# SR geodesics

# 6/2/26

* last results about geodesics

## Non-holonomic tangent space

* non-holonomic order
* first order approximation of a vector field

• **Theorem**: $\gamma : [0,1] \to M$ normal extremal with no abnormal segments

and $\exists\, t_0 < 1$ st $\gamma(t_0)$ is a cut point to $\gamma(0)$ along $\gamma$

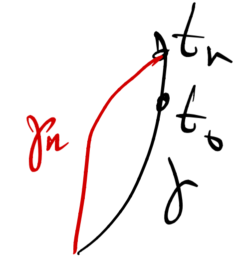
$\Rightarrow$ one or both

(i) $\gamma(t_0)$ is the first conj. point to $\gamma(0)$ on $\gamma$

(ii) $\exists\, \hat{\gamma} \neq \gamma$ which is a length minimizer joining $\gamma(0)$ and $\gamma(t_0)$

→ <u>Proof</u>: assume NOT (i). We are going to prove (ii)

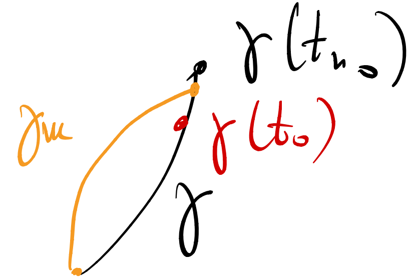
In $\gamma|_{[0,t_0]}$ there are no conj. points.
If not by what we saw $\gamma|_{[0,t_0]}$ can't be length-min.
fix a sequence $t_n \searrow t_0$ such that $\gamma|_{[0,t_n]}$ contains no conj. points.
And take $\gamma_n$ length min btw $\gamma(0)$ and $\gamma(t_n)$




We saw that $\gamma|_{[0,t_n]}$ is locally (in $W^{1,2}$) a length minimizer

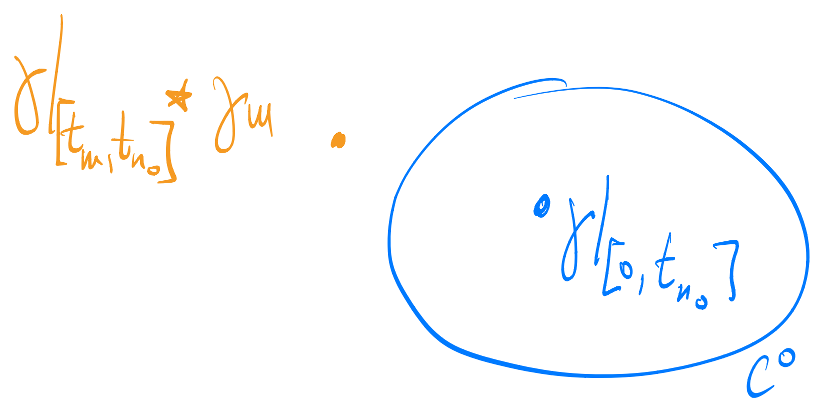
Therefore $\exists n_0$ : $\forall m \geq n_0$ , $\gamma_m$ must stay outside some neighborhood of $\gamma|_{[0,t_{n_0}]}$ (wrt the $W^{1,2}$-top)

$\Rightarrow$ lim $\gamma_m$ is outside the blue neighborhood




$$\lim \gamma_m = \hat{\gamma} \neq \gamma$$

□



**Converse.** if (i) or (ii) are true, then $t_* \leq t_0$

$t_*$: time of the first cut point

$t_0$: time in (i) or (ii)

→ <u>Proof</u>: if (i), $\exists\, t_0 > 0$ st $\gamma(t_0)$ is the first conj. point to $\gamma(0)$ along $\gamma$, then if $t_* > t_0$ the $\gamma|_{[0,t_*]}$ is a local length-min.

