SR geodesics

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* Recap of the Pontryagin extremals
* Endpoint map
* Volterra "chronological" expansion
* Endpoint map differential

We proved that $\exists$ length minimizer $\forall q_0, q_1 \in \overline{B(q_0, r)}$ for r suff. small

In general the lack of compactness comes from the non-completness of M, but M is locally cpt because $\forall q_0$ $\exists$ U neigh. of q_0 suff. small st U is complete (diffeomorphic to a disk in $\mathbb{R}^n$)

On a SR manifold the following are equivalent

(1) (M, d_{SR}) is complete

(2) $\overline{B(x,r)}$ is compact $\forall r>0$

(3) $\exists\, \varepsilon>0$ st $\forall x$, $\overline{B(x,\varepsilon)}$ cpt

- (Riemannian) Hopf-Rinow theorem

tfae

(1) (M,d) complete

(2) Closed + bounded = cpt

(3) M is geodesically complete $\exp_q : T_qM \longrightarrow M$

on SR-mfds, if there are no abn. minimizers and

$\exists\, x : \exp_x : T_x^*M \longrightarrow M$

$\Longrightarrow M$ is complete

Notational remark: if $X \in Vec(M)$, it acts on TM and T^*M via e^{-tX}_* and $(e^{-tX})^*$

This last action induces a flow on T^*M

$$(e^{-tX})^* = e^{t\vec{h}_X} \quad \text{where } h_X(\lambda) = \langle \lambda, X \rangle$$

we also stated that if $P_t^{X_t}$ is the flow for X_t

$\Rightarrow (P_{t,0}^{X_t})^* = P_{0,t}^{\vec{h}_{X_t}}$ the flow associated to $\vec{h}_{X_t}$ on T^*M

⚠ We use $P^{X_t}_{0,t}$ for the flow of a non-aut v.f.
because the notations $P^{X_t}_t$ and e^{tX_t} are misleading

Recap of the Pontryagin extremals

$\{X_1, \dots, X_m\}$
generating family

if γ admissible

$u = (u_1(t), u_2(t), \dots, u_m(t))$ is a minimal control for γ $\Rightarrow u \in L^\infty([0,1] \to \mathbb{R}^m)$

γ extremal of the J functional $\left(\begin{array}{ll}\text{with the} & \gamma(0) = q_0 \\ \text{constraint} & \gamma(1) = q_1\end{array}\right)$

iff

(N) $\langle \lambda(t), X_i \rangle_{\gamma(t)} = u_i(t)$

(A) $\langle \lambda(t), X_i \rangle_{\gamma(t)} = 0$

$$\lambda(t) = \left(P^{\overline{X}}_{t,0}\right)^* \lambda_0$$

$$\overline{X} = \sum u_i(t) X_i$$

Observe that normal extremals are smooth

Indeed, $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ and H is a smooth function on $C^\infty(T^*M) \Rightarrow \lambda(t)$ smooth

(even if $\dot{\lambda}(t) = \vec{H}$ a.e.)

— " —

Regularity of abnormal extremals $\longrightarrow$ open question

The Endpoint map is defined on a broader class of controls

$$J(\gamma_u) = \int \sum u_i^2(t)\, dt$$

$$u \in L^2([0,1] \to \mathbb{R}^m)$$

γ_u = maximal solution of

$$\begin{cases} \dot{\gamma}_u = \sum u_i X_i \\ \gamma_u(0) = q_0 \end{cases}$$

$$\mathcal{U}_{q_0} = \{ u \in L^2 : \gamma_u \text{ is defined on } [0,1] \}$$

- **Lemma**: $\mathcal{U}_{q_0}$ is open in L^2

→ Proof (Sketch): first observe $\ell(\gamma_u) \leq |u|_{L^2}$, therefore for ε sufficiently small $B_{L^2}(0, \varepsilon) \subset \mathcal{U}_{q_0}$

