High-order Levi curvatures and classification results

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Abstract In this paper we write some differential formulas involving the high-order Levi curvatures of a real hypersurface in a complex space form. As application we get a classification result under suitable assumptions.

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1 Introduction

In this paper we are going to consider a real hypersurface M^{2n+1} in a complex space form $K(c)$, i.e. a Kähler manifold with constant holomorphic sectional curvature $4c$ (see for instance [8]).
The problem of classifying real hypersurfaces in complex space forms has been widely investigated by many authors giving rise to a huge literature on this subject: we refer the reader to the expository article [20] and the references therein.
Here we will make assumptions only on the horizontal distribution of M. In particular for our purpose we will assume that $T_{1,0}M$ is H-parallel, i.e.

(H) $\nabla_Z W$ is tangent to M for any $Z, W \in T_{1,0}M$,

where ∇ denotes the Levi-Civita connection of $K(c)$: equivalently, this means that the symmetric part of the Levi form vanishes (see next sections for definitions and remarks).
Hypotheses stronger than (H) have already appeared in the literature (e.g. in [5]), and in different forms (see for instance [20], [4], and Lemma 3.4 below for further explanations).

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We will consider, for any $k \in \{1, \dots, n\}$, the k-th Levi curvature of M that is the k-th elementary symmetric function $\sigma_k(\ell(M))$ of the eigenvalues of the Levi form. We will prove, under hypothesis (H), our main formulas (3) in Lemma 2.3. The proof is based on some explicit computations for the derivatives of $\sigma_k(\ell(M))$ obtained from the Codazzi equations.
Then we will deduce the main result:

Theorem 1.1. *Let M be a real pseudoconvex hypersurface in $K(c)$ satisfying the hypothesis (H). If $\sigma_k(\ell(M))$ is a positive constant, for some $k \in \{1, \dots, n\}$ then M is a Hopf hypersurface. Moreover all the principal curvatures are constant.*

From the proof it will be clear that the hypothesis of pseudoconvexity can be relaxed depending on k. In particular, in Corollary 3.2, we will drop this assumption for the case $k = n$.
We want to explicitly stress that our aim is to recover the condition on the characteristic direction of being of Hopf type by exploiting just *horizontal* assumptions. We will clarify this issue in the last section.
Finally, by using some structural properties (Lemma 3.4), we will get as corollary a classification result saying that, under the hypotheses of Theorem 1.1, the hypersurface M is of type A referring to the "Takagi's list" and "Montiel's list" (see [20], [23], [24], [25], [6], [18], [1]). These classifications concern the case of non-vanishing sectional curvature c. On the other hand, in $\mathbb{C}^{n+1}$ (that is for $c = 0$) our Theorem infers that M has to be contained in a sphere or a spherical tube. In the case of constant mean Levi curvature ($k = 1$) in $\mathbb{C}^{n+1}$, this result has been proved by Monti and Morbidelli in [17]. In the compact case, these Aleksandrov's type results have been investigated with different techniques by several authors (see [5], [9], [14], [26], [13], [12], [11]).

2 Preliminaries

Let us fix a Kähler manifold K of dimension $2n + 2$. We denote by J the complex structure and g the Riemannian metric that are compatible in the following sense (see for instance [8]):

$$\omega(X, Y) = g(X, JY)$$

for every pair of vector fields $X, Y \in TK$, where ω is the fundamental symplectic 2-form of K. We also denote by ∇ the Levi-Civita connection of K.
Let us consider a smooth real orientable and connected embedded manifold M of codimension 1 on K, with induced metric and connection that we will indicate again g and ∇ (see also [5] for an exhaustive notion of real hypersurface in a Kähler manifolds). We denote by N a local choice for the unit normal to M and by $T = JN$ the characteristic or structure vector field. Let A be the Weingarten or shape operator, namely

$$A : TM \to TM, \quad AX := -\nabla_X N.$$

The Second Fundamental Form of M is defined by

$$h(\cdot,\cdot) := g(A\cdot,\cdot).$$

We also recall that both ∇ and g are compatible with the complex structure J, i.e.

