Existence and Concentration of Positive Solutions for a Super-critical Fourth Order Equation

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Abstract In this paper we investigate the problem of existence of solutions for a super-critical fourth order Yamabe type equation and we exhibit a family of solutions concentrating at two points, provided the domain contains one hole and we give a multiplicity result if we are given multiple holes.

1 Introduction and main results

In this paper we will study the existence of positive solutions for a homogeneous super-critical problem of the form

$$\begin{cases} \Delta^2 u = |u|^{p-1+\varepsilon} u & \text{on} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{P_ε}$$

where Ω is a smooth bounded set of $\mathbb{R}^n$, with $n \geq 5$, and $p = \frac{n+4}{n-4}$ is the critical exponent. This problem was studied in the case of the Laplacian by Del Pino et al. in [11], [10], where they use the finite dimensional reduction

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method. Our work will be in the same spirit. Let us recall that problem (P_ε) was studied in [5] where the authors show that there is no one-bubble solution to the problem and there is a one-bubble solution to the slightly sub-critical case under some suitable conditions.

Recall that for $\varepsilon = 0$, this problem has a deep geometrical meaning, in fact if (M, g) is an $n-$dimensional compact closed riemannian manifold with $n \geq 5$, we can define the $Q-$curvature (see for instance [18])

$$Q := \frac{n^3 - 4n^2 + 16n - 16}{8\left(n-2\right)^2\left(n-1\right)^2}R^2 - \frac{2}{(n-2)^2}\left|Ric\right|^2 + \frac{1}{2(n-1)}\Delta R,$$

then after a conformal change of the metric, one gets for $\widetilde{g} = u^{\frac{4}{n-4}}g$,

$$Q_{\widetilde{g}}u^{\frac{n+4}{n-4}} = P_g u, \tag{1}$$

where P_g is the Paneitz operator, defined by

$$P_g u := \Delta_g^2 u - div\left(\left(\frac{\left(n-2\right)^2+4}{2\left(n-2\right)\left(n-1\right)}Rg - \frac{4}{n-2}Ric\right)du\right) + \frac{n-4}{2}Qu.$$

Hence prescribing the $Q-$curvature problem is analogous to the scalar curvature prescribing problem. Now remark that in the flat case, for instance if we consider an open set of $\mathbb{R}^n$, the problem of prescribing constant Q-curvature coincides with (P_ε) with $\varepsilon = 0$ that is

$$\Delta^2 u = \left|u\right|^{p-1}u. \tag{2}$$

The variational formulation of (2) under Navier boundary conditions in a bounded set was deeply studied, especially from the perspective of the theory of critical points at infinity, introduced by Bahri [1] (see [9], [14] and [13]), and this reveals more interesting analytical phenomena involving the topology of the underlying set Ω. We bring the attention of the reader to the fact

that this problem is not compact, that is for the case $\varepsilon = 0$ it corresponds exactly to the limiting case of the Sobolev embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-4}}$, (see [21]), and thus we loose the compact embedding. The case $\varepsilon > 0$ is even worse since the continuous embedding is also violated, so the variational setting in the classical spaces fails to show existence of solutions: in fact as in the case of the Laplacian, if the domain is star shaped we know that it has no solutions in the super-critical case and no positive solutions in the critical case ([21], [22]). In this work we will show the existence of positive solutions of (P_ε) having two concentration points in a domain with holes and those solutions do not survive when $\varepsilon \longrightarrow 0$.
We also recall that the authors in [17] proved also existence and multiplicity results for the problem (P_ε) with $\varepsilon = 0$ and with a non-homogeneous term. The main result of this paper reads as follow :

Theorem 1.1. *Let $\mathcal{D}$ be a bounded smooth open domain of $\mathbb{R}^n$, and let $P \in \mathcal{D}$, then there exists $\mu_0 > 0$ such that if $0 < \mu < \mu_0$ and $\Omega = \mathcal{D} - B(P, \mu)$, then there exists $\varepsilon_0 > 0$ and a family of solutions u_ε for problem (P_ε) with $0 < \varepsilon < \varepsilon_0$. Moreover u_ε reads as follows :*

$$u_\varepsilon(x) = \left(\frac{\alpha_n \lambda_1 \varepsilon^{\frac{1}{n-4}}}{\varepsilon^{\frac{2}{n-4}} \lambda_1^2 + |x - \xi_1^\varepsilon|^2} \right)^{\frac{n-4}{2}} + \left(\frac{\alpha_n \lambda_2 \varepsilon^{\frac{1}{n-4}}}{\lambda_2^2 \varepsilon^{\frac{2}{n-4}} + |x - \xi_2^\varepsilon|^2} \right)^{\frac{n-4}{2}} + \varphi_\varepsilon(x)$$

where $\varphi_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ uniformly, and α_n is a constant depending on n. λ_i and ξ_i^ε are critical points of a function that will be determined later and there exists $0 < c < C$ such that

$$c\mu < |\xi_i^\varepsilon - P| < C\mu, \; i = 1, 2.$$

Using the same idea and the estimates in the proof of the theorem, one can then show:

Corollary 1.2. *Let $\mathcal{D}$ be a bounded smooth open domain of $\mathcal{D} - \cup_{1\leq i\leq m}\overline{B}(P_i, \mu_i)$, then there exists $\varepsilon_0 > 0$ and $2^m - 1$ solutions for problem (P_ε) for $0 < \varepsilon < \varepsilon_0$, moreover those solutions read as follows :*

$$u_{k,\varepsilon}(x) = \sum_{j=1}^{k} \left(\frac{\alpha_n \lambda_{1,j} \varepsilon^{\frac{1}{n-4}}}{\varepsilon^{\frac{2}{n-4}} \lambda_{1,j}^2 + \left| x - \xi_{1,j}^{\varepsilon} \right|^2} \right)^{\frac{n-4}{2}} + \left(\frac{\alpha_n \lambda_{2,j} \varepsilon^{\frac{1}{n-4}}}{\lambda_{2,j}^2 \varepsilon^{\frac{2}{n-4}} + \left| x - \xi_{2,j}^{\varepsilon} \right|^2} \right)^{\frac{n-4}{2}} + \varphi_\varepsilon(x)$$

where $\varphi_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ uniformly, $1 \leq k \leq m$ and α_n is a constant depending on n. $\lambda_{i,j}$ and $\xi_{i,j}^{\varepsilon}$ are critical points of a function built using the Green's function and its regular part. Also, there exist $0 < c < C$ such that

$$c\mu_j < \left| \xi_{i,j}^{\varepsilon} - P_j \right| < C\mu_j, \ i = 1, 2 \text{ and } 1 \leq j \leq m.$$

As remarked in the paper [10], from the proof one can see that there is no need for the excised domains to be balls: in fact the scheme of the proof applies in the same way if one considers any holes contained in some small balls (see also Corollary 2.1 in [11]). Moreover we believe that our result can be generalized with a condition similar to that of Theorem 1.1 in [11], where the authors consider general holes with assumptions on the cohomology groups: in fact one uses estimates on the expansions of the Green's and Robin's functions in the abstract min-max argument, and for the bi-Laplacian these last functions are essentially the same as in the classical case, just by taking into account the related exponent.

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2 Preliminaries

Let us start by defining the following functions

$$\overline{U}_{(\xi,\lambda)}(x) = \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{n-4}{2}},$$

where $\lambda > 0$ and $\xi \in \Omega$. For $u \in D^{2,2}(\Omega)$, we will denote by Pu the projection on $H^2(\Omega) \cap H_0^1(\Omega)$, defined as the unique solution of the problem

$$\begin{cases} \Delta^2 v = u^p & \text{on} \quad \Omega \\ v = \Delta v = 0 & \text{on} \quad \partial\Omega \end{cases},$$

We recall that for the bi-Laplacian operator, the Green's function of a set Ω with Navier boundary conditions is defined to be the solution of

$$\begin{cases} \Delta_x^2 G(x,y) = \delta_y & \text{on} \quad \Omega \\ G(x,y) = \Delta_x G(x,y) = 0 & \text{on} \quad \partial\Omega \end{cases}.$$

This function can be written as

$$G(x,y) = \frac{a_n}{|x-y|^{n-4}} - H(x,y), \ \forall x,y \in \Omega \text{ and } x \neq y,$$

where a_n is a positive constant depending on n and H is the positive smooth solution of

$$\begin{cases} \Delta_x^2 H(x,y) = 0 & \text{on} \quad \Omega \\ H(x,y) = \frac{1}{|x-y|^{n-4}}, \quad \Delta H(x,y) = \Delta \frac{1}{|x-y|^{n-4}} & \text{on} \quad \partial\Omega \end{cases} \tag{3}$$

Now let ξ_1, ξ_2 be two points in Ω, and $\lambda_1, \lambda_2 > 0$, we will write $\overline{U}_i = \overline{U}_{(\xi_i,\lambda_i)}$ and $U_i = P\overline{U}_i$. Then one has $U_i = \overline{U}_i - \varphi_i$ and

$$\varphi_i(x) = H(x,\xi_i)\lambda_i^{\frac{n-4}{2}} \int_{\mathbb{R}^n} \overline{U}^p(y)dy + o(\lambda_i^{\frac{n-4}{2}}). \tag{4}$$

Away from $x = \xi$, we have

$$U_i(x) = G(x,\xi_i)\lambda_i^{\frac{n-4}{2}} \int_{\mathbb{R}^n} \overline{U}^p(y)dy + o(\lambda_i^{\frac{n-4}{2}}). \tag{5}$$

For more details about these estimates we refer to the Appendix.

