A Tribute to Abbas Bahri (1955–2016)

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The eminent mathematician Abbas Bahri passed away on 10 January 2016 at the age of 61 years. He was a leading figure in nonlinear analysis and conformal geometry. Indeed, he played a fundamental role in the understanding of the lack of compactness arising in certain variational problems. For example, his book *Critical Points at Infinity in Some Variational Problems* exerted a tremendous influence on researchers working in the field of nonlinear partial differential equations involving critical Sobolev exponents. In particular, the book included finite dimensional reduction for Yamabe-type problems and the related shadow flow for an appropriate pseudogradient, as well as the accurate expansion of the Euler-Lagrange functional and its gradient, which later became tools widely used in this field.

Abbas Bahri was born and grew up in Tunisia where he attended elementary school and high school. At the age of 16, he moved to Paris, where he was admitted to the prestigious *Ecole Normale Supérieure, Rue d'Ulm* at the age of 19. He later defended a *Thèse d'Etat* under the direction of Professor Haim Brezis in 1981 at the age of 26.

Abbas' remarkable achievements have been widely recognised. He was awarded the Langevin and Fermat prizes[1] in 1989 for "introducing new tools in the calculus of variation" and in 1990 he received the Board of Trustees Award for Excellence, the Rutgers University highest honour for outstanding research.

Abbas Bahri's mathematical interests were very broad, ranging from nonlinear partial differential equations arising from geometry and physics to systems of differential equations of celestial mechanics. However, his research focus was mainly on fundamental problems in contact form and conformal geometry. The aim of this short note is to outline some of the milestones of Abbas' mathematical legacy.

1 The critical points at infinity approach to non-compact variational problems

Many partial differential equations (PDEs) enjoy a variational structure, that is, one can see their solutions as critical points of functionals. The space where the functionals are defined depends on the PDEs. For example, to study the following nonlinear PDE:

$$-\Delta u = |u|^{q-2}u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega \quad (\text{with } q > 2), \tag{1}$$

where Ω is a bounded subset of $\mathbb{R}^n$, $n \geq 3$, one can define

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q}\int_\Omega |u|^q, \quad u \in H_0^1(\Omega),$$

where $H_0^1(\Omega)$ denotes the Sobolev space (the space of the functions $u \in L^2(\Omega)$ with $\nabla u \in (L^2(\Omega))^n$ and $u = 0$ on $\partial\Omega$). It is clear that the solutions of (1) are in a one-to-one correspondence with the critical points of I.

Recall that the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ is compact for $q < \frac{2n}{n-2}$ and it is only continuous if $q = \frac{2n}{n-2}$. Hence, the exponent $q - 1 = \frac{n+2}{n-2}$ is said to be a *critical exponent* for equations of type (1). Note that, for $1 < q - 1 < \frac{n+2}{n-2}$, using the compactness of the embedding, one can show that I has at least one critical point by maximising I on the space $\Sigma := \{u \in H_0^1(\Omega) : \|u\| = 1\}$.

PDEs with critical nonlinearities exhibit exceptional features such as blowups, loss of compactness, energy quantisation and formation of singularities, emerging from the critical balance between the model linear PDE and strongly nonlinear terms. At the frontier of stability, these behaviours present great challenges to mathematical analysis and examples of such phenomena may be found in geometry (prescribed curvature problems, conformal deformation laws), physics (mean field equations, Chern-Simons-Higgs models, electroweak theory, Yang-Mills equation) and general relativity (quasi-local mass, static metrics). The understanding of the nonlinear features of these equations (bubbling off phenomena, existence mechanisms) is one of the main concerns of nonlinear geometric analysis and occupied a central position in Abbas' research. In a recent interview (Fifth Saudi Science Conference, An Interview with Professor Abbas Bahri, 1 Dec 2011) when asked the question: "What are the most fascinating discoveries scientists have made in your area during the last 20 years?" he answered laconically: "The understanding of non-compact phenomena." Indeed, Abbas became fascinated by variational problems arising in contact geometry at the beginning of his career and kept working on that topic all his life. He was, in particular, motivated by the Weinstein conjecture about the existence of periodic orbits of the Reeb vector field of a contact form. Although this problem features a variational structure, its corresponding variational form is neither compact nor Fredholm. It is in this framework that Abbas developed the concept of *critical points at infinity* [3]. These are accumulating points of non-compact orbits of the gradient flow. In fact, he discovered that the ω-limit set of non-compact orbits of the gradient flow behaves like a *usual critical point* once a Morse reduction in the neighbourhood of such geometric objects is performed. In particular, one can associate to such *asymptotes* a Morse index (counting the number of decreasing directions in the normal form given by the Morse reduction) as well as a stable and an unstable