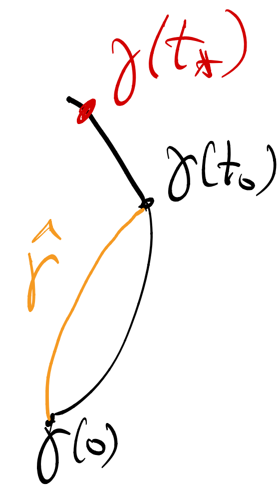
ABSURD

if $\exists\ \hat{\gamma} \neq \gamma$ length minimizers btw $\gamma(0)$ and $\gamma(t_0)$

Suppose $t_* > t_0$

then $\gamma|_{[t_0,t_*]} * \hat{\gamma}$ and $\gamma|_{[0,t_*]}$ are two

extremals corresponding to two different lifts




$\Rightarrow$ they correspond to 2 different liftings

of $\gamma|_{[t_0,t_*]}$, $\underbrace{\lambda^{(1)}(t) \text{ and } \lambda^{(2)}(t)}_{\text{normal trajectories}}$

But if you have 2 different normal liftings of a non-trivial extremal, then you have also an abnormal one

$$\langle \lambda^{(1)}(t), X_i \rangle = u_i(t) \qquad \forall t \in [t_0, t_*]$$

$$\langle \lambda^{(2)}(t), X_i \rangle = u_i(t) \qquad \forall t \in [t_0, t_*]$$

$$\langle (\lambda^{(1)} - \lambda^{(2)})(t), X_i \rangle \equiv 0 \qquad \forall t \in [t_0, t_*] \qquad \text{ABSURD}$$

$\square$

Let's call $c: H^{-1}(1/2) \to \mathbb{R} \cup \{+\infty\}$ cut time

$$c(\lambda) = \inf \{ s : \exp(s\lambda) \text{ is a cut point for } \exp(t\lambda) \}$$

If there are no abn. length-min (in our SR mfd) and the mfd is complete

then $c: H^{-1}(1/2) \subset T^*M \to \mathbb{R} \cup \{+\infty\}$

is continuous

· **Theorem**: normal extremals are geodesic (locally length-minimizers)

→ Sketch of the proof: $a \in C^\infty(M)$

consider $\mathcal{L}_0 := \{ d_q a \mid q \in M \} \subset T^*M$

$\mathcal{L}_t := e^{t\vec{H}}(\mathcal{L}_0)$

$\mathcal{L} := \{ (t, e^{t\vec{H}}(\lambda_0)) \mid \lambda_0 \in \mathcal{L}_0 \} \subset [0,1] \times T^*M$

Consider the 1-form $s - H\,dt$

$H$ the Hamiltonian

$s$ the Liouville form

If $\omega \in T_\lambda T^*M$ then $s(\omega) = \langle s, \omega \rangle = \langle \lambda, \pi_* \omega \rangle_q$

$q = \pi(\lambda)$

$\pi : T^*M \to M$

$s|_{\mathcal{L}_0} = d(a \circ \pi)$. Indeed, $\lambda \in \mathcal{L}_0 \Rightarrow \lambda = d_q a$

$\Rightarrow \langle s, \omega \rangle = \langle d_q a, \pi_* \omega \rangle = d_q a(d\pi(\omega)) = d(a \circ \pi)(\omega)$

Facts: (1) $s - H\,dt$ is closed ($d(s - H\,dt) = 0$)

(2) $s - H\,dt$ is exact, $\exists \beta : s - H\,dt = d\beta$

• **Theorem**: if $\exists\, a \in C^\infty(M)$ st $\pi|_{\mathcal{L}_t}$ is a diffeomorphism $\forall t \in [0,1]$

and we take $\lambda_0 \in \mathcal{L}_0$.

Then $\gamma(t) = \exp(t\lambda_0) = \pi \circ e^{t\vec{H}}(\lambda_0)$

is a strict minimizer of

$$J_a(\gamma) = a(\gamma(0)) + J(\gamma)$$

among curves with the same final point

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**Rmk**: if this is true, reasoning "locally" we get normal $\Rightarrow$ geodesic

→ Proof: take $\gamma'(t) \neq \gamma(t)$ and lift it on $\mathcal{L}_t$

$\Rightarrow \lambda'(t)$ with controls $u'(t)$

Define $\Gamma(t) = (t, \lambda(t))$ and $\Gamma'(t) = (t, \lambda'(t))$ in $[0,1] \times T^*M$

$\Gamma(1) = \Gamma'(1)$.

by def.