To prove that $\mathcal{U}_{q_0}$ is open we want to show that

$d(\gamma_{u+v}(t), \gamma_u(t)) \to 0$ uniformly wrt $|v|_{L^2}$. (Exercise)

because we choose a cpt neigh K of γ_u, we set $\forall t$

$d_0 = \min_t (\partial K, \gamma_u(t))$ and choose $|v|_{L^2}$ suff. small that $d(\gamma_{u+v}, \gamma_u) < d_0$

$\square$

Endpoint map

$$E_{q_0} : \mathcal{U}_{q_0} \subset L^2 \longrightarrow M \quad \text{st} \quad E_{q_0}(u) = \gamma_u(1)$$

analogously $E^t_{q_0}(u) = \gamma_u(t)$

* E_{q_0} is weakly continuous

(weak L^2 topology
$u_n \to u$ if $\langle u_n, v \rangle \to \langle u, v \rangle$ $\forall v \in L^2$

* $\forall q_0 \in M$ if M is complete
$\Longrightarrow \forall q \; \exists$ length minimizer from q_0 to q

• **Theorem**: E_{q_0} is weakly continuous

We want to prove that $u_n \rightharpoonup u \Rightarrow E_{q_0}(u_n) \to E_{q_0}(u)$

By the UBP (Unif. Boundedness Principle), $|u_n|_{L^2}$ is bounded

$|u_n|_{L^2} < r_0 \quad \forall n \Rightarrow \gamma_n = \gamma_{u_n} \subset \overline{B(q_0, r_0)}$

if we repar. the γ_n by constant speed, they are unif. Lipschitz $\Rightarrow$ by A.A., $\{\gamma_n\}$ has cpt closure in the sense of unif. convergence

$\exists\, \gamma : [0,1] \to M$ st $\gamma_n \to \gamma$ uniformly

$$\gamma_n(t) = q_0 + \int_0^t \underline{\sum u_{n,i}(t)}\; \underline{X_i(\gamma_n(t))} \xrightarrow[\text{then}]{} q_0 + \int_0^t \sum u_i(t)\, X_i\,\gamma(t))$$

$\gamma_n(t)$

$\downarrow$

$\gamma(t)$

this conv. weakly to $u_i(t)$ for $n \to \infty$

this conv. strongly to $X_i(\gamma(t))$ because $\gamma_n(t) \to \gamma(t)$ uniformly

So $\gamma(t)$ must be $\gamma_u(t)$ $\square$

A control $u \in \mathcal{U}_{q_0}$ is a minimizer if $|u|_{L^2} = d(q_0, E_{q_0}(u))$

- **Theorem:** $K \subset M$ cpt $\Rightarrow$ $\mathcal{M}_K = \{u$ minimizers : $E_q(u) \in K\}$ is compact in the strong L^2 topology (is the topology ass. to $|\cdot|_{L^2}$)

$\rightarrow$ Proof: $\{u_n\} \subset \mathcal{M}_K$, as before $|u_n|_{L^2}$ is bounded

$\Rightarrow \{u_n\}$ is weakly cpt. $\Rightarrow$ (up to subseq.) $u_n \xrightarrow[L^2]{} u \in L^2$

therefore $E_{q_0}(u_n) \longrightarrow E_{q_0}(u) \in K$

$$|u_n|_{L^2} = d(q_0, E_{q_0}(u_n)) \longrightarrow d(q_0, E_{q_0}(u))$$

because they are minimizers

weak semicont. of the L^2-norm $\Longrightarrow$

$$|u|_{L^2} \leq \liminf |u_n|_{L^2} = d(q_0, E_{q_0}(u)) \leq |u|_{L^2}$$

u is a minimizer, $u \in \mathcal{M}_K$ and $|u_n|_{L^2} \to |u|_{L^2}$

this implies $|u_n - u|_{L^2} \to 0$ $\square$

- **Theorem**: E_{q_0} is smooth on $\mathcal{U}_{q_0}$ and

$$\forall u \in \mathcal{U}_{q_0} \qquad D_u E_{q_0}(v) = \int_0^1 \left(P^u_{t,1}\right)_* \left(\sum v_i(t) X_i\right)\Big|_{\gamma_u(1)} dt$$

$$v \in L^2([0,1] \to \mathbb{R}^m) \longrightarrow T_{\gamma_u(1)} M$$

Volterra expansion

We would like to have something like $\gamma(t) = \gamma(0) + \int_0^t \dot{\gamma}(s)\,ds$

which is true on $\mathbb{R}^n$, and is true in coordinates

on any smooth mfd M.

