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Moduli of curves with principal and spin bundles: singularities and global geometry

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Introduction

The moduli space $\overline{\mathcal{M}}_g$ of genus g stable curves is a central object in algebraic geometry. From the point of view of birational geometry, it is natural to ask if $\overline{\mathcal{M}}_g$ is of general type. Harris and Mumford [24] and Eisenbud and Harris [14] address this question for $g \geq 4$, and find that $\overline{\mathcal{M}}_g$ is of general type for genus $g \geq 24$ and $g = 22$. For lower genus many cases are completely described, for instance $\overline{\mathcal{M}}_{14}$ and $\overline{\mathcal{M}}_{16}$ are unirational (see [8] and [17]). For $g = 23$, we know a bound for the Kodaira dimension, $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$, due to Farkas [16], but this case keeps being mysterious.

In the last decade, in an attempt to clarify this, a new approach emerged inspired by work of Farkas and his collaborators Chiodo, Eisenbud, Ludwig, Schreyer, Verra. The idea is to consider finite covers of $\overline{\mathcal{M}}_g$ that are moduli spaces of stable curves equipped with additional structure as $\boldsymbol{\mu}_\ell$-covers (ℓth roots of the trivial bundle) or $\boldsymbol{\mu}_\ell$-spin bundles (ℓth roots of the canonical bundle). These spaces have the property that the transition to general type happens in lower genus. For instance Farkas [18] showed that the moduli space of even $\boldsymbol{\mu}_2$-spin curves (L square root of ω and even dimensional space of sections $h^0(L)$) is of general type for $g \geq 9$. Farkas and Verra also showed, [21], that the moduli space of odd $\boldsymbol{\mu}_2$-spin curves (L square root of ω and odd dimensional space of sections $h^0(L)$) is of general type for $g \geq 12$. Furthermore, Farkas and Ludwig proved in [19] that the moduli space of curves with a $\boldsymbol{\mu}_2$-cover (also called *Prym* curves) is of general type for $g \geq 14$, and, in [11], Farkas, Chiodo, Eisenbud, Schreyer proved the same in the case of $\boldsymbol{\mu}_3$-covers for $g \geq 12$. In this work we intend to generalize this approach in two ways:

1. a classification of the singular locus for the moduli space of curves with any root of the canonical bundle and any of its powers;
2. a study of the moduli space of curves with G-covers for any finite group G.

Singularities of moduli of curves and roots. Our first direction consists in analyzing $\boldsymbol{\mu}_\ell$-root structures for any line bundle on the universal family $\mathcal{C}_g \to \overline{\mathcal{M}}_g$. By the Franchetta conjecture, every line bundle on $\mathcal{C}_g$ is a power of the relative canonical bundle $\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}$ up to pullbacks from $\overline{\mathcal{M}}_g$ itself, it is sufficient to consider roots of these powers. This is the approach of the author's paper [22], *"Singularities of moduli of curves with a universal root"*. In the first part of this work we briefly present its main results.

We write $\mathcal{R}^k_{g,\ell}$ for the moduli space of smooth curves of genus g equipped with a line bundle L such that $\mathsf{L}^{\otimes \ell} \cong \omega^{\otimes k}$. This space comes with a natural proper forgetful morphism $\pi\colon \mathcal{R}^k_{g,\ell} \to \mathcal{M}_g$. In order to extend π to a proper morphism to $\overline{\mathcal{M}}_g$, we will use the notion of twisted curve as done by Chiodo-Farkas in [12]. A twisted curve C is a Deligne-Mumford stack with a stable curve C as coarse space and possibly non-trivial stabilizers at its nodes. We consider twisted curves and line bundles over them. In this way we obtain a compactification $\overline{\mathcal{R}}^k_{g,\ell}$.

For every rooted curve (C,L) the local picture of $\overline{\mathcal{R}}^k_{g,\ell}$ at the geometric point $[\mathsf{C},\mathsf{L}]$ is isomorphic to $\mathbb{C}^{3g-3}/\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L})$, where $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L})$ is the group of C automorphisms lifting to L (see discussion in Section 2.2.2 where we mod out quasi-trivial automorphism). Our key tool is the Age Criterion 3.8 of Reid, Shepherd-Barron and Tai detecting non-canonical singularities quotients in terms of the *age* of any automorphism h in $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L})$ acting on $\mathbb{C}^{3g-3}$. Following Reid's terminology we write *junior* and *senior* for age less than 1 or ≥ 1.

The singular locus of $\overline{\mathcal{R}}^k_{g,\ell}$ is a union of two components,

$$\operatorname{Sing} \overline{\mathcal{R}}^k_{g,\ell} = N^k_{g,\ell} \cup H^k_{g,\ell}.$$

The $N^k_{g,\ell}$ component is the locus of singularities "coming" from $\overline{\mathcal{M}}_g$, that is

$$N^k_{g,\ell} \subset \pi^{-1} \operatorname{Sing} \overline{\mathcal{M}}_g.$$

Given a twisted curve C, the dual graph $\Gamma(\mathsf{C})$ is the graph whose vertices are the irreducible components of C and whose edges are the nodes of C. Moreover, we label the edges of $\Gamma(\mathsf{C})$ with the orders of the stabilizers at the associated nodes. The dual graph encodes a fundamental behavior of the point $[\mathsf{C},\mathsf{L}]$. The $H^k_{g,\ell}$ component, the locus of "new" singularities inside $\operatorname{Sing}\overline{\mathcal{R}}^k_{g,\ell}$, can be entirely described in terms of dual graphs, as shown in Theorem 3.13. We recall that a graph is *tree-like* if every cycle is a loop.

Theorem. *For every twisted curve C and every prime number p, we note $\Gamma_p(\mathsf{C})$ the graph obtained from $\Gamma(\mathsf{C})$ by contracting the edges with stabilizer order divisible by p. For any g,k,ℓ non-negative integers with $g \geq 4$ and $\ell \geq 1$, any geometric point $[\mathsf{C},\mathsf{L}]$ of $\overline{\mathcal{R}}^k_{g,\ell}$ is in $H^k_{g,\ell}$ if at least one of the contracted graphs $\Gamma_p(\mathsf{C})$, for any prime number p dividing ℓ, is not tree-like.*

In order to describe the singular non-canonical locus $\operatorname{Sing}^{\mathrm{nc}}\overline{\mathcal{R}}^k_{g,\ell}$ we generalize the notions of T-curve and J-curve introduced by Chiodo and Farkas in [12]: a T-curve is a rooted curve (C,L) with an elliptic tail admitting an elliptic tail automorphism of order 6; a J-curve is a rooted curve such that the group of *ghost* automorphisms, *i.e* those automorphisms acting trivially on the coarse space C of C, is a junior group. Theorem 4.36 shows that these curves fill the non-canonical locus.

Theorem. *The non-canonical locus of $\overline{\mathcal{R}}^k_{g,\ell}$ is the union*

$$\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}^k_{g,\ell} = T^k_{g,\ell} \cup J^k_{g,\ell},$$

where $T^k_{g,\ell}$ and $J^k_{g,\ell}$ are the locus of T-curves and the locus of J-curves respectively.