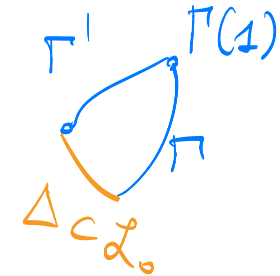
$$H(\lambda'(t)) \geq \langle \lambda'(t), \sum_i u_i'(t) X_i \rangle - \frac{1}{2}|u'|_{L^2}^2$$

with $>$ on a set $\subset [0,1]$ of positive measure

Exercise

We use the exactness of $s - H\,dt$


$$\int_\Gamma s - H\,dt = \int_\Delta s - H\,dt + \int_{\Gamma'} s - H\,dt$$




$$\int_\Delta s = \int_\Delta d(a \circ \pi) = a(\gamma'(0)) - a(\gamma(0))$$

$$\int_0^1 \langle \lambda', \dot{\gamma}'(t) \rangle - H(\lambda'(t))\,dt < \frac{1}{2}\int_0^1 |u'(t)|^2_{L^2}\,dt = J(\gamma')$$

$$= \frac{1}{2}\int |u(t)|^2_{L^2}\,dt = J(\gamma) \quad \square$$

# Non-holonomic order

We focus $q \in M$ and a neighborhood $q \in U \subset M$ where $M$ is a SR-mfd

<u>Definition</u>: $f : M \to \mathbb{R}$ continuous. The non-holonomic order of at $q$

$$\mathrm{ord}_q(f) := \sup \left\{ s \in \mathbb{R} : f(p) = O\left(d_{SR}(q,p)^s\right) \right\}$$

i.e. $f(p) \leq C \cdot d(q,p)^s$ for $p \to q$

In Euclidean setting, if $f(x) = \sum_\alpha c_\alpha \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\alpha = (\alpha_1, \dots, \alpha_n)$

$q = 0$

$p = x \in \mathbb{R}^n$

$$\mathrm{ord}_0(f) = \min \{ |\alpha| : c_\alpha \neq 0 \}$$

Let's take $f \in C^\infty(q)$, we consider the non-holonomic derivatives $X_{i_1} X_{i_2} \cdots X_{i_k} f$

( $i_j \in \{1, 2, \dots, m\}$

$X_1, \dots, X_m$ gen. family )

• **Lemma:** $\mathrm{ord}_q(f) = \max \{ \ell \in \mathbb{N} :$ every non-holonomic der. $X_{i_1} \cdots X_{i_k} f(q) = 0$ if $k < \ell \}$

Moreover, $f(p) = O\left(d(q,p)^{\mathrm{ord}(f)}\right)$

(the sup in the definition is attained)

We are going to prove

(i) if $\ell \in \mathbb{Z}$ st $\ell < \mathrm{ord}(f)$, then $X_{i_1} \cdots X_{i_k} f(q) = 0$ $\forall k < \ell$

(ii) if $X_{i_1} \cdots X_{i_k} f(q) = 0$ $\forall k \leq \ell$ $\Rightarrow f(p) = O\left(d(q,p)^{\ell+1}\right)$

(i) We first consider the case $l = 1 \Rightarrow ord(f) > 1$

We consider $p = \exp(tX)(q)$, $X_i f(q) = \frac{d}{dt} f(\exp(tX_i)(q))\Big|_{t=0}$

$\Rightarrow d(q,p) \leq |t|$ since $ord(f) > 1$, $\exists s > 0$

st by definition $f(p) = O(d(q,p)^{1+s}) = O(|t|^{1+s})$

being $f(q) = 0 \Rightarrow \left|\frac{\partial f}{\partial t}(\exp(tX_i)(q))\right| = |X_i f(q)| \leq \left|\frac{\partial}{\partial t}(t^{1+s})\Big|_{t=0}\right| = 0$

The same reasoning works for every $l$ by taking

$$X_{i_1} \cdots X_{i_k} f(q) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} f(\exp(t_1 X_{i_1}) \exp(t_2 X_{i_2}) \cdots \exp(t_k X_{i_k})(q))$$

$$d(q,p) \leq |t_1| + |t_2| \cdots + |t_k| \quad \text{and} \quad \text{since } k < l < \text{ord}(f)$$

$$\exists\, s>0 : \quad f(p) = O\left(d(q,p)^{k+s}\right) = O\left(\left(|t_1| + \cdots + |t_k|\right)^{k+s}\right)$$

we conclude the same way observing that $f(q) = 0$

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(ii) take a Euclidean chart around $q$, $f \in C^\infty(M)$