1 Prix Fermat 1989, Annales de la faculté des sciences de Toulouse, vol. XI, no. 2, 1990, pp. 7–8.

manifold. This strategy turns out to be very useful in handling non-compactness in other variational problems (such as Yamabe-type equations, prescribed scalar curvature equations, the n-body problem in celestial mechanics and mean field equations).

The following short sections highlight some of Abbas Bahri's main contributions in this field.

The Yamabe-type problem and the Bahri-Coron topological argument

In [13], Abbas Bahri and Jean-Michel Coron studied the following Yamabe-type equation on domains $\Omega \subset \mathbb{R}^n$, $\quad n \geq 3$:

$$-\Delta u = u^{\frac{n+2}{n-2}}, u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2}$$

This nonlinear PDE involving a critical Sobolev exponent does not have a solution when the domain is star-shaped but it has a radial solution on annular domains. Such behaviour suggests that the topology of the domain should play a crucial role in the condition that ensures the existence of solutions. Indeed, by exploring the impact of the topology of the domain, Bahri and Coron proved that if the group homology $H_d(\Omega, \mathbb{Z}_2)$ is not trivial for some $d \geq 1$ then the equation (2) always has at least one solution. In proving such a result, they discovered a topological argument, which turns out to be very useful in the study of some other Yamabe-type problems (such as Yamabe-type problems on manifolds with or without boundaries [4, 12, 1], the $CR-$Yamabe problem [26, 27], and mean field-type equations on unbounded domains [15]).

To explain the strategy of the proof, we need to set up the variational framework and recall some preliminaries.

Equation (2) has a very nice variational structure. Indeed, its solutions are in a one-to-one correspondence with the critical points of the functional

$$J(u) := \frac{1}{\int_\Omega |u|^{\frac{2n}{n-2}}},$$

defined on $\Sigma_+ := \{u \in H_0^1(\Omega); \|u\| = 1; u \geq 0\}$, where $H_0^1(\Omega)$ stands for the Sobolev space and $\|.\|$ its norm.

Note that the Palais-Smale (PS) condition is a crucial property to extend the standard tools of the calculus of variation. A function f is said to satisfy the PS condition in $[a, b]$ if each sequence (x_n) such that $a \leq f(x_n) \leq b$ and $\nabla f(x_n)$ tends to 0 possesses a convergent subsequence.

Although J does not satisfy the PS condition, the flow lines of the gradient violating this condition are well understood (see [30], [20]). Namely, such flow lines should enter an $\varepsilon-$neighbourhood $V(p, \varepsilon)$ of some highly concentrated bubbles. Moreover, the limit energy levels of non-compact ones are given by $p^{2/(n-2)}S$, where p denotes the number of bubbles and S the Sobolev constant.

By setting

$$W_p := \{u \in \Sigma_+; J(u) < (p+1)^{2/(n-2)}S\}$$

from a classical deformation lemma, it follows, under the assumption that there are no critical points between the levels $p^{2/(n-2)}S$ and $(p+1)^{2/(n-2)}S$, that the sublevel W_p retracts by deformation onto $W_{p-1} \cup A$, where $A \subset V(p, \varepsilon)$.

To study the equation above, Bahri and Coron argued as follows. Firstly, they observed, under the assumption that $H_d(\Omega, \mathbb{Z}_2) \neq 0$, that it follows from a well known result of R. Thom [31] that this nontrivial homology class can be realised by a $d-$dimensional compact manifold without boundary V. That is, denoting by $\iota : V \to \Omega$ the related embedding and by ω the orientation class of the manifold V, we have that $0 \neq \iota_*(\omega) \in H_d(\Omega, \mathbb{Z}_2)$.

Next, let

$$B_p(V) := \left\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i}; \quad a_i \in V, \quad \alpha_i \geq 0; \quad \sum_{i=1}^{p} \alpha_i = 1 \right\}$$

denote the p-set of formal barycentres of V endowed with the weak topology of measures. These are stratified sets of top dimension $dp + p - 1$. Let $\omega_p \in H_{dp+p-1}(B_p(V), B_{p-1}(V))$ denote the top homology class of the pair $(B_p(V), B_{p-1}(V))$.