It is possible to describe the J-locus in terms of dual graph informations, as we do in Section 3.3 for ℓ prime number. This analysis is new and completes the description by Chiodo and Farkas of $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}^0_{g,\ell}$ for $\ell = 5$. Furthermore, when the J-locus is empty, we have a fundamental pluricanonical form extension property, as proven in Remark 4.45 and Theorem 4.43.

Theorem. *For any g, k, ℓ non-negative integers with $g \geq 4$ and $\ell \geq 1$, consider a desingularization $\widehat{\mathcal{R}}^k_{g,\ell} \to \overline{\mathcal{R}}^k_{g,\ell}$, if the J-locus of $\overline{\mathcal{R}}^k_{g,\ell}$ is empty, then*

$$H^0\left(\overline{\mathcal{R}}^{k,\mathrm{reg}}_{g,\ell}, nK_{\overline{\mathcal{R}}^k_{g,\ell}}\right) = H^0\left(\widehat{\mathcal{R}}^k_{g,\ell}, nK_{\widehat{\mathcal{R}}^k_{g,\ell}}\right),$$

for n sufficiently big and divisible, where $\overline{\mathcal{R}}^{k,\mathrm{reg}}_{g,\ell}$ is the locus of regular points.

Moduli of curves and principal bundles. In the second part, for any finite group G we consider curves with a principal G-bundle. We note $\mathcal{R}_{g,G}$ the moduli space of smooth curves equipped with a principal G-bundle. This space comes with a natural forgetful morphism $\pi\colon \mathcal{R}_{g,G} \to \mathcal{M}_g$. In order to properly extend π over $\overline{\mathcal{M}}_g$, we will use again the notion of twisted curve, and also the notion of *admissible* G-cover, following Bertin-Romagny in [5]. Abramovich-Corti-Vistoli proved in [1] that these two approaches are equivalent, and this allows to define the compactification $\overline{\mathcal{R}}_{g,G}$.

Given a twisted G-cover (C, ϕ), the local picture of $\overline{\mathcal{R}}_{g,G}$ at the geometric point $[\mathsf{C}, \phi]$ is isomorphic to the quotient $\mathbb{C}^{3g-3}/\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$, where $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ is the automorphism group of the twisted G-cover. As in the case before, we can describe the singular locus $\mathrm{Sing}\,\overline{\mathcal{R}}_{g,G}$. In particular, the notions of T-curves and J-curves generalize to this case too, and again they fill the non-canonical singular locus $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G}$.

Theorem. *The non-canonical locus of $\overline{\mathcal{R}}_{g,G}$ is the union*

$$\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G} = T_{g,G} \cup J_{g,G},$$

where $T_{g,G}$ and $J_{g,G}$ are the locus of T-curves and the locus of J-curves respectively.

Furthermore, in the case $G = S_3$ the symmetric group of order 3, we prove in Theorem 4.46 that the J-locus is empty.

The final part of this work focuses on the case of the symmetric group $G = S_3$, with the goal of estimating the Kodaira dimension of $\overline{\mathcal{R}}_{g,S_3}$ and finding for which genus g this space is of general type. In this thesis, we provide an adapted version of the Grothendieck-Riemann-Roch tool by Chiodo in [10] (see Proposition 6.3). For odd genus $g = 2i + 1$, we follow [11] and study a virtual divisor $\mathcal{U}_g$ in view of a possible proof of the fact that the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}}$ is big. The construction of the divisor relies in Koszul cohomology techniques: $\mathcal{U}_g$ is the jumping loci for the Koszul cohomology group associated to a rank 2 vector bundle (see Definition 6.7). This brings us near to the following conjecture concerning the connected component $\overline{\mathcal{R}}^{S_3}_{g,S_3} \subset \overline{\mathcal{R}}_{g,S_3}$ parametrizing curves with a connected S_3-cover.

Conjecture. *The connected component $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ of the moduli space $\overline{\mathcal{R}}_{g,S_3}$ is of general type for every odd genus $g \geq 13$.*

In order to achieve the result it remains to prove two points: an extension result over the T-locus of $\overline{\mathcal{R}}_{g,S_3}$ analogous to Theorem 4.43, and the effectiveness of the virtual divisor $\mathcal{U}_g$.

Structure of this thesis. In Chapter 1 we introduce the notions of twisted G-cover and admissible G-cover for any finite group G, we describe their local structures and state the important Abramovich-Corti-Vistoli result (see [1]) that the two constructions are equivalent. In Chapter 2 we introduce spin bundles, *i.e.* curves with a line bundle which is a root of some power of the canonical bundle. A reader which is interested mainly at moduli spaces of such rooted curves can start from here. Chapter 3 focuses on the techniques to analyze quotient singularities, and gives a classification of the singular locus and the non-canonical singular locus for any moduli of rooted curves. In Chapter 4 we apply the same local analysis to the moduli space of twisted G-covers $\overline{\mathcal{R}}_{g,G}$. Finally, in the last two chapters we set the main tools to approach the conjecture above, in particular we describe the connected components of $\overline{\mathcal{R}}_{g,S_3}$, we evaluate the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}}$ and we write it down as the linear combination of the virtual divisor $\mathcal{U}_g$ and other effective divisors. The proof the the bigness of this divisor is the last subject tackled by this work, and it remains as the principal open question.

Chapter 1

Moduli of curves with a principal G-bundle

We start this chapter by recalling the definition of the moduli space $\mathcal{M}_g$ of smooth curves of genus g, and its compactification $\overline{\mathcal{M}}_g$, the moduli space of stable curves of genus g, as defined by Deligne and Mumford in [13].

Thereafter we introduce smooth curves with a connected principal G-bundle, for G a finite group, and their moduli space $\mathcal{R}_{g,G}$. This space comes with a natural forgetful proper morphism $\pi\colon \mathcal{R}_{g,G} \to \mathcal{M}_g$. In the case of principal connected G-bundles over stable curves, the nodal singularities prevent the forgetful projection to $\overline{\mathcal{M}}_g$ to be proper. To find a compactification of $\mathcal{R}_{g,G}$ which is proper over $\overline{\mathcal{M}}_g$, we introduce two equivalent stacks: the one of twisted G-covers of genus g, noted $\mathcal{B}_g^{\mathrm{bal}}(G)$, and the one of admissible G-covers of genus g, noted Adm_g^G. These stacks are Deligne-Mumford and are proven to be isomorphic by Abramovich, Corti and Vistoli ([1]), we introduce them both because we will use different insights given by both point of views. We will note $\overline{\mathbf{R}}_{g,G}$ the stack of connected twisted G-covers. As shown in Proposition 1.47, it generalizes the notion of *level* structure on curves. The coarse space $\overline{\mathcal{R}}_{g,G}$ of $\overline{\mathbf{R}}_{g,G}$ has an open and dense subset isomorphic to $\mathcal{R}_{g,G}$, and it comes with a proper forgetful morphism $\overline{\pi}\colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ which extends π.

1.1 Structure of the moduli space

1.1.1 Smooth and stable curves

In this work a curve is always a proper reduced scheme of dimension 1 over the field of complex numbers. For any scheme S, a smooth S-curve is a flat morphism $X \to S$ such that every geometric fiber is a curve. An n-marking on X is the datum of n non-intersecting sections $\sigma_1, \dots, \sigma_n\colon S \to X$.