Let us set now J to be the functional defined by

$$J(u) = \frac{1}{2}\int_{\Omega} |\Delta u|^2 - \frac{1}{p+1}\int_{\Omega} |u|^p \,, \tag{6}$$

and let us find an expansion of

$$J(U_1+U_2) = \frac{1}{2}\int_{\Omega} |\Delta\left(U_1+U_2\right)|^2 - \frac{1}{p+1}\int_{\Omega} (U_1+U_2)^p \,. \tag{7}$$

We define the set

$$O_\delta\left(\Omega\right) = \left\{(\xi_1,\xi_2)\in\Omega\times\Omega; |\xi_1-\xi_2|>\delta \text{ and } d(\xi_i,\partial\Omega)>\delta\right\}, \tag{8}$$

where $\delta > 0$ is a small fixed number and let

$$C_n = \frac{1}{2}\int_{\Omega} \left|\nabla\overline{U}\right|^2 - \frac{1}{p+1}\int_{\Omega} \overline{U}^p.$$

Then we have the following

Lemma 2.1. *For (ξ_1,ξ_2) in $O_\delta\left(\Omega\right)$ we get*

$$\begin{aligned} J(U_1+U_2) \;&=\; 2C_n + \frac{1}{2}\left(\int_{\mathbb{R}^n}\overline{U}^p\right)\left(H\left(\xi_1,\xi_1\right)\lambda_1^{n-4} + H\left(\xi_2,\xi_2\right)\lambda_2^{n-4} - 2\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}G\left(\xi_1,\xi_2\right)\right) \\ &\quad + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right). \end{aligned}$$

The proof follows from the following estimates (see Appendix):

$$\int_{\Omega}|\Delta U_i|^2 = \int_{\mathbb{R}^n}\left|\Delta\overline{U}\right|^2 - \left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2 H\left(\xi_i,\xi_i\right)\lambda_i^{n-4} + o(\lambda_i^{n-4}),$$

$$\int_{\Omega}\Delta U_1\Delta U_2 = \left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2 \lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}G\left(\xi_1,\xi_2\right) + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right),$$

$$\frac{1}{p+1}\int_{\Omega} U_i^{p+1} = \frac{1}{p+1}\int_{\Omega}\overline{U}^{p+1} - \left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2 H\left(\xi_i,\xi_i\right)\lambda_i^{n-4} + o(\lambda_i^{n-4}),$$

and

$$\frac{1}{p+1}\int_{\Omega}(U_1+U_2)^{p+1} - U_1^{p+1} - U_2^{p+1} =$$

$$= 2\left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 \lambda_1^{\frac{n-4}{2}} \lambda_2^{\frac{n-4}{2}} G(\xi_1, \xi_2) + o\left(\max(\lambda_1, \lambda_2)^{n-4}\right).$$

Therefore one has

$$\begin{aligned}
J(U_1+U_2) &= \frac{1}{2}\int_\Omega |\Delta (U_1+U_2)|^2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^p \\
&= \sum_{i=1}^{2}\left(\frac{1}{2}\int_\Omega |\Delta U_i|^2 - \frac{1}{p+1}U_i^{p+1}\right) + \int_\Omega \Delta U_1 \Delta U_2 \\
&\quad -\frac{1}{p+1}\int_\Omega (U_1+U_2)^{p+1} - U_1^{p+1} - U_2^{p+1} \\
&= \sum_{i=1}^{2}\frac{1}{2}\left(\int_{\mathbb{R}^n} |\Delta \overline{U}|^2 - \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 H(\xi_i,\xi_i)\lambda_i^{n-4}\right) - \frac{1}{p+1}\int_\Omega \overline{U}^{p+1} \\
&\quad + \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 H(\xi_i,\xi_i)\lambda_i^{n-4} + \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 \lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}} G(\xi_1,\xi_2) \\
&\quad -2\left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 \lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}} G(\xi_1,\xi_2) + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right) \\
&= 2C_n + \frac{1}{2}\left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 \left(H(\xi_1,\xi_1)\lambda_1^{n-4} + H(\xi_2,\xi_2)\lambda_2^{n-4} - 2\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}} G(\xi_1,\xi_2)\right) \\
&\quad + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right).
\end{aligned}$$

Consider now the perturbed energy functional J_ε defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1+\varepsilon}\int_\Omega u^{p+1+\varepsilon},$$

and assume that $(\lambda_i)^{n-4} = c_n \Lambda_i^2 \varepsilon$. Hence we have

$$J_\varepsilon(U_1+U_2) = J(U_1+U_2) + \frac{\varepsilon}{(p+1)^2}\int_\Omega (U_1+U_2)^{p+1} - \frac{\varepsilon}{p+1}\int_\Omega (U_1+U_2)^{p+1}\ln(U_1+U_2) + o(\varepsilon)$$

Using the fact that

$$\int_\Omega (U_1+U_2)^{p+1} = 2\int_{\mathbb{R}^n} \overline{U}^{p+1} + o(1).$$

and for $\rho > 0$ small, we have

$$\int_\Omega (U_1+U_2)^{p+1}\ln(U_1+U_2) = \int_{|x-\xi_1|<\rho}(U_1+U_2)^{p+1}\ln(U_1+U_2) + \int_{|x-\xi_2|<\rho}(U_1+U_2)^{p+1}\ln(U_1+U_2)+o\left(\varepsilon\right),$$

Notice that

$$\int_{|x-\xi_1|<\rho}(U_1+U_2)^{p+1}\ln(U_1+U_2) = -\ln(\lambda_1^{\frac{n-4}{2}})\int_{|x-\xi_1|<\rho}(U_1+U_2)^{p+1}\ln(U_1+U_2)+$$

$$+\int_{|x-\xi_1|<\rho}(U_1+U_2)^{p+1}\ln(\lambda_1^{\frac{n-4}{2}}U_1+\lambda_1^{\frac{n-4}{2}}U_2)$$

$$= -\ln(\lambda_1^{\frac{n-4}{2}})\left(\int_{\mathbb{R}^n}\overline{U}^{p+1}+O(\lambda_1^n)\right)+\int_{\mathbb{R}^n}\overline{U}^{p+1}\ln\overline{U}+o(1),$$

so that

$$\int_\Omega (U_1+U_2)^{p+1}\ln(U_1+U_2) = -\frac{n-4}{2}\ln(\lambda_1\lambda_2)\int_{\mathbb{R}^n}\overline{U}^{p+1}+2\int_{\mathbb{R}^n}\overline{U}^{p+1}\ln\overline{U}+o(1) \tag{9}$$

Thus

$$J_\varepsilon(U_1+U_2) = J(U_1+U_2)+$$

$$+\varepsilon\left(\frac{2}{(p+1)^2}\int_{\mathbb{R}^n}\overline{U}^{p+1}+\frac{n-4}{2\,(p+1)}\ln(\lambda_1\lambda_2)\int_{\mathbb{R}^n}\overline{U}^{p+1}-\frac{2}{p+1}\int_{\mathbb{R}^n}\overline{U}^{p+1}\ln\overline{U}\right)+o(\varepsilon)$$

Using the previous lemma we have the following

Lemma 2.2. *Let us set* $(\lambda_i)^{n-4} = c_n\Lambda_i^2\varepsilon$. *Then we get*

$$J_\varepsilon(U_1+U_2) = 2C_n+\gamma_n\varepsilon+w_n\varepsilon\ln\left(\varepsilon\right)+w_n\varepsilon\Psi\left(\xi_1,\xi_2,\Lambda_1,\Lambda_2\right)+o(\varepsilon)$$

for every $\left(\xi_1,\xi_2,\Lambda_1,\Lambda_2\right)\in O_\delta\left(\Omega\right)\times\left(\delta,\delta^{-1}\right)^2$. *Where:*

$$\gamma_n = \frac{2}{(p+1)^2}\int_{\mathbb{R}^n}\overline{U}^{p+1}-\frac{2}{p+1}\int_{\mathbb{R}^n}\overline{U}^{p+1}\ln\overline{U}+w_n\ln(c_n),$$

$$w_n = \frac{1}{(p+1)^2}\int_{\mathbb{R}^n}\overline{U}^{p+1},$$

and finally

$$\Psi\left(\xi_1,\xi_2,\Lambda_1,\Lambda_2\right) = \frac{1}{2}\left(H\left(\xi_1,\xi_1\right)\Lambda_1^2+H\left(\xi_2,\xi_2\right)\Lambda_2^2-2\Lambda_1\Lambda_2G\left(\xi_1,\xi_2\right)\right)+\ln\left(\Lambda_1\Lambda_2\right).$$

