SR geodesics

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10/2/26

* privileged coordinates
* first order approximations
* Ball-Box theorem

The interplay btw Riemannian and SubRiemannian distance

$\forall q \in M$, SR-mfd, regular of step r, $\exists U \ni q$ neigh. st d_{Riem} and d_{SR} respect the following

$$C_1 \cdot d_R(q,p) \leq d_{SR}(q,p) \leq C_2 \cdot d(q,p)^{1/r}$$

$\forall p \in U$, for some $C_1, C_2 > 0$

Consider $\phi_1, \phi_2 \in \mathrm{Diff}(M)$, we define

$$[\phi_1, \phi_2] := \phi_2^{-1} \circ \phi_1^{-1} \circ \phi_2 \circ \phi_1$$

$$\Rightarrow [\exp(tX_1), \exp(tX_2)] = \mathrm{id} + t^2 [X_1, X_2] + o(t^2)$$

Indeed, if $u(t,s) := \exp(-tX_2)\exp(-sX_1)\exp(tX_2)\exp(sX_1)$

$$\Rightarrow \frac{\partial u}{\partial t} = -X_2 + \mathrm{Ad}_{\exp(-sX_1)}(X_2) \; ; \; \frac{\partial u}{\partial s} = \mathrm{Ad}_{\exp(-tX_2)}(-X_1) + X_1$$

$$\frac{\partial^2 u}{\partial t \partial s} = [X_1, X_2] \; ; \; \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial s^2} = 0$$

with a similar reasoning, if $X = [\cdots[[X_1, X_2], X_3]\cdots], X_k]$

$$\Rightarrow [\cdots[[\exp(tX_1), \exp(tX_2)], \exp(tX_3)]\cdots], \exp(tX_k)] = id + t^k X + o(t^k)$$

we call the left side $\phi_X(t)$ $\Rightarrow \phi_X(t^{1/k}) = id + tX + o(t)$

Euclidean distance traveled $= t^k \cdot |X|$

SR distance traveled $\leq N \cdot t \cdot |X|$

$$\Rightarrow \boxed{d_{SR} \leq C \cdot d_E^{1/k}}$$

There exists a basis of TM in a neigh of q

such that $\underbrace{X_1, \dots, X_{m}}_{D_q = \text{Step } 1}, \underbrace{X_{m_1+1}, \dots, X_{m_2}}_{\text{Step } 2}, \underbrace{X_{m_2+1} \dots}_{\text{Step } 3}, \dots, \underbrace{X_n}_{\text{Step } r}$

We call s_i the step of X_i, and define $\varphi : \mathbb{R}^n \to M$

$$\varphi(t_1, \dots, t_n) = \phi_{X_n}(t_n^{1/s_n}) \circ \dots \circ \phi_{X_2}(t_2^{1/s_2}) \circ \phi_{X_1}(t_1)(q)$$

is C^1 near $0 \Rightarrow$ local C^1 diff.

$$\boxed{s_n = r}$$

$$d_{SR}(q, \varphi(t)) \leq C \cdot \left(|t_1| + \dots + |t_i|^{1/s_i} + \dots + |t_n|^{1/s_n} \right)$$

this would be a better bound than $\left(d_{\mathbb{R}^n}(0,t)\right)^{1/r}$

But these coordinates are not smooth

If we take some smooth coordinates $(y_1, \dots, y_n)$ on U, st $\left.\frac{\partial}{\partial y_i}\right|_q = X_i$

if $\varphi^y = y \circ \varphi \Rightarrow d_0 \varphi^y = id_{\mathbb{R}^n}$ and $y_i(t) = t_i + o(|t|_{\mathbb{R}^n})$

The best estimate we can get from

$y_i(t) = t_i + o(|t|_{\mathbb{R}^n})$ (this is $o(|t|_{\mathbb{R}^n})$, not $o(|t_i|)$)