$$J\nabla = \nabla J, \qquad g(\cdot,\cdot) = g(J\cdot, J\cdot). \tag{1}$$

The horizontal distribution or Levi distribution HM is the biggest subspace in TM invariant under the action of J, that is

$$HM = TM \cap JTM.$$

Then TM splits in the g-orthogonal direct sum:

$$TM = HM \oplus \mathbb{R}T.$$

In addition, we denote by φ the endomorphism

$$JX = \varphi X - g(X,T)N \qquad \text{for } X \in TM.$$

We are going to use also some standard complex notation, namely:

$$T_{1,0}M := T^{1,0}K \cap T^{\mathbb{C}}M \qquad \text{and} \qquad T_{0,1}M := \overline{T_{1,0}M},$$

where $T^{1,0}K$ is the holomorphic tangent space of K (i.e. the complex space generated by the eigenvalue $+i$ of J) and $T^{\mathbb{C}}M$ is the complexified tangent space of M. Moreover we set

$$H^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M \quad \text{ and we have } \quad T^{\mathbb{C}}M = H^{\mathbb{C}}M \oplus \mathbb{C}T.$$

Finally we will still denote by the same symbols the metric, the complex structure, etc., that we will extend by $\mathbb{C}$-linearity, as no ambiguity will occur. Let us just remind that in this setting the $\mathbb{C}$-linear extension of the Levi-Civita connection coincides with the Chern connection on $T^{\mathbb{C}}K$, so that it is compatible with the complex structure and it parallelizes the holomorphic bundle.

The Levi form ℓ is the hermitian and $\mathbb{C}$-bilinear operator defined on $T_{1,0}M$ in the following way:

$$\ell(Z,\overline{W}) := g(\nabla_Z \overline{W}, N) \qquad \text{for } Z, W \in T_{1,0}M.$$

We can compare the Levi form with the Second Fundamental Form by using the following identity (see [2], Chap.10, Theorem 2):

$$2\ell(Z,\overline{Z}) = h(X,X) + h(JX,JX)$$

with $X \in HM$, $Z \in T_{1,0}M$, and $Z = \frac{1}{\sqrt{2}}(X - iJX)$.

Definition 2.1. *We will say that M is Levi flat if ℓ identically vanishes. Moreover, we will say that M is (Levi) pseudoconvex if ℓ is positive semi-definite as quadratic form.*

Now let us extend the Levi form to the whole $H^{\mathbb{C}}M$, namely we define

$$\ell(Z,W) := g(\nabla_Z W, N) \qquad \text{for } Z, W \in H^{\mathbb{C}}M$$

and we will refer to the symmetric part of the Levi form if both Z and W belong to $T_{1,0}M$ (or both to $T_{0,1}M$).
Therefore with these notations we can restate our assumption (H) in the following way:

(H) the symmetric part of the Levi form vanishes; i.e.

$$\ell(Z,W) = g(\nabla_Z W, N) = 0, \qquad \text{for any} \quad Z, W \in T_{1,0}M.$$

For the sake of simplicity, for a given basis $\{Z_1, \dots, Z_n\}$ of $T_{1,0}M$, we will denote by

$$\ell_{js} = g(\nabla_{Z_j} Z_s, N), \quad \ell_{j\bar{s}} = g(\nabla_{Z_j} \bar{Z}_s, N), \quad j, s \in \{1, \dots, n\}.$$

Next we define the k-th Levi curvature (we refer the reader also to [10] and [14]).

Definition 2.2. *Let $\{Z_1, \dots, Z_n\}$ be an orthonormal basis for $T_{1,0}M$. We denote the Levi matrix in this basis by $\mathcal{L} = \{\ell_{j\bar{s}}\}_{j,s=1,\dots,n}$. For any $k \in \{1, \dots, n\}$, we define*

$$\sigma_k(\ell(M)) = \sigma_k(\lambda_1, \dots, \lambda_n) := \sum_{1 \le i_1 < \dots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathcal{L}$ and σ_k is the k-th elementary symmetric function in $\mathbb{R}^n$.