3 Linear Problem

From now on let $\Omega_\varepsilon = \varepsilon^{-\frac{1}{n-4}}\Omega$ and we will consider points $\xi_i' \in \Omega_\varepsilon$ and numbers $\Lambda_i > 0$ for $i = 1,2$ such that $|\xi_1' - \xi_2'| > \delta\varepsilon^{-\frac{1}{n-4}}$, $d(\xi_i', \partial\Omega_\varepsilon) > \delta\varepsilon^{-\frac{1}{n-4}}$ and $\delta < \Lambda_i < \delta^{-1}$. For the sake of simplicity we will adopt the same notations as in [10], that is $\overline{V}_i(x) = \overline{U}_{\xi_i',\Lambda_i^*}$ for $\Lambda_i^* = \left(c_n\Lambda_i^2\right)^{\frac{1}{n-4}}$. The projections on $H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ will be denoted by V_i. Consider the functions

$$\overline{Z}_{ij} = \frac{\partial \overline{V}_i}{\partial \xi_{ij}},\ i = 1, ..., n \text{ and } \overline{Z}_{in+1} = \frac{\partial \overline{V}_i}{\partial \Lambda_i^*}$$

and their projections $Z_{ij} = P\overline{Z}_{ij}$. Let $V = V_1 + V_2$ and $\overline{V} = \overline{V}_1 + \overline{V}_2$. Now for a smooth function h, we want to solve the following linear problem :

$$\begin{cases} \Delta^2\varphi - (p+\varepsilon)V^{p+\varepsilon-1}\varphi = h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{on } \Omega_\varepsilon \\ \varphi = \Delta\varphi = 0 & \text{on } \partial\Omega_\varepsilon \\ \left\langle V_i^{p-1} Z_{ij}, \varphi \right\rangle = 0 & \text{for } i = 1,2\ ;\ j = 1, ..., n+1 \end{cases}, \tag{10}$$

We define the following weighted L^∞ norms : for a function u defined on Ω_ε

$$\|u\|_* = \left\| (w_1 + w_2)^{-\beta} u \right\|_{L^\infty} + \left\| (w_1 + w_2)^{-\beta - \frac{1}{n-4}} \nabla u \right\|_{L^\infty}$$

where $w_i = \left(\frac{1}{1+|x-\xi_i'|^2}\right)^{\frac{n-4}{2}}$, $\beta = \frac{4}{n-4}$, and

$$\|u\|_{**} = \left\| (w_1 + w_2)^{-\gamma} u \right\|_{L^\infty}$$

with $\gamma = \frac{8}{n-4}$.

Proposition 3.1. *There exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\Omega_\varepsilon)$ the problem (10) admits a unique solution $\varphi = L_\varepsilon(h)$. Moreover we have*

$$\|L_\varepsilon(h)\|_* \leq C \|h\|_{**}$$

and

$$|c_{ij}| \leq C \|h\|_{**}.$$

We need the following

Lemma 3.2. *Assume there exists a sequence $\varepsilon = \varepsilon_n$ such that there are functions φ_ε and h_ε such that*

$$\begin{cases} \Delta^2\varphi_\varepsilon - (p+\varepsilon)V^{p+\varepsilon-1}\varphi_\varepsilon & = & h_\varepsilon + \sum_{i,j} c_{ij}V_i^{p-1}Z_{ij} & on & \Omega_\varepsilon \\ \varphi_\varepsilon = \Delta\varphi_\varepsilon & = & 0 & on & \partial\Omega_\varepsilon \\ \left\langle V_i^{p-1}Z_{ij}, \varphi_\varepsilon \right\rangle & = & 0 & for & i=1,2\ ;\ j=1,...,n+1 \end{cases}$$

for certain constants c_{ij} depending on ε, with $\|h_\varepsilon\|_{**} = o(1)$ if $n \neq 6, 8$ and $\ln(\varepsilon)^4 \|h\|_{**} = o(1)$ *in dimension* 6 *and* 8*. Then $\|\varphi_\varepsilon\|_* \longrightarrow 0$.*

Proof. Take $\rho > 0$ and define

$$\|\varphi\|_\rho = \left\|(w_1+w_2)^{-(\beta-\rho)}u\right\|_{L^\infty} + \left\|(w_1+w_2)^{-(\beta-\rho)-\frac{2}{n-4}}\nabla u\right\|_{L^\infty}$$

Assume first that $\|\varphi_\varepsilon\|_\rho = 1$. Then testing by Z_{ij} we get

$$\sum c_{ij}\left\langle V_i^{p-1}Z_{ij}, Z_{lk}\right\rangle = \left\langle \varphi_\varepsilon, \Delta^2 Z_{lk} - (p+\varepsilon)V^{p+\varepsilon-1}Z_{lk}\right\rangle - \langle h_\varepsilon, Z_{lk}\rangle$$

which is an almost diagonal system (see Appendix). Since

$$\Delta^2 Z_{lk} = p\overline{V}_l^{p-1}\overline{Z}_{lk},$$

we get

$$\left\langle \varphi_\varepsilon, \Delta^2 Z_{lk} - (p+\varepsilon)V^{p+\varepsilon-1}Z_{lk}\right\rangle = o(1)\|\varphi_\varepsilon\|_\rho.$$

And since

$$|\langle h_\varepsilon, Z_{lk}\rangle| \leq C\|h_\varepsilon\|_{**},$$

we can deduce that $c_{ij} = o(1)$. Now let us estimate φ_ε. Using the Green's representation formula one has :

$$\varphi_\varepsilon(x) = (p+\varepsilon)\int_{\Omega_\varepsilon} G(x,y)V^{p+\varepsilon-1}\varphi_\varepsilon + \int_{\Omega_\varepsilon} G(x,y)h_\varepsilon + \sum_{i,j} c_{ij}\int_{\Omega_\varepsilon} G(x,y)V_i^{p-1}Z_{ij},$$

We recall that

$$\int_{\Omega_\varepsilon} G(x,y)V^{p+\varepsilon-1}|\varphi_\varepsilon| \leq C\|\varphi_\varepsilon\|_\rho \int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-4}}\overline{V}^{p+\varepsilon-1+\beta} \leq C\|\varphi_\varepsilon\|_\rho (w_1(x)+w_2(x))^\beta$$

and

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} G(x,y)h_\varepsilon\right| &\leq C\|h_\varepsilon\|_{**}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-4}}\left(\left(1+|y-\xi_1'|^2\right)^{-4}+\left(1+|y-\xi_1'|^2\right)^{-4}\right)\\ &\leq C\|h_\varepsilon\|_{**}\ln(\varepsilon)^m (w_1(x)+w_2(x))^\beta,\end{aligned}$$

where $m=1$ if $n=6,8$ and $m=0$ elsewhere. For the last term we have

$$\begin{aligned}\int_{\Omega_\varepsilon} G(x,y)\left|V_i^{p-1}Z_{ij}\right| &\leq C\sum_i\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-4}}\left|\overline{V}_i^{p-1}\overline{Z}_{ij}\right|\\ &\leq C\sum_i\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-4}}\left(1+|y-\xi_1'|^2\right)^{-\frac{n+7}{2}}\\ &\leq C(w_1(x)+w_2(x))^\beta\end{aligned}$$

Now, we also recall that

$$\frac{\partial\varphi_\varepsilon}{\partial x_i}(x) = (p+\varepsilon)\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}G(x,y)V^{p+\varepsilon-1}\varphi_\varepsilon + \int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}G(x,y)h_\varepsilon + \sum_{i,j}c_{ij}\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}G(x,y)V_i^{p-1}Z_{ij}.$$

and one has the following

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}G(x,y)h_\varepsilon\right| &\leq C\|h_\varepsilon\|_{**}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-3}}\left(\left(1+|y-\xi_1'|^2\right)^{-4}+\left(1+|y-\xi_1'|^2\right)^{-4}\right)\\ &\leq C\|h_\varepsilon\|_{**}\ln(\varepsilon)^m (w_1(x)+w_2(x))^{\beta+\frac{1}{n-4}}\end{aligned}$$

In the same way, for the other terms we get

$$\left|\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}G(x,y)V^{p+\varepsilon-1}\varphi_\varepsilon\right| \leq C\|\varphi_\varepsilon\|_\rho (w_1(x)+w_2(x))^{\beta+\frac{1}{n-4}}$$

and

$$\left|\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_i}G(x,y)V_i^{p-1}Z_{ij}\right|\le C\left(w_1(x)+w_2(x)\right)^{\beta+\frac{1}{n-4}}$$

Hence one has

$$\begin{aligned}\left|\varphi_\varepsilon\left(x\right)\right| &\le C\left(\left\|\varphi_\varepsilon\right\|_\rho+\ln(\varepsilon)^m\left\|h_\varepsilon\right\|_{**}\right)\left(w_1(x)+w_2(x)\right)^\beta \\ \left|\nabla\varphi_\varepsilon\left(x\right)\right| &\le C\left(\left\|\varphi_\varepsilon\right\|_\rho+\ln(\varepsilon)^m\left\|h_\varepsilon\right\|_{**}\right)\left(w_1(x)+w_2(x)\right)^{\beta+\frac{1}{n-4}}\end{aligned} \tag{11}$$