$d_{SR}(q, \varphi(t)) \leq C \cdot \left(|t_1| + \dots + |t_i|^{1/s_i} + \dots + |t_n| \right)^{1/r}$

is $d(q, p) \leq C_2 \cdot |y(p)|_{\mathbb{R}^n}^{1/r}$

Last lesson resumé

$$\mathrm{ord}_q(f) = \min\{\ell \in \mathbb{N} : \exists\, X_{i_1} \cdots X_{i_\ell} f(q) \neq 0\}$$

$$\mathrm{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \mathrm{ord}_q(Xf) \geq \sigma + \mathrm{ord}_q(f) \quad \forall f \in C^\infty(q)\}$$

$$f(p) = O\left(d(q,p)^{\mathrm{ord}_q(f)}\right)$$

(for $p \to q$)

Privileged coordinates

$$\Delta^1 \subset \Delta^2 \subset \dots \subset \Delta^r = TM$$

$$\Delta^1 = \mathrm{Span}\{X_1,\dots,X_m\}, \quad \Delta^{i+1} = [\Delta^1, \Delta^i]$$

$$\Delta^r = TM$$

r = step of M = degree of non-holonomy

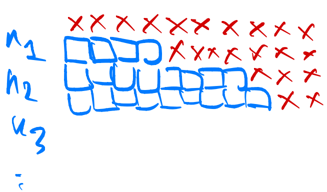
$n_i(q) = \dim \Delta^i(q)$, q is regular if the growth vector $(n_1, n_2, \dots, n_r)$ is locally constant

Also we define the weights w_i such that

$w_1, \dots, w_{n_1} = 1$

$w_{n_1+1}, \dots, w_{n_2} = 2$

$\vdots$



Example: the Martinet case on $\mathbb{R}^3$

$$X_1 = \partial_x$$

$$X_2 = \partial_y + \frac{x^2}{2}\partial_z$$

$$X_3 = \partial_z$$

GROWTH VECTOR $\begin{cases} \overset{n_1\, n_2}{(2,3)} & \text{if } x \neq 0 \qquad w_1 = w_2 = 1,\ w_3 = 2 \\ \underset{n_1\, n_2\, n_3}{(2,2,3)} & \text{if } x = 0 \qquad w_1 = w_2 = 1,\ w_3 = 3 \end{cases}$

Remark:

$q \longmapsto n_s(q)$ is lower sc $M \longrightarrow \mathbb{N}$ (Exercise)

$q \longmapsto w_i(q)$ is upper sc $M \longrightarrow \mathbb{N}$

$r = w_n$ is bounded on compacts

Definition: a frame of vector fields $Y_1, \dots, Y_n$ is adapted (to the flag $\{\Delta^i\}$) if

$Y_1(q), \dots, Y_n(q)$ is a basis of T_qM

and $Y_i \in \Delta^{w_i} \quad \forall i = 1, \dots, r$

Observe that if $X \in \Delta^j \setminus \Delta^{j-1}$ then $\mathrm{ord}(X) = -j$

We choose coordinates $(y_1, \dots, y_n)$ around q such that

$$dy_j(\Delta^s / \Delta^{s-1}) \neq 0 \quad \text{for } s = w_j \quad (n_{s-1} < j \leq n_s)$$

$$\Longrightarrow \quad \mathrm{ord}(y_j) \leq w_j \quad \text{by definition of order}$$

If $\quad dy_j(\Delta^{w_j}) \neq 0 \quad$ and $\quad dy_j(\Delta^{w_j - 1}) = 0 \qquad \forall j = 1, \dots, n$

the y_j are **linearly adapted**

$$\Uparrow \qquad \not\Downarrow$$

if $\quad \mathrm{ord}(y_j) = w_j$

the y_j are **privileged**

Example: on $\mathbb{R}^3$, $X_1 = \partial_x$, $X_2 = \partial_y + (x^2+y)\partial_z$

$\Rightarrow$ weights at 0 (is a property of the flag $\{\Delta^i\}$)

$$w_1 = 1, \quad w_2 = 1, \quad w_3 = 3$$

orders of (x, y, z) at 0

$$\mathrm{ord}(x) = 1, \quad \mathrm{ord}(y) = 1, \quad \mathrm{ord}(z) = 2$$

because $X_2 X_2 (z) = 1$

$\Rightarrow$ (x,y,z) not privileged at 0

it is linearly ad. to $\Delta^1, \Delta^2, \Delta^3$ because $Xz = 0$ $\forall X \in \Delta^2(0) = \Delta^1(0)$

if $(z_1, \dots, z_n)$ are privileged coordinates around q

if $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $w(\alpha) = \sum_i w_i \cdot \alpha_i$