The moduli $\mathbf{M}_{g,n}$ of smooth n-marked curves of genus g is the category of smooth S-curves of genus g with n markings, for any scheme S. We refer to Abramovich and Vistoli paper [2] for definitions and notations. A morphism of n-marked smooth curves from $X' \to S'$ to $X \to S$ is a pair of scheme morphisms

$$\tilde{f}\colon X' \to X, \quad f\colon S' \to S$$

such that they form a cartesian square

$$\begin{tikzcd} X' \arrow[r, "\tilde{f}"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "f"] & S, \end{tikzcd}$$

and $\tilde{f}$ and f commute with the markings. We note $\mathfrak{F}\colon \mathsf{M}_{g,n} \to Sch$ the canonical forgetful functor, and $\mathsf{M}_{g,n}(S)$ the fiber of $\mathsf{M}_{g,n}$ over S, *i.e.* the category whose objects are smooth n-marked S-curves, and morphisms are morphisms of S-curves over the identity morphism id_S.

Definition 1.1 (Stable curve)**.** A stable curve C is a curve whose singularities are of type node, and whose automorphism group is a finite group.

Definition 1.2. A stable n-marked S-curve is a flat morphism $X \to S$ such that every geometric fiber is a stable curve and there exists n markings $\sigma_1, \dots, \sigma_n\colon S \to X$ such that $\sigma_i(s) \neq \sigma_j(s)$ if $i \neq j$ and such that $\sigma_i(s)$ is in the smooth locus of C_s for every i and any geometric point $s \in S$.

Remark 1.3. Given a stable n-marked curve C, we denote by $C_1, C_2, \dots, C_V$ its irreducible component. Let $\mathsf{nor}\colon \overline{C} \to C$ the normalization morphism of C, and denote by $\overline{C}_i$ the normalization of component C_i for every i, then $\overline{C} = \sqcup_i \overline{C}_i$. We call special point the preimage on $\overline{C}$ of any marked point or node via nor morphism. We denote by g_i the genus of $\overline{C}_i$ for any i, and by n_i the number of special points on $\overline{C}_i$. The stability condition for C is therefore equivalent to $2g_i - 2 + n_i > 0$ for all i.

Remark 1.4. Given a nodal curve C with some marked points, we call C_{gen} the *generic locus* of C, *i.e.* the complementary set of marked and nodal points.

1.1.2 Local structure of $\overline{\mathcal{M}}_{g,n}$

The category $\overline{\mathsf{M}}_{g,n}$ of stable n-marked curves, is a Deligne-Mumford stack, containing $\mathsf{M}_{g,n}$ as an open and dense substack. In [13], Deligne and Mumford proved its coarse space $\overline{\mathcal{M}}_{g,n}$ to be a proper projective scheme, thus containing the coarse space $\mathcal{M}_{g,n}$ of $\mathsf{M}_{g,n}$ as an open and dense subset.

In the same paper [13], Deligne and Mumford carry a local analysis of stack $\overline{\mathsf{M}}_{g,n}$ based on deformation theory. For every n-marked stable curve $(C; p_1, \dots, p_n)$, the deformation functor is representable (see [31] and [4, §11]) and it is represented by a smooth scheme $\mathrm{Def}(C; p_1, \dots, p_n)$ of dimension $3g - 3 + n$ with one distinguished point noted p. The deformation scheme comes with a universal family

$$X \to \mathrm{Def}(C; p_1, \dots, p_n)$$

whose central fiber X_{p} is identified with $(C; p_1, \dots, p_n)$. Every automorphism of the central fiber naturally extends to the whole family X by the universal property of the deformation scheme, thus $\mathrm{Aut}(C; p_1, \dots, p_n)$ naturally acts on $\mathrm{Def}(C; p_1, \dots, p_n)$ and has p as fixed point.

The strict henselization of $\overline{\mathsf{M}}_{g,n}$ at the geometric point $[C; p_1, \ldots, p_n]$ on $\overline{\mathsf{M}}_{g,n}$ is the same as the Deligne-Mumford stack

$$[\mathrm{Def}(C; p_1, \ldots, p_n)/ \operatorname{Aut}(C; p_1, \ldots, p_n)]$$

at p. As a consequence, for every geometric point $[C; p_1, \ldots, p_n]$ of the coarse space $\overline{\mathcal{M}}_{g,n}$, the strict Henselization of $\overline{\mathcal{M}}_{g,n}$ at $[C; p_1, \ldots, p_n]$ is the quotient

$$\mathrm{Def}(C; p_1, \ldots, p_n)/ \operatorname{Aut}(C; p_1, \ldots, p_n).$$

This implies that every singularity of $\overline{\mathcal{M}}_{g,n}$ is a quotient singularity, and it will be useful in our analysis on Chapter 3. From now on, we will refer to the strict henselization of a scheme X at a geometric point p as the *local picture* of X at p.

Remark 1.5. We give a detailed description of the scheme $\mathrm{Def}(C; p_1, \ldots, p_n)$. As showed in [4, §11.2], given a smooth curve C with n marked points $p_1, \ldots, p_n$, we have

$$\mathrm{Def}(C; p_1, \ldots, p_n) \cong H^1(C, T_C(-p_1 - \cdots - p_n)),$$

where T_C is the tangent bundle to curve C. This gives to $\mathrm{Def}(C; p_1, \ldots, p_n)$ a natural structure of vector space.

In the case of a nodal curve C, we follow [12] and consider $\mathrm{Def}(C; \operatorname{Sing} C)$, the universal deformation of curve C alongside with its nodes. We impose $n = 0$ in this for sake of simplicity, the $n > 0$ case is similar. The normalization morphism $\mathsf{nor}\colon \overline{C} \to C$ exists for every nodal curve. If we call $C_1, C_2, \ldots, C_V$ the irreducible components of C and $\overline{C}_i$ their normalizations, then as we already said

$$\bigsqcup_{i=1}^{V} \overline{C}_i = \overline{C}.$$

We note $q_1, q_2, \ldots, q_\delta$ the nodes of C, and we mark the preimages of the nodes via nor on the normalization $\overline{C}$. We call D_i the divisor of marked points on the curve $\overline{C}_i$ for every i, then we have a canonical decomposition

$$\mathrm{Def}(C; \operatorname{Sing} C) = \bigoplus_{i=1}^{V} \mathrm{Def}(\overline{C}_i; D_i) \cong \bigoplus_{i=1}^{V} H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i)). \tag{1.1}$$

Furthermore, if we consider the quotient $\mathrm{Def}(C)/ \mathrm{Def}(C; \operatorname{Sing} C)$, we have a canonical splitting

$$\mathrm{Def}(C)/ \mathrm{Def}(C; \operatorname{Sing} C) = \bigoplus_{j=1}^{\delta} M_j, \tag{1.2}$$

where $M_j \cong \mathbb{A}^1$ is the deformation scheme of node q_j of C. The isomorphism $M_j \to \mathbb{A}^1$ is non-canonical and choosing one isomorphism is equivalent to choose a smoothing of the node.