In particular since $\left\|\varphi_\varepsilon\right\|_\rho=1$, we have

$$\left|\varphi_\varepsilon\left(x\right)\right|\left(w_1(x)+w_2(x)\right)^{-\beta-\rho}+\left|\nabla\varphi_\varepsilon\left(x\right)\right|\left(w_1(x)+w_2(x)\right)^{-\beta-\frac{1}{n-4}-\rho}\le C\left(w_1(x)+w_2(x)\right)^\rho.$$

Thus there exists $R>0$ and $\gamma>0$ such that $\left\|\varphi_\varepsilon\right\|_{L^\infty\left(B(\xi_i',R)\right)}\ge\gamma$ for $i=1,2$. Also using elliptic regularity theory, one has that $\varphi_\varepsilon\left(x-\xi_i'\right)$ converges uniformly on every compact set to a function $\widetilde{\varphi}$ solution of the following equation :

$$\Delta^2\widetilde{\varphi}=p\overline{U}_{\Lambda,0}^{p-1}\widetilde{\varphi}\text{ on }\mathbb{R}^n,$$

for a certain $\Lambda>0$, and using the fact that

$$|\widetilde{\varphi}|\le\frac{C}{|x|^{(n-4)\beta}},$$

a simple boot-strap argument yields to

$$|\widetilde{\varphi}|\le\frac{C}{|x|^{(n-4)}}$$

Thus using the classification of solutions in [16] one finds that $\widetilde{\varphi}$ is a linear combination of $\frac{\partial}{\partial x_i}\overline{U}_{\Lambda,0}$, $i=1,...,n$ and $\frac{\partial}{\partial\Lambda}\overline{U}_{\Lambda,0}$. But passing to the limit in the orthogonality conditions, it yields to $\widetilde{\varphi}=0$ which contradicts the fact that $\left\|\varphi_\varepsilon\right\|_{L^\infty\left(B(\xi_1',R)\right)}\ge\gamma$. Now to finish the proof, notice that from (11) we get that

$$\left\|\varphi_\varepsilon\right\|_*\le C\left(\left\|\varphi_\varepsilon\right\|_\rho+\ln(\varepsilon)^m\left\|h_\varepsilon\right\|_{**}\right).$$

□

Proof. of proposition (3.1):

Consider the Space

$$H=\left\{\varphi\in H^{2}\left(\Omega_{\varepsilon}\right)\cap H_{0}^{1}\left(\Omega_{\varepsilon}\right);\ \left\langle V_{i}^{p-1}Z_{ij},\varphi\right\rangle=0,\forall i,j\right\}$$

endowed with the $H^{2}\left(\Omega_{\varepsilon}\right)\cap H_{0}^{1}\left(\Omega_{\varepsilon}\right)$ inner product, namely

$$(u,v)=\int_{\Omega_{\varepsilon}}\Delta u\Delta v$$

The weak formulation of the problem then becomes

$$(\varphi,v)=\left\langle(p+\varepsilon)\,V^{p+\varepsilon-1}\varphi-h,v\right\rangle,\ \ \forall v\in H.$$

Therefore using Riesz representation theorem, we get that

$$\varphi=T_{\varepsilon}\left(\varphi\right)+\widetilde{h}$$

where T_{ε} is a linear operator that is compact on H because of the elliptic regularity and Sobolev embedding, hence using the Fredholm alternative we have existence of a unique solution if and only if the kernel of the operator $Id-T_{\varepsilon}$ is trivial. One is led to consider then the solutions of $\varphi=T_{\varepsilon}\left(\varphi\right)$, but this is equivalent to solving the problem

$$\begin{cases}\Delta^{2}\varphi-(p+\varepsilon)V^{p+\varepsilon-1}\varphi & = & \sum_{i,j}c_{ij}V_{i}^{p-1}Z_{ij} & \text{on} & \Omega_{\varepsilon}\\ \varphi=\Delta\varphi & = & 0 & \text{on} & \partial\Omega_{\varepsilon}\\ \left\langle V_{i}^{p-1}Z_{ij},\varphi\right\rangle & = & 0 & \text{for} & i=1,2\ ;\ j=1,...,n+1\end{cases},$$

The conclusion thus follows from the lemma, and the unique solution for this is $\varphi=0$. Also the fact that $\|\varphi\|_{*}\leq C\,\|h\|_{**}$ follows easily from the previous lemma. □

Now using the same strategy and following the argument in [10] we get

Proposition 3.3. *Under the assumption of Proposition 3.1, we have*

$$\left\|\nabla_{\xi',\Lambda}L_{\varepsilon}\left(h\right)\right\|_{*}\leq C\,\|h\|_{**}\,.$$

4 Finite dimensional reduction

In this section we want to reduce the resolution of the problem to the study of critical points of a function defined on a finite dimensional manifold. So here we will look for a solution of the form $u = V + \varphi$ where $\varphi \in H$, the Hilbert space defined in the previous section. We will split the difficulties in several steps. First we will start by looking for a solution of the following intermediate problem

$$\begin{cases} \Delta^2(V+\overline{\varphi}) - (V+\overline{\varphi})_+^{p+\varepsilon} & = & \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{on} & \Omega_\varepsilon \\ \overline{\varphi} = \Delta\overline{\varphi} & = & 0 & \text{on} & \partial\Omega_\varepsilon \\ \left\langle V_i^{p-1} Z_{ij}, \overline{\varphi} \right\rangle & = & 0 & \text{for} & i=1,2\ ;\ j=1,...,n+1 \end{cases}, \tag{12}$$

Notice that this problem is equivalent to

$$\Delta^2\overline{\varphi} - (p+\varepsilon)V^{p+\varepsilon-1}\overline{\varphi} = N_\varepsilon(\overline{\varphi}) - R_\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \tag{13}$$

where $\overline{\varphi} \in H$ and

$$N_\varepsilon(\overline{\varphi}) = (V+\overline{\varphi})_+^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\overline{\varphi} - V^{p+\varepsilon}$$

and

$$R_\varepsilon = V^{p+\varepsilon} - \overline{U}_1^p - \overline{U}_2^p.$$

We will split the problem and then we will use a fixed point argument to find a solution. If we take $\psi = -L_\varepsilon(R_\varepsilon)$ then one is looking for a solution to (13) of the form $\overline{\varphi} = \varphi + \psi$ and thus φ will satisfy

$$\varphi = L_\varepsilon(N_\varepsilon(\varphi+\psi)). \tag{14}$$

Consider the fixed point problem

$$\varphi = L_\varepsilon(N_\varepsilon(\varphi+\psi)) = A_\varepsilon(\varphi),$$

We want to show that A_ε satisfies the contraction mapping theorem in an appropriate complete set. Let us estimate $N_\varepsilon(u)$ for $\|u\|_* \leq 1$. Using a Taylor expansion we get the existence of $t \in [0,1]$ such that

$$N_\varepsilon(u) = \frac{(p+\varepsilon)(p+\varepsilon-1)}{2}(V+tu)^{p-2+\varepsilon}u^2,$$

with $p-2 = \frac{n+4}{n-4} - 2 = \frac{8-n+4}{n-4} = \frac{12-n}{n-4}$. So if $n \leq 12$ one gets

$$\begin{aligned}
\left|N_\varepsilon(u)\overline{V}^{-\frac{8}{n-4}}\right| &= \frac{(p+\varepsilon)(p+\varepsilon-1)}{2}\overline{V}^{-\frac{8}{n-4}}(V+tu)^{p-2+\varepsilon}u^2 \\
&\leq C\overline{V}^{-\frac{8}{n-4}+2\beta}(V+tu)^{p-2+\varepsilon}\|u\|_*^2 \\
&\leq C\overline{V}^{-\frac{8}{n-4}+2\beta+(p-2)\beta}\|u\|_*^2 \\
&\leq C\overline{V}^{(p-2)\beta}\|u\|_*^2.
\end{aligned}$$

For $n > 12$, the proof is more involved, since we have to distinguish two cases. First consider $\delta > 0$ and take the region $d(y,\partial\Omega_\varepsilon) > \delta\varepsilon^{-\frac{1}{n-4}}$, then one has the existence of $C_\delta > 0$ such that $V > C_\delta\overline{V}$ and therefore we get

$$\begin{aligned}
\left|N_\varepsilon(u)\overline{V}^{-\frac{8}{n-4}}\right| &= \frac{(p+\varepsilon)(p+\varepsilon-1)}{2}\overline{V}^{-\frac{8}{n-4}}(V+tu)^{p-2+\varepsilon}u^2 \\
&\leq \overline{V}^{2\beta-\frac{8}{n-4}}C\|u\|_*^2 \\
&\leq C\overline{V}^{2\beta-1}\|u\|_*^2 \leq C\varepsilon^{2\beta-1}\|u\|_*^2
\end{aligned}$$