Also it is true $\forall a \in C^\infty(M)$,

$$a(\gamma(t)) = a(\gamma(0)) + \int_0^t \langle da, \dot{\gamma}(s) \rangle\, ds = \widehat{\gamma(0)}\,a + \int_0^t \widehat{\dot{\gamma}(s)}\,a\, ds$$

$\parallel$

$\widehat{\gamma(t)}\,a$

~~Hope~~: $\widehat{\gamma(t)} = \widehat{\gamma(0)} + \int_0^t \widehat{\dot\gamma(s)}\,ds$ to make sense for diffeomorphisms

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We need to set up a functional analysis setting for the operators acting on $C^\infty(M)$

if $f \in C^\infty(M)$, $\alpha \in \mathbb{N}$, $K \subset M$ cpt

$$\|f\|_{\alpha,K} := \sup_K \{ X_{i_1} \cdots X_{i_\ell} f : 0 \leq \ell \leq \alpha \}$$

the set of semi-norms $\|\cdot\|_{\alpha,K}$, $\alpha \in \mathbb{N}$, $K \subset M$ subsets

induce a topology on $C^\infty(M)$ does not depend on the basis that we choose on TM

⚠ $\{X_1, \ldots, X_n\}$ a basis of TM not a generating family

<u>Fact</u>: with this topology, $C^\infty(M)$ is Frechet

(complete, metrizable, loc. convex, topological vec. space)

if $P \in \mathrm{Diff}(M)$, <u>exercise</u>: $\forall K \subset M$ cpt $\exists\, C_{\alpha,P,K} > 0$

st $\forall f \in C^\infty(M)$, $\|\hat{P} f\|_{\alpha,K} \leq C \cdot \|f\|_{\alpha, P(K)}$

this allows to define $\|P\|_{\alpha,K} := \sup \{ \|\hat{P} f\| : \|f\| \leq 1 \}$

the same can be done with v.f.

* f_t is **continuous**: if f_t is continuous as a function of t wrt the topology

* f_t is **measurable**: if $t \longmapsto \hat{q} f_t = f_t(q)$ is measurable $\forall q \in M$

* f_t is **loc. integrable**: if $\int_{t_0}^{t_1} \| f_t \|_{\alpha, K} \, dt < +\infty$ $\forall \alpha \in \mathbb{N}$ $\forall K \subset M$ cpt

* f_t is **abs. integrable**: if $\exists\, b_t$ loc integrable st

$$f_t = f_0 + \int_0^t b_s \, ds \qquad \forall t$$

* f_t is loc. Lipschitz: if $\forall \alpha \in \mathbb{N}$, $K \subset M$ cpt

$\exists\, C = C(\alpha, k)$ st

$$\| f_t - f_s \|_{\alpha, k} \leq C \cdot |t-s|$$

A family of op. $\{A_t\}$ on $C^\infty(\mathcal{M})$ is continuous/measurable

if $t \longmapsto A_t f$ is continuous/measurable $\forall f \in C^\infty(\mathcal{M})$

non-autonomous v.f. is a family of v.f. X_t
which is measurable and loc. bounded

non-autonomous flow is a family of diffeomorphism P_t
which is AC

the flow of a non-aut v.f. is a non-aut flow

If A_t, B_t are families of operators on $C^\infty(M)$

$$\int_0^t B_s\,ds \; : \quad a \in C^\infty(M) \longmapsto \int_0^t B_s a\,ds$$

$$\frac{d}{dt} B_t \; : \quad a \longmapsto \frac{d}{dt}(B_t a)$$

if the families are derivable $\Rightarrow$

$$\boxed{\frac{d}{dt}(A_t \circ B_t) = A_t' \circ B_t + A_t \circ B_t'}$$

Goal: We would like to write a flow $P_{s,t}$ in terms of an associated (non-aut) v.f. X_t "as an integral"