$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$

- Theorem:

if $f(z) \sim \sum_i c_\alpha \cdot z^\alpha$ around $q \Rightarrow \mathrm{ord}_q(f) = $ least $w(\alpha)$ st $c_\alpha \neq 0$

if $X(z) \sim \sum_i a_{\alpha,j} z^\alpha \partial_{z_j} \Rightarrow \mathrm{ord}_q(X) = $ least $w(\alpha) - w_j$ st $a_{\alpha,j} \neq 0$

Dilations : if $(z_1, \_\_, z_n)$ are priv. coord. around q

we introduce $\delta_\lambda : M \to M$, $\lambda \in \mathbb{R}_{\geq 0}$

$$\delta_\lambda : (z_1, \_\_, z_n) = (\lambda^{w_1} z_1, \lambda^{w_2} z_2, \dots, \lambda^{w_n} z_n)$$

$\delta_\lambda^* f := f \circ \delta_\lambda$, f is **homogeneous of deg. s**

if $\delta_\lambda^* f = \lambda^s \cdot f$

$$(\delta_\lambda^* X) f := X(f \circ \delta_\lambda^{-1}) \circ \delta_\lambda$$

X is **homogeneous of deg. s**

if $\delta_\lambda^* X = \lambda^s \cdot X$

Exercise: if f homogeneous of deg s $\Rightarrow$ $\mathrm{ord}_q(f) = s$

Example: $z = (z_1, \dots, z_n)$, $\|z\|_q = \sum_i |z_i|^{1/w_i}$

$\|\cdot\|_q$ homogeneous of deg 1

$\Rightarrow \|z(p)\|_q = O(d(q,p))$

and $\mathrm{ord}_q(f) = \min \{ s \in \mathbb{Z} : f(p) = O(\|z(p)\|_q^s) \}$

The exponential coordinates of the first and second kind are privileged coordinates around q

with $Y_1, \dots, Y_n$ an adapted frame at q

$$(z_1, \dots, z_n) \longmapsto \exp(z_1 Y_1 + \dots + z_n Y_n)(q)$$

$$(z_1, \dots, z_n) \longmapsto \exp(z_n Y_n) \circ \dots \circ \exp(z_1 Y_1)(q)$$

There exists an "algebraic" way (meaning by polynomes) of building a set of privileged coordinates from any set of coordinates

First order approximations

$\{X_i\}$ generating Δ^1, $\quad \operatorname{ord}(X_i) = -1 \quad \forall i = 1, \dots, m = n_1$

$(z_1, \dots, z_n)$ p.c.

$$\Rightarrow X_i(z) \sim \sum_{\alpha, j} a_{\alpha, j} z^\alpha \partial_{z_j}$$

$w(\alpha) \geqslant w_j - 1 \quad \forall \alpha, j$ st $a_{\alpha,j} \neq 0$

grouping terms of the same order we get,

$$X_i = \underbrace{X_i^{(-1)}}_{\text{first order approx. of } X_i} + X_i^{(0)} + X_i^{(1)} + \dots$$

• **Theorem**: $\hat{X}_i := X_i^{(-1)}$ is a first order approx. of X_i and the $\hat{X}_i$s generate a nilpotent Lie algebra of step $w_n = r$

→ Proof: every homogeneous v.f. such $\deg(X) < -w_n = -r$ is 0

Small proof: (1) $\deg(\partial_{z_j}) = -w_j$

(2) use the Taylor expansion

Moreover $\operatorname{ord}([X,Y]) = \operatorname{ord}(X) + \operatorname{ord}(Y)$

for any $X, Y \in \operatorname{Lie}(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$

because $\delta_\lambda^\# [X,Y] = [\delta_\lambda^\# X, \delta_\lambda^\# Y] = \lambda^{-w_X - w_Y} [X,Y]$

