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Anno Accademico 2012-13

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# Existence result for the CR-Yamabe equation.

18 Aprile 2013

Abstract

In this note we will prove that the CR-Yamabe equation has infinitely many changing-sign solutions. The problem is variational but the associated functional does not satisfy the Palais-Smale compactness condition; by mean of a suitable group action we will define a subspace on which we can apply the minimax argument of Ambrosetti-Rabinowitz. The result solves a question left open from the classification results of positive solutions by Jerison-Lee in the '80s.

## 1. Introduction

The results in this note are parts of a joint paper written with Ali Maalaoui [12].

Here we will prove that the Yamabe equation on the standard CR-sphere $(\mathbb{S}^{2n+1}, \theta_0)$ has infinitely many changing-sign solutions. However, for the sake of simplicity we will start our presentation on the Heisenberg group $\mathbb{H}^n$, then we will move to the sphere by using the Cayley transform.
Hence let us consider the following CR-Yamabe equation on the whole Heisenberg group

$$-\Delta_{\mathbb{H}} u = |u|^{\frac{4}{q-2}} u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{1}$$

Here $\Delta_{\mathbb{H}}$ denotes the sub-Laplacian of the group, $q = 2n+2$ is the homogenous dimension of $\mathbb{H}^n$, and $S_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev type space on $\mathbb{H}^n$ (see [9]).
We recall that for the positive solutions of (1), Jerison and Lee in [11] gave a complete classification. The problem is variational but, as in the Riemannian case, the functional associated with the equation (1) fails to satisfy the Palais-Smale compactness condition. For the classical Yamabe equation on $\mathbb{R}^n$, the first result in this direction was proved by Ding in [8]: following the analysis by Ambrosetti and Rabinowitz [1], he found a suitable subspace $X$ of the space of the variations for the related functional, on which he performed the minimax argument. The same argument was then used by Saintier in [14] for the Yamabe equation on $\mathbb{R}^n$ involving the bi-Laplacian operator.
Later on, many authors proved the existence of changing-sign solutions using other kinds of variational methods (see [2], [3] and the references therein). Finally in a couple of recent works [6], [7], M. del Pino, M.Musso, F.Pacard and A.Pistoia found changing-sign solutions, different from those of Ding, by using a superposition of positive and negative *bubbles* arranged on some special sets.
We are going to use the approach of Ding. Therefore, by mean of the Cayley transform we will set the problem on the sphere $\mathbb{S}^{2n+1}$ and with the help of the group of the isometries generated by the Reeb vector field of the standard contact form on $\mathbb{S}^{2n+1}$, we will be able to exhibit a suitable closed subspace on which we can apply the minimax argument for the restriction of the functional associated with the equation (1).

## 2. Preliminaries

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ be the Heisenberg group. If we denote by $\xi = (z,t) = (x+iy,t) \simeq (x,y,t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ then the group law is given by

$$\xi_0 \centerdot \xi = (x + x_0, y + y_0, t + t_0 + 2(x \cdot y_0 - x_0 \cdot y)),\ \forall \xi, \xi_0 \in \mathbb{H}^n$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. The left translations are defined by

$$\tau_{\xi_0}(\xi) := \xi_0 \centerdot \xi$$

Finally the dilations of the group are

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$$

for any $\lambda > 0$. Moreover we will denote by $q = 2n + 2$ the homogeneous dimension of the group. The canonical left-invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

The horizontal (or intrinsic) gradient of the group is

$$D_{\mathbb{H}} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$$

The Kohn Laplacian (or sub-Laplacian) on $\mathbb{H}^n$ is the following second order operator invariant with respect to the left translations and homogeneous of degree two with respect to the dilations:

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{n} X_j^2 + Y_j^2$$

Let now

$$q^* = \frac{2q}{q-2}$$

then the following Sobolev-type inequality holds

$$\|\varphi\|_{q^*}^2 = \left( \int_{\mathbb{H}^n} |\varphi|^{q^*} \right)^{\frac{2}{q^*}} \le C \int_{\mathbb{H}^n} |D_{\mathbb{H}}\varphi|^2 = C\|D_{\mathbb{H}}\varphi\|_2^2, \qquad \forall \varphi \in C_0^\infty(\mathbb{H}^n)$$

with $C$ a positive constant. For every domain $\Omega \subseteq \mathbb{H}^n$, the Folland-Stein Sobolev space $S_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\| = \|D_{\mathbb{H}}\varphi\|_2$$

The exponent $q^*$ is called critical since the embedding

$$S_0^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$$

is continuous but not compact for every domain $\Omega \subseteq \mathbb{H}^n$.
Let us now consider the following Yamabe type equation on the Heisenberg group $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}} u = |u|^{q^*-2}u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{2}$$

The idea is to consider the problem, after the Cayley transform, on the sphere $\mathbb{S}^{2n+1}$ and there we will be able to find a suitable closed subspace $X \subseteq S^1(\mathbb{S}^{2n+1})$, compactly embedded in $L^{q^*}(\mathbb{S}^{2n+1})$, on which we can apply the minimax argument for the functional associated with the equation.
We recall that a solution of the problem (2) on $\mathbb{H}^n$ can be found as critical point of the following functional

$$J : S_0^1(\mathbb{H}^n) \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\int_{\mathbb{H}^n} |D_{\mathbb{H}} u|^2 - \frac{1}{q^*}\int_{\mathbb{H}^n} |u|^{q^*}$$

Moreover any variational solution is actually a classical solution [9].
We will prove the following

**Theorem 2.1.** *There exists a sequence of solutions $\{u_k\}$ of (2), with*

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u_k|^2 \longrightarrow \infty, \qquad as \quad k \to \infty$$

Theorem (2.1) will imply then that equation (2) has infinitely many changing-sign solutions: in fact by a classification result by Jerison and Lee [11] all the positive solutions of the equation (2) are in the form

$$u = \omega_{\lambda,\xi} = \lambda^{\frac{2-q}{2}} \omega \circ \delta_{\frac{1}{\lambda}} \circ \tau_{\xi^{-1}}$$

for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$, where

$$\omega(x, y, t) = \frac{c_0}{\left((1 + |x|^2 + |y|^2)^2 + t^2\right)^{\frac{q-2}{4}}}$$

with $c_0$ a positive constant; in particular all the solutions $\omega_{\lambda,\xi}$ have the same energy.

## 3. Proof of Theorem (2.1)

Let us consider the sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ defined by

$$\mathbb{S}^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \ s.t. \quad |z_1|^2 + \ldots + |z_{n+1}|^2 = 1\}$$

As in the Riemannian case, we will use an analogous of the stereographic projection. The Cayley transform is the CR-diffeomorphism between the sphere minus a point and the Heisenberg group

$$F : \mathbb{S}^{2n+1} \setminus \{0, \ldots, 0, -1\} \to \mathbb{H}^n$$

$$F(z_1, \ldots, z_{n+1}) = \Big(\frac{z_1}{1+z_{n+1}}, \ldots, \frac{z_n}{1+z_{n+1}}, Re\big(i\frac{1-z_{n+1}}{1+z_{n+1}}\big)\Big)$$

Denoting by $\theta_0$ the standard contact form on $\mathbb{S}^{2n+1}$ and by $\Delta_{\theta_0}$ the related sub-Laplacian, a direct computation show that equation (2) becomes

$$-\Delta_{\theta_0} v + c_n v = |v|^{\frac{4}{q-2}} v, \qquad v \in S^1(\mathbb{S}^{2n+1}) \tag{3}$$

with $c_n$ a suitable positive dimensional constant related to the (constant) Webster curvature of the sphere (see [10] for a full detailed exposition); in particular by setting $u \circ F = v\varphi$ (where $\varphi$ is the function that gives the conformal factor in the change of the contact form) we have that every solution $u$ of (2) corresponds to a solution $v$ of (3) and it holds

$$\int_{\mathbb{H}^n} |D_{\mathbb{H}} u|^2 = \int_{\mathbb{S}^{2n+1}} |v|^{q^*}$$