If P is the flow associated to X_t

$\Rightarrow$ the ODE of X_t is $\frac{d}{dt}q(t) = q(t) \odot X_t$

$$\Rightarrow \frac{d}{dt} P_{s,t} = P_{s,t} \odot X_t \quad \text{with} \quad P_{s,s} = id \quad \forall s \in \mathbb{R}$$

A notation like the following is often used

$$P_{s,t} = \overrightarrow{\int_s^t} X_\tau \, d\tau \quad \text{or} \quad \overrightarrow{\exp} \int_s^t X_\tau \, d\tau$$

look at the difference with $\displaystyle\int_s^t X_\tau \, d\tau$

For now we set $P_{0,t} = P_t$

$$\begin{cases} \dot{P}_t = P_t \circ X_t \\ P_0 = \mathrm{id} \end{cases} \implies P_t = \overrightarrow{\int_0^t} X_s \, ds = \mathrm{id} + \int_0^t P_s \circ X_s \, ds$$

$$P_t = \mathrm{id} + \int_0^t \left(\mathrm{id} + \int_0^{s_1} P_{s_2} \circ X_{s_2} \, ds_2 \right) \circ X_{s_1} \, ds_1 =$$

$$= \mathrm{id} + \int_0^t X_{s_1} \, ds_1 + \int_{\boxed{0 \le s_2 \le s_1 \le t}} P_{s_2} \circ X_{s_2} \circ X_{s_1} \, ds_1 \, ds_2$$

In the end

$$\overrightarrow{\int_0^t X_s\, ds} = \sum_{k=0}^{\infty} \int_{\Delta_k(t)} X_{s_k} \odot X_{s_{k-1}} \odot \cdots \odot X_{s_1}\, ds_1 \cdots ds_k$$

$$\Delta_k(t) = \{ (s_1, \ldots, s_k) : 0 \le s_k \le \cdots \le s_1 \le t \}$$

<u>Observation 1</u>: if X_t is autonomous $\overrightarrow{\int X_s\, ds} = \sum \underbrace{\mathrm{Vol}(\Delta_k(t))}_{\frac{t^k}{k!}} \cdot X^k = e^{tX}$

<u>Exercise</u>: if $[X_{t_1}, X_{t_2}] = 0 \ \forall t_1, t_2 \Rightarrow \overrightarrow{\int_0^t X_s\, ds} = e^{Y_t}$, $Y_t = \int_0^t X_s\, ds$

The expression we stated formally has good convergence properties

- **Theorem:** if $(L, \|\cdot\|) \subset C^\infty(M)$ is a Banach space

$X_t a \in L$, $\forall a \in L$ & $t \in I$

and X_t is bounded for $t \in I$ wrt $\|\cdot\|$

$$\Rightarrow \left\| \int_{\Delta_k(t)} X_{s_k} \circ \cdots \circ X_{s_1} a \right\| \leq \int_{\Delta_k(t)} \|X_{s_k}\| \cdots \|X_{s_1}\| \cdot \|a\| = \frac{1}{k!} \left(\int_0^t \|X_s\| \right)^k \cdot \|a\|$$

this gives convergence

For a more precise result we call $S_N(t) = \sum_{k=0}^{N} \int X_{s_k} \circ \dots \circ X_{s_1}$

