Concentrating Solutions for a Sub-Critical Sub-Elliptic Problem

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Abstract In this paper we prove the existence of concentrating solutions for a slightly sub-critical problem involving the Kohn Laplacian on a bounded domain of the Heisenberg group, under the assumption that the Robin's function of the domain has a non-degenerate critical point.

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1 Introduction

Let Ω be a bounded domain in the Heisenberg group $\mathbb{H}^n$. In this work we are interested in solving the following sub-critical problem, for $\varepsilon > 0$:

$$\begin{cases} -\Delta_{\mathbb{H}} u = u^{q^*-1-\varepsilon}, & \text{in}\, \Omega, \\ u > 0, & \text{in}\, \Omega, \\ u = 0, & \text{on}\, \partial\Omega \end{cases} \tag{1}$$

Here $\Delta_{\mathbb{H}}$ denotes the sub-Laplacian of the group and $q^* = (2n+2)/2$. When $\varepsilon = 0$, the problem (1) coincides with the CR-Yamabe equation on Ω which has been intensively studied in the last years (see for instance [10], [8], [1] and the references therein). Regarding perturbation results on bounded

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domains, we recall the result obtained by Garagnani and Uguzzoni in [7]: they consider the homogeneous equation

$$-\Delta_{\mathbb{H}} u = |u|^{q^*-2}u + \lambda u, \quad \text{in}\,\Omega$$

with zero Dirichlet boundary conditions; under suitable hypotheses on the boundary of Ω, they provide a multiplicity result for positive solutions, involving the Lujsternik-Schnirelmann category.
In a recent paper [12] instead, the first two authors found multiple solutions of the following nonhomogeneous Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{H}} u = |u|^{q^*-2}u + f, & \text{in}\,\Omega, \\ u > 0, & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega, \\ f \in C(\bar{\Omega}),\ f \not\equiv 0\ f \geq 0 & \end{cases} \tag{2}$$

They used a min-max argument on the homology groups of Ω, as in the work of Hirano [9], in which he solves the analogous problem for the standard Laplacian on bounded domains in $\mathbb{R}^n$.

For our purpose we will also need an hypothesis on $\partial\Omega$, in particular we will require that the boundary of Ω has no characteristic points (see definition (2.2) in the next section). The condition on Ω is needed in order to overcome some technical difficulties in proving some estimates. We explicitly note that if we consider $\mathbb{H}^1$ for instance, then the boundary of the standard Heisenberg ball defined by using the homogeneous distance has two characteristic points: in particular any contractible domain in $\mathbb{H}^1$ with smooth boundary has characteristic points. Instead, the torus in $\mathbb{H}^1$ defined by $\{(R - \sqrt{x^2+y^2})^2 + t^2 - r^2 < 0,\ R > r > 0\}$ is an example of domain whose boundary does not have any characteristic point.

Now if we define the Robin's function as $\varphi(\xi) = H(\xi, \xi)$, where H denotes the regular part of the Green's function G of Ω, then the main result of this paper can be stated as follows :

Theorem 1.1. *Let $\Omega \subseteq \mathbb{H}^n$ be a bounded domain with smooth boundary with no characteristic points and let us set*

$$\alpha = \begin{cases} \frac{2q}{(q-2)(q+2)} & \text{if } q > 4 \\ \\ \frac{1}{2q} & \text{if } q = 4 \end{cases}$$

If φ has a non-degenerate critical point ξ_0 in Ω, then there exists a sequence (u_ε) of solutions of (1) that concentrates at ξ_0, that is

$$u_\varepsilon = P_\varepsilon \omega_{\lambda_\varepsilon,\xi_\varepsilon} + \phi_\varepsilon,$$

where $\|\phi_\varepsilon\| < c\varepsilon$ and, as ε approaches zero,

$$\xi_\varepsilon \longrightarrow \xi_0, \qquad \varepsilon^{\alpha(q-2)-1}\lambda_\varepsilon^{q-2} \longrightarrow \frac{B}{A^2\varphi(\xi_0)}$$

Here A and B denote two constants to be determined later, $\omega_{\lambda_\varepsilon,\xi_\varepsilon}$ denotes the so-called "bubble" and $\|\cdot\|$ is a suitable norm (see formula (4) and (2.1) respectively in the next section); P_ε is the natural projection of $\Delta_{\mathbb{H}}$ on a suitable rescaled domain Ω_ε.