At this point we can consider the variational problem on the sphere

$$I : S^1(\mathbb{S}^{2n+1}) \to \mathbb{R}, \qquad I(v) = \frac{1}{2}\int_{\mathbb{S}^{2n+1}} \big(|D_{\theta_0} v|^2 + c_n v^2\big) - \frac{1}{q^*}\int_{\mathbb{S}^{2n+1}} |v|^{q^*}$$

Now we notice that $I$ meets the structural conditions of the Mountain-Pass Lemma, but since the embedding

$$S^1(\mathbb{S}^{2n+1}) \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$$

is not compact, the functional $I$ does not satisfy the Palais-Smale condition. The following lemma by Ambrosetti and Rabinowitz gives a condition on some particular subspaces of the space of variations on which it is allowed to perform the minimax argument; we will omit the proof (see Theorems 3.13 and 3.14 in [1])

**Lemma 3.1.** *Let $X$ be a closed subspace of $S^1(\mathbb{S}^{2n+1})$. Suppose that the embedding $X \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$ is compact. Then $I_{|X}$, the restriction of $I$ on $X$, satisfies the Palais-Smale condition. Furthermore, if $X$ is infinite-dimensional, then $I_{|X}$ has a sequence of critical points $\{v_k\}$ in $X$, such that*

$$\int_{\mathbb{S}^{2n+1}} |v_k|^{q^*} \longrightarrow \infty, \qquad as \quad k \to \infty$$

Now let us suppose that we can find a closed and compact group $G$ such that the functional $I$ is invariant under the action of $G$, namely:

$$I(v) = I(v \circ g), \qquad \forall\, g \in G$$

Let us set

$$X_G = \{v \in S^1(\mathbb{S}^{2n+1})\ s.t. \quad v = v \circ g,\ \forall\, g \in G\}$$

Then if $X_G$ satisfies the condition of Lemma (3.1), by the Principle of Symmetric Criticality [13], any critical point of the restriction $I_{|X_G}$ is also a critical point of $I$ on the whole space of variations. We are going to prove that such a $X_G$ exists. First we observe the following fact:

**Lemma 3.2.** *The Reeb vector field $T$ related to the standard contact form $\theta_0$ on $\mathbb{S}^{2n+1}$ is a Killing vector field.*

*Proof.* The proof is straightforward. The vector field $T$ is Killing if

$$g_0(\nabla_V T, W) + g_0(V, \nabla_W T) = 0, \qquad \forall \quad V, W \in T(\mathbb{S}^{2n+1}) \tag{4}$$

where $g_0$ is the metric induced by $\mathbb{C}^{n+1}$ and $\nabla$ is the Levi-Civita connection (we will call $g_0$ and $\nabla$ also the standard metric and the related Levi-Civita connection in $\mathbb{C}^{n+1}$). Since we are on the sphere we can consider an orthonormal basis on $T(\mathbb{S}^{2n+1})$ of eigenvectors of the Weingarten operator $A$ (we recall that for every $V \in T(\mathbb{S}^{2n+1})$, then $A(V) := -\nabla_V N$) with eigenvalues all equal to 1:

$$E = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}, \qquad Y_j = JX_j, \quad j = 1, \ldots, n$$

where $J$ is the standard complex structure on $\mathbb{C}^{n+1}$. Moreover $T = JN$, with $N$ the inward unit normal to $\mathbb{S}^{2n+1}$. We have that

$$g_0(\cdot,\cdot) = g_0(J\cdot, J\cdot), \qquad J\nabla_{(\cdot)}\cdot = \nabla_{(\cdot)}J\cdot$$

