The visual V_1 cortex action as a heterogeneous differential operator

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Visual stimulus and perceived image

The **visual stimulus** is a function defined on the retinal plan

$$I\colon \mathbb{R}^2 \to [0,1].$$

The the cortical **receptive profiles** (RPs) act on the visual stimulus

$$\underbrace{O(x,y)}_{\text{Cortical output}} = \underbrace{RP(x,y)}_{\text{Receptive profile filter}} \star \underbrace{I(x,y)}_{\text{Retinal input}}$$

The RPs are modeled by $\mathcal{L}G$ where $\mathcal{L}$ is a differential operator and G is a Gaussian filter. Therefore if I_σ is the **smoothing** of the input,

$$O = \mathcal{L}G \star I = \mathcal{L}(G \star I) = \mathcal{L}I_\sigma$$

Therefore cortical RPs, changing from point to point, act as a differential operator on the visual stimulus [3]

$$I \mapsto \mathcal{L}I_\sigma.$$

Cortical connectivity inverts the problem and recovers an image called **perceived image**. This consists in solving the inverse problem

$$\mathcal{L}u = \mathcal{L}I_\sigma.$$

In the **retinex** model, $\mathcal{L} = \Delta$ the laplacian operator [5]. Recovering the perceived image is a Poisson's problem $\Delta u = \Delta I_\sigma$. In particular we recover (the smoothed version of) I up to an harmonic function.

Receptive profile and cortical connectivity action

Examples of the action of receptive profils

- **Mexican hat**

$$RP_h = \Delta G(x,y)$$

- **Primate simple cells**

$$X_\theta = \cos(\theta)\partial_x + \sin(\theta)\partial_y$$
$$RP_s = X_\theta^* X_\theta G(x,y)$$

where $\theta\colon \mathbb{R}^2 \to [0,\pi]$ is an orientation map depending on the case study and the model.

In our works, *e.g.* [1], we analyze the transform of the visual stimulus **performed by the RPs on V_1 in order to reconstruct the image** perceived by the subject. We consider the

- **Heterogeneous Poisson's operator**

$$\mathcal{L}_\Lambda = a_1\Delta + a_2 X_\theta^* X_\theta + a_3 (X_\theta^2)^* X_\theta^2.$$

It depends on the coefficient and the orientation map θ

The solution of the steepest descent

$$\partial_t u = \mathcal{L}_\Lambda I_\sigma - \mathcal{L}_\Lambda u$$

exists (under opportune regularity conditions) and it weakly converges to a solution of the heterogeneous Poisson's equation $\mathcal{L}_\Lambda u = \mathcal{L}_\Lambda I_\sigma$.

Image reconstruction

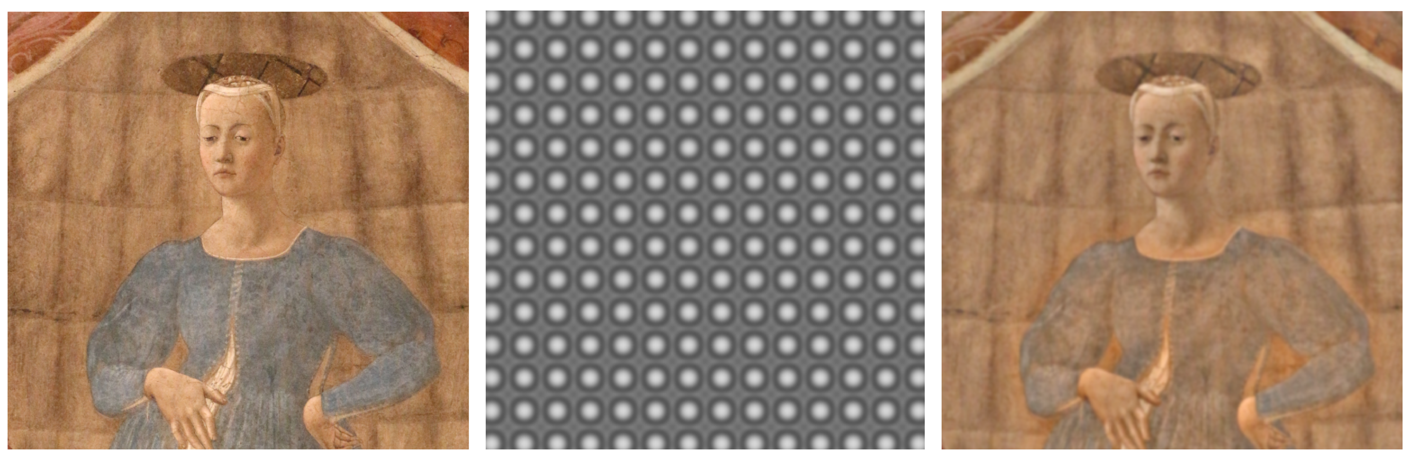

Figure 1:In this case the operator is the classic Laplacian Δ. (Left) Original image by Piero Della Francesca (Center) The RPs are visualized in a subsampled set of points. (Right) Reconstructed image using the steepest descent method. The slight change in color is due to the modulation effect of the cortical action; in fact the input is reconstructed up to an harmonic function.
Stopping criterion depends on the error of convergence $dt = 0.1,\ \varepsilon_c = 10^{-4}$