This is the first result of existence of blowing up solutions for a sub-elliptic problem. A similar result for the standard Laplacian in the Euclidean case was obtained by Rey [16]. We think that our approach allows to extend to the sub-Laplacian many results obtained for the standard Laplacian. In particular, existence of positive and sign changing solutions which blow-up at different points proved in Bahri-Li-Rey [2] and Bartsch-Micheletti-Pistoia [3] can be also proved for our problem.

The proof of our results relies on a very well known Ljapunov-Schmidt reduction. In particular, we will often refer to the Appendix of [15], in which they explicitly prove some estimates that we will use in our proofs; the only technical assumption that we will add is that $\partial\Omega$ is without characteristic points.

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2 Setting of the problem

Let $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \centerdot)$ be the Heisenberg group. If we denote by $\xi = (x, y, t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ then the group law is

$$\xi_0 \centerdot \xi = (x + x_0, y + y_0, t + t_0 + 2(x \cdot y_0 - x_0 \cdot y)), \ \forall \xi, \xi_0 \in \mathbb{H}^n$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. The left translations are then given by

$$\tau_{\xi_0}(\xi) := \xi_0 \centerdot \xi$$

The dilations of the group are

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$$

for any $\lambda > 0$. We define the homogeneous norm

$$\rho(\xi) = \left((|x|^2 + |y|^2)^2 + t^2\right)^{1/4},$$

and the distance

$$d(\xi, \xi_0) = \rho(\xi_0^{-1} \cdot \xi).$$

It holds

$$d(\delta_\lambda \xi, \delta_\lambda \xi_0) = \lambda\, d(\xi, \xi_0).$$

We will denote by $B_d(\xi, r)$ the ball with respect to the distance d, of center ξ and radius r. We have

$$B_d(\xi, r) = \tau_\xi(B_d(0, r)), \qquad B_d(0, r) = \delta_r(B_d(0, 1))$$

The canonical left-invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n$$

The (intrinsic) gradient of the group is

$$D_H = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$

The Kohn Laplacian (or sub-Laplacian) on $\mathbb{H}^n$ is the following second order operator invariant with respect to the left translations and homogeneous of degree two with respect to the dilations:

$$\Delta_H = \sum_{j=1}^{n} X_j^2 + Y_j^2$$

By a result in [4], the fundamental solution on $\mathbb{H}^n$ of $-\Delta_H$ with pole at the origin is

$$\Gamma(\xi) = \frac{c_q}{\rho(\xi)^{q-2}}$$

where c_q is a suitable positive constant and $q = 2n + 2$ is the homogeneous dimension of the group. The fundamental solution on $\mathbb{H}^n$ of $-\Delta_H$ with pole at the ξ will be then

$$\Gamma(\xi, \eta) = \frac{c_q}{d(\xi, \eta)^{q-2}}$$

Let us now set
$$q^* = \frac{2q}{q-2}$$
then the following Sobolev-type inequality holds
$$\|\varphi\|_{q^*}^2 = \Big(\int_{\mathbb{H}^n} |\varphi|^{q^*}\Big)^{\frac{2}{q^*}} \le C \int_{\mathbb{H}^n} |D_{\mathbb{H}}\varphi|^2 = C\|D_{\mathbb{H}}\varphi\|_2^2, \qquad \forall \varphi \in C_0^\infty(\mathbb{H}^n)$$
with C a positive constant.