If $d(y,\partial\Omega_\varepsilon) \leq \delta\varepsilon^{-\frac{1}{n-4}}$, by using Hopf lemma, we have that for δ sufficiently small $V(y) \sim \frac{\partial V}{\partial\nu}d(y,\partial\Omega_\varepsilon)$. Then we recall that $|\nabla V| = \left|\nabla\overline{V}\right| + o(1)$,

$$|\nabla V| \geq C\varepsilon^{\frac{n-3}{n-4}} \text{ for } \varepsilon \text{ small enough}$$

and thus $V(y) \geq C\varepsilon^{\frac{n-3}{n-4}}d(y,\partial\Omega_\varepsilon)$. Therefore

$$\begin{aligned}
\left|N_\varepsilon(u)\overline{V}^{-\frac{8}{n-4}}\right| &\leq C\overline{V}^{-\frac{8}{n-4}}\left(\varepsilon^{\frac{n-3}{n-4}}d(y,\partial\Omega_\varepsilon)\right)^{p-2}u^2 \\
&\leq C\varepsilon^{\frac{n-3}{n-4}(p-2)-\frac{8}{n-4}}d(y,\partial\Omega_\varepsilon)^p\|u\|_*^2 \\
&\leq C\varepsilon^{\frac{n-3}{n-4}(p-2)-\frac{8}{n-4}-\frac{p}{n-4}+2\beta+\frac{2}{n-4}}\|u\|_*^2 \\
&\leq C\varepsilon^{2\beta-1}\|u\|_*^2,
\end{aligned}$$

hence

$$\|N_\varepsilon(u)\|_{**} \le \begin{cases} C\overline{V}^{(p-2)\beta}\|u\|_*^2, \text{ if } n \le 12 \\ C\varepsilon^{2\beta-1}\|u\|_*^2 \text{ if } n > 12 \end{cases} \tag{15}$$

Now consider

$$\begin{aligned} |R_\varepsilon| &= \overline{V}^{p+\varepsilon} - \overline{V}_1^p - \overline{V}_2^p + V^{p+\varepsilon} - \overline{V}^{p+\varepsilon} = \overline{V}^{p+\varepsilon} - \overline{V}_1^p - \overline{V}_2^p + o(\varepsilon^p) \\ &\le \sum_i C\varepsilon\overline{V}_i^p \ln(\overline{V}_i) + o(\varepsilon^p), \end{aligned}$$

thus

$$\|R_\varepsilon\|_{**} \le C\varepsilon.$$

We get then the following

Lemma 4.1. *There exists $C > 0$ such that for ε small enough and for $\|u\|_* \le 1$ we have*

$$\|N_\varepsilon(u+\psi)\|_{**} \le \begin{cases} C\left(\|u\|_*^2 + \varepsilon^2\right) \text{ if } n \le 12 \\ C\left(\varepsilon^{2\beta-1}\|u\|_*^2 + \varepsilon^{2\beta+1}\right) \text{ if } n > 12 \end{cases}. \tag{16}$$

Now, we can state the following

Proposition 4.2. *There exists $C > 0$ such that for ε small enough, the problem (14) has a unique solution φ with $\|\varphi\|_* < C\varepsilon$.*

Proof. Let

$$F = \left\{u \in H^2(\Omega) \cap H_0^1(\Omega), \|u\|_* < \varepsilon\right\},$$

and then consider $A_\varepsilon : F \longrightarrow H^2(\Omega) \cap H_0^1(\Omega)$. By using the previous lemma and proposition 3.1 we get

$$\begin{aligned}
\|A_\varepsilon(u)\|_* &\leq C\,\|N_\varepsilon(u+\psi)\|_{**} \\
&\leq \begin{cases} C\left(\|u\|_*^2+\varepsilon^2\right) \text{ if } n\leq 12 \\ C\left(\varepsilon^{2\beta-1}\|u\|_*^2+\varepsilon^{2\beta+1}\right) \text{ if } n>12 \end{cases} \\
&\leq \begin{cases} C\varepsilon^2 \text{ if } n\leq 12 \\ C\varepsilon^{2\beta+1} \text{ if } n>12 \end{cases},
\end{aligned}$$

so for $\varepsilon > 0$ small enough, we have that A_ε maps F into itself. Now let us estimate $\|A_\varepsilon(u) - A_\varepsilon(v)\|_*$ for $u, v \in F$. Since

$$\|A_\varepsilon(u) - A_\varepsilon(v)\|_* \leq C\,\|N_\varepsilon(u+\psi) - N_\varepsilon(v+\psi)\|_{**},$$

it suffices to show that N_ε is a contraction to finish the proof of the proposition. Notice that by construction

$$D_u N_\varepsilon(u) = (p+\varepsilon)\left((V+u)_+^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right),$$

thus

$$|N_\varepsilon(u+\psi) - N_\varepsilon(v+\psi)| \leq C\overline{V}^{p-2}\,|h|\,|u-v|,$$

where h belongs to the segment $u+\psi$, $v+\psi$. Hence

$$\overline{V}^{-\frac{8}{n-4}}\,|N_\varepsilon(u+\psi) - N_\varepsilon(v+\psi)| \leq \overline{V}^{p-2}C\,\|h\|_*\,\|u-v\|_*,$$

and this leads to

$$\overline{V}^{-\frac{8}{n-4}}\,|N_\varepsilon(u+\psi) - N_\varepsilon(v+\psi)| \leq C\overline{V}^{p-2}\left(\|u\|_*+\|v\|_*+\|\psi\|_*\right)\|u-v\|_*.$$

Therefore

$$\begin{aligned}
\|N_\varepsilon(u+\psi) - N_\varepsilon(v+\psi)\|_{**} &\leq \begin{cases} C\left(\|u\|_*+\|v\|_*+\|\psi\|_*\right)\|u-v\|_* \text{ if } n\leq 12 \\ C\varepsilon^{p-2}\left(\|u\|_*+\|v\|_*+\|\psi\|_*\right)\|u-v\|_* \text{ if } n>12 \end{cases} \\
&\leq C\varepsilon^{\min(1,p-1)}\,\|u-v\|_*,
\end{aligned}$$

and thus for ε small it is a contraction, and that finishes the proof. □

Lemma 4.3. *The map* $(\xi', \Lambda) \longrightarrow \varphi(\xi', \Lambda)$ *is of class* C^1 *for the norm* $\|\|_*$ *and there exists* $C > 0$ *such that*

$$\left\|\nabla_{\xi',\Lambda}\widetilde{\varphi}\right\|_* \leq C\varepsilon.$$

Proof. Let K be the map defined by

$$K\left(\xi', \Lambda, \widetilde{\varphi}\right) = \widetilde{\varphi} - A_\varepsilon(\widetilde{\varphi})$$

We recall that

$$D_u N_\varepsilon(u) = (p+\varepsilon)\left((V+u)_+^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right)$$

and

$$D_{\xi'} N_\varepsilon(u) = (p+\varepsilon)\left[(V+u)_+^{p+\varepsilon-1} - (p+\varepsilon-1)V^{p+\varepsilon-2}u - V^{p+\varepsilon-1}\right] D_{\xi'}V.$$

The same holds for $D_\Lambda N_\varepsilon(u)$. Also,

$$D_u K\left(\xi', \Lambda, u\right) h = h + L_\varepsilon(D_u N_\varepsilon(u+\psi)h) = h + M(h).$$

Now

$$\begin{aligned} \|M(h)\|_* &\leq C\,\|D_u N_\varepsilon(u+\psi)h\|_{**} \\ &\leq C\left\|\overline{V}^{-\frac{8}{n-4}+\beta} D_u N_\varepsilon(u+\psi)\right\|_\infty \|h\|_* \end{aligned}$$

and since

$$\left|\overline{V}^{-\frac{8}{n-4}+\beta} D_u N_\varepsilon(u+\psi)\right| \leq C\overline{V}^{2\beta-1}\,\|u+\psi\|_*\,,$$

we get

$$\left\|\overline{V}^{-\frac{8}{n-4}+\beta} D_u N_\varepsilon(u+\psi)\right\|_\infty \leq C\begin{cases} \varepsilon \text{ if } n \leq 12 \\ \varepsilon^{2\beta} \text{ if } n > 12 \end{cases}$$

Therefore

$$\|M(h)\|_* \leq C\varepsilon^{\min(1,2\beta)}\,\|h\|_*$$

and by using the implicit function theorem, φ depends continuously on the parameter (ξ', Λ).