$\Rightarrow$ if we take $r+1$ Lie brackets of the $\hat{X}_i$'s

we must get 0

$\square$

Example: $X_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y$

$\mathbb{R}^3_{x,y,\theta}$

$X_2 = \partial_\theta$

$X_3 = [X_1, X_2] = \sin\theta\,\partial_x - \cos\theta\,\partial_y$

$\Delta^1 = \langle X_1, X_2 \rangle$

$\Delta^2 = \langle X_1, X_2, X_3 \rangle$

$n = (2,3)$, $w = (1,1,2)$, everywhere

$q = (0,0,0) \Rightarrow (x,\theta,y)$ are privileged coordinates around q

$$X_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y = \partial_x - \frac{\theta^2}{2}\partial_x + \cdots + \theta\,\partial_y - \frac{\vartheta^3}{6}\partial_y + \cdots$$

$$= \underbrace{\partial_x + \theta\,\partial_y}_{\hat{X}_1 = X_1^{(-1)}} + \underbrace{\left(-\frac{\vartheta^2}{2}\partial_x - \frac{\vartheta^3}{6}\partial_y\right)}_{X_1^{(1)}} + \cdots$$

$$X_2 = \partial_\theta$$

$$\hat{X}_2 = \partial_\theta$$

$\mathrm{Lie}(\hat{X}_1, \hat{X}_2)$ has step 2

$$[\hat{X}_1, \hat{X}_2] = \hat{X}_3$$

$$[X_1, \hat{X}_3] = 0$$

<u>Remark</u>: in general the nilp. approx depends on the set of pc but $\mathrm{Lie}(\hat{X}_1, \dots, \hat{X}_m)$ does not

If q is regular

$$\Rightarrow \mathrm{Lie}(\hat{X}_1, \dots) = \Delta^1(q) + \left(\frac{\Delta^2}{\Delta^1}\right)(q) + \dots + \left(\frac{\Delta^r}{\Delta^{r-1}}\right)(q)$$

$$\hat{X}_i = \sum f_{ij}(z)\, \partial_{z_j}$$

$\deg(\hat{X}_i) = -1$

$f_{ij}(z)$ hom. polynomials of degree $w_j - 1$

therefore in f_{ij} there are no variables of deg $> w_j - 1$

$$\hat{X}_i(z) = \sum f_{ij}(\underbrace{z_1, \dots, z_{j-1}}_{})\, \partial_{z_j}$$

in fact we can stop at $z_{n_{w_j}-1}$

Consider, given some ODE on M, $\dot{q} = \sum_i u_i X_i$,

the non-holonomic associated ODE

$$\dot{z} = \sum_i u_i \hat{X}_i$$

A solution is easy to compute because

$$\dot{z}_j = \sum_i u_i(t)\, f_{ij}(z_1, \dots, z_{j-1})$$

$\forall u(t)$ the system is complete ($\Rightarrow$ the $\hat{X}_i$s are complete) the $\hat{X}_1, \dots, \hat{X}_m$ induce a Sub-Riemannian distance on $\mathbb{R}^n$, $\hat{d}_{SR}$ is homogeneous (of deg. 1) wrt δ_λ

• **Theorem**: (1) $\hat{X}_1, \dots, \hat{X}_m$ satisfy the Hörmander condition

(2) GW of the $\hat{X}_i$s at 0 is the same of the X_is at q

(3) $\hat{d}_{SR}$ is homogeneous of degree 1

(4) $\exists C>0$ st $\forall z \in \mathbb{R}^n$, $\frac{1}{C} \cdot \|z\|_q \leq \hat{d}(0,z) \leq C \cdot \|z\|_q$

→ **Proof**: we call $\hat{\Delta}^k$ the flag associated to the $\hat{X}_i$'s

we recall that $[\cdots[[\hat{X}_{i_1}, \hat{X}_{i_2}]\cdots], \hat{X}_{i_k}]$ is homogeneous of degree $-k$. If $I = (i_1, \dots, i_k)$ we use the notation $\hat{X}_I$ for the result of the Lie bracket above (analogously for X_I)

And $|I| = k$

$\Rightarrow X_I$ has $\mathrm{ord} = -k$ and $X_I = \hat{X}_I + (\text{terms of ord} > -k)$