We explicitly note that the previous definition does not depend on the choice of the orthonormal basis.
We put $\sigma_0(\lambda_1, \dots, \lambda_n) = 1$. Moreover, the notation

$$\sigma_k(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_n)$$

with $k \le n-1$ stands for the k-th elementary symmetric function in $\mathbb{R}^{n-1}$ evaluated at $(\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_n)$. It is easy to check that

$$\sigma_k(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_n) = \sigma_k(\lambda_1, \dots, \lambda_n) - \lambda_j \sigma_{k-1}(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_n) \qquad (2)$$

for any $k \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, n\}$.
In the following lemma we prove the key formulas.

Lemma 2.3. *Suppose M is a real hypersurface in the complex space form $K(c)$ satisfying the hypothesis (H). Then, for any $k \in \{1, \dots, n\}$, we have*

$$\nabla^Z(\sigma_k(\ell(M))) = i\left(k\sigma_k(\ell(M))\mathbb{I}_n + 2\sum_{j=1}^{k}(-1)^{j-1}\sigma_{k-j}(\ell(M))\mathcal{L}^j\right)\gamma \tag{3}$$

where $\mathbb{I}_n$ denotes the $n \times n$ identity matrix and, for any fixed orthonormal basis $\{Z_1, \dots, Z_n\}$ of $T_{1,0}M$, $\mathcal{L}^j$ is the j-th power of the Levi matrix; whereas $\nabla^Z(\sigma_k(\ell(M)))$ and γ are vectors of components $Z_j(\sigma_k(\ell(M)))$ and $g(\nabla_T Z_j, N)$ respectively.

Before starting with the proof, let us make some remarks on complex space forms. It is known that the models for such settings are the standard complex space $\mathbb{C}^{n+1}$ endowed with the standard hermitian metric, the complex projective space $\mathbb{C}P^{n+1}$ with the Fubini-Study metric, and the complex hyperbolic space $\mathbb{C}H^{n+1}$ with the Bergman metric (see for instance [8]). These three prototypes differ in the sign of the holomorphic sectional curvature (respectively zero, positive, and negative). The reason why we consider the ambient space with constant holomorphic sectional curvature relies on the fact that we are going to use the Codazzi equations in the proof, and these equations become considerably simpler in this situation since the metrics of the model spaces are explicit. Moreover, all the characterization results that we know in literature make use of this assumption, so that we will be able to relate our theorem to other classification results.
Thus, let us assume $K(c)$ has constant holomorphic sectional curvature $4c$. In this situation, Codazzi equations appear as follows (see [20])

$$(\nabla_X A)Y - (\nabla_Y A)X = -c\left(g(X,T)\varphi Y - g(Y,T)\varphi X + 2g(X,\varphi Y)T\right),$$

for any $X, Y \in TM$. We can rewrite it as:

$$g((\nabla_Z A)W - (\nabla_W A)Z, V) = 0 \qquad \text{for any } Z, W, V \in H^{\mathbb{C}}M, \tag{4}$$

since the characteristic direction T is orthogonal to $H^{\mathbb{C}}M$.
By making use of the Bortolotti derivative, the relation (4) is equivalent to the following

$$\begin{aligned} Z(\ell(W,V)) \quad &- \quad W(\ell(Z,V)) = \\ = \ell(\nabla_Z W, V) + \ell(W, \nabla_Z V) \quad &- \quad \ell(\nabla_W Z, V) + \ell(Z, \nabla_W V) \end{aligned} \tag{5}$$

for any $Z, W, V \in H^{\mathbb{C}}M$. Now, we can start the proof of Lemma 2.3.

Proof. Fix $k \in \{1, \dots, n\}$. Since the formula is tensorial, it is enough to prove it for a particular choice of the basis. Thus we fix an orthonormal basis $\{Z_1, \dots, Z_n\}$ of $T_{1,0}M$ such that the Levi matrix $\mathcal{L}$ is diagonal. We are going to introduce some more notations. For $s, j, m \in \{1, \dots, n\}$, we put

$$\Gamma^m_{s,j} = g(\nabla_{Z_s} Z_j, Z_m).$$

We will write $\bar{s}$ or 0 instead of s or other indices, meaning that we will substitute $\overline{Z}_s$ or T in the correspondent spot. By (1) and the fact that Z_j's and $\overline{Z}_j$'s are eigenvectors for J we get

$$\Gamma^m_{j,s} = \Gamma^{\bar{m}}_{j,\bar{s}} = \Gamma^m_{\bar{j},s} = 0 \qquad \text{for any } s, j, m \in \{1, \dots, n\}.$$