Definition 2.1. *For every domain $\Omega \subseteq \mathbb{H}^n$, the Folland-Stein Sobolev space $S_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm*
$$\|\cdot\| = \|D_H \cdot\|_2$$
The exponent q^* is called critical since the embedding
$$S_0^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$$
is continuous but not compact for every domain Ω. Moreover, by defining the inner product on $S_0^1(\Omega)$
$$\langle u, v\rangle = \int_\Omega \langle D_H u, D_H v\rangle$$
then there exists a natural orthogonal projection, defined by
$$P : S_0^1(\mathbb{H}^n) \longrightarrow S_0^1(\Omega)$$
$$\begin{cases} \Delta_{\mathbb{H}} Pu = \Delta_{\mathbb{H}} u, & \text{in}\,\Omega, \\ Pu = 0, & \text{on}\,\partial\Omega \end{cases}$$
Next we define the function
$$\omega(x,y,t) = \frac{c_0}{\big((1+|x|^2+|y|^2)^2+t^2\big)^{\frac{q-2}{4}}}$$
with c_0 a suitable positive constant; then Jerison and Lee showed in [11] that all the positive solutions to the problem
$$-\Delta_{\mathbb{H}} u = u^{q^*-1}, \qquad u \in S_0^1(\mathbb{H}^n) \tag{3}$$
are in the form
$$\omega_{\lambda,\xi} = \lambda^{\frac{2-q}{2}} \omega \circ \delta_{\frac{1}{\lambda}} \circ \tau_{\xi^{-1}} \tag{4}$$
for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$.
Next is the definition of characteristic points.

Definition 2.2. *Let $\varphi : \mathbb{H}^n \to \mathbb{R}$ be a smooth defining function for Ω, namely*

$$\Omega = \{\xi \in \mathbb{H}^n \ : \ \varphi(\xi) < 0\}, \qquad \partial\Omega = \{\xi \in \mathbb{H}^n \ : \ \varphi(\xi) = 0\}$$

A point $\xi_0 \in \partial\Omega$ is said to be characteristic if $D_{\mathbb{H}}\varphi(\xi_0) = 0$.

Let now $\alpha > 0$ to be fixed, we set $\Omega_\varepsilon = \delta_{\varepsilon^{-\alpha}}(\Omega)$ and $\xi_\varepsilon = \delta_{\varepsilon^{-\alpha}}(\xi)$, $\xi \in \Omega$. We are going to consider the rescaled problem:

$$\begin{cases} -\Delta_{\mathbb{H}} v = v^{q^*-1-\varepsilon}, & \text{in}\, \Omega_\varepsilon, \\ v > 0, & \text{in}\, \Omega_\varepsilon, \\ v = 0, & \text{on}\, \partial\Omega_\varepsilon \end{cases} \tag{5}$$

In fact, if $u(\xi)$ is a solution for (1), then

$$v(\eta) = \varepsilon^{\alpha\beta} u\big(\delta_{\varepsilon^\alpha}(\eta)\big)$$

is a solution of (5), with $\beta = \frac{2}{q^*-2-\varepsilon}$ and $\alpha > 0$.
For the sake of simplicity we will also set $f_\varepsilon(s) = (s^+)^{q^*-1-\varepsilon}$, with $f(s) = f_0(s)$.

Definition 2.3. *Let $\bar{q}^* = \frac{2q}{q+2}$ be the conjugate exponent to q^*, and let*

$$i_\varepsilon^* : L^{\bar{q}^*}(\Omega_\varepsilon) \to S_0^1(\Omega_\varepsilon)$$

be the adjoint operator of the immersion $i_\varepsilon : S_0^1(\Omega_\varepsilon) \hookrightarrow L^{q^*}(\Omega_\varepsilon)$, namely:

$$i_\varepsilon^*(u) = v \ \Leftrightarrow \ -\Delta_{\mathbb{H}} v = u$$

We have:

Lemma 2.1. *The map i_ε^* is continuous, uniformly with respect to ε, and*

$$\|i_\varepsilon^*(u)\|_{q^*} \leq \|u\|_{\bar{q}^*}, \qquad \forall\, u \in L^{\bar{q}^*}(\Omega_\varepsilon)$$

Proof. Let S be the best constant for the Sobolev type embedding related to the Folland-Stein space $S_0^1(\Omega_\varepsilon)$:

$$\|v\|_{q^*} \leq S^{\frac{1}{2}} \|v\|, \ \forall\, v \in S_0^1(\Omega_\varepsilon)$$