1.1.3 Group actions

To introduce curves equipped with a principal G-bundle, we need some basic tools of group theory.

Given any finite group G and an element h in it, we note $b_h\colon G \to G$ the multiplication map such that $b_h\colon s \mapsto h \cdot s$ for all s in G. Moreover, we note $c_h\colon G \to G$ the conjugation automorphism such that $c_h\colon s \mapsto h \cdot s \cdot h^{-1}$ for all s in G. The subgroup of conjugation automorphisms, inside $\mathrm{Aut}(G)$, is called group of the *inner automorphisms* and noted $\mathrm{Inn}(G)$.

We call $\mathrm{Sub}(G)$ the set of G subgroups and, for any subgroup H in $\mathrm{Sub}(G)$, we call $Z_G(H)$ its centralizer

$$Z_G(H) := \{s \in G |\ sh = hs\ \forall h \in H\}.$$

We note simply by Z_G the center of the whole group. The group $\mathrm{Inn}(G)$ acts naturally on $\mathrm{Sub}(G)$.

Definition 1.6. We call $\mathcal{T}(G)$ the set of the orbits of the $\mathrm{Inn}(G)$-action in $\mathrm{Sub}(G)$. Equivalently, $\mathcal{T}(G)$ is the set of conjugacy classes of G subgroups.

Definition 1.7. Consider two subgroup conjugacy classes $\mathcal{H}_1, \mathcal{H}_2$ in $\mathcal{T}(G)$, we say that $\mathcal{H}_2$ is a subclass of $\mathcal{H}_1$, noted $\mathcal{H}_2 \leq \mathcal{H}_1$, if for one element (and hence for all) $H_2 \in \mathcal{H}_2$, there exists H_1 in $\mathcal{H}_1$ such that H_2 is a subgroup of H_1. If the inclusion is strict, then $\mathcal{H}_2$ is a strict subclass of $\mathcal{H}_1$ and the notation is $\mathcal{H}_2 < \mathcal{H}_1$.

Consider a set $\mathcal{T}$ with a transitive left G-action

$$\psi\colon G \times \mathcal{T} \to \mathcal{T}.$$

Any map $\eta\colon \mathcal{T} \to G$ induces, via ψ, a map $\mathcal{T} \to \mathcal{T}$. In particular,

$$E \mapsto \psi(\eta(E), E), \quad \forall E \in \mathcal{T}.$$

This resumes in a map

$$\psi_*\colon G^{\mathcal{T}} \to \mathcal{T}^{\mathcal{T}}.$$

Noting $S_{\mathcal{T}}$ the set of permutations inside $\mathcal{T}^{\mathcal{T}}$, we obtain that $\psi_*^{-1}(S_{\mathcal{T}})$ is the subset of maps $\mathcal{T} \to G$ inducing a $\mathcal{T}$ permutation.

Consider an element E in $\mathcal{T}$. We note, H_E its stabilizer, *i.e.* the G subset fixing E. Given any other element $\psi(s, E)$ for some s in G, its stabilizer is

$$H_{\psi(s,E)} = s \cdot H_E \cdot s^{-1},$$

this proves the following lemma.

Lemma 1.8. *Given any set $\mathcal{T}$ with a transitive G-action, there exists a canonical conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$, and a canonical surjection*

$$\mathcal{T} \to \mathcal{H}$$

sending any object on its stabilizer.

Given $\mathcal{T}$ with its G-action and the group G itself with the $\mathrm{Inn}(G)$-action, we consider the set of G-equivariant maps $\mathrm{Hom}^G(\mathcal{T}, G)$.

Lemma 1.9. *For any object E in $\mathcal{T}$, and any map η in $\mathrm{Hom}^G(\mathcal{T}, G)$, we have*

$$\eta(E) \in Z_G(H_E).$$

Proof. The equivariance condition means that

$$\eta(\psi(h, E)) = c_h(\eta(E)) = h \cdot \eta(E) \cdot h^{-1}$$

for all h in G. If h is in H_E, the left hand side of the equality above is simply $\eta(E)$, therefore $c_h(\eta(E)) = \eta(E)$ for all h in H_E, and this is possible if and only if $\eta(E)$ is in $Z_G(H_E)$. □

The last two lemmata yield the following crucial proposition.

Proposition 1.10. *The set of equivariant maps $\mathrm{Hom}^G(\mathcal{T}, G)$ is uniquely determined by the canonical class $\mathcal{H}$ associated to $\mathcal{T}$ (see Lemma 1.8). In particular given any object E in $\mathcal{T}$, there exists a canonical isomorphism*

$$\mathrm{Hom}^G(\mathcal{T}, G) \cong Z_G(H_E).$$

Proof. The second part of the proposition follows from Lemma 1.9. We observe that if we consider another object $E' = \psi(s, E)$, then $H_{E'} = s \cdot H_E \cdot s^{-1}$ and

$$Z_G(H_{E'}) = s \cdot Z_G(H_E) \cdot s^{-1}.$$

Therefore the inclusion $\mathrm{Hom}^G(\mathcal{T}, G) \hookrightarrow G$ is determined, up to conjugation, by the class $\mathcal{H}$ of H_E. □

1.1.4 Principal G-bundles

We introduce the stack $\mathbf{R}_{g,G}$ of smooth curves of genus g with a principal G-bundle.

Definition 1.11 (principal G-bundle)**.** If G is a finite group, a principal G-bundle over a scheme X is a fiber bundle $F \to X$ together with a left action $\psi\colon G \times F \to F$ such that the induced morphism

$$\widetilde{\psi}\colon G \times F \to F \times_X F,$$

is an isomorphism. Here $\widetilde{\psi}\colon (h, z) \mapsto (\psi(h, z), z)$ for all h in G and z point of F.

Remark 1.12. As a direct consequence of the definition, every geometric fiber of $F \to X$ is isomorphic to the group G itself. In our work X is always a (smooth or nodal) curve.

Remark 1.13. The category of principal G-bundles is noted BG and comes with a natural forgetful functor $BG \to Sch$.

Definition 1.14. The objects of the category $\mathsf{R}_{g,G}$ are smooth S-curves $X \to S$ of genus g, equipped with a principal G-bundle $F \to X$, for any scheme S. The morphisms of $\mathsf{R}_{g,G}$ are commutative diagrams as

$$\begin{tikzcd} F' \arrow[r, "b"] \arrow[d] & F \arrow[d] \\ X' \arrow[r, "\Phi"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "\phi"] & S \end{tikzcd}$$

such that the two squares are cartesian and b is G-equivariant with respect to the natural G-actions.

For any category $\mathcal{G}$ with a functor $\mathfrak{F}$ to the scheme category, the datum of a morphism $S \to \mathcal{G}$ from a scheme S to $\mathcal{G}$ is equivalent to the datum of an object X over S, *i.e.* such that $\mathfrak{F}(X) = S$. In the setting of principal G-bundles, this means that a principal G-bundle over a smooth curve C is uniquely determined by a morphism $C \to BG$.

The objects of category $\mathsf{R}_{g,G}$ are morphisms $\phi\colon C \to BG$ where C is a smooth S-curve for any scheme S.