Moreover, by hypothesis (H), we have

$$\Gamma^0_{s,j} = 0 \qquad \text{and} \qquad \ell(Z_s, Z_j) = 0 \quad \text{for any } s, j \in \{1, \dots, n\}.$$

For fixed $s, j \in \{1, \dots, n\}$, we want to compute $Z_s(\ell(Z_j, \overline{Z}_j)) = Z_s(\mathcal{L}_{j\bar{j}})$. We can do it by using (5) with $Z = Z_s$, $W = \overline{Z}_j$, and $V = Z_j$. Therefore, recalling that

$$\gamma_m = g(\nabla_T Z_m, N) \quad \text{ for } m \in \{1, \dots, n\},$$

we get

$$Z_s(\ell_{j\bar{j}}) = \sum_{m=1}^{n} (\Gamma^{\bar{m}}_{s,j} + \Gamma^m_{s,\bar{j}})\ell_{j\bar{m}} + \Gamma^0_{s,\bar{j}}\gamma_j - \Gamma^0_{\bar{j},s}\gamma_j - \Gamma^0_{\bar{j},j}\gamma_s$$

where we have exploited again the hypothesis (H). We note that

$$\Gamma^0_{s,\bar{j}} = i\ell_{s,\bar{j}} = -\Gamma^0_{\bar{j},s}.$$

By keeping in mind that the matrix $\mathcal{L}$ is diagonal with our choices and $\Gamma^{\bar{j}}_{s,j} = -\Gamma^j_{s,\bar{j}}$, we deduce that

$$Z_s(\ell_{j,\bar{j}}) = i(1 + 2\delta_{sj})\mathcal{L}_{jj}\gamma_s, \tag{6}$$

where δ_{sj} stands for the Kronecker delta. Hence, we have proved that

$$Z_s(\mathrm{Tr}(\mathcal{L})) = i(\mathrm{Tr}(\mathcal{L})\gamma_s + 2(\mathcal{L}\gamma)_s),$$

which is the desired formula for $k = 1$. In the general case, since $\lambda_j := \mathcal{L}_{j\bar{j}}$ are the eigenvalues of $\mathcal{L}$, we have by (6)

$$\begin{aligned} Z_s(\sigma_k(\ell(M))) &= \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{j=1}^{k} Z_s(\lambda_{i_j}) \prod_{m \neq j} \lambda_{i_m} \\ &= i\sum_{j=1}^{k} \sigma_k(\ell(M))\gamma_s + 2i\gamma_s \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{j=1}^{k} \delta_{si_j}\lambda_{i_j} \prod_{m \neq j} \lambda_{i_m} \\ &= ik\sigma_k(\ell(M))\gamma_s + 2i\lambda_s\sigma_{k-1}(\lambda_1, \dots, \hat{\lambda}_s, \dots, \lambda_n)\gamma_s. \end{aligned} \tag{7}$$

By using iteratively (2) we recognize that

$$\sigma_{k-1}(\lambda_1, \dots, \hat{\lambda}_s, \dots, \lambda_n) = \sum_{j=1}^{k} (-1)^{j-1}\sigma_{k-j}(\mathcal{L})\lambda_s^{j-1}. \tag{8}$$

Hence, for any $s \in \{1, \dots, n\}$, we finally get

$$\begin{aligned} Z_s(\sigma_k(\ell(M))) &= ik\sigma_k(\ell(M))\gamma_s + 2i\sum_{j=1}^{k}(-1)^{j-1}\sigma_{k-j}(\mathcal{L})\lambda_s^j\gamma_s \\ &= ik\sigma_k(\ell(M))\gamma_s + 2i\sum_{j=1}^{k}(-1)^{j-1}\sigma_{k-j}(\mathcal{L})(\mathcal{L}^j\gamma)_s. \end{aligned}$$

□

We want to remark explicitly the expressions of formula (3) in the special cases $k = 1, n$. We have

$$\nabla^Z(\mathrm{Tr}(\ell(M))) = i\,(\mathrm{Tr}(\ell(M))\mathbb{I}_n + 2\mathcal{L})\,\gamma \tag{9}$$

and

$$\nabla^Z(\det(\ell(M))) = (n+2)i\det(\ell(M))\gamma. \tag{10}$$

The second formula follows from the fact

$$\det(\mathcal{L}) - \sum_{j=1}^{n}(-1)^{j-1}\sigma_{n-j}(\mathcal{L})\lambda_s^j = \det(\mathcal{L} - \lambda_s\mathbb{I}_n) = 0$$

for any λ_s eigenvalue of $\mathcal{L}$.

