SR geodesics

6

4/2/26

* SR Hopf-Rinow theorem
* conjugate points
* geodesics

What have we proved?

$u \longmapsto \gamma_u(1)$

$E_{q_0} : \mathcal{U}_{q_0} \subset L^2([0,1] \to \mathbb{R}^m) \longrightarrow M$ is smooth

and $\quad D_u E_{q_0}(v) = \displaystyle\int_0^1 \left(P^u_{t,1}\right)_* \left(\sum v_i(t) X_i\right)\Big|_{\gamma_u(1)} dt$

$\dot{\gamma}_u(t) = \sum u_i(t) X_i(\gamma_u(t))$

$P^u_{s,t}(q) = \gamma_u(t)$ if $\gamma_u(s) = q$

Sub-Riemannian Hopf-Rinow theorem

$$\exp(\lambda) := \pi\left(e^{\vec{H}}(\lambda)\right)$$

here $\pi : T^*M \longrightarrow M$

$\lambda \in T^*M$

$H \in C^\infty(M)$

$H(\lambda) = \frac{1}{2}\sum_i \langle \lambda, X_i \rangle^2$

$\exp : T^*M \longrightarrow T^*M$

when we set the starting point we write $\exp_{q_0}(\lambda)$

We call $A_{q_0} := \{ \lambda_0 \in T^*_{q_0}M : \exp_{q_0}(t\lambda_0) \text{ is def for } t \in [0,1] \}$

$\underset{\text{open}}{\subset} T^*_{q_0}M$

usually we want to work with trajectories

st $\|\dot{\gamma}\| = 1$ $\forall t \in [0,1]$

therefore we take $\lambda_0 \in H^{-1}\left(\frac{1}{2}\right)$

- **Theorem**: if (M, d_{SR}) is complete, then $\mathcal{A}_{q_0} = T^*_{q_0} M$

$\rightarrow$ Sketch of the proof: by completness if $\gamma(t)$ normal traj. is defined on $[0,T)$ then it is defined on $[0,T]$

$\Rightarrow$ I can extend γ, and therefore γ is defined $\forall t \in \mathbb{R}$

Indeed, looking at the hamiltonian system

$$\begin{cases} \dot{p} = \frac{\partial H}{\partial x} \\ \dot{x} = -\frac{\partial H}{\partial p} \end{cases}$$

I can prove that $|p(t)|$ is bounded on a neighborhood of T

and so the system for $(x(t), p(t))$ is solvable in a neighborhood of t $\square$

If there are no abnormal extremals $\Rightarrow$ all the extremals must be normal $\Rightarrow$ $\exp_{q_0}$ must be surjective

There exist extremals of $J(u)$ with the constraint that $u \in E_{q_0}^{-1}(q_1) \subset \mathcal{U}_{q_0}$

The LM method can be applied to Hilbert spaces

$$\Rightarrow \exists\, \bar{\lambda} = (\nu, \lambda) \in \mathbb{R} \times T_{q_1}^{*}M \neq (0,0)$$

st $\nu D_u J + \lambda \circ D_u E = 0$ with $\nu = 0$ or -1

<u>Observation 1</u> : $J(u) = \frac{1}{2}|u|_{L^2} \Rightarrow D_u J = \langle u, - \rangle_{L^2}$

$$D_u J(v) = \int_0^1 u \cdot v$$

if $\nu = -1 \Rightarrow \lambda_0 D_u E_{q_0} =_{L^2} u$

if $\nu = 0 \Rightarrow \lambda_0 D_u E_{q_0} =_{L^2} 0$

<u>Observation 2</u> : take an extremal $(u(t), \lambda(t))$ (Pontryagin notation)

here $\lambda(t) = (P^u_{t,0})^* \lambda_0 = (P^u_{t,1})^* \lambda(1)$

(N) if $\langle \lambda(t), X_i \rangle = u_i(t) \quad \forall i = 1, \dots, m$

then $\langle \lambda(1), D_u E_{q_0} \rangle = \int_0^1 \langle \lambda(1), (P_{t,1})_* \left(\sum v_i X_i\right) \rangle_{q_1} dt$