The category $\mathsf{R}_{g,G}$ comes with a forgetful functor

$$\pi\colon \mathsf{R}_{g,G} \to \mathsf{M}_g.$$

Indeed, every object or morphism of $\mathsf{R}_{g,G}$ is sent by π on the underlying curve or curve morphism.

Remark 1.15. Consider a smooth curve C of genus g and a connected principal G-bundle $F \to C$, the scheme F is a smooth curve of genus $g' = |G| \cdot (g-1) + 1$, with a free and faithful G-action. The moduli stack $\mathsf{H}_{g',G}$ of smooth curves of genus g' with a free and faithful G-action, has been studied by Bertin and Romagny in [5]. The category $\mathsf{H}_{g',G}$ is a subcategory of $\overline{\mathsf{R}}_{g,G}$, and the inclusion functor is faithful.

We consider a connected normal scheme X and a principal G-bundle $F \to X$. We note $\mathrm{Aut}_{\mathrm{Cov}}(X,F)$ its automorphism group in the category of coverings, that is the automorphisms of F commuting with the projection $F \to X$. We note $\mathrm{Aut}_{BG}(X,F)$ its automorphism group in the category of principal G-bundles, that is the covering automorphisms of F compatible with the natural G-action.

We call $\mathcal{T}(F)$ the set of connected components of any principal G-bundle $F \to X$. This comes equipped with a natural projection $T\colon F \to \mathcal{T}(F)$. The group G acts transitively on $\mathcal{T}(F)$, and by abuse of notation we call

$$\psi\colon G \times \mathcal{T}(F) \to \mathcal{T}(F)$$

this action. As explained in Section 1.1.3, this action induces a map

$$\psi_*\colon G^{\mathcal{T}(F)} \to \mathcal{T}(F)^{\mathcal{T}(F)}.$$

Proposition 1.16. *If X is a connected normal scheme, and $F \to X$ a principal G-bundle, then we have the following canonical identifications:*

1. $\mathrm{Aut}_{\mathrm{Cov}}(X, F) = \psi_*^{-1}(S_{\mathcal{T}(F)})$;
2. $\mathrm{Aut}_{BG}(X, F) = \mathrm{Hom}^G(\mathcal{T}(F), G)$.

Proof. We consider any morphism

$$b \colon F \to F$$

commuting with the projection to X. Given the morphism

$$\widetilde{\psi} \colon G \times F \to F \times_X F$$

introduced in Definition 1.11, we consider the chain of maps

$$F \xrightarrow{b \times \mathrm{id}} F \times_X F \xrightarrow{\widetilde{\psi}^{-1}} G \times F \xrightarrow{\pi_1} G, \tag{1.3}$$

where π_1 is the first projection. As G is discrete, the map above is constant on the connected components and therefore there exists

$$\eta = \eta(b) \colon \mathcal{T}(F) \to G$$

such that

$$b = \psi \circ ((\eta \circ T) \times \mathrm{id}).$$

The morphism b is an automorphism if and only if it is bijective on the connected components set, *i.e.* if and only if $\psi_*(\eta(b)) \in S_{\mathcal{T}(G)}$.

The automorphisms of F as a principal G-bundle must moreover preserve the G-action, *i.e.* we must have

$$b \circ b_h = b_h \circ b \quad \forall h \in G,$$

and therefore

$$\eta \circ b_h = c_h \circ \eta.$$

This is the exact definition of η being in $\mathrm{Hom}^G(\mathcal{T}(F), G)$. □

Remark 1.17. In the case of a connected principal G-bundle $F \to X$, the proposition above resumes in

$$\mathrm{Aut}_{\mathrm{Cov}}(X, F) = G \quad \text{and} \quad \mathrm{Aut}_{BG}(X, F) = Z_G.$$

The set of connected components $\mathcal{T}(F)$ of a general G-bundle $F \to X$, has a transitive G-action. By Lemma 1.8, this induces a canonical conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$.

Definition 1.18. We call principal $\mathcal{H}$-bundle, a principal G-bundle whose canonical associated class in $\mathcal{T}(G)$ is $\mathcal{H}$. Equivalently, the stabilizer of every connected component in $\mathcal{T}(F)$ is a G subgroup in $\mathcal{H}$.

Remark 1.19. By Proposition 1.10, the automorphism group of any principal $\mathcal{H}$-bundle, is isomorphic to $Z_G(H)$, where H is any G subgroup in the $\mathcal{H}$ class.

It is possible to describe principal G-bundles over curves by the monodromy action, as done for example in [5]. Consider a smooth curve C, a marked point p_* on it and a principal G-bundle $F \to C$. We note $\tilde{p}_1, \dots, \tilde{p}_{|G|}$ the preimages of p_* on F. There exists a natural morphism $\pi_1(C, p_*) \to S_{|G|}$ from the fundamental group of C to the permutation set of the fiber F_{p_*}, the morphism is well defined up to relabelling the points $\tilde{p}_i$. It is known that the monodromy group, *i.e.* the image of $\pi_1(C, p_*)$ inside $S_{|G|}$, identifies canonically with group G, and this gives a surjection

$$\varpi \colon \pi_1(C, p_*) \to G,$$

well defined up to conjugation. The following proposition is a rephrasing of [5, Lemma 2.6] in the case of a smooth non-marked curve C.

Proposition 1.20. *We consider a smooth curve C and a point p_* on it. The set of isomorphism classes of connected principal G-bundles over C, is canonically in bijection with the set of conjugacy classes of surjections $\pi_1(C, p_*) \to G$.*

Remark 1.21. If $F \to C$ is any principal G-bundle, without the connectedness hypothesis, it is still possible to define a morphism $\varpi \colon \pi_1(C, p_*) \to G$, but without the surjectivity hypothesis. As we will also se later, the image of ϖ is the stabilizer of any connected component of F, and moreover the cokernel of ϖ is in bijection with the set of F connected components.

Remark 1.22. The fundamental group for a genus g smooth curve C is well known to be the group freely generated by $2g$ elements

$$\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \dots, \beta_g$$

with the additional relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdot \alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1. \tag{1.4}$$

We observe that Propostion 1.20 gives an easy counting tool. The number of principal G-bundles over a given curve C, equals the number of $2g$-uples of not necessarily distinct elements in G, defined up to conjugation, respecting relation (1.4) and generating G. For example, in any abelian group, relation (1.4) is automatically verified, therefore there exists $|G|^{2g}$ principal G-bundles over any smooth curve of genus g.

Changing the base point p_* to p'_* induces a natural isomorphism $\pi_1(C, p_*) \to \pi_1(C, p'_*)$. The identification between $\pi_1(C, p_*)$ and the abstract group described above, is well defined up to conjugation.

If we simply use the same definition of principal G-bundles to extend the stack $\mathsf{R}_{g,G}$ over nodal curves, the extended functor $\overline{\pi} \colon \overline{\mathsf{R}}_{g,G} \to \overline{\mathsf{M}}_g$ would not be proper. Indeed, we observe for example that in the case of $G = \boldsymbol{\mu}_\ell$ a cyclic group of order prime ℓ, the length of $\overline{\pi}$ fibers is not locally constant. If a curve C has δ nodes and v irreducible components, its fundamental groups $\pi_1(C)$ is freely generated by

$2g-\delta+v-1$ elements. The fiber over $[C]$ has length equal to $\ell^{2g-\delta+v-1}-1$, but in any neighborhood of $[C]$ there exists a smooth curve C' with fiber length $\ell^{2g}-1$ as shown above. We have to enlarge the $\overline{\mathsf{R}}_{g,G}$ category, allowing over nodal curves a more general notion than principal G-bundle.