To perform their topological argument, Bahri and Coron constructed, for a large real parameter λ, a continuous sequence of functions

$$f_p(\lambda) : (B_p(V), B_{p-1}(V)) \to (W_p, W_{p-1}).$$

These functions induce homeomorphisms in the homology

$$(f_p(\lambda))_* : H_*(B_p(V), B_{p-}(V)) \to H_*(W_p, W_{p-1}).$$

Moreover, under the assumption that the functional J does not have any critical point under the level $2^{2/(n-2)}S$, we have that

$$(f_1(\lambda))* : H_d(V) \to H_d(W_1)$$

is a homotopy equivalence. On the other hand, for large p, all $(f_p(\lambda))_*$ are trivial in homology. Indeed, through a careful expansion of J using the interaction terms, one can prove that $f_p(\lambda)$ is homotopic to a map with its values in W_{p-1}.

Next, for a small number $\varepsilon > 0$, let $q \in \mathbb{N}$ be the smallest integer such that

$$(f_q(\lambda))_*(\omega_q) = 0 \text{ in } H_{dq+q-1}(W_q, W_{q-1})$$
$$\text{and} \quad (f_q(\lambda))_*(\omega_q) \neq 0 \text{ in } H_{dq+q-1}(J^{qS+\varepsilon}, W_{q-1}).$$

It then follows, from an exact sequence argument in homology of the triple $(W_p, J^{pS+\varepsilon}, W_{p-1})$, that

$$H_{dq+q}(J^{(q+1)^{2/(n-2)}S}, J^{q^{2/(n-2)}S+\varepsilon}) \neq 0.$$

Since the PS condition is satisfied in the range

$$\left(q^{2/(n-2)}S + \varepsilon, (q+1)^{2/(n-2)}S\right) \text{ for } \varepsilon \text{ small},$$

one can derive the existence of a critical point of generalised Morse index $qd + q$.

We point out that the topological argument above is of an abstract nature; in fact, it is based on the following dynamical features of the gradient in the neighbourhood at infinity $V(p, \varepsilon)$. Indeed, one can use the following two properties in a crucial way.

- The unstable manifold $W_u(V(p, \varepsilon))$ of $V(p, \varepsilon)$ does not intersect $V(m, \varepsilon)$ with $m > p$.
- The flow lines leaving $V(p, \varepsilon)$ through the unstable manifold of a *critical point at infinity* do not re-enter $V(p, \varepsilon)$ through the stable manifold of another *critical point at infinity*.

Finally, we point out that a necessary and sufficient condition for the existence of solutions to equation (2) is still missing. Moreover, the geometry of the domain should play an important role in such a condition. Indeed, Wei Yue Ding [23] constructed solutions for (2) on a contractible domain. In order to investigate the role of geometry, Abbas conjectured (private communication) that the following result would be true.

Denoting by G the Green's function of the Laplace operator under Dirichlet boundary conditions and by H its regular part, and setting $\varrho_p(x) := \varrho_p(x_1, \cdots, x_p)$ to be the least eigenvalue of the matrix $M := (a_{ij})$ given by $a_{ii} := H(x_i, x_i)$ and $a_{ij} := -G(x_i, x_j)$ for $i \neq j$, one can define the set:

$$I_p := \{x = (x_1, \cdots, x_p) \in \Omega^p; \varrho(x_1, \cdots, x_p) < 0\}.$$

Then, the following should hold true.

If Ω is contractible but for some $p \geq 2$ the set I_p is not contractible then equation (2) *has at least one solution.*

The reason behind the above conjecture is that the contribution of critical points at infinity to the topology of the pair

$$(J^{p^{2/(n-2)}S+\varepsilon}, J^{p^{2/(n-2)}S-\varepsilon})$$

is described by the pair (Ω^p, I_p) (see [19]).

The prescribed scalar curvature problem

A natural extension of the Yamabe problem is to ask the following question. Given a compact closed Riemannian manifold (M, g) and a smooth function $K \in C^\infty(M)$, does there exist a conformal metric $\tilde{g}$ such that the scalar curvature with respect to $\tilde{g}$ is given by the function K?