In the other hand if we differentiate with respect to ξ' we get

$$D_{\xi'}K\left(\xi', \Lambda, u\right) = D_{\xi'}u + D_{\xi'}L_{\varepsilon}(N_{\varepsilon}(u+\psi))$$

From proposition (3.3) we have that

$$\left\|D_{\xi'}L_{\varepsilon}(h)\right\|_{*} \leq C\left\|h\right\|_{**}$$

and thus we need to compute

$$D_{\xi'}\psi = (D_{\xi'}L_{\varepsilon})(R_{\varepsilon}) + L_{\varepsilon}(D_{\xi'}R_{\varepsilon})$$

But

$$D_{\xi_1'}R_{\varepsilon} = (p+\varepsilon)V^{p+\varepsilon-1}D_{\xi_1'}V - p\overline{V}_1^{p-1}D_{\xi_1'}\overline{V}_1$$

which depends continuously on the parameters, and this is enough to prove that $\widetilde{\varphi}$ is C^1 with respect to the parameters (ξ', Λ). Moreover we have

$$\begin{aligned} D_{\xi'}\varphi = &-\left(D_{\varphi}K\left(\xi', \Lambda, \varphi\right)\right)^{-1}\left[\left(D_{\xi'}L_{\varepsilon}\right)\left(N_{\varepsilon}\left(\varphi+\psi\right)\right)+\right. \\ &\left.+L_{\varepsilon}\left(D_{\xi'}(N_{\varepsilon}\left(\varphi+\psi\right))\right)+L_{\varepsilon}\left(D_{\varphi}(N_{\varepsilon})\left(\varphi+\psi\right)D_{\xi'}\psi\right)\right], \end{aligned} \tag{17}$$

hence

$$\left\|D_{\xi'}\varphi\right\|_{*} \leq C\left(\left\|N_{\varepsilon}\left(\varphi+\psi\right)\right\|_{**} + \left\|D_{\xi'}(N_{\varepsilon}\left(\varphi+\psi\right))\right\|_{**} + \left\|D_{\varphi}(N_{\varepsilon})\left(\varphi+\psi\right)D_{\xi'}\psi\right\|_{**}\right).$$

Now we know that from (16)

$$\left\|N_{\varepsilon}(\varphi+\psi)\right\|_{**} \leq \begin{cases} C\varepsilon^2 \text{ if } n \leq 12 \\ C\varepsilon^{2\beta+1} \text{ if } n > 12 \end{cases}$$

and

$$\begin{aligned} \left|D_{\xi'}(N_{\varepsilon}\left(u\right))\right| &= (p+\varepsilon)\left|\left[(V+u)_+^{p+\varepsilon-1} - (p+\varepsilon-1)V^{p+\varepsilon-2}u - V^{p+\varepsilon-1}\right]D_{\xi'}V\right| \\ &\leq CV^{p-2}\left|D_{\xi'}V\right|\left|u\right| \\ &\leq C\overline{V}^{p-2+\frac{n-3}{n-4}+\beta}\left\|u\right\|_{*}. \end{aligned}$$

From this we get

$$\overline{V}^{-\frac{8}{n-4}} \left| D_{\xi'}(N_\varepsilon(u)) \right| \leq C\overline{V}^{\frac{n-3}{n-4}+\beta-1} \|u\|_*,$$

therefore

$$\left\| D_{\xi'}(N_\varepsilon(\widetilde{\varphi}+\psi)) \right\|_{**} \leq C\varepsilon,$$

and a similar estimate gives

$$\left\| D_{\widetilde{\varphi}}(N_\varepsilon)(\widetilde{\varphi}+\psi) D_{\xi'}\psi \right\|_{**} \leq C\varepsilon.$$

Since there is no difference for the case of the differentiation with respect to Λ, we omit that proof. □

5 The reduced functional

Here we will use the same notations for the rescaled parameters and domain, and we recall that so far we have $\overline{\varphi} = \varphi + \psi$ is the unique solution of

$$\begin{cases} \Delta^2(V+\overline{\varphi}) - (V+\overline{\varphi})_+^{p+\varepsilon} = \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{on} \quad \Omega_\varepsilon \\ \overline{\varphi} = \Delta\overline{\varphi} = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ \left\langle V_i^{p-1} Z_{ij}, \overline{\varphi} \right\rangle = 0 & \text{for} \quad i=1,2\ ;\ j=1,...,n+1 \end{cases},$$

with $\|\varphi\|_* \leq \varepsilon$ and it is smooth with respect to (ξ', Λ) and more than that $\left\| \nabla_{\xi',\Lambda}(\varphi) \right\|_* \leq C\varepsilon$. So, now we want to go back to our original set Ω, therefore we will denote $\xi_i' = \varepsilon^{-\frac{1}{n-4}} \xi_i$ where $\xi_i \in \Omega$ and we recall that if we take ξ_i and Λ so that it $c_{ij} = 0$ then we have a solution of our original problem.

Let $\mathcal{I}_\varepsilon$ be the functional defined by

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |\Delta v|^2 - \frac{1}{p+\varepsilon+1} \int_{\Omega_\varepsilon} |v|^{p+\varepsilon+1}$$

so that $v = V + \overline{\varphi}$ is a solution to our problem if and only if it is a critical point for this functional.

Let us set $\kappa = \frac{8}{8+\varepsilon(n-4)}$ and consider the functions defined on Ω by

$$\widehat{\varphi}(\xi,\Lambda)(x) = \varepsilon^{-\frac{\kappa}{2}}\varphi\left(\varepsilon^{-\frac{1}{n-4}}\xi,\Lambda\right)\left(\varepsilon^{-\frac{1}{n-4}}x\right),$$

$$\widehat{\psi}(x) = \varepsilon^{-\frac{\kappa}{2}}\psi\left(\varepsilon^{-\frac{1}{n-4}}x\right),$$

and

$$\widehat{U}_i(x) = \varepsilon^{-\frac{\kappa}{2}}V_i\left(\varepsilon^{-\frac{1}{n-4}}x\right).$$

Therefore if we set $\widehat{U}(x) = \widehat{U}_2(x) + \widehat{U}_1(x)$ and

$$I(\xi,\Lambda) = J_\varepsilon\left(\widehat{U} + \widehat{\psi} + \widehat{\varphi}(\xi,\Lambda)\right)$$

then

$$I(\xi,\Lambda) = \varepsilon^{1-\kappa}\mathcal{I}_\varepsilon\left(V + \psi + \widehat{\varphi}\right).$$

Lemma 5.1. *$u = \widehat{U} + \widehat{\psi} + \widehat{\varphi}(\xi,\Lambda)$ is a solution of the problem P_ε if and only if (ξ,Λ) is a critical point of I.*

Proof. Notice that

$$DI_\varepsilon(V+\varphi)Z_{ij} = \sum_{k,l} c_{kl}\left\langle Z_{kl}V_l^{p-1}, Z_{ij}\right\rangle$$

hence using the fact that the system is almost diagonal, we get that

$$DI_\varepsilon(V+\varphi)Z_{ij} = 0$$

for every i,j if and only if $c_{ij} = 0$ for every i,j. Notice also that if we assume that $\frac{\partial}{\partial\xi_{kl}}I(\xi,\Lambda) = 0$ then

$$\frac{\partial}{\partial\xi'_{kl}}\mathcal{I}_\varepsilon\left(V + \overline{\varphi}\right) = 0.$$

Namely if and only if

$$D\mathcal{I}_\varepsilon(V+\overline{\varphi})\left(\frac{\partial}{\partial\xi'_{kl}}V+\frac{\partial}{\partial\xi'_{kl}}\varphi\right)=0$$

Now it is easy to see that

$$\frac{\partial}{\partial\xi'_{kl}}V=Z_{kl}+o(1)$$

hence

$$D\mathcal{I}_\varepsilon(V+\overline{\varphi})\left(Z_{kl}+o(1)\right)=0. \tag{18}$$

Now for a given a smooth function u, we can find constants a_{ij} such that

$$\left\langle u-\sum_{ij}a_{ij}Z_{ij},Z_{kl}V_l^{p-1}\right\rangle=0,\ \text{for every } k,l$$

But by construction $D\mathcal{I}_\varepsilon(V+\overline{\varphi})v=0$ for every $v\in H$, the space defined in section 3. Notice that one can show that $|a_{ij}|=O(\|u\|_*)$. Combining this fact with (18) we get

$$D\mathcal{I}_\varepsilon(V+\overline{\varphi})\left(Z_{kl}+wo(1)\right)=0,$$

where w is a uniformly bounded function in the space spanned by Z_{ij}, and hence

$$D\mathcal{I}_\varepsilon(V+\overline{\varphi})Z_{kl}=0,$$

which finishes the proof. □

Proposition 5.2. *We have the following expansion*

$$\varepsilon^{\kappa-1}I(V+\overline{\varphi})=2C_n+\gamma_n\varepsilon+w_n\varepsilon\Psi\left(\xi,\Lambda\right)+o(\varepsilon),$$

where $o(\varepsilon)\longrightarrow 0$ *as* $\varepsilon\longrightarrow 0$ *in the* C^1 *sense, uniformly in* $O_\delta\left(\Omega\right)\times\left(\delta,\delta^{-1}\right)^2$.