1.2 Admissible G-covers

In this section we introduce the possibility of ramified covers over stable curves, by defining admissible G-covers. This notion was firstly developed by Abramovich, Corti and Vistoli in [1], and also by Jarvis, Kaufmann and Kimura in [25].

1.2.1 Definitions

Definition 1.23 (Admissible cover)**.** Given a nodal S-curve $X \to S$ with some marked points, an admissible cover $u\colon F \to X$ is a morphism such that:

1. $F \to X$ is a nodal S-curve;
2. every node of F maps to a node of X via u;
3. the restriction $F|_{X_{\mathrm{gen}}} \to X_{\mathrm{gen}}$ is an étale cover of degree d;
4. the local picture of $F \xrightarrow{u} X \to S$ at a point of F over a marked point of X is isomorphic to
$$\operatorname{Spec} A[x'] \to \operatorname{Spec} A[x] \to \operatorname{Spec} A,$$
for some normal ring A, an integer $r>0$, and with $u^*x = (x')^r$;
5. the local picture of $F \xrightarrow{u} X \to S$ at a point of F over a node of X is isomorphic to
$$\operatorname{Spec}\left(\frac{A[x',y']}{(x'y'-a)}\right) \to \operatorname{Spec}\left(\frac{A[x,y]}{(xy-a^r)}\right) \to \operatorname{Spec} A,$$
for some integer $r>0$ and an element $a \in A$, with $u^*x=(x')^r$ and $u^*y=(y')^r$.

The category $\mathrm{Adm}_{g,n,d}$ of n-pointed stable curves of genus g with an admissible cover of degree d, is a proper Deligne-Mumford stack.

Using the notion of admissible cover, we can generalize principal G-bundles over nodal curves, by defining admissible G-covers. We start by giving a local description of the G-action at nodes and marked points of admissible covers.

Lemma 1.24. *Consider $u\colon F \to C$ an admissible cover of a nodal curve C such that $F|_{C_{\mathrm{gen}}} \to C_{\mathrm{gen}}$ is a principal G-bundle. If $\tilde p \in F$ is one of the preimages of a node or a marked point, then the stabilizer $H_{\tilde p}$ is a cyclic group.*

Proof. If $\tilde p$ is the preimage of a marked point, the local picture of morphism u at $\tilde p$ is
$$\operatorname{Spec} A[x'] \to \operatorname{Spec} A[x],$$

where $x' = x^r$ for some integer $r > 0$. This local description induces an action of $H_{\tilde{p}}$ on $U := \operatorname{Spec} A[x']$ which is free and transitive on $U \backslash \{\tilde{p}\}$. The group of automorphisms of $U \backslash \{\tilde{p}\}$ preserving r is exactly $\boldsymbol{\mu}_r$, therefore $H_{\tilde{p}}$ must be cyclic too.

In the case of a node $\tilde{p}$ we observe that r is locally isomorphic to

$$\operatorname{Spec}\left(\frac{A[x', y']}{(x'y' - a^r)}\right) \to \operatorname{Spec}\left(\frac{A[x, y]}{(xy - a)}\right),$$

where $x' = x^r$ and $y' = y^r$, for an integer $r > 0$ and an element $a \in A$. The scheme $U' := \operatorname{Spec}\left(A[x', y']/(x'y' - a^r)\right)$ is the union of two irreducible components U_1, U_2, and we can apply the deduction above to $U_i \backslash \{\tilde{p}\}$ for $i = 1, 2$. □

Remark 1.25. Focusing on the case of the node, we observe that $H_{\tilde{p}}$ acts independently on the local charts U_1 and U_2. We say that its action is balanced when the character on the tangent space to U_1 is opposite to the character on the tangent space to U_2.

Definition 1.26 (Admissible G-cover)**.** Given any finite group G, an admissible cover $r \colon F \to C$ of a nodal curve C is an admissible G-cover if

1. the restriction $u|_{C_{\text{gen}}} \colon F|_{C_{\text{gen}}} \to C_{\text{gen}}$ is a principal G-bundle. This implies, by Lemma 1.24, that for every node or marked point $\tilde{p} \in F$, the stabilizer $H_{\tilde{p}}$ is a cyclic group;
2. the action of $H_{\tilde{p}}$ is balanced for every node $\tilde{p} \in F$.

Definition 1.27. We call $\operatorname{Adm}^G_{g,n}$ the stack of stable curves of genus g with n marked points and equipped with an admissible G-cover.

1.2.2 Local structure of an admissible G-cover

Consider an admissible G-cover $F \to C$, and a marked point $\tilde{p}$ on F. We note $H_{\tilde{p}}$ the cyclic stabilizer at $\tilde{p}$, and we observe that by definition of admissible G-cover, the G-action induces a primitive character

$$\chi_{\tilde{p}} \colon H_{\tilde{p}} \to \operatorname{GL}(T_{\tilde{p}}F) = \mathbb{C}^*,$$

where $T_{\tilde{p}}F$ is the tangent space of F at $\tilde{p}$.

Given a subgroup H of G, for any primitive character $\chi \colon H \to \mathbb{C}^*$ for any $s \in G$ we note χ^s the conjugated character $\chi^s \colon sHs^{-1} \to \mathbb{C}^*$ such that $\chi^s(h) = \chi(s^{-1}hs)$ for all $h \in G$. In the set of pairs (H, χ), with H a cyclic G subgroup and $\chi \colon H \to \mathbb{C}^*$ a character, we introduce the equivalence relation

$$(H, \chi) \sim (H', \chi') \text{ iff there exists } s \in G \text{ such that } H' = sHs^{-1} \text{ and } \chi' = \chi^s.$$

Consider a point $\tilde{p}$ on the admissible G-cover F, such that it has a non-trivial stabilizer $H_{\tilde{p}}$ with associated character $\chi_{\tilde{p}}$. We observe that for any point $s \cdot \tilde{p}$ of the same singular fiber, the pair $(H_{s\cdot\tilde{p}}, \chi_{s\cdot\tilde{p}})$ is equivalent to $(H_{\tilde{p}}, \chi_{\tilde{p}})$:

$$H_{s\cdot\tilde{p}} = sH_{\tilde{p}}s^{-1}; \quad \chi_{s\cdot\tilde{p}} = \chi^s_{\tilde{p}}.$$

Therefore the equivalence class of the pair only depends on the fiber F_p.

Definition 1.28. For any point $\tilde{p}$ on F, we call *local index* the associated pair $(H_{\tilde{p}}, \chi_{\tilde{p}})$. For any point $p \in C$ the conjugacy class of the local index of any $\tilde{p}$ in its fiber is the same. Following [5], we call the conjugacy class the *G-type* of the fiber F_p, and note it

$$[\![H_p, \chi_p]\!],$$

where H_p is the stabilizer of one of the points in F_p, and χ_p the associated character.