Note that $\tilde{g}$ is a conformal metric to g if there exists a function f such that $\tilde{g} = e^f g$. Setting $\tilde{g} := u^{\frac{4}{n-2}} g$, this problem amounts to solving the following nonlinear PDE involving a critical Sobolev exponent:

$$L_g u = K u^{\frac{n+2}{n-2}}, \; u > 0 \text{ in } M, \tag{3}$$

where L_g stands for the conformal Laplacian.

Just as in the Yamabe case, problem (3) has a variational formulation. However, the associated Euler-Lagrange functional does not satisfy the PS condition. Moreover, there are some obstructions to the existence of solutions (see [25] and [21]).

Using the positive mass theorem of Schoen and Yau [28], [29], Escobar and Schoen [24] proved, for a three-dimensional closed Riemannian manifold, that as long as the manifold is not conformally equivalent to the three round sphere $\mathbb{S}^3$, problem (3) is always solvable under the assumption that K is a positive function.

In [14], Bahri and Coron, inspired by earlier work of Abbas on contact form geometry [3], developed a *critical points at infinity* approach for the scalar curvature problem on $\mathbb{S}^3$ under the non-degeneracy assumption that at a critical point a of K we have $\Delta K(a) \neq 0$. Their strategy consists of studying the ω-limit set of non-compact orbits of the gradient flow. Following the work of Michael Struwe [30], we know that such non-compact orbits will be trapped in an ε-neighbourhood $V(p, \varepsilon)$ of highly concentrated p-bubbles δ_{a_i,λ_i}. The strategy adopted by Bahri and Coron consists of tracking down such flow lines that remain indefinitely in $V(p, \varepsilon)$ by showing that they decompose into an infinite part, which will vanish as time tends to infinity, and a *shadow flow*, which is a finite dimensional part and which splits in a canonical way into an ordinary differential equation, whose variables are: matching parameters $\alpha_i(t)$, concentration points $a_i(t) \in \mathbb{S}^3$ and concentration rates $\lambda_i(t)$. Namely, we have that, as $t \to +\infty$,

$$\begin{aligned} -\frac{\dot{\lambda}_i}{\lambda_i} &= A_1 \frac{\Delta K(a_i)}{\lambda_i^2 K(a_i)^{\frac{5}{4}}} - A_2 \sum_{j \neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \text{l.o.t.}, \\ \dot{a}_i &= B_1 \frac{\nabla K(a_i)}{\lambda_i^2 K(a_i)^{\frac{5}{4}}} + B_2 \sum_{j \neq i} \frac{\partial \varepsilon_{ij}}{\partial a_i} + \text{l.o.t.}, \end{aligned} \tag{4}$$

where A_i, B_i are positive constants and $\varepsilon_{ij} :=< \delta_i, \delta_j >$, which behaves like $\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{1/2}}$, denotes the interactions of two bubbles.

It then follows from the system of equations above that such a flow line will exit any set $V(p, \varepsilon)$ for $p \geq 2$ since the interaction terms ε_{ij}, which are the leading ones, would bring the flow lines down. Hence, the only possible non-compact flow lines are those that concentrate at single points. Moreover, analysing the above ordinary differential equation in the set $V(1, \varepsilon)$, one can derive that the only possibility for a flow line to build a *critical point at infinity* is that $a_i(t)$ converges to a critical point a_i of K such that $\Delta K(a_i) < 0$. Conversely, any gradient flow line, starting from a bubble $\delta_{b,\lambda}$ where $b \in \mathbb{S}^3$ is a critical point of K such that $\Delta K(b) < 0$, would remain forever in $V(1, \varepsilon)$ for $\varepsilon \to 0$. Hence, the only singularities of the gradient flows are those of the above type. The index of such a critical point at infinity is defined as the coindex of K at the concentration point a. It follows then, from an Euler-Poincaré-type argument, that:

$$\sum_{a \in \mathbb{S}^3; \nabla K(a)=0, \Delta K(a)<0} (-1)^{3-Morse(K,a)} = 1.$$

Hence, any function violating such an equality can be realised as a scalar curvature of a Riemannian metric conformally equivalent to the standard metric on $\mathbb{S}^3$.