We also recall that $T$ is a geodesic vector field, namely

$$g_0(\nabla_T T, V) = 0, \qquad \forall \quad V \in T(\mathbb{S}^{2n+1})$$

Since the formula (4) is linear in $V$ and $W$, we can check it on the basis $E$. Let us first fix $W = T$. Then $g_0(\nabla_V T, T) = 0$ since $T$ is unitary, and $g_0(V, \nabla_T T) = 0$ since $T$ is geodesic. Now suppose $(V, W) = (X_j, X_k)$ or $(V, W) = (Y_j, Y_k)$. Then

$$g_0(\nabla_{X_j} T, X_k) = -g_0(\nabla_{X_j} N, Y_k) = 0$$

$$g_0(\nabla_{Y_j} T, Y_k) = g_0(\nabla_{Y_j} N, X_k) = 0$$

as $X_j$ and $Y_j$ are eigenvectors for $A$. The same holds for $(V, W) = (X_j, Y_k)$ with $j \neq k$. Finally let us consider $(V, W) = (X_j, Y_j)$. Then we have

$$g_0(\nabla_{X_j} T, Y_j) + g_0(X_j, \nabla_{Y_j} T) = g_0(\nabla_{X_j} N, X_j) - g_0(Y_j, \nabla_{Y_j} N) = 0$$

□

Then $T$ generates a one-parameter family of diffeomorphisms, namely a closed group $G$, and since $T$ is Killing these diffeomorphisms are isometries. Moreover on $\mathbb{S}^{2n+1}$ we have the following crucial property of commutation

$$T\,\Delta_{\theta_0} = \Delta_{\theta_0}\,T \tag{5}$$

We notice that the previous formula (5) is true for every $K$-contact manifold (see Lemma 4.3. in[15]) and the sphere $(\mathbb{S}^{2n+1}, \theta_0)$ is a particular case of $K$-contact manifold according the definition in [4].

Hence we can conclude that the the functional $I$ is invariant under the action of $G$. Moreover $G$ is a closed subgroup of the compact Lie group $O(2n+2)$ of the isometries on $\mathbb{S}^{2n+1}$, in particular $G$ is compact. Now we will call basic a function that belongs to the following set:

$$X_G := \{v \in S^1(\mathbb{S}^{2n+1}) \ s.t. \quad Tv = 0\}$$

In other words, a basic function is a function on $\mathbb{S}^{2n+1}$ invariant for $T$. There is an explicit way to characterize the basic functions on $\mathbb{S}^{2n+1}$: since the orbits of $T$ are circles, we can consider the following Hopf fibration:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

where the fibers are exactly the orbits of $T$. Therefore a function $v$ is basic if and only if can be written as

$$v = w \circ \pi$$

for some function $w : \mathbb{C}P^n \to \mathbb{R}$. Let us consider now the subspace of functions $X_G \subseteq S^1(\mathbb{S}^{2n+1})$. We have

**Lemma 3.3.** *The embedding*

$$X_G \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$$

*is compact.*

*Proof.* The proof is based on the following simple observation. First note that the *Riemannian* critical exponent

$$p^* = \frac{2m}{m-2}, \qquad m = 2n+1$$

is always greater than the *sub-Riemannian* critical exponent $q^*$. In addition for every basic function, the horizontal gradient (and the sub-Laplacian as well) coincides with the usual Riemannian gradient (the Laplacian, respectively). Therefore, denoting by $H^1(\mathbb{S}^{2n+1})$ the usual Sobolev space on $\mathbb{S}^{2n+1}$, we have the following chain of embeddings

$$X_G \hookrightarrow H^1(\mathbb{S}^{2n+1}) \hookrightarrow L^{q^*}(\mathbb{S}^{2n+1})$$

and the second one is compact since $q^*$ is subcritical for the (Riemannian) Sobolev embedding. $\square$

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