By duality it holds:

$$\|i_\varepsilon^*(u)\| \leq S^{-\frac{1}{2}} \|u\|_{\bar{q}^*}, \ \forall\, u \in L^{\bar{q}^*}(\Omega_\varepsilon)$$

therefore the continuity follows. By combining the previous inequalities, with $v = i_\varepsilon^*(u)$, we get the claim. □

We can thus rewrite the problem (5) in the following equivalent form:

$$u = i_\varepsilon^*(f_\varepsilon(u)), \quad u \in S_0^1(\Omega_\varepsilon) \tag{6}$$

Let us now denote by

$$P_\varepsilon \omega_{\lambda,\xi_\varepsilon} = \omega_{\lambda,\xi_\varepsilon} - h_{\lambda,\xi_\varepsilon}$$

the $S_0^1(\Omega_\varepsilon)$ projection of $\omega_{\lambda,\xi_\varepsilon}$, defined by

$$\begin{cases} -\Delta_{\mathbb{H}} P_\varepsilon \omega_{\lambda,\xi_\varepsilon} = -\Delta_{\mathbb{H}} \omega_{\lambda,\xi_\varepsilon} = \omega_{\lambda,\xi_\varepsilon}^{q^*-1}, & \text{in } \Omega_\varepsilon, \\ P_\varepsilon \omega_{\lambda,\xi_\varepsilon} = 0, & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and

$$\begin{cases} -\Delta_H h_{\lambda,\xi_\varepsilon} = 0, & \text{in } \Omega_\varepsilon, \\ h_{\lambda,\xi_\varepsilon} = \omega_{\lambda,\xi_\varepsilon}, & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

Equivalently $P_\varepsilon \omega_{\lambda,\xi_\varepsilon} = i_\varepsilon^*(\omega_{\lambda,\xi_\varepsilon}^{q^*-1})$. The Green's function G and its regular part H are defined by

$$G(\xi,\eta) = \Gamma(\xi,\eta) - H(\xi,\eta)$$

and

$$\begin{cases} -\Delta_H H(\xi,\cdot) = 0, & \text{in } \Omega, \\ H(\xi,\cdot) = \Gamma(\xi,\cdot), & \text{on } \partial\Omega, \end{cases}$$

where $\Gamma(\xi,\cdot)$ is the fundamental solution of $-\Delta_H$ with pole at ξ. By using similar estimates as in the Euclidian case (see for instance [1] and [6]), it holds

$$h_{\lambda,\xi}(\eta) = H(\xi,\eta)\lambda^{\frac{q-2}{2}} \int_{\mathbb{H}^n} \omega_{1,0}^{q^*-1} + o(\lambda^{\frac{q-2}{2}})$$

and, for $\xi \neq \eta$

$$P_\varepsilon \omega_{\lambda,\xi}(\eta) = G(\xi,\eta)\lambda^{\frac{q-2}{2}} \int_{\mathbb{H}^n} \omega_{1,0}^{q^*-1} + o(\lambda^{\frac{q-2}{2}}) \tag{7}$$

We are looking for solutions of (6) of the form $V_\varepsilon + \phi_\varepsilon$, where for sake of notation we have set

$$V_\varepsilon := P_\varepsilon \omega_{\lambda,\xi_\varepsilon}$$

for some couple (λ,ξ) and with ϕ_ε belonging to a suitable space. It is known (see [13]) that a solution of the linearized problem

$$-\Delta_{\mathbb{H}} u = (q^*-1)\omega_{\lambda,\xi}^{q^*-2} u, \qquad u \in S_0^1(\mathbb{H}^n) \tag{8}$$

belongs to the following set

$$span\Big\{\psi^0_{\lambda,\xi}=\frac{\partial\omega_{\lambda,\xi}}{\partial\lambda},\quad \psi^j_{\lambda,\xi}=\frac{\partial\omega_{\lambda,\xi}}{\partial\xi_j},\quad j=1,\dots,2n+1\Big\}$$