$$f(p) \leq C \cdot d_{Eu}(q,p) \leq C \cdot C' \cdot d_{SR}(q,p)$$

$p \to q$

locally by smoothness

locally a general fact

Induction. take $f$ st all the derivatives

((ii) true up to $\ell$) of order $\leq \ell+1$ are $0$

$$X_{i_1} \cdots X_{i_\ell} X_{i_{\ell+1}} f = X_{i_1} \cdots X_{i_\ell} \left( X_{i_{\ell+1}} f \right) = 0$$

therefore by induction hypothesis $\forall X \in D_q$

$$Xf(p) = O\left( d(q,p)^{\ell+1} \right)$$

Now consider a minimizing curve $\gamma$ from $q$ to $p$

$$\dot{\gamma}(t) = \sum u_i(t) X_i \,, \qquad \sum u_i^2(t) = 1 \quad \text{a.e.}$$

$$\gamma : [0,T] \to M, \qquad \gamma(0) = q \,, \quad \gamma(T) = p \,, \quad T = d_{SR}(q,p)$$

$$\Rightarrow \quad f(p) = f(\gamma(T)) = \int_0^T \sum u_i(t) X_i f(\gamma(t)) \, dt$$

$$\leq C''' \cdot \int_0^T t^{l+1} dt = \frac{C''' \cdot T^{l+2}}{l+2}$$

$\square$

$$\mathrm{ord}_q(f) = \min\{\ell \in \mathbb{N} : \exists\, X_{i_1} \cdots X_{i_\ell} f(q) \neq 0\}$$

Properties (Exercise)

$$\mathrm{ord}_q(fg) \geq \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$$

(is in fact an equality)

$$\mathrm{ord}_q(c \cdot f) = \mathrm{ord}_q(f) \quad \text{for } c \in \mathbb{R} \setminus \{0\}$$

$$\mathrm{ord}_q(f+g) \geq \min(\mathrm{ord}_p(f), \mathrm{ord}_q(g))$$

Take $X \in \mathrm{Vec}(q)$

$$\mathrm{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \mathrm{ord}_q(Xf) \geq \sigma + \mathrm{ord}_q(f) \quad \forall f \in C^\infty(q)\}$$

Properties

(Exercise)

$$\mathrm{ord}_q([X,Y]) \geq \mathrm{ord}_q(X) + \mathrm{ord}_q(Y)$$

$$\mathrm{ord}_q(fX) \geq \mathrm{ord}_q(f) + \mathrm{ord}_q(X)$$

(as before is an equality)

$$\mathrm{ord}_q(X) \leq \mathrm{ord}_q(Xf) - \mathrm{ord}_q(f)$$

$$\mathrm{ord}_q(X+Y) \geq \mathrm{ord}_q(X) + \mathrm{ord}_q(Y)$$

Example: Heisenberg $\mathbb{H}^1$

$$X_1 = \partial_x - \frac{y}{2}\partial_z \; ; \; X_2 = \partial_y + \frac{x}{2}\partial_z$$

$\Rightarrow$ $\mathrm{ord}(x) = \mathrm{ord}(y) = 1$ (because $X_1 x = 1$, $X_2 y = 1$)

$\mathrm{ord}(z) = 2$

$$\mathrm{ord}(X_1) = \mathrm{ord}(X_2) = -1$$

$$\mathrm{ord}([X_1, X_2] = \partial_z) = -2$$

# First order approximation of a vector field

take $X \in \mathrm{Vec}(q)$ then $\hat{X}$ is a f.o.a. of $X$

if $\mathrm{ord}(X - \hat{X}) \geq 0$ $\quad (\forall X \in D_q)$

Direct consequence

$$X_{i_1} \cdots X_{i_k} f(p) = \hat{X}_{i_1} \cdots \hat{X}_{i_k} f(p) + O\left(d(q,p)^{\mathrm{ord}_q(f) - k + 1}\right)$$

$$\Delta^1 = \text{span}\{X_1, \ldots, X_m\}$$

$$\Delta^{i+1} = [\Delta^1, \Delta^i]$$

$r$ = step of $M$ at $q$ st $\Delta^r(q) = T_q M$

$n_i(q) = \dim \Delta^i(q)$

regular if
- $r(q)$ is constant
- $n_i(q)$ are constant