$\Rightarrow$ At $z=0$, the terms of order $>-k$ that "survive" are some ∂_{z_j} with $w_j < k$

$$\hat{X}_I(0) = X_I(q) \mod \left(\operatorname{span}\{z_j \mid w_j < k\}\right)$$

$$= X_I(q) \mod \Delta^{k-1}(q)$$

therefore, by induction, $\dim \hat{\Delta}^k(0) = \dim \Delta^k(q)$, same growth vector (2)

Let $X_{I_1}, \dots, X_{I_n}$ be an adapted frame at q

i.e. $X_{I_j} \in \Delta^{w_j} \quad \forall j$

$\Rightarrow \hat{X}_{I_1}(0), \dots, \hat{X}_{I_n}(0)$ has rank $= n$

$\Rightarrow$ non-zero determinant, but the det of this is a homogeneous polynomial of deg $= 0$ i.e. a constant $\neq 0$

$\Rightarrow$ the Hormander condition is respected everywhere (1) for (3) consider a traj. $\hat{\gamma}$ of the nilpotent system

$$\frac{d}{dt}\hat{\gamma}(t) = \sum_i u_i(t)\, \hat{X}_i(\hat{\gamma}(t))$$

, and its dilation $\delta_\lambda \hat{\gamma}$ at

$$\frac{d}{dt}(\delta_\lambda \hat{\gamma})(t) = \sum_i \lambda u_i(t)\, \hat{X}_i(\delta_\lambda \hat{\gamma}(t))$$

by construction $\delta_\lambda^* \hat{X}_i = \lambda^{-1} \hat{X}_i$

<u>Exercise:</u>

$$\frac{d}{dt}(\delta_\lambda \circ \gamma) = (\delta_\lambda^{-1})^* \dot{\gamma}(\delta_\lambda \circ \gamma)$$

by the ex in blue, $\delta_\lambda \hat{\gamma}$ is an admissible trajectory between $\delta_\lambda(\hat{\gamma}(0))$ and $\delta_\lambda(\hat{\gamma}(T))$, and $\ell(\delta_\lambda \hat{\gamma}) = \lambda \cdot \ell(\hat{\gamma})$

this proves (3)

For the point (4), observe that both $d(0,\cdot)$ and $\|\cdot\|_q$ are homogeneous of degree 1 then we can choose C st $\frac{1}{C} \leq \hat{d}(0,z) \leq C$

on $\{\|z\|_q = 1\}$ $\square$

Ball - Box theorem

$(z_1, \ldots, z_n)$ p-c. at $q \Rightarrow \exists\, C, \varepsilon_0 > 0$

st if $d(q,p) < \varepsilon_0$, then

$$\frac{1}{C} \| z(p) \|_q \leq d_{SR}(q,p) \leq C \cdot \| z(p) \|_q$$

Corollary: for ε sufficiently small

$$\mathrm{Box}\left(\frac{\varepsilon}{nC}\right) \subset B(q,\varepsilon) \subset \mathrm{Box}(C \cdot \varepsilon)$$

where $\mathrm{Box}(\varepsilon) =$

$= [-\varepsilon^{w_1}, \varepsilon^{w_1}] \times \cdots \times [-\varepsilon^{w_n}, \varepsilon^{w_n}]$

for the proof we need that

Lemma: if $z(t)$ and $\hat{z}(t)$ are trajectories

satisfying $\begin{cases} \dot{z} = \sum u_i X_i \\ \dot{\hat{z}} = \sum u_i \hat{X}_i \end{cases}$ for the same u st $\|u(t)\| = 1$ a.e.

and $z(0) = \hat{z}(0) = z_0$

$$\Rightarrow \|z(t) - \hat{z}(t)\|_q \leq C \cdot \tau \cdot t^{1/r}$$

$\tau = \max(t, \|z_0\|_q) < \varepsilon$

→ Proof BBT: we already know that $C \cdot \|z\|_q \leq d(q,p)$
because of def. of ord