$$= \int_0^1 \langle \lambda(t), \sum_i v_i X_i \rangle_{\gamma(t)} dt$$

$$= \int_0^1 \sum u_i(t)\, v_i(t)\, dt = \langle u, v \rangle_{L^2}$$

we proved that $\lambda(1) \circ D_u E_{q_0} \stackrel{L^2}{=} u$

(A) if $\langle \lambda(t), X_i \rangle = 0 \quad \forall i = 1, \dots, m$

then $\lambda(1) \circ D_u E_{p_0} \overset{L^2}{=} 0$

(and only if)

• **Lemma** if $K \subset M$ cpt st every point of K is reached from q_0 by only strictly normal minimizers (no abnormal lifts)

$$C = \left\{ \lambda_0 \in T^*_{q_0}M \;\middle|\; \begin{array}{l} \exp_{q_0}(t\lambda_0) \text{ is a minimizer} \\ \exp_{q_0}(\lambda_0) \in K \end{array} \right\} \subset \mathcal{A}_{q_0}$$

is compact

→ <u>Proof</u>: if $\lambda_n \to \lambda_0$ with $\{\lambda_n\} \subset C$, then the corresponding traj $\gamma_n : [0,1] \to M$ conv. unif to $\gamma_0(t) = \exp(t\lambda_0)$

$\Rightarrow \gamma_0(t)$ is itself a minimizer $\Rightarrow$ C is closed

---

Boundedness. Suppose $\exists \{\lambda_n\}$ st $|\lambda_n| \longrightarrow +\infty$

take u_n the minimal control of the traj. γ_n

$$\Rightarrow \lambda_n \circ D_{u_n} E_{q_0} = u_n \Rightarrow \frac{\lambda_n}{|\lambda_n|} \circ D_u E_{q_0} = \frac{u_n}{|\lambda_n|} \quad \forall n$$

we proved that minimizers with endpoints in a cpt K are themselves a cpt set, $\mathcal{U}_K$

therefore (up to subseq.) $u_n \to u$

$\frac{\lambda_n}{|\lambda_n|} \to \lambda$

moreover $D_{u_n} E_{q_0} \to D_u E_{q_0}$

we can conclude $\lambda \circ D_u E_{q_0} = 0$ (for some λ)

$\Rightarrow$ γ_u is abnormal

$\square$

• Sub-Riemannian Hopf-Rinow $\quad M \quad$ SR-mfd

if M have no abnormal length-minimizers

& $\exists x \in M : \mathcal{A}_x = T_x^*M \implies (M, d_{SR})$ is complete

$\longrightarrow$ Proof: $\quad A := \{ r > 0 \mid \overline{B(x,r)} \text{ is cpt} \}, \quad R := \sup(A)$

we know that $(0,R) \subset A$ and A is open by loc.

compactness (exercise) $\Rightarrow$ it suffices to prove $R \in A$

Suppose $\{y_i\} \subset M$ st $d(x,y_i) =: r_i \longrightarrow R$

$\exists\, \lambda_i \in H^{-1}(1/2) \cap T_x^*M$ st $\underset{\overset{\shortparallel}{\gamma_i(t)}}{\exp_x(t\lambda_i)}$ is the length-minimizing from x to y_i

by hypothesis every $\gamma_i : [0, r_i] \to M$ can be extended to $[0,R]$,

moreover $\overline{B(x, R-\varepsilon)}$ is cpt and so, by the Lemma before,

$\{(R-\varepsilon)\lambda_i\} \subset \mathcal{M}_{\overline{B(x,R-\varepsilon)}}$ compact $\Rightarrow$ $(R-\varepsilon)\lambda \in \mathcal{M}_{\overline{B(x,R-\varepsilon)}}$ such that

$\lambda_i \to \lambda \implies r_k\lambda_k \longrightarrow R\lambda$ and by the continuity of exp

$y_i = \exp(r_i\lambda_i) \longrightarrow \exp(R\lambda) = y$ $\square$

Conjugate point

$q_0 \in M$, $q \in M$ is conjugate to q_0 if $\exists \lambda_0 \in T^*_{q_0}M$ & $s>0$
$\lambda_0 \in H^{-1}(1/2)$

