, *Nonsmooth hypersurfaces with smooth Levi curvature* Nonlinear Analysis 76 (2013) 115–121.

[8] S. Kolodziej, *The complex Monge-Ampère equation.* Acta Math. 180(1), (1998) 69–117.

[9] Krylov, N.V., *Nonlinear elliptic and parabolic equations of the second order,* Dordrecht Reidel, 1987.

[10] H. Ishii, P.L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations,* J. Differential Equations 83, 1, (1990), 26–78.

[11] S. Y. Li, *On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian.* Asian J. Math. 8 (2004), no. 1, 87–106.

[12] V. Martino, A. Montanari, *Nonsmooth solutions for a class of fully nonlinear PDE's on Lie groups,* Nonlinear Analysis: Theory, Methods & Applications. (2015) doi:10.1016/j.na.2015.02.009

[13] N.C. Nguyen, *Hölder continuous solutions to complex Hessian equations.* Potential Anal. 41 (2014), no. 3, 887–902.

[14] D.H. Phong, J. Song, J. Sturm, *Complex Monge-Ampère equations.* Surv. Differ. Geom. 17 (2012), 327–411

[15] N.S. Trudinger, *Fully nonlinear, uniformly elliptic equations under natural structurale conditions.* Trans. Amer. Math. Soc. 278 (1983) 751-769.

[16] John I. E. Urbas, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J. 39 (1990), no. 2, 355–382

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