Let $P_\varepsilon\psi^j_{\lambda,\xi_\varepsilon}$ be the $S^1_0(\Omega_\varepsilon)$ projection of $\psi^j_{\lambda,\xi_\varepsilon}$, $j=0,1,\dots,2n+1$, and let us define the following finite dimensional subspace of $S^1_0(\Omega_\varepsilon)$:

$$K_\varepsilon=K_{\varepsilon,\lambda,\xi}=span\Big\{P_\varepsilon\psi^j_{\lambda,\xi_\varepsilon},\quad j=0,1,\dots,2n+1\Big\}$$

and its orthogonal

$$K_\varepsilon^\perp=\Big\{\phi\in S^1_0(\Omega_\varepsilon):\ \langle\phi,P_\varepsilon\psi^j_{\lambda,\xi_\varepsilon}\rangle=0,\quad j=0,1,\dots,2n+1\Big\}$$

Then let Π_ε and $\Pi_\varepsilon^\perp$ be the projection operators on K_ε and $K_\varepsilon^\perp$ respectively. We want to split the problem (6) and solve both the following equations:

$$\Pi_\varepsilon\big((V_\varepsilon+\phi_\varepsilon)-i^*_\varepsilon(f_\varepsilon(V_\varepsilon+\phi_\varepsilon))\big)=0 \tag{9}$$

$$\Pi_\varepsilon^\perp\big((V_\varepsilon+\phi_\varepsilon)-i^*_\varepsilon(f_\varepsilon(V_\varepsilon+\phi_\varepsilon))\big)=0 \tag{10}$$

3 Solving in $K_\varepsilon^\perp$

Here we solve (10). Let then $\gamma\in(0,1)$ and let us define:

$$\mathcal{O}_\gamma=\big\{(\lambda,\xi)\in\mathbb{R}\times\Omega \text{ such that } \lambda\in(\gamma,\gamma^{-1}),\ d(\xi,\partial\Omega)>\gamma,\big\}$$

First we have:

Lemma 3.1. *For any $\gamma\in(0,1)$ there exist $\varepsilon_0>0$ and $c>0$ such that*

$$\|\Pi_\varepsilon(u)\|\le c\|u\|,\quad u\in S^1_0(\Omega_\varepsilon)$$

for every $\varepsilon\in(0,\varepsilon_0)$ and $(\lambda,\xi)\in\mathcal{O}_\gamma$.

Proof. By definition of Π_ε:

$$\|\Pi_\varepsilon(u)\|\le\sum_{j=0}^{2n+1}\big|\langle u,P_\varepsilon\psi^j_{\lambda,\xi_\varepsilon}\rangle\big|\,\|P_\varepsilon\psi^j_{\lambda,\xi_\varepsilon}\|$$

Moreover by Lemma (2.1), we have for every j:

$$\|P_\varepsilon\psi^j_{\lambda,\xi_\varepsilon}\|\le S^{-\frac12}\|\omega^{q^*-2}_{\lambda,\xi_\varepsilon}\,\psi^j_{\lambda,\xi_\varepsilon}\|_{\bar q^*}\le c$$

for every $(\lambda,\xi)\in\mathcal{O}_\gamma$. □

Remark 3.1. *Since $\Pi_\varepsilon^\perp = Id - \Pi_\varepsilon$, we have that the map $\Pi_\varepsilon^\perp$ is continuous as well.*

Let us now define the following operator:

$$L_\varepsilon = L_{\varepsilon,\lambda,\xi} : K_\varepsilon^\perp \longrightarrow K_\varepsilon^\perp$$

$$L_\varepsilon(\phi) = \Pi_\varepsilon^\perp\big(\phi - i_\varepsilon^*[f_\varepsilon'(V_\varepsilon)\phi]\big)$$

We have then:

Lemma 3.2. *For any $\gamma \in (0,1)$ and $\phi \in K_\varepsilon^\perp$, there exist $\varepsilon_1 > 0$ and $C > 0$ such that L_ε is invertible and $\|L_\varepsilon(\phi)\| \geq C\|\phi\|$, for every $\varepsilon \in (0, \varepsilon_1)$ and $(\lambda, \xi) \in \mathcal{O}_\gamma$.*