We need to prove $d(q,p) \leq C \cdot \|z(p)\|_q$

We will show that for $\|z_0\|_q$ small enoug

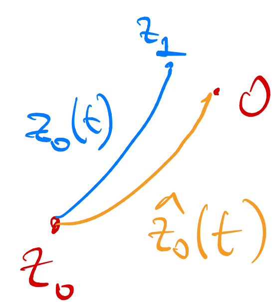
$d(0,z_0) \leq 2 \cdot \hat{d}(0,z_0)$ and we know $\hat{d}(0,z_0) \sim \|z_0\|_q$

We take the minimizing curve $\hat{z}_0(t)$ of speed 1 wrt $\hat{d}$, $\hat{z}_0 : [0, T_0] \longrightarrow \mathbb{R}^n$ and $T_0 = \hat{d}(0, z_0)$

such a curve exists if ε is small enough

let's call $u_0(t)$ the associated control.

Consider $z_0(t)$ the curve with control $u_0(t)$ but wrt $X_1, \ldots, X_m$



$$z_0(t) : [0, T_0] \longrightarrow \mathbb{R}^n$$

$$\ell(z_0(t)) \leq T_0$$

from the lemma

$$\|z_1\|_q = \| \overbrace{z_0(T_0)}^{z_1} - \overbrace{\hat{z}_0(T_0)}^{0} \|_q \leq C \, \overbrace{\tau}^{\text{bounded by } \varepsilon} \cdot T_0^{1/r}$$

in fact by definition we can bound

$$\tau \leq C' \cdot T_0 = C' \cdot \hat{d}(0, z_0)$$

$$\Rightarrow \quad \hat{d}(0, z_1) \leq D \cdot \|z_1\|_q \leq D' \cdot \hat{d}(0, z_0)^{\frac{r+1}{r}}$$

As before we build

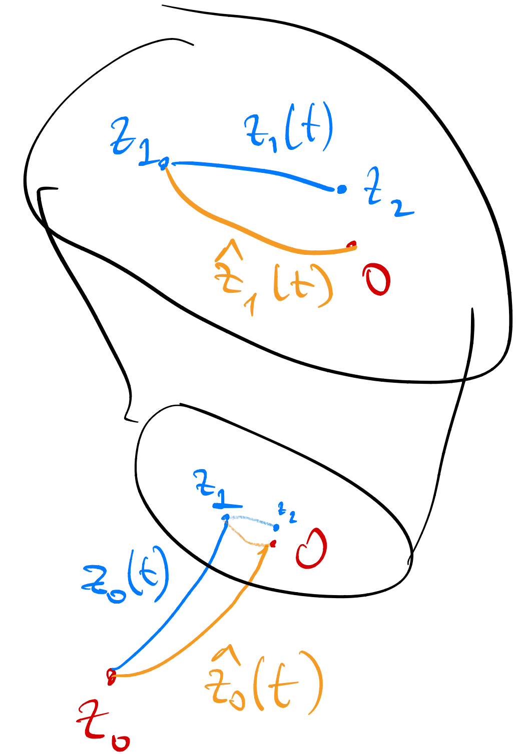
$\hat{z}_1(t)$ minimizing for $\hat{d}(z_1, 0)$

and $z_1(t)$

$$z_2 := z_1(T_1), \quad \ell(z_1(t)) \leq T_1$$

$$\hat{d}(0, z_2) \leq D' \cdot \hat{d}(0, z_1)^{\frac{r+1}{r}}$$

and so on at every step



With $\|z_0\|$ small enough, $D' \cdot \hat{d}(0, z_0)^{1/r} \leq \frac{1}{2}$

$$\Rightarrow \quad \hat{d}(0, z_1) \leq \frac{\hat{d}(0, z_0)}{2}$$

$$\hat{d}(0, z_2) \leq \frac{\hat{d}(0, z_0)}{4}$$

$$\hat{d}(0, z_k) \leq \frac{\hat{d}(0, z_0)}{2^k}$$

and $z_k \longrightarrow 0$ as $k \to +\infty$

$$d(0,z_0) \leq \sum_k l(z_k(t)) \leq \sum_k T_k \leq \sum_k \hat{d}(0,z_k)$$

$$\leq 2 \cdot \hat{d}(0,z_0)$$

$\square$