Remark 1.29. For any cyclic subgroup $H \subset G$, the image of a primitive character $\chi\colon H \to \mathbb{C}^*$ is the group of $|G|$th roots of the unit, $\boldsymbol{\mu}_{|H|}$. We choose a privileged root in this set, which is $\exp(2\pi i/|H|)$. After this choice, The datum of (H, χ), is equivalent to the datum of the H generator $h_p = \chi^{-1}(e^{2\pi i/|H|})$. As a consequence the conjugacy class $[\![H, \chi]\!]$ is naturally associated to the conjugacy class $[\![h_p]\!]$ of h in G. Given p point of C, we note the G-type of F_p by this last conjugacy class.

Consider a node $\tilde{q}$ on the admissible G-cover F, and denote by q its node image on curve C. Consider

$$V \cong \{xy = 0\} \subset \mathbb{A}^2_{x,y}.$$

By construction $V = \mathbb{A}^1_x \cup \mathbb{A}^1_y$ and the local picture of C at q is isomorphic to V, with the node at the origin. We note $F' := F|_{\mathbb{A}^1_x}$ and $F'' := F|_{\mathbb{A}^1_y}$. We note $\tilde{q}'$ and $\tilde{q}''$ the preimages of $\tilde{q}$ on F' and F'' respectively. By definition, $F' \to \mathbb{A}^1_x$ and $F'' \to \mathbb{A}^1_y$ are admissible G-covers of $\mathbb{A}^1$ with a marking at the origin. By the balancing condition, the local index $\tilde{h}'$ of $\tilde{q}'$ is the inverse of the local index of $\tilde{q}''$,

$$\tilde{h}' = (\tilde{h}'')^{-1}.$$

Definition 1.30. Given an admissible G-cover $F \to C$, if q is a node of C and $\tilde{q}$ one of its preimages on F, then the local index of $\tilde{q}$ and the G-type of q are well defined once we fix a privileged branch of q. Switching the branch sends the local index and the G-type in their inverses.

Definition 1.31. Given an admissible G-cover $F \to C$, the series

$$[\![h_1]\!],\ [\![h_2]\!], \dots,\ [\![h_n]\!]$$

of the G-types of the singular fibers over the marked points, is called *Hurwitz datum* of the cover. The stack of admissible G-covers of genus g with a given Hurwitz datum is noted $\mathrm{Adm}^G_{g,[\![h_1]\!],\dots,[\![h_n]\!]}$.

Remark 1.32. Given an admissible G-cover $F \to C$, F is a stable curve with a G-action on it which is free on the generic locus F_{gen} and transitive on every geometric fiber F_s. The genus g' of F is uniquely determined via the Hurwitz formula by the genus g of C, the degree $|G|$ of the covering and the cardinalities $|H_{p_i}|$ of the stabilizers of points on the singular fibers. We generalize Remark 1.15. If we fix the G-types $[\![h_1]\!], \dots, [\![h_n]\!]$ of the singular orbits, we can define the moduli stack $\mathsf{H}_{g',G,[\![h_1]\!],\dots,[\![h_n]\!]}$ of connected stable curves of genus g' with a faithful G-action and fixed G-types on the singular orbits. This stack is the object of [5] and a full substack of $\mathrm{Adm}^G_{g,[\![h_1]\!],\dots,[\![h_n]\!]}$.

In the case of a smooth non-marked curve C, an admissible G-cover $F \to C$ is simply a principal G-bundle as before. If we add one marked point p, we observe that $F|_{C\backslash\{p\}}$ is a principal G-bundle over $C\backslash\{p\}$. The datum of $F|_{C\backslash\{p\}}$ completely determines the admissible G-cover by taking the closure $F := \overline{F|_{C\backslash\{p\}}}$. Also, the G-action extends over the marked fiber F_p, and because of the ramification over p, a non-trivial stabilizer $H_{\tilde{p}}$ could eventually appear at any point $\tilde{p}$ of the fiber.

This suggests that if we have an admissible G-cover $F \to C$ over a smooth curve C, with singular orbits over the n marked points $p_1, \dots, p_n \in C$, in order to generalize Proposition 1.20 we have to consider the fundamental group of C_{gen}. This is indeed the result of [5, Lemma 2.6].

Proposition 1.33. *We consider a point p_* on the generic locus C_{gen} of a smooth n-marked curve $(C; p_1, \dots, p_n)$. The set of isomorphism classes of admissible G-covers of the curve is naturally in bijection with the set of conjugacy classes of maps $\pi_1(C_{\mathrm{gen}}, p_*) \to G$.*

Remark 1.34. If the smooth curve C of genus g has n-marked points $p_1, \dots, p_n$, the fundamental group of $C_{\mathrm{gen}} = C\backslash\{p_1, \dots, p_n\}$ has $2g + n$ generators

$$\alpha_1, \dots, \beta_{2g}, \gamma_1, \dots, \gamma_n,$$

respecting the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_{2g}\beta_{2g}\alpha_{2g}^{-1}\beta_{2g}^{-1} \cdot \gamma_1 \cdots \gamma_n = 1. \tag{1.5}$$

In particular for every generator γ_i, its image in G is in the G-type of marked point p_i (see [5, §2.3.1]).

In the case of a non-connected admissible G-cover $F \to C$ over a smooth n-marked curve $(C; p_1, \dots, p_n)$, we consider the stabilizer $H \subset G$ of any connected component $E \subset F$. Clearly the component $\psi(s, E)$, for some element s of G, has stabilizer $s \cdot H \cdot s^{-1}$. Therefore the conjugacy class of the stabilizer is independent on the choice of E. We note $\mathcal{T}(F)$ the set of connected components of F. As in the case of principal G-bundles, for every admissible G-cover there exists a canonical class $\mathcal{H}$ in $\mathcal{T}(G)$ such that the stabilizer of every E in $\mathcal{T}(F)$ is a subgroup H_E in $\mathcal{H}$. Moreover, we have a canonical surjective map

$$\mathcal{T}(F) \to \mathcal{H}.$$

Definition 1.35. Given the set $\mathcal{T}(G)$ of subgroup conjugacy classes in G, and a class $\mathcal{H}$ in it, an admissible $\mathcal{H}$-cover is an admissible G-cover such that every connected component has stabilizer in $\mathcal{H}$.

Definition 1.36. We note $\mathrm{Adm}_g^{G,\mathcal{H}}$ the stack of admissible $\mathcal{H}$-covers over stable curves of genus g, and we note $\mathrm{Adm}_{g,[\![h_1]\!],\dots,[\![h_n]\!]}^{G,\mathcal{H}}$ the stack of admissible $\mathcal{H}$-cover with Hurwitz datum $[\![h_1]\!], \dots, [\![h_n]\!]$ over the n marked points.

Remark 1.37. It is important to point out that the moduli stack $\mathrm{Adm}_{g,[\![h_1]\!],\dots,[\![h_n]\!]}^{G,\mathcal{H}}$ has a non-empty set of objects if and only if one element (and therefore all the elements) of any class $[\![h_i]\!]$ is in the commutator group of G.