Proof. We argue as in [14]: in particular all the regularity results hold, since the boundary of Ω has no characteristic points. □

Next we introduce the operator:

$$T_\varepsilon = T_{\varepsilon,\lambda,\xi} : K_\varepsilon^\perp \longrightarrow K_\varepsilon^\perp$$

$$T_\varepsilon(\phi) = L_\varepsilon^{-1}\Pi_\varepsilon^\perp i_\varepsilon^*\big[f_\varepsilon(V_\varepsilon + \phi) - f(\omega_{\lambda,\xi_\varepsilon}) - f_\varepsilon'(V_\varepsilon)\phi\big]$$

It holds:

Lemma 3.3. *Let $\alpha = \frac{2q}{(q-2)(q+2)}$ if $q \geq 6$ and $\alpha = \frac{1}{2q}$ if $q = 4$. For any $\gamma \in (0,1)$ there exist positive constants ε_2, μ such that for every $\varepsilon \in (0, \varepsilon_2)$ and $(\lambda, \xi) \in \mathcal{O}_\gamma$, T_ε is a contraction on the set $\mathcal{A} \subseteq K_\varepsilon^\perp$ defined by:*

$$\mathcal{A} = \{\phi \in K_\varepsilon^\perp : \ \|\phi\| \leq \mu\varepsilon\}$$

Proof. All the estimates we will use are similar to those in the Appendix of [15]. First we prove that T_ε maps $\mathcal{A}$ into itself. We know

$$\|T_\varepsilon(\phi)\| \leq c\|f_\varepsilon(V_\varepsilon + \phi) - f_\varepsilon(V_\varepsilon) - f_\varepsilon'(V_\varepsilon)\phi\|_{\bar{q}^*} +$$

$$+ c\|f_\varepsilon(V_\varepsilon) - f(V_\varepsilon)\|_{\bar{q}^*} + c\|f(V_\varepsilon) - f(\omega_{\lambda,\xi_\varepsilon})\|_{\bar{q}^*}$$

Now

$$\|f_\varepsilon(V_\varepsilon + \phi) - f_\varepsilon(V_\varepsilon) - f_\varepsilon'(V_\varepsilon)\phi\|_{\bar{q}^*} \leq c\|\phi\|^{\min\{2, q^*-1-\varepsilon\}}$$

and

$$\|f_\varepsilon(V_\varepsilon) - f(V_\varepsilon)\|_{\bar{q}^*} \leq c\varepsilon$$

Last we have the estimate

$$\|f(V_\varepsilon) - f(\omega_{\lambda,\xi_\varepsilon})\|_{\bar{q}^*} \leq \begin{cases} c\varepsilon^{\alpha\frac{(q-2)(q+2)}{2q}}, & q \geq 6 \\ c\varepsilon^{\alpha 2q}, & q = 4 \end{cases}$$

With our choice of α and with a suitable positive constant μ, therefore T_ε maps $\mathcal{A}$ into itself. Next we prove that T_ε is a contraction. By using the mean value theorem, with $\theta \in (0,1)$:

$$\|T_\varepsilon(\phi_2) - T_\varepsilon(\phi_1)\| \leq c\|f_\varepsilon(V_\varepsilon + \phi_2) - f_\varepsilon(V_\varepsilon + \phi_1) - f'_\varepsilon(V_\varepsilon)(\phi_2 - \phi_1)\|_{\bar{q}^*} \leq$$

$$c\big\|\big\{f'_\varepsilon(V_\varepsilon + \phi_2 + \theta(\phi_2 - \phi_1)) - f'_\varepsilon(V_\varepsilon)\big\}(\phi_2 - \phi_1)\big\|_{\bar{q}^*} \leq$$

$$\leq c\big(\|\phi_2 - \phi_1\|_{q^*}^{q^*-1} + \|\phi_2\|_{q^*}^{q^*-2}\|\phi_2 - \phi_1\|_{q^*}\big) \leq c\varepsilon\|\phi_2 - \phi_1\|$$

□

As corollary we obtain:

Proposition 3.1. *For any $\gamma \in (0,1)$ there exist positive constants ε_2, μ such that for every $\varepsilon \in (0, \varepsilon_2)$ and $(\lambda, \xi) \in \mathcal{O}_\gamma$, there exists a unique $\phi_\varepsilon := \phi_{\varepsilon,\lambda,\xi} \in \mathcal{A} \subseteq K_\varepsilon^\perp$ such that $V_\varepsilon + \phi_\varepsilon$ is a solution for the problem (10), with $\|\phi_\varepsilon\| \leq \mu\varepsilon$.*

Proof. Since T_ε is a contraction on $\mathcal{A}$, then there exists a unique fixed point $\phi_\varepsilon = \phi_{\varepsilon,\lambda,\xi} \in \mathcal{A} \subseteq K_\varepsilon^\perp$ for T_ε, namely:

$$\phi_\varepsilon = L_\varepsilon^{-1}\Pi_\varepsilon^\perp i_\varepsilon^*\big[f_\varepsilon(V_\varepsilon + \phi_\varepsilon) - f(\omega_{\lambda,\xi_\varepsilon}) - f'_\varepsilon(V_\varepsilon)\phi_\varepsilon\big]$$

By applying L_ε to both sides we get a solution of (10). □

4 Solving in K_ε

Here we solve equation (9): we want to find (λ, ξ) such that for every $j = 0, \ldots, 2n+1$, it holds

$$\langle V_\varepsilon + \phi_\varepsilon - i_\varepsilon^*\big[f_\varepsilon(V_\varepsilon + \phi_\varepsilon)\big], P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon}\rangle = 0$$

We define the constants

$$A = \int_{\mathbb{H}^n} \omega_{1,0}^{q^*-1}$$

and

$$B = \frac{1}{q^*}\int_{\mathbb{H}^n} \omega_{1,0}^{q^*}.$$

It holds:

Proposition 4.1. *Let ϕ_ε the function found in Proposition (3.1). We have, in the case $j = 1, \dots, 2n+1$:*

$$\langle V_\varepsilon + \phi_\varepsilon - i^*_\varepsilon\big[f_\varepsilon(V_\varepsilon + \phi_\varepsilon)\big], P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon}\rangle =$$

$$= -A^2\lambda^{q-2}\frac{\partial H}{\partial \xi^j}(\xi,\xi)\varepsilon^{\alpha(q-1)} + o(\varepsilon^{\alpha(q-1)}) \tag{11}$$

Moreover, if $j = 0$:

$$\langle V_\varepsilon + \phi_\varepsilon - i^*_\varepsilon\big[f_\varepsilon(V_\varepsilon + \phi_\varepsilon)\big], P_\varepsilon \psi^0_{\lambda,\xi_\varepsilon}\rangle =$$

$$= -\frac{q-2}{2}A^2\lambda^{q-3}H(\xi,\xi)\varepsilon^{\alpha(q-2)} + \frac{q-2}{2\lambda}\varepsilon B + o(\varepsilon^{\alpha(q-2)}) \tag{12}$$

Proof. Since $\phi_\varepsilon \in K_\varepsilon^\perp$, we have:

$$-\int_{\Omega_\varepsilon} D_{\mathbb{H}}\big(V_\varepsilon + \phi_\varepsilon - i^*_\varepsilon\big[f_\varepsilon(V_\varepsilon+\phi_\varepsilon)\big]\big) D_{\mathbb{H}} P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} = \int_{\Omega_\varepsilon} \big(\Delta_{\mathbb{H}} V_\varepsilon + f_\varepsilon(V_\varepsilon+\phi_\varepsilon)\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} =$$

$$\int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon+\phi_\varepsilon) - f(\omega_{\lambda,\xi_\varepsilon})\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} = \int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon+\phi_\varepsilon) - f_\varepsilon(V_\varepsilon) - f'_\varepsilon(V_\varepsilon)\phi_\varepsilon\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} +$$

$$\int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon) - f(V_\varepsilon)\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} + \int_{\Omega_\varepsilon} f'_\varepsilon(V_\varepsilon)\phi_\varepsilon P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} +$$

$$+ \int_{\Omega_\varepsilon} f(V_\varepsilon) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} - \int_{\Omega_\varepsilon} f(\omega_{\lambda,\xi_\varepsilon}) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon}$$