3 Proof of the results

Let us first recall the following definition.

Definition 3.1. *A real hypersurface M in a Kähler manifold K is said to be a Hopf hypersurface if the structure vector $T = JN$ is an eigenvector for the shape operator A.*

With our notation this is equivalent to $\gamma = 0$. Also, we address the reader to the articles [3], [7], [15], [16], concerning Hopf hypersurfaces.

Proof of Theorem 1.1. Besides hypothesis (H), we are assuming that

$$M \text{ is Levi pseudoconvex and } \sigma_k(\ell(M)) \equiv c_0 = \mathit{const} > 0 \text{ in } M$$

for some $k \in \{1, \dots, n\}$. By Definition 2.1, all the eigenvalues λ_s's of $\ell(M)$ are non-negative. By the relation (8), it is clear that also the matrix

$$P := 2\sum_{j=1}^{k}(-1)^{j-1}\sigma_{k-j}(\ell(M))\mathcal{L}^j$$

is positive semi-definite. Thus we get

$$i(kc_0\mathbb{I}_n + P)\gamma = 0 \qquad \Longrightarrow \qquad \gamma = 0$$

and M is an hypersurface of Hopf type. Therefore

$$Z(\sigma_k(\ell(M))) = 0, \quad \forall\, Z \in H^{\mathbb{C}}M, \quad \forall\, k \in \{1, \dots, n\},$$

i.e. all the k-th Levi curvatures are constant along the horizontal directions. Now, since by hypothesis there exists $k \in \{1, \dots, n\}$ such that $\sigma_k(\ell(M)) > 0$, then by Definition 2.1 M is non Levi flat. In particular, there exists at least one eigenvalue $\lambda > 0$: let us call $Z \in T_{1,0}M$ the eigenvector associated to λ. We have, if $Z = \frac{1}{\sqrt{2}}(X - iY)$, $Y = JX$, that

$$g([X,Y],T) = g(\nabla_X Y - \nabla_Y X, T) = g(\nabla_X X + \nabla_Y Y, N) = 2\lambda > 0.$$

Hence the Hörmander's rank condition holds on HM, namely: for every basis $\{X_j, j = 1, \dots, 2n\}$ of HM it holds true the following

$$\dim\Big(span\big\{X_j, [X_\ell, X_k], \quad j,k,\ell = 1, \dots, 2n\big\}\Big) = 2n+1. \tag{11}$$

By Chow's theorem we then get the H-connectivity property: for every couple of points $p, q \in M$ there exists an H-admissible curve r, that is

$$r : [0,1] \to M, \qquad r(0) = p, \quad r(1) = q$$

$$\dot{r}(t) \in HM \qquad \forall\, t \in [0,1].$$

Now, for every couple of points $p, q \in M$, let us take an H-admissible curve r and we have

$$\frac{d}{dt}\Big(\sigma_k\big(\ell(M)(r(t))\big)\Big) = 0, \quad \forall\, k \in \{1, \dots, n\}.$$

Therefore $\sigma_k(\ell(M))$ is constant along r and therefore is constant on M for any $k \in \{1, \dots, n\}$. In particular all the eigenvalues of the Levi form are constant. In order to conclude the proof, let us denote by $a = h(T,T)$, so that if M is a Hopf hypersurface a is the principal curvature (the eigenvalue) related to the characteristic direction T. We know that if M is a Hopf hypersurface in a complex space form $K(c)$ with $c \neq 0$ then a must be a constant. On the other hand, if $c = 0$ then a need not to be constant in general, but here M is non Levi flat and so a must be constant as well (see for instance [13]). Finally, by using again hypothesis (H), we deduce that all the principal curvatures of the second fundamental form are constant. □