Proof. Let us show first that

$$I\left(\xi,\Lambda\right)-J_{\varepsilon}\left(\widehat{U}\right)=o(\varepsilon).$$

Indeed, using a Taylor expansion we have

$$J_{\varepsilon}\left(\widehat{U}+\widehat{\psi}\right)-J_{\varepsilon}\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)=\int_{0}^{1}tD^{2}J_{\varepsilon}\left(\widehat{U}+\widehat{\psi}+t\widehat{\varphi}\right)[\widehat{\varphi},\widehat{\varphi}]\,dt$$

and this holds since $DJ_{\varepsilon}\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)=0$. Therefore, we have

$$\int_{0}^{1}tD^{2}J_{\varepsilon}\left(\widehat{U}+\widehat{\psi}+t\widehat{\varphi}\right)[\widehat{\varphi},\widehat{\varphi}]\,dt=\varepsilon^{1-\kappa}\int_{0}^{1}tD^{2}\mathcal{I}_{\varepsilon}\left(\widehat{U}+\widehat{\psi}+t\widehat{\varphi}\right)[\widehat{\varphi},\widehat{\varphi}]\,dt$$

$$=\varepsilon^{1-\kappa}\int_{0}^{1}t\left[\int_{\Omega_{\varepsilon}}\left|\nabla\varphi\right|^{2}-(p+\varepsilon)\left(V+\psi+t\varphi\right)^{p+\varepsilon-1}\varphi^{2}\right]dt$$

$$=\varepsilon^{1-\kappa}\int_{0}^{1}t\left[\int_{\Omega_{\varepsilon}}(p+\varepsilon)\left[V^{p+\varepsilon-1}-\left(V+\psi+t\varphi\right)^{p+\varepsilon-1}\right]\varphi^{2}+N_{\varepsilon}\left(\varphi+\psi\right)\varphi\right]dt.$$

But we have $\left\|\varphi\right\|_{*}+\left\|\psi\right\|_{*}=O\left(\varepsilon\right)$, using (16), we get

$$\int_{\Omega_{\varepsilon}}N_{\varepsilon}\left(\varphi+\psi\right)\varphi\leq\int_{\Omega_{\varepsilon}}\overline{V}^{p-1+\beta}\left\|N_{\varepsilon}\left(\varphi+\psi\right)\right\|_{**}\left\|\varphi\right\|_{*}\leq C\varepsilon^{3}\int_{\Omega_{\varepsilon}}\overline{V}^{p-1+\beta}\leq C\varepsilon^{3}.$$

Now, the remaining part can be estimated as follows

$$\begin{aligned}\int_{\Omega_{\varepsilon}}\left[V^{p+\varepsilon-1}-\left(V+\psi+t\varphi\right)^{p+\varepsilon-1}\right]\varphi^{2}&\leq C\varepsilon^{2}\int_{\Omega_{\varepsilon}}\overline{V}^{2\beta}\left[V^{p+\varepsilon-1}-\left(V+\psi+t\varphi\right)^{p+\varepsilon-1}\right]\\&\leq C\varepsilon^{2},\end{aligned}$$

therefore

$$I\left(\xi,\Lambda\right)-J_{\varepsilon}(\widehat{U}+\widehat{\psi})=o\left(\varepsilon\right).$$

For the second stage we consider the derivative with respect to ξ of the difference, then one gets

$$D_{\xi}\left[I\left(\xi,\Lambda\right)-J_{\varepsilon}(\widehat{U}+\widehat{\psi})\right]=$$

$$\varepsilon^{1-\kappa-\frac{1}{n-4}} \int_0^1 t \left[\int_{\Omega_\varepsilon} D_\xi \left[N_\varepsilon \left(\varphi + \psi \right) \varphi \right] + \left(p + \varepsilon \right) D_\xi \left(\left[V^{p+\varepsilon-1} - (V + \psi + t\varphi)^{p+\varepsilon-1} \right] \varphi^2 \right) \right] dt,$$

hence using the same argument in [10] and the rescaling for ξ, we get the desired result.

Now we want to estimate the difference between $J_\varepsilon(\widehat{U} + \widehat{\psi})$ and $J_\varepsilon(\widehat{U})$. In fact one has

$$J_\varepsilon(\widehat{U} + \widehat{\psi}) - J_\varepsilon(\widehat{U}) = \varepsilon^{1-\kappa} \left(I_\varepsilon(V + \psi) - I_\varepsilon(V) \right)$$

$$= \varepsilon^{1-\kappa} \left[\int_0^1 (1-t) \left(\left[(p+\varepsilon) \int_{\Omega_\varepsilon} \left((V + t\psi)^{p+\varepsilon-1} - V^{p+\varepsilon-1} \right) \psi^2 \right] - 2 \int_{\Omega_\varepsilon} R^\varepsilon \psi \right) dt \right].$$

By using the same estimates as above we get

$$J_\varepsilon(\widehat{U} + \widehat{\psi}) - J_\varepsilon(\widehat{U}) = o\left(\varepsilon\right)$$

Now, to finish, we need to estimate the derivative of the difference :

$$D_\xi \left[J_\varepsilon(\widehat{U} + \widehat{\psi}) - J_\varepsilon(\widehat{U}) \right] = \varepsilon^{1-\kappa-\frac{1}{n-4}} D_{\xi'} \left[\int_0^1 (1-t) \cdot \right.$$

$$\left. \left(\left[(p+\varepsilon) \int_{\Omega_\varepsilon} \left((V + t\psi)^{p+\varepsilon-1} - V^{p+\varepsilon-1} \right) \psi^2 \right] - 2 \int_{\Omega_\varepsilon} R^\varepsilon \psi \right) dt \right],$$

hence

$$D_\xi \left[J_\varepsilon(\widehat{U} + \widehat{\psi}) - J_\varepsilon(\widehat{U}) \right] = o\left(\varepsilon\right) - 2\varepsilon^{1-\kappa-\frac{1}{n-4}} D_{\xi'} \int_{\Omega_\varepsilon} R^\varepsilon \psi.$$

Thus an argument similar to [11] adapted to our problem yields to the result, that is

$$D_\xi \left[J_\varepsilon(\widehat{U} + \widehat{\psi}) - J_\varepsilon(\widehat{U}) \right] = o(\varepsilon).$$

We also notice that if we set $\overline{\varsigma} = \frac{1}{2} - \frac{\kappa}{2}$ then we get

$$J_\varepsilon\left(\widehat{U}\right) = J_\varepsilon\left(\varepsilon^{\overline{\varsigma}}\left(U_1 + U_2\right)\right).$$

Hence

$$\varepsilon^{-2\overline{\varsigma}} J_{\varepsilon}\left(\varepsilon^{\overline{\varsigma}}\left(U_1+U_2\right)\right)=J_{\varepsilon}\left(U_1+U_2\right)+\frac{1-\varepsilon^{\frac{\varepsilon}{2}}}{p+1+\varepsilon}\int_{\Omega_{\varepsilon}}\left(U_1+U_2\right)^{p+1+\varepsilon}$$

Now we remark that

$$\frac{1-\varepsilon^{\frac{\varepsilon}{2}}}{p+1+\varepsilon}\int_{\Omega_{\varepsilon}}\left(U_1+U_2\right)^{p+1+\varepsilon}=\frac{1}{p+1}\left(-\varepsilon\ln\left(\varepsilon\right)+o\left(\varepsilon\right)\right)\left(\int_{\Omega}\varepsilon\ln(U_1+U_2)\left(U_1+U_2\right)^{p+1}+\right.$$

$$\left.\int_{\Omega}\left(U_1+U_2\right)^{p+1}+o\left(\varepsilon\right)\right)=\frac{-2\varepsilon\ln\left(\varepsilon\right)}{p+1}\int_{\mathbb{R}^n}\overline{U}^{p+1}+o\left(\varepsilon\right)$$

and therefore the proposition follows. □

6 The exterior domain case

In this case let us consider the domain $D=\mathbb{R}^n-B(0,1)$, and we recall that the regular part of the Green's function of the exterior of a the unit ball is given by:

$$H_*\left(x,y\right)=\frac{a_n}{\left|\left|y\right|\left(x-\overline{y}\right)\right|^{n-4}},$$

where $\overline{y}=\frac{y}{|y|^2}$ is the reflection with respect to the unit ball. Hence one can see that if

$$\rho_*\left(x,y\right)=H_*\left(x,x\right)^{\frac{1}{2}}H_*\left(y,y\right)^{\frac{1}{2}}-G_*(x,y)$$

then

$$\begin{aligned} a_n^{-1}\rho_*\left(x,y\right) &= \frac{1}{\left(\left|x\right|^2-1\right)^{\frac{n-4}{2}}\left(\left|y\right|^2-1\right)^{\frac{n-4}{2}}}+\frac{1}{\left(1+\left|x\right|^2\left|y\right|^2-2\left|x\right|\left|y\right|\cos\left(\theta\right)\right)^{\frac{n-4}{2}}} \\ &\quad -\frac{1}{\left(\left|x\right|^2+\left|y\right|^2-2\left|x\right|\left|y\right|\cos\left(\theta\right)\right)^{\frac{n-4}{2}}} \end{aligned}$$