Now we estimate the last expression term by term. One has

$$\int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon + \phi_\varepsilon) - f_\varepsilon(V_\varepsilon) - f'_\varepsilon(V_\varepsilon)\phi_\varepsilon\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} \le c\|\phi_\varepsilon\|^2$$

and also

$$\int_{\Omega_\varepsilon} f'_\varepsilon(V_\varepsilon)\phi_\varepsilon P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} = \int_{\Omega_\varepsilon} (f'_\varepsilon(V_\varepsilon) - f'(\omega_{\lambda,\xi_\varepsilon})\phi_\varepsilon P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} \le c\varepsilon\|\phi_\varepsilon\|$$

A term slightly different that we need to estimate is the following one

$$\int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon) - f(V_\varepsilon)\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon}$$

For that, we have by using the mean value theorem

$$\int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon) - f(V_\varepsilon)\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} = -\varepsilon \int_{\Omega_\varepsilon} \log(V_\varepsilon) f(V_\varepsilon)\partial_j V_\varepsilon + h.o.t.$$

Now

$$\int_{\Omega_\varepsilon} \log(V_\varepsilon) f(V_\varepsilon)\partial_j V_\varepsilon = \frac{1}{q^*}\int_{\Omega_\varepsilon} \partial_j \left(\log(V_\varepsilon) V_\varepsilon^{q^*}\right) + h.o.t.$$

and by using (7) again, we get

$$\int_{\Omega_\varepsilon} \big(f_\varepsilon(V_\varepsilon) - f(V_\varepsilon)\big) P_\varepsilon \psi^j_{\lambda,\xi_\varepsilon} = \begin{cases} h.o.t. & \text{if j=1,...,2n+1} \\ \\ \varepsilon\frac{q-2}{2}\frac{1}{\lambda}B + h.o.t. & \text{if j=0} \end{cases}$$

The expansion of the last two terms is exactly the same of the ones in the Appendix of [15]: in particular we refer the reader to formulas (B.9), (B.10) and (B.5), (B.7), noticing that we have only the part that involves H, since we are dealing with only one bubble.
By putting all the expansions together we get the claim. □

Proof. of Theorem (1.1)
Firs of all we can assume without loss of generality that $\xi_0 = 0$. Since

$$\varphi(\xi) = \varphi(0) + \varphi''(0)(\xi,\xi) + o(|\xi|^2)$$

and recalling that $\varphi(\xi) = H(\xi,\xi)$ is the diagonal of the regular part of the Green's function, we notice that solving equations (11) and (12) is equivalent to solve the following system

$$\begin{cases} -A^2\lambda_\varepsilon^{q-2}\varphi''(0)(\xi_\varepsilon) = o(1) + o(|\xi_\varepsilon|) \\ \\ -A^2\lambda_\varepsilon^{q-3}(\varphi(0) + O(|\xi_\varepsilon|^2))\varepsilon^{\alpha(q-2)} + \frac{1}{\lambda_\varepsilon}\varepsilon B + o(\varepsilon^{\alpha(q-2)}) = 0 \end{cases}$$

Since $\xi_0 = 0$ is a non-degenerate critical point for φ, that is $\varphi''(0)$ is invertible, then the previous system is always solvable by using the implicit function theorem. Therefore we obtain a sequence of solutions of the form

$$u_\varepsilon = P_\varepsilon \omega_{\lambda_\varepsilon,\xi_\varepsilon} + \phi_\varepsilon$$

with $\phi_\varepsilon \in \mathcal{A}$, and with the couple $(\lambda_\varepsilon, \xi_\varepsilon)$ satisfying

$$\xi_\varepsilon \longrightarrow \xi_0, \qquad \varepsilon^{\alpha(q-2)-1}\lambda_\varepsilon^{q-2} \longrightarrow \frac{B}{A^2\varphi(0)}$$

as ε approaches zero. □

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