Formula (9) suggests that the hypothesis of pseudoconvexity is needed in the case of the mean-Levi curvature. On the other hand, it can be dropped in the case of the total-Levi curvature ($k = n$), so we get:

Corollary 3.2. *Let M be a real hypersurface in $K(c)$ satisfying the hypothesis (H). Suppose that $\det(\ell(M))$ is a non-zero constant in M. Then M is a hypersurface of Hopf type (and all the principal curvatures are constant).*

Proof. It follows directly from formula (10) and from the conclusion of the last proof. □

Remark 3.3. *If* $\det(\ell(M))$ *vanishes two situations may occur: there are only some eigenvalues equal to zero or all of them are zero. In the first case it could be some* $\sigma_k(\ell(M))$ *non constant, in the second one the hypersurface is Levi-flat giving rise to foliations.*

We want now to relate our assumption (H) to a general condition occurring in many classification results of real hypersurfaces in complex space forms. It is a structural condition on the invariance of the shape operator A with respect to the endomorphism φ, namely

$$(S) \qquad \varphi A(X) = A(\varphi X), \qquad \forall\, X \in TM.$$

In the next lemma we compare the two conditions.

Lemma 3.4. *Let* M *be a real hypersurface in any Kähler manifold* K*. Then* M *satisfies condition* (S) *if and only if* M *satisfies condition* (H) *and it is a Hopf hypersurface.*

Proof. We will use the $\mathbb{C}$-linear extension of φ, so that by definition

$$\varphi(Z) = iZ, \quad \text{if } Z \in T_{1,0}M, \qquad \varphi(Z) = -iZ, \quad \text{if } Z \in T_{0,1}M$$

and T generates the kernel of φ.
Let us call $S := \varphi A - A\varphi$. We get, for any $Z, W \in T_{1,0}M$,

$$\begin{aligned} g(S(Z), \overline{W}) &= g(\varphi A(Z) - iA(Z), \overline{W}) \\ &= g(A(Z), i\overline{W}) - ig(A(Z), \overline{W}) = 0. \end{aligned} \tag{12}$$

Let us first assume condition (S), i.e. $S \equiv 0$. We have in particular

$$\varphi A(T) - A(\varphi(T)) = \varphi A(T) = 0$$

which implies that $A(T)$ belongs to the kernel of φ. Thus M is of Hopf type. Moreover, for any $Z, W \in T_{1,0}M$, we get

$$\begin{aligned} 0 = g(S(Z), W) &= g(\varphi A(Z) - A(\varphi Z), W) = g(\varphi A(Z) - iA(Z), W) = \\ &= g(-A(Z), iW) - ig(A(Z), W) = -2i\ell(Z, W). \end{aligned}$$

Therefore M satisfies condition (H). Viceversa, if we assume that M satisfies condition (H) and it is a Hopf hypersurface, by arguing in the same way and by using (12), we get that M satisfies condition (S). □

By mean of the previous lemma we see that hypothesis (S) is a condition on both the subbundles of the tangent space HM and $\mathbb{R}T$. Hence, with Theorem 1.1 we have proved that we can recover the condition on the characteristic direction T of being of Hopf type by using only assumptions on the horizontal space, i.e. hypothesis (H) and the conditions on the k-th Levi curvatures.
We finally get the following result of characterization:

Corollary 3.5. *Let M be a real pseudoconvex hypersurface in $K(c)$, satisfying the hypothesis (H) and suppose that $\sigma_k(\ell(M))$ is a positive constant, for some $k \in \{1, \ldots, n\}$. Then,*

- *if $c > 0$, M is contained in a hypersurface of type A in the Takagi's list in $\mathbb{C}P^{n+1}$;*
- *if $c < 0$, M is contained in a hypersurface of type A in the Montiel's list in $\mathbb{C}H^{n+1}$;*
- *if $c = 0$, M is contained in a sphere or a spherical tube in the standard complex space $\mathbb{C}^{n+1}$.*

Proof. From Theorem 1.1 we have that M is of Hopf type, then by Lemma 3.4 we get condition (S). The classification result follows (see [20], [22], [21], [19], [17]). □

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