By this formula one can see that critical points of $\rho_*\left(x,y\right)$ are located so that x and y point in opposite directions, that is when $\sin(\theta)=0$. Therefore we will take $x = s\mathbf{e}$ and $y = -t\mathbf{e}$, where $\mathbf{e}$ is a unit vector and s and t are real numbers greater than 1. Hence our function reads as

$$\widetilde{\rho}(s,t) = a_n \left(\frac{1}{(t^2-1)^{\frac{n-4}{2}}(s^2-1)^{\frac{n-4}{2}}} + \frac{1}{(1+st)^{n-4}} - \frac{1}{(s+t)^{n-4}} \right)$$

and it is easy to see that it has a negative absolute minimum in a point of the form (k,k). Hence we can write

$$c_* = -\widetilde{\rho}(k,k) = -\min_{x,y\in D} \rho_*(x,y) \tag{19}$$

and consider the set

$$A = \left\{(x,y) \in D^2; \rho_*(|x|,|y|) < -r\right\}$$

where r is a small positive real number so that $\widetilde{\rho} = -r$ is a closed curve on which $\nabla\widetilde{\rho} \neq 0$. Observe then that two situations might happen on ∂A: either there exists a tangential direction τ so that $\nabla\rho\cdot\tau \neq 0$ or x and y point in two different directions and $\nabla\rho_*(x,y) \neq 0$ points in the normal direction. Also, it is important to observe that if we consider $D_\mu = \mathbb{R}^n - B(0,\mu)$ then $G_\mu(x,y) = \mu^{4-n}G_*\left(\mu^{-1}x, \mu^{-1}y\right)$ and hence we set $A_\mu = \mu A$ and that corresponds to the set $\rho_\mu(|x|,|y|) < -\mu^{4-n}r$.
For a general domain $\Omega = \mathcal{D} - B(p,\mu)$ one has

$$G(x,y) = G_\mu(x-p, y-p) + O(1),$$

for $(x,y) \in (p,p) + A_\mu$, and $O(1)$ is bounded in the C^1 sense independently of μ.

7 Main Theorem

Since the function Ψ defined in section 2 is singular in the diagonal of $\Omega\times\Omega$, we replace the term $G(\xi_1,\xi_2)$ by $G_M(\xi_1,\xi_2) = \min(G(\xi_1,\xi_2), M)$ for a constant $M > 0$ to be fixed later. We will restrict our study to $\Omega^{\alpha} \times \Omega^{\alpha} \times \mathbb{R}_{+}^{2}$ where

$$\Omega^{\alpha} = \{\xi \in \Omega; d(\xi, \partial\Omega) > \alpha\}$$

and in fact we will restrict more our function to the set $A_{\mu}^{2} \times \mathbb{R}_{+}^{2}$.

Recall that using this restriction we get that $\rho < 0$ and hence the principal part of Ψ, which is a quadratic form, has a negative direction, and we will set $\mathbf{e}(\xi_1, \xi_2)$ the vector defining it. In fact if $\mathbf{e}(\xi_1, \xi_2) = (\mathbf{e}_1(\xi_1, \xi_2), \mathbf{e}_2(\xi_1, \xi_2))$, then

$$\mathbf{e}(\xi_1, \xi_2) = \left(\frac{H(\xi_1, \xi_1)^{\frac{1}{2}}}{H(\xi_2, \xi_2)^{\frac{1}{2}} \rho(\xi_1, \xi_2)}, \frac{H(\xi_2, \xi_2)^{\frac{1}{2}}}{H(\xi_1, \xi_1)^{\frac{1}{2}} \rho(\xi_1, \xi_2)} \right).$$

Hence we have

$$\Psi((\xi_1, \xi_2), \mathbf{e}(\xi_1, \xi_2)) = -\frac{1}{2} - \ln(|\rho(\xi_1, \xi_2)|),$$

Now let k be the number defined in (19), then we set

$$S = S(0, \mu k)$$

and define the class of curves $\gamma : S^2 \times [s, s^{-1}] \times [0, 1] \longrightarrow A_{\mu} \times \mathbb{R}_{+}^{2}$ defined by:

i) for $(\xi_1, \xi_2) \in S^2$, $t \in [0, 1]$, the following holds

$$\gamma(\xi_1, \xi_2, s, t) = (\xi_1, \xi_2, s\mathbf{e}(\xi_1, \xi_2)),$$

$$\gamma(\xi_1, \xi_2, s^{-1}, t) = (\xi_1, \xi_2, s^{-1}\mathbf{e}(\xi_1, \xi_2))$$

ii) $\gamma(\xi_1, \xi_2, t, 0) = (\xi_1, \xi_2, t\mathbf{e}(\xi_1, \xi_2))$, for all $(\xi_1, \xi_2, t) \in S^2 \times [s, s^{-1}]$.

Now we have the following

Proposition 7.1. *The min-max value defined by*

$$C(\Omega) = \inf_{\gamma} \sup_{(\xi_1,\xi_2,t)\in S^2\times[s,s^{-1}]} \Psi(\gamma(\xi_1,\xi_2,t,1)),$$

is a critical value of Ψ.

The proof of this proposition is similar to the one in [11], therefore it will be omitted. The main result is then proved.

8 Appendix

Here we will give a list of estimates used in some of the proofs.

Let $\overline{U}_{(\xi,\lambda)}(x) = \left(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\right)^{\frac{n-4}{2}}$ and for $i = 1,2$ we will set $\overline{U}_i = \overline{U}_{(\xi_i,\lambda_i)}$. Also using the same notation as in section 1, we set $U_i = P\overline{U}_i$, $\varepsilon_{12} = \frac{1}{\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}+\lambda_1\lambda_2|\xi_1-\xi_2|^2}$ and $d_i = dis(\xi_i, \partial\Omega)$. Here the O is for $\frac{d_i}{\lambda_i} \longrightarrow \infty$ and $\varepsilon_{12} \longrightarrow 0$.

Proposition 8.1. *Let* $\theta_1 = \overline{U}_1 - U_1$, *then :*

i) $0 \le \theta_1 \le \overline{U}_1$,

ii) $\theta_1(x) = H(\xi_1,x)\lambda_1^{\frac{n-4}{2}} + f_1(x)$

iii) $f_1(x) = O\left(\frac{\lambda_1^{\frac{n}{2}}}{d_1^{n-2}}\right)$, $\frac{\partial}{\partial\lambda_1} f_1(x) = O\left(\frac{\lambda_1^{\frac{n}{2}+1}}{d_1^{n-2}}\right)$

iv) $\frac{\partial}{\partial\xi_1} f_1(x) = O\left(\frac{\lambda_1^{\frac{n}{2}}}{d_1^{n-1}}\right)$

Lemma 8.2. *i)* $\|U_1\|^2 = \langle U_1,U_1\rangle = C_n - c_1 H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$

ii) $\langle U_2,U_1\rangle = c_1\left(\varepsilon_{12} - H(\xi_1,\xi_2)\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}\right) + O\left(\varepsilon_{12}^{\frac{n-2}{n-4}} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right)$

iii) $\int_\Omega U_1^{\frac{2n}{n-4}} = C_n - \frac{2n}{n-4} H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$

iv) $$\int_\Omega U_1^{\frac{n+4}{n-4}} U_2 = \langle U_2,U_1\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \ge 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}} \frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \le 7 \end{cases}.$$

Lemma 8.3. *We have the following estimates on* $\frac{\partial}{\partial\lambda}U_1$.

$i)\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle = \frac{n-4}{2}c_1 H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$

$ii)\int_\Omega U_1^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1 = 2\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$

$iii)\left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle = c_1\left(\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda_1}\varepsilon_{12} + \frac{n-4}{2}H(\xi_1,\xi_2)\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}\right) + O\left(\varepsilon_{12}^{\frac{n-2}{n-4}} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right)$

$iv)\int_\Omega U_2^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1 = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7 \end{cases}$

$v)\int_\Omega U_2\frac{1}{\lambda_1}\left(\frac{\partial}{\partial\lambda}U_1\right)^{\frac{n+4}{n-4}} = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7 \end{cases}$

Lemma 8.4. *We have the following estimates on* $\frac{\partial}{\partial\xi}U_1$

$i)\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle = -\frac{1}{2}c_1 H(\xi_1,\xi_1)\lambda_1^{n-3} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$

$ii)\int_\Omega U_1^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1 = 2\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$

$iii)\left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle = c_1\left(\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}\varepsilon_{12} - \frac{\partial}{\partial\xi_1}H(\xi_1,\xi_2)\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}\right) + O\left(\varepsilon_{12}^{\frac{n-1}{n-4}}\frac{|\xi_1-\xi_2|}{\lambda_2} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right)$

$iv)\int_\Omega U_2^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1 = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7 \end{cases}$

$v)\int_\Omega U_2\frac{1}{\lambda_1}\left(\frac{\partial}{\partial\xi_1}U_1\right)^{\frac{n+4}{n-4}} = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7 \end{cases}$

The proofs of these estimates are similar to the ones in [1] and we refer also to [5], [6] and [13] for more details.

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