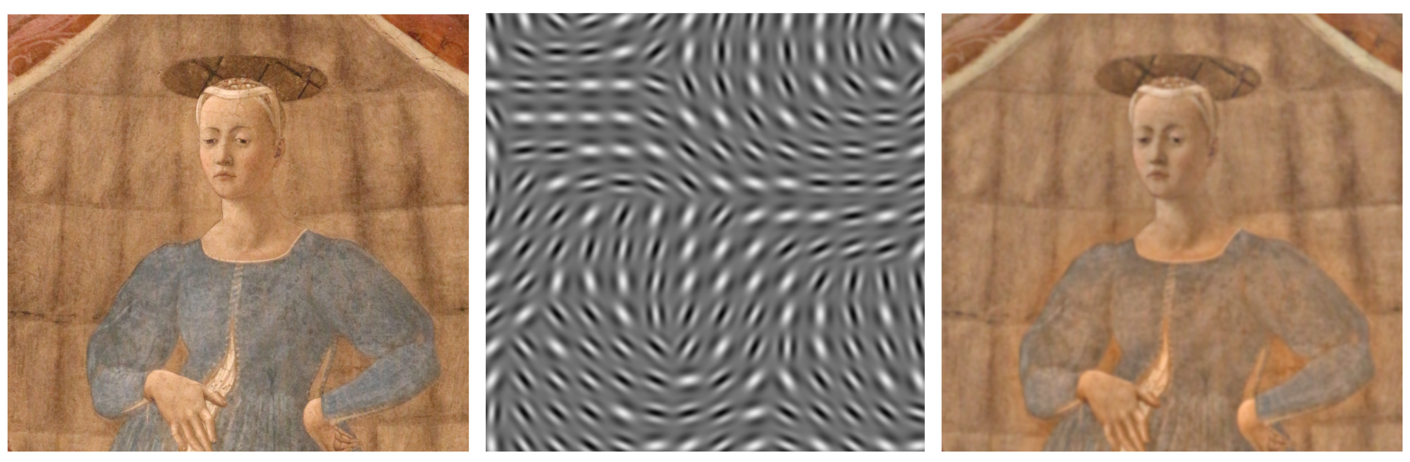

Figure 2:In this case the operator is heterogeneous of second or fourth order, meaning that $\mathcal{L}_\Lambda = a_2 X_\theta^* X_\theta + a_3 (X_\theta^2)^* X_\theta^2$ with $\mathbb{P}(a_i = 1) = 1/2$ for $i = 1,2$ and orientation given by the map $\theta(x,y) = \arg \sum_{k=1}^N c_k e^{2\pi i (x\cos(2\pi k/N) + y \sin(2\pi k/N))}$ also called *pinwheel distribution*, see [6].
Stopping criterion depends on the error of convergence $dt = 0.001,\ \varepsilon_c = 10^{-4}$

Figure 3:Example with a test image from [2]. (Left Left): Original image. (Center Left): Our "cortical" transform. (Centee Right and Right Right): The retinex algorithm proposed in [2] with different choice of the parameters.

H-convergence

Consider a series of discrete Dirichlet problems

$$\mathcal{L}^\varepsilon u^\varepsilon = \sum_{z\in\Lambda} X^\varepsilon_{-z}(p^\varepsilon_z X^\varepsilon_z u^\varepsilon) = f^\varepsilon$$

and a classic Dirichlet problem

$$\mathcal{L}^0 u^0 = \sum_{z\in\Lambda} -\partial_z(p^0_z \partial_z u^0) = f^0.$$

Where $\Lambda \subset \mathbb{Z}^d$ is a finite set symmetric with respect to 0.
The sequence $\mathcal{L}^\varepsilon$ **H-converges** to $\mathcal{L}^0$, $\mathcal{L}^\varepsilon \xrightarrow{H} \mathcal{L}^0$, if for any sequence $f^\varepsilon \to f^0$ in $W^{-1,2}(Q)$, the u^ε weakly converge to u^0 in $W^{-1,2}(Q_\varepsilon)$ and analogously for their first order derivatives in $L^2(Q_\varepsilon)$.
The notion of H-convergence and the associated homogenization techniques are an important tool, see [4]

Main result

If the functions p^ε_z satisfy

- $\sum_z p^\varepsilon_z(x) = 1\ \forall x$;
- $\exists \delta > 0$ such that $p^\varepsilon_z \geq \delta\ \forall z \in \Lambda$;
- $p^\varepsilon_z(x) = p^\varepsilon_{-z}(x + \varepsilon z)\ \forall z \in \Lambda,\ \forall x$.

then a.s.

$$\mathcal{L}^\varepsilon \xrightarrow{H} p^0 \cdot \Delta,$$

a multiple of the Laplace operator with $p^0 \in \mathbb{R}$.

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