A Sub-Riemannian version of Benamou-Brenier theorem

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Kantorovich formulation of Optimal Transport

Given a Polish space (M, d) and a Borel cost function

$$c \colon M \times M \to \mathbb{R} \cup \{+\infty\},$$

the classic formulation of **Optimal Transport** for $\mu_0, \mu_1 \in \mathcal{P}(M)$, is the minimization problem

$$C_{\mathrm{Mon}}(\mu_0, \mu_1) = \inf_{T_\# \mu_0 = \mu_1} \int_X c(x, T(x)) d\mu_0(x).$$

The condition $T_\# \mu_0 = \mu_1$ lack closedness with respect to the principal weak topologies.

To avoid this problem, we introduce the more general **formulation by Kantorovich**. The set of **admissible transport plans** is

$$\Pi(\mu_0, \mu_1) := \left\{ \gamma \in \mathcal{P}(M \times M) : \ \pi^{(0)}_\# \gamma = \mu_0, \ \pi^{(1)}_\# \gamma = \mu_1 \right\}$$

$$C_{\mathrm{Kan}}(\mu_0, \mu_1) = \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{M \times M} c(x, y) \, d\gamma(x, y).$$

We work with cost $c(x, y) = d^2(x, y)$. If M is a Riemannian manifold, the cost is bounded from below and (lower semi)continuous, therefore for any $\mu_0, \mu_1 \in \mathcal{P}(M)$ there exists a minimizer of $C_{\mathrm{Kan}}(\mu_0, \mu_1)$.

The Kantorovich cost induces the so called **Wasserstein distance** on the space $\mathcal{P}_2(M)$ of probability measures with finite 2-momentum.

We point out that if M is geodesic (such as a Riemannian or Sub-Riemannian manifold), then $\mathcal{P}_2(M)$ is geodesic as well. If μ_t is a geodesic between μ_0 and μ_1 this means that

$$\sqrt{C_{\mathrm{Kan}}(\mu_t, \mu_s)} = |t - s| \cdot \sqrt{C_{\mathrm{Kan}}(\mu_0, \mu_1)} \quad \forall s, t \in [0, 1].$$

See for example [2].

Benamou-Brenier formulation(s) of Optimal transport

Let μ_t a narrowly continuous family of probabilities, where $t \in [0, 1]$, and v_t a Borel family of vector fields on M such that $\iint ||v_t|| d\mu_t dt < +\infty$. They respect the **Continuity Equation** if

$$(\clubsuit) \quad \iint \left(\partial_t \phi + \langle v_t, \nabla \phi \rangle_x\right) d\mu_t(x) dt = \int \phi(1, \cdot) d\mu_1 - \int \phi(0, \cdot) d\mu_0 \qquad \forall \phi \in C^\infty_c([0, 1] \times M)$$

The Continuity Equation describes curves of measures μ_t that **follow the flow** of v_t (for example see [1]). Furthermore, it is the constraint for the Benamou-Brenier dynamic formulation of Optimal Transport.

$$C_{\mathrm{BB}}(\mu_0, \mu_1) = \inf_{(\clubsuit)} \iint ||v_t||^2 \, d\mu_t dt.$$

The problems C_{Kan} and C_{BB} are equivalent in Riemannian setting.

We want to generalize to a **sub-Riemannian** manifold M. As done in [4] for a non-linear control systems in $\mathbb{R}^n$, we define a **relaxed version of the Benamou-Brenier formulation**. We consider the horizontal bundle $HM \subset TM$, v must lie in HM and ∇_H is the horizontal gradient.

$$(\spadesuit) \quad \int \left(\partial_t \phi + \langle v, \nabla_H \phi \rangle\right) d\eta(t, v) = \int \phi(1, \cdot) d\mu_1 - \int \phi(0, \cdot) d\mu_0 \qquad \forall \phi \in C^\infty_c([0, 1] \times M)$$