- <u>Theorem</u>: $\forall t>0$, $\alpha, N \in \mathbb{N}$, $K \subset M$ cpt, $a \in C^\infty(M)$,

$$\left\| \left(\overrightarrow{\int_0^t X_s \, ds} - S_N(t) \right) a \right\|_{\alpha, K} \leq \frac{C}{N!} \cdot e^{C \cdot D_t} \cdot \left(\int_0^t \| X_s \|_{\alpha+N-1, K'} \right)^N \cdot \| a \|_{\alpha+N, K'}$$

for some $K \subset_{cpt} K' \subset M$ & $C = C_{\alpha, N, K'} > 0$

with $D_t = \int_0^t \| X_s \|_{\alpha, K'}$

In our case $X_t = \sum u_i(t) X_i$, if we set $|u|_{1,t} = |u|_{L^1([0,t])}$

$$LHS \leq \frac{C}{N!} \cdot e^{C \cdot |u|_{1,t}} \cdot |u|_{1,t}^{N} \cdot \|a\|_{\alpha+N,K'}$$

If $Q_t = P_{t,0} = P_t^{-1} \Rightarrow \begin{cases} \dot{Q}_t = -X_t \odot P_t \\ Q_0 = 0 \end{cases}$

$$Q_t =: \int_0^t -X_s \, ds$$

you can obtain similar expressions as the ones for P_t

Differential of E_{q_0}

In this formalism we recall that $P_* X = P^{-1} \circ X \circ P$
$$= Ad(P^{-1}) X$$

we will use $Ad(P^{-1})$ instead of P_*

We want to study perturbations of a flow

suppose $\dot{q}(t) = X_t + Y_t$ we look at Y_t as a perturbation of X_t

If $P_t = \overrightarrow{\int X_s\, ds}$ then

$$\overrightarrow{\int_0^t (X_s + Y_s)\, ds} = \left(\overrightarrow{\int_0^t Ad(P_s)\, Y_s\, ds} \right) \odot P_t$$

$\longrightarrow$ <u>Proof</u>: $S_t := \overrightarrow{\int (X_s+Y_s)\, ds} = R_t \odot P_t \Rightarrow R_t = S_t \odot P_t^{-1}$

by definition $\dot{S}_t = S_t \odot (X_t + Y_t) = \dot{R}_t \odot P_t + R_t \odot \dot{P}_t$

$$\Rightarrow S_t \odot Y_t = \dot{R}_t \odot P_t \Rightarrow \dot{R}_t = S_t \odot Y_t \odot P_t^{-1}$$

$$\dot{R}_t = \delta_t \odot \psi_t \odot P_t^{-1} = R_t \odot \mathrm{Ad}(P_t)\,\psi_t$$

$$\Longrightarrow \quad R_t = \overrightarrow{\int \mathrm{Ad}(P_s)\,\psi_s \, ds}$$

$$\square$$

Theorem

$$D_u E_{q_0}(v) = \int_0^1 \left(P^u_{t,1}\right)^{-1}_{*}\left(\sum v_i X_i\right)\Big|_{\gamma_u(1)} dt$$

→ Proof: by what we said, we know

$$E_{q_0}(u) = q_0 \odot \overrightarrow{\int_0^1 \sum u_i X_i}$$

$$\Rightarrow E_{q_0}(u+v) = q_0 \odot \overrightarrow{\int \sum u_i X_i + \sum v_i X_i} = q_0 \odot \overrightarrow{\int_0^1 Ad(P^u_s)\left(\sum v_i X_i\right)} \odot P^u_1$$

let's work it out for $u=0$

$$E_{q_0}(v) = q_0 \odot \overrightarrow{\int_0^1 \sum_i v_i X_i}$$ we use the estimates of before

$$\left\| \left(\overrightarrow{\int_0^1 \sum_i v_i X_i} - \int_0^1 \sum v_i X_i \right) a \right\|_{\alpha,k} \leq C \cdot e^{C \cdot |v|_2} \cdot |v|_2^2 \cdot \|a\|_{\alpha+1,k'}$$

the RHS is an $o(|v|_{L^2}^2)$

therefore taking the differential only the first order remains for other $u \in L^2$, observe that $\mathrm{Ad}(P_s^u) = (P_{0,s}^u)_*$

and you have to make an additional composition

with P_s^u

$\square$