To study the non-compactness in the case of spheres of dimension $n \geq 4$, Abbas introduced a family of bounded *pseudogradients* in the neighbourhood at infinity. A vector field W is said to be a pseudogradient for J if it satisfies the following condition: there exists a positive constant c such that

$$\langle \nabla J(u), W(u) \rangle \geq c \|\nabla J(u)\|^2, \quad \forall u.$$

This family allowed him to determine the end-points of the flow lines and he proved that they are in a one-to-one correspondence with the *critical points at infinity*. He then associated to this family a topological invariant $\mathcal{I}(V)$ (see [11]), a kind of degree that has to be non-zero, to ensure the existence of solutions. Such an invariant has been extended by Ben Ayed, Chtioui and Hammami [18] and has been used to investigate the problem of prescribed scalar curvature on high dimensional spheres.

2 The lack of compactness and Fredholm structure in contact form geometry

Bahri has made many contributions in the field of contact geometry, which is in some sense the counterpart of symplectic geometry for the odd dimensional case. One can see this duality as follows. Let us consider a particle q moving on the plane under a force field $-\nabla V$. Then, Newton's equation reads as

$$m\ddot{q} = -\nabla V(q).$$

If we assume for simplicity that $m = 1$ and we set $p = \dot{q}$ then we have the system

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\nabla V(q). \end{cases}$$

So, if we denote by $H(p,q) = \frac{1}{2}|p|^2 + V(q)$ the total energy (kinetic and potential), the previous system reads as

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q}. \end{cases}$$

Such a system is called Hamiltonian. Now, let us suppose that we are looking for periodic solutions of such a system, namely periodic orbits of the vector field $\left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$. We have two choices. Either we fix the period T, therefore working on a manifold of dimension four and meeting the theory of symplectic manifolds and the Arnold conjecture, or we fix the energy level H, therefore being constrained on a manifold of dimension three and meeting the theory of contact manifolds and the Weinstein conjecture. The study of such problems is variational, i.e. there exists a functional J on a suitable space such that the critical points of J are solutions to our system. Clearly, since we are looking for periodic orbits, the space of variations should be in the space of loops.

This approach can be formalised for the general setting of a compact three-dimensional manifold (M,α), where α is a 1-form on M such that $\alpha \wedge d\alpha$ is a volume form (never vanishing). Such a form has a very intrinsic vector field associated to it, called the Reeb vector field, which we denote by ξ and which satisfies

$$\begin{cases} d\alpha(\xi,\cdot) = 0 \\ \alpha(\xi) = 1. \end{cases}$$

Now, the problem turns to looking for periodic orbits of the vector field ξ. This is equivalent to studying the critical points of the functional $J : H^1(S^1; M) \to \mathbb{R}$ defined by

$$J(x) = \int_{S^1} \alpha(\dot{x}) .$$

Here, we denote by $H^1(S^1; M)$ the H^1-loops space on M, that is, the space of the curves $x : [0,1] \to M$, $x(0) = x(1)$, that have the regularity of the Sobolev space H^1. This functional basically measures the contribution of ξ along $\dot{x}$. Indeed, if we assume that the manifold has a global frame of the form (ξ, v, w), where v and w are in $\ker(\alpha)$, then for a loop x we have $\dot{x} = a\xi + bv + cw$ for some functions a, b, c, and thus $J(x) = \int_0^1 a(t)dt$. It is important to notice that this functional defined on $H^1(S^1; M)$ is unbounded from below and above, since a can take any form; moreover, it is not Morse.

Bahri first focused on finding a better space of variations for the functional J. In order to restrict the functional on a smaller space and avoid the degeneracy, he introduced a sort of Legendre transform similar to the classical one. Indeed, the standard contact form α_0 on S^3 is a pull-back from the standard contact form on $P(\mathbb{R}^3)$, i.e. the unit sphere cotangent bundle of S^2. Therefore, it is equipped with its Liouville form: the Legendre duality can be completed for this Liouville form. This transform can be viewed as the data of a vector field v in $ker(\alpha_0)$ such that $\beta_0(\cdot) := d\alpha_0(v,\cdot)$ is a contact form with the same orientation as α_0. This Legendre transform allows one to move from a Hamiltonian problem on the cotangent sphere of S^2 to a Lagrangian problem.