We minimize among $\eta \in \mathcal{P}([0, 1] \times HM)$ that disintegrate over $[0, 1]$, *i.e.* η is a so called **Young measure** (see [3]).

$$C^\star_{\mathrm{BB}}(\mu_0, \mu_1) = \inf_{(\spadesuit)} \iint ||v||^2 \, d\eta(t, v).$$

Just relax, the solution will come

We know that $C_{\mathrm{BB}}(\mu_0, \mu_1) \geq C^\star_{\mathrm{BB}}(\mu_0, \mu_1)$. Under the opportune assumptions, by a compactness result, if $C^\star_{\mathrm{BB}}(\mu_0, \mu_1) < +\infty$, then a minimizer η for the relaxed problem exists.

Consider $\pi \colon HM \to M$ the canonical projection. From η respecting the modified Continuity Equation ($\spadesuit$), we build

$$\mu^\eta_t := \pi_\# \eta_t, \quad v^\eta_t(x) := \int_{H_x M} v \, d\eta_{t,x}(v).$$

The pair (μ^η_t, v^η_t) respects the (original) Continuity Equation ($\clubsuit$).

If η is a minimizer of $C^\star_{\mathrm{BB}}(\mu_0, \mu_1)$, by convexity we prove

$$\int ||v^\eta_t||^2 d\mu^\eta_t \, dt \leq \iint ||v||^2 \, d\eta = C^\star_{\mathrm{BB}}(\mu_0, \mu_1)$$

and therefore the BB and $\mathrm{BB}^\star$ formulations are equivalent

In order to prove the equivalence with the Kantorovich formulation, we consider the measurable map

$$\omega \colon M \times M \to \mathrm{Geod}(M)$$

and the map $F \colon (t, x, y) \mapsto (t, \partial_t \omega(x, y)) \in [0, 1] \times HM$. Then, from a Kantorovich minimizer $\gamma \in \Pi(\mu_0, \mu_1)$ we build

$$\eta^\gamma := F_\#(\mathcal{L} \otimes \gamma)$$

the associated measure that **follows the geodesics** and respects ($\spadesuit$).

We prove that

$$\int ||v_t||^2 \, d\eta^\gamma = \int_{M \times M} d^2(x, y) \, d\gamma,$$

and therefore

$$C^\star_{\mathrm{BB}}(\mu_0, \mu_1) \leq C_{\mathrm{Kan}}(\mu_0, \mu_1).$$

Equivalence result

Finally, to prove the equivalence between Kantorovich and Benamou-Brenier, for any family of vector fields v_t we consider the space of admissible curves satisfying the following ODE,

$$\begin{cases} \partial_t \omega_t(x) = v_t(\omega_t(x))) \\ \omega_0(x) \quad = x \end{cases}$$

$$\Omega_v := \{\omega \colon [0, 1] \times M \to M : \text{ horizontal curve satisfying the ODE}\}$$

Consider the map $e_t \colon \Omega_v \to M$ such that $\omega \mapsto \omega_t$. For any (μ_t, v_t) satisfying ($\clubsuit$), by a **superposition principle** there exists a probability $\widetilde{\mu} \in \mathcal{P}(\Omega_v)$ such that $\mu_t = (e_t)_\# \widetilde{\mu}$.

If we define $E \colon \omega \mapsto (\omega_0, \omega_1) \in M \times M$, then $E_\# \widetilde{\mu} \in \Pi(\mu_0, \mu_1)$ and we can prove

$$\int d^2(x, y) \, dE_\# \widetilde{\mu} \leq \int ||v_t||^2 \, d\mu_t dt.$$

Thus $C_{\mathrm{Kan}}(\mu_0, \mu_1) \leq C_{\mathrm{BB}}(\mu_0, \mu_1)$. We can conlude

$$C_{\mathrm{Kan}} \equiv C_{\mathrm{BB}} \equiv C^\star_{\mathrm{BB}},$$

the three formulations are equivalent.

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