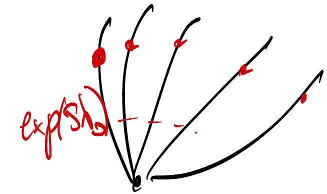
such that $q = \exp_{q_0}(s\lambda_0)$

and $s\lambda_0$ is a critical point of $\exp_{q_0}$ ($D_{s\lambda_0}\exp_{q_0}$ is non-surjective)

"first" conj point to q_0 wrt λ_0

is $q = \exp(s\lambda_0)$ st $s = \inf\{t>0 \mid t\lambda_0 \text{ is critical for } \exp_{q_0}\}$

$\mathrm{Conj}(q_0) = \{ \text{first conj. points to } q_0 \}$



q_0

<u>Remark</u>: if $\gamma(t) = \exp(t\lambda_0)$ admits an abnormal lift

then $\exp_{q_0}(\lambda_0) = E_{q_0}(u)$ and if we vary λ_0

we stay in $\mathrm{Im}(E_{q_0})$ by definition

$$\Rightarrow \mathrm{Im}(D_{\lambda_0} \exp_{q_0}) \subset \mathrm{Im}(D_u E_{q_0}) \subsetneq T_{\gamma(1)} M$$

THEREFORE λ_0 IS CRITICAL

$\Rightarrow$ $\gamma(1)$ is conjugate to q_0

If we re-scale $\gamma(t)$, everything works the same

$\Rightarrow$ every point on an abnormal - normal traj. is conjugate to q_0

<u>Rmk</u>: $D_0 \exp_{q_0} = D_{q_0}$

- Sort of converse of the previous result:

if $\exists\, t_n \downarrow t_0$ (or $t_n \nearrow t_0$) such that $\gamma(t_n)$ is conjugate to $\gamma(0)$ $\forall n$

with γ normal extremal path

$\Longrightarrow \exists\, \varepsilon > 0$: (1) $\forall t \in [t_0, t_0+\varepsilon]$, $\gamma(t)$ conj to $\gamma(0)$

(2) $\gamma\big|_{[t_0,\, t_0+\varepsilon]}$ is abnormal

Corollary if $\gamma : [0,1] \to M$ normal extremal with no abnormal segments

$$\Rightarrow \quad \mathcal{T}_c := \{ t > 0 : \gamma(t) \text{ conj to } \gamma(0) \} \text{ is discrete}$$

In general $e^{\vec{H}} : T^*M \longrightarrow T^*M$ $\qquad \dim(T^*M) = 2n$

$\exp = \pi \circ e^{\vec{H}} : T^*M \longrightarrow M$ $\qquad \dim(M) = n$

$D\exp_{q_0} = d\pi \circ de^{\vec{H}}\Big|_{T^*_{q_0}M}$ but $e^{\vec{H}}$ is a diffeomorphism

$T_{\lambda_0}T^*M \Rightarrow de^{\vec{H}}\left(T_{\lambda_0}T^*_{q_0}M\right)$ has rank n in $T_{\lambda(t)}T^*M$

when is $d\pi \circ de^{\vec{H}}\Big|_{T_{\lambda_0}T^*M}$ non surjective?

when $de^{\vec{H}}\left(T_{\lambda_0}T^*_{q_0}M\right) \cap T_{\lambda(1)}T^*_{\gamma(1)}M \neq 0$

GEOMETRIC MEANING OF exp CRITICALITY

the set of horizontal trajectories is endowed with two topologies

C^0 topology $\left(\gamma_n \to \gamma \text{ if } d(\gamma_n(t), \gamma(t)) \to 0 \text{ uniformly} \right)$

$W^{1,2}$ topology $\left(\|\gamma_{u'} - \gamma_u\| = |u' - u|_{L^2} + \sup_{t \in [0,1]} d(\gamma_u(t), \gamma(t)) \right)$

$W^{1,2}$ is stronger

- **Theorem:** if γ is a strict normal extremal traj. (no abnormal segments), then