This duality was then extended by Bahri-Bennequin in [10] to the more general framework of a contact form α on a three-dimensional compact orientable manifold without boundary M, leading to a variational problem on a space of curves. In fact, assuming that v is a non-vanishing vector field in $ker(\alpha)$, we say that the Legendre transform of α with respect to v can be completed if the dual non-singular one-form $\beta(\cdot) := d\alpha(v,\cdot)$ is again a contact form on M with the same orientation as α. In this situation, by restricting the functional J on the subspace of the H^1-loops on M:

$$C_\beta = \{x \in H^1(S^1; M) \ s.t.\ \beta(\dot{x}) = 0;\ \alpha(\dot{x}) = a\},$$

where a is a positive constant, the following result by Bahri-Bennequin holds [10]:

J is a C^2 functional on C_β whose critical points are periodic orbits of ξ and they are of finite Morse index.

We notice that the curves in C_β can be expressed in a simple way, that is, if $x \in C_\beta$ then $\dot{x} = a\xi + bv$, where a is now just a constant (eventually depending on x) and therefore $J(x) = a > 0$. The understanding of this variational problem, such as the study of the existence and the multiplicity of critical points of J, is closely related to problems such as the Weinstein conjecture and the definition and well-posedness of a contact homology. However, this variational formulation has two main difficulties: the lack of compactness and the Fredholm assumption.

Non-compactness. The functional J defined on C_β is non-compact, in the sense that it does not satisfy the PS condition. In fact, one can see this fact directly from the functional: if we have a curve $x \in C_\beta$ with $\dot{x} = a\xi + bv$ then, as was mentioned above, the functional just controls the value a of the curve but the b-component along v is free. Therefore, it can have any behaviour along a PS sequence.

In his works [10, 6], Bahri constructed a flow in order to give a precise description of the violation of the PS assumption. Moreover, the introduced flow has many important topological properties on the curves, such as the decreasingness of the linking number, and it has asymptotes going to critical points at infinity. The description of the PS sequences is made by means of curves that lie on a stratified space and that are made of pieces of ξ and pieces of $\pm v$ (since the functional does not catch the variation along v), in terms of *conjugate points* and *characteristic pieces*. In order to describe this fact, let x be a curve belonging to C_β. Then, $\dot{x} = a\xi + bv$ for some function b and positive constant a. Then, one defines the set

$$\Gamma_{2k} = \left\{\gamma \in C_\beta, \quad ab = 0\right\},$$

where the curve γ in C_β is made by k pieces along ξ and k *jumps* on $\pm v$ (see Figure 1 for a typical element of Γ_{10}). This set will be the limiting set of the flow and we can again consider the functional J on it (after passing to the limiting process). The set of variations at infinity is defined as $\bigcup_{k\geq 0} \Gamma_{2k}$ and on this set the functional at infinity reads as

$$J_\infty(\gamma) = \sum_{k=0}^{k=\infty} a_k.$$

The critical points of this functional are what Bahri called critical points at infinity. Next, he gave an exact geometric definition of these critical points by introducing the following definitions. First, let ϕ_s be the transport map of v, namely the one parameter group of diffeomorphisms generated by the flow

$$\frac{d}{ds}(\phi_s(x)) = v_{\phi_s(x)}, \qquad \phi_0(x) = x.$$

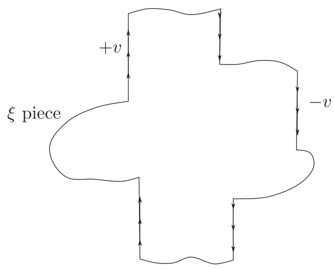

Figure 1. Typical element of Γ_{10}

Then, one says that a v-jump between two points x_0 and $x_1 = x(s_1)$, $s_1 \neq 0$, is a v-jump between *conjugate points* if the following holds:

$$(\phi^*_{s_1}\alpha)_{x_1} = \alpha_{x_0}.$$

In other words, conjugate points are points on the same v-orbit such that the form α is transported onto itself by the transport map along v.

Also, a ξ-piece $[x_0; x_1]$ of an orbit is *characteristic* if v completes an exact number $k \in \mathbb{Z}$ of half revolutions from x_0 to x_1.

Now, Bahri gave the following characterisation for critical points at infinity.

A curve in $\bigcup_{k\geq 0}\Gamma_{2k}$ is a critical point at infinity if it satisfies one of the following assertions:

(1) The v-jumps are between conjugate points. These critical points are called *true* critical points at infinity.

(2) The ξ-pieces have characteristic length and, in addition, the v-jumps send $ker(\alpha)$ to itself.

The main result of Bahri, in this setting, is the understanding of how these critical points at infinity contribute to the change of topology in the variational problem.

A major difference to the Yamabe-type problems is that, in this situation, there can be characteristic pieces where a single curve can behave as many superposed critical points at infinity, with different indices, and they can interact [7].

Violation of Fredholm. The second difficulty that Bahri tackled in this variational problem was the violation of the Fredholm assumption [7, 17]. Let us briefly recall the definition of a Fredholm operator and a basic example. Let X and Y be two Banach spaces and $F : X \to Y$ be a bounded linear operator. F is called Fredholm if its kernel and co-kernel are of finite dimension and, in that case, the Fredholm index is the difference between these two dimensions. Fredholm operators are close cousins of invertible operators, in the sense that the non-invertibility of F is mild. As an example, if we take $X = Y$ separable Hilbert spaces and $F = Id + K$, where Id is the identity and K is a compact operator (in this case F has index 0), then the *Fredholm alternative* says that the problem $F(x) = y$ is solvable if and only if y is orthogonal to the kernel of F, that is, y satisfies a finite number of orthogonality conditions. This allows, in many cases, the reduction from an infinite dimensional problem to a finite dimensional one: first solving orthogonally to the kernel (since we have invertibility) and then solving in the kernel, which is a finite dimensional space. Now, we say that a nonlinear operator F is Fredholm if its differential dF is Fredholm at every point. For these operators, one has a version of the implicit function theorem, which is needed in order to apply a Morse theory approach to variational problems and make use of transversality, gluing and perturbations.

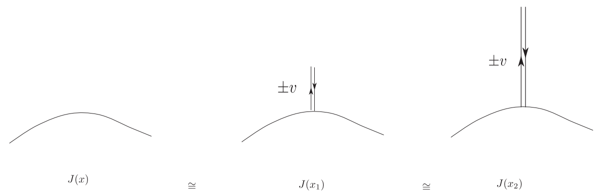

Figure 2. The functional and the v-jumps

In his work, Bahri found simple a criterion to check if violation of the Fredholm property occurs or not based on some properties of the transport map ϕ. The main idea comes from the fact that the functional does not control b properly. Therefore, it might be an infinite space of perturbations that are *invisible* to the linearised operator. In fact, by looking at the functional J in the larger space

$$C^+_\beta = \{x \in H^1(S^1; M) \ s.t. \ \beta(\dot{x}) = 0; \ \alpha(\dot{x}) \geq 0\},$$

one can see that it remains insensitive to the introduction of a $\pm v$ piece (see Figure 2). Moreover, the modified functional is Fredholm in the following way:

$$\tilde{J}(x) = \int_0^1 \alpha(\dot{x})(t)dt + \delta \log(1 + \int_0^1 |b(t)|dt),$$

since one has control of b, as shown in [7]. Now, let x be a curve that is transverse to v and, at a point $x(t_0)$, one introduces a "back and forth" v piece of length s. Let x_ϵ be the curve obtained by introducing a small "opening" piece of length ϵ between the two v pieces. Then, one has

$$J(x_\epsilon) = J(x) - \epsilon(\alpha_{x(t_0)}(d\phi_{-s}(\xi)) - 1) + o(\epsilon).$$

Thus, if there exists $s > 0$ such that $\alpha(\phi_{-s}(\xi)) > 1$ then one would have a decreasing direction from the level $J(x)$ and one would be able to bypass a critical point without changing the topology, even though it has a finite Morse index, and this is due exactly to the violation of the Fredholm condition. Therefore, the criterion is the following.

If $\phi^*_{-s}(\alpha)(\xi) < 1$, for every $s \neq 0$, then J satisfies the Fredholm condition.

For instance, the Fredholm assumption is violated for the standard contact structure α_0 and the first exotic structure of Gonzalo and Varela defined on S^3 and for a family of tight contact structures on the torus T^3.

In particular, Bahri pointed out the *point to circle* relations, where a circle of critical points, under the S^1-action, can have Morse relations with a single point. The fact is that, during the perturbation procedure, the S^1-action could be lost. In order to explain and give some applications of these phenomena, Bahri made explicit computations on two main examples: the sphere equipped with its standard contact form and with the first overtwisted contact form of Gonzalo-Varela. In his works [5], he showed how it is possible to overcome the Fredholm violation by carefully studying the Fadell-Rabinowitz index of different sets bounding the critical points. In particular, it led to another proof of the Weinstein conjecture on the sphere.

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