(1) $t_c := \inf\{\text{conj times}\} > 0$

(2) $\forall \underset{t > t_0}{t < t_c}$, $\gamma\big|_{[t_0,t]}$ is a local length min. in $W^{1,2}$

(3) $\forall t > t_c$, $\gamma\big|_{[t_0,t]}$ is <u>not</u> a length min in C^0

$\psi : N \to \mathbb{R}$, $\quad N$ smooth manifold

smooth

then for any curve $\gamma : [0,1] \to N$

$$\frac{d}{ds}\psi(\gamma(s)) = D_{\gamma(0)}\psi \cdot \dot\gamma(0) \quad \text{it just depends on } \dot\gamma(0)$$

while $$\frac{d^2}{ds^2}\psi(\gamma(s)) = D^2_{\gamma(0)}\psi(\dot\gamma(0), \dot\gamma(0)) + D_{\gamma(0)}\psi \cdot \ddot\gamma(0)$$

the second derivative depends on the coordinates (connection) except when $\gamma(0)$ is a critical point of ψ

This is the Hessian of ψ at $\gamma(0) = q_0$

$$\mathrm{Hess}_{q_0}(\psi)(v) = D^2_{q_0}\psi(v, v)$$

$v \in T_{q_0}N$

$$\underbrace{\qquad\qquad}_{} \quad '' \quad \underbrace{\qquad\qquad\qquad\qquad}_{}$$

$$J : \mathcal{U}_{q_0} \to \mathbb{R} \quad , \quad J\big|_{E^{-1}_{q_0}(q_1)} : E^{-1}_{q_0}(q_1) \cap \mathcal{U}_{q_0} \to \mathbb{R}$$

what happens at a critical point?

$$D_u J\big|_{E^{-1}_{\varphi_0}(\varphi)} = D_u J - \lambda \circ D_u E \qquad \text{for } \lambda \in T^*_\varphi M$$

associated to u

$J(u) = \frac{1}{2}|u|_{L^2}$

$D_u J(v) = \langle u, v\rangle_{L^2}$

$D^2_u J(v,v) = |v|^2_{L^2}$

$$\mathrm{Hess}\left(J\big|_{E^{-1}_{\varphi_0}(\varphi)}\right)(v) = D^2_u J - \lambda \circ D^2_u E$$

$$= |v|^2 - \lambda \circ D^2_u E(v)$$

Given a control $u:[0,1] \longrightarrow \mathbb{R}^m$, $\gamma:[0,1] \longrightarrow M$

we define $u^s(t) = s \cdot u(st)$

$$\gamma^s(t) = \gamma(st)$$

$\mathrm{Hess}_{u^s}\left(J \Big|_{E_{q_0}^{-1}(\gamma^s(1))} \right)$ is degenerate

iff $\gamma_u(s)$ conj to $\gamma_u(0)$

Geodesics

A curve is a geodesic if locally (in t) it is a length minimizer.

i.e. $\forall t \ \exists \varepsilon$ st $\ell\left(\gamma|_{[t,t+\varepsilon]}\right) = d(\gamma(t), \gamma(t+\varepsilon))$

normal extremals are always geodesics

If γ is a geodesic st $\gamma(0) = q_0$

$$A = \{ t > 0 \mid \gamma|_{[0,t]} \text{ is a length-minimizer} \}$$

then $A = (0, t_*]$ or $(0, +\infty)$ (exercise)

$\gamma(t_*)$ is a cut point for q_0

$\mathrm{Cut}(q_0) = \{$ cut points for $q_0 \}$

• **Theorem**: $\gamma : [0,1] \to M$ normal extremal with no abnormal segments

and $\exists\, t_0 < 1$ st $\gamma(t_0)$ is a cut point to $\gamma(0)$ along γ

$\Rightarrow$ one or both
- (i) $\gamma(t_0)$ is the first conj. point to $\gamma(0)$ on γ
- (ii) $\exists\, \hat{\gamma} \neq \gamma$ which is a length minimizer joining $\gamma(0)$ and $\gamma(t_0)$