Furthermore, we observe that it is possible to generalize point (2) of Proposition 1.16 too. We note $\mathrm{Aut}_{\mathrm{Adm}}(C, F)$ the set of automorphisms of an admissible G-cover $F \to C$.

Proposition 1.38. *Consider $(C; p_1, \dots, p_n)$ a nodal n-marked curve, and $F \to C$ an admissible $\mathcal{H}$-cover for some class $\mathcal{H}$ in $\mathcal{T}(G)$, then*

$$\mathrm{Aut}_{\mathrm{Adm}}(C, F) = \mathrm{Hom}^G(\mathcal{T}(F), G).$$

Proof. In the case of a smooth curve C, we consider the general locus $C_{\mathrm{gen}} = C \backslash \{p_1, \dots, p_n\}$. The restriction $F_{\mathrm{gen}} = F|_{C_{\mathrm{gen}}}$ is a principal G-bundle, therefore by Proposition 1.16,

$$\mathrm{Aut}_{\mathrm{Adm}}(C, F) \subset \mathrm{Aut}_{BG}(C_{\mathrm{gen}}, F_{\mathrm{gen}}) = \mathrm{Hom}^G(\mathcal{T}(F_{\mathrm{gen}}, C).$$

Since $\mathcal{T}(F_{\mathrm{gen}}) = \mathcal{T}(F)$ and every automorphism of $F_{\mathrm{gen}} \to C_{\mathrm{gen}}$ extends to the whole F, then the thesis follows in this case.

In the case of a general stable curve C, we call $C_1, \dots, C_V$ its connected components, and F_i the restriction of F to the components C_i for all i. As a consequence of the first part, we have

$$\mathrm{Aut}_{\mathrm{Adm}}(C_i, F_i) = \mathrm{Hom}^G(\mathcal{T}(F_i), C_i).$$

The balancing condition at the nodes imposes that any automorphism in $\mathrm{Aut}_{\mathrm{Adm}}(C, F)$ acts as the same multiplicative factor on two touching components. This means that a sequence of functions in $\prod_i \mathrm{Hom}^G(\mathcal{T}(F_i), G)$, induces a global automorphism if and only if it is the sequence of restrictions of a global function $\mathrm{Hom}^G(\mathcal{T}(F), G)$. □

Remark 1.39. As a consequence of Proposition 1.10, for every connected component $E \subset F$, there exists a canonical isomorphism

$$\mathrm{Aut}_{\mathrm{Adm}}(C, F) \cong Z_G(H_E).$$

Remark 1.40. We state also the case of a non connected nodal curve $\tilde{C}$ with an admissible G-cover $\tilde{F} \to \tilde{C}$. We note $\tilde{C}_1, \dots, \tilde{C}_m$ the connected components of $\tilde{C}$, and $\tilde{F}_i := \tilde{F}|_{\tilde{C}_i}$ the $\tilde{F}$ restrictions. In this case,

$$\mathrm{Aut}_{\mathrm{Adm}}(\tilde{C}, \tilde{F}) = \prod_i \mathrm{Aut}_{\mathrm{Adm}}(\tilde{C}_i, \tilde{F}_i) = \prod_i \mathrm{Hom}^G(\mathcal{T}(\tilde{F}_i), G).$$

1.3 Twisted G-covers

In our effort to enlarge the notion of principal G-bundles over stable curves, in the previous section we admitted ramified covers of stable curves. Here we admit non-trivial stabilizers at the nodes of a stable curve, by defining twisted curves. On this new setting, principal G-bundles generalize to twisted G-covers. The twisting techniques are widely discussed in [2] and [1], in particular twisted curves are introduced in [12] in the case of a $\boldsymbol{\mu}_\ell$-level structure on stable curves.

1.3.1 Definitions

Definition 1.41 (Twisted curve)**.** A twisted n-marked S-curve is a diagram

$$\begin{array}{ccc} \Sigma_1, \Sigma_2, \dots, \Sigma_n & \subset & \mathsf{C} \\ & & \downarrow \\ & & C \\ & & \downarrow \\ & & S. \end{array}$$

Where:

1. C is a Deligne-Mumford stack, proper over S, and étale locally it is a nodal curve over S;
2. the $\Sigma_i \subset \mathsf{C}$ are disjoint closed substacks in the smooth locus of $\mathsf{C} \to S$ for all i;
3. $\Sigma_i \to S$ is an étale gerbe for all i;
4. $\mathsf{C} \to C$ exhibits C as the coarse space of C, and it is an isomorphism over C_{gen}.

We recall that, given a scheme U and a finite abelian group $\boldsymbol{\mu}$ acting on U, the stack $[U/\boldsymbol{\mu}]$ is the category of principal $\boldsymbol{\mu}$-bundles $E \to T$, for any scheme T, equipped with a $\boldsymbol{\mu}$-equivariant morphism $f\colon E \to U$. The stack $[U/\boldsymbol{\mu}]$ is a proper Deligne-Mumford stack and has a natural morphism to its coarse scheme $U/\boldsymbol{\mu}$. By the definition of twisted curve we get the local picture.

At a marking, morphism $\mathsf{C} \to C \to S$ is locally isomorphic to

$$[\operatorname{Spec} A[x']/\boldsymbol{\mu}_r] \to \operatorname{Spec} A[x] \to \operatorname{Spec} A$$

for some normal ring A and some integer $r > 0$. Here $x = (x')^r$, and $\boldsymbol{\mu}_r$ is the cyclic group of order r acting on $\operatorname{Spec} A[x']$ by the action $\xi\colon x' \mapsto \xi x'$ for any $\xi \in \boldsymbol{\mu}_r$.

At a node, morphism $\mathsf{C} \to C \to S$ is locally isomorphic to

$$\left[\operatorname{Spec}\left(\frac{A[x',y']}{(x'y'-a)}\right)/\boldsymbol{\mu}_r\right] \to \operatorname{Spec}\left(\frac{A[x,y]}{(xy-a^\ell)}\right) \to \operatorname{Spec} A$$

for some integer $r > 0$ and $a \in A$. Here $x = (x')^\ell,\ y = (y')^\ell$. The group $\boldsymbol{\mu}_\ell$ acts by the action

$$\xi_r\colon (x', y') \mapsto (\xi_r x', \xi_r^m y')$$

where m is an element of $\mathbb{Z}/r$ and ξ_r is a primitive rth root of the unit. The action is called *balanced* if $m \equiv -1 \mod r$. A curve with balanced action at every node is called a balanced curve.

Definition 1.42 (Twisted G-cover)**.** Given an n-marked twisted curve

$$(\Sigma_1, \dots, \Sigma_n;\ \mathsf{C} \to C \to S),$$

a twisted G-cover is a representable stack morphism $\phi\colon \mathsf{C} \to BG$, *i.e.* an object of category

$$\mathrm{Fun}(\mathsf{C}, BG)$$

which moreover is representable.

Definition 1.43. We introduce category $\mathcal{B}_{g,n}(G)$. Objects of $\mathcal{B}_{g,n}(G)$ are twisted n-marked S-curves of genus g with a twisted G-cover, for any scheme S.

Consider two twisted G-covers $\phi'\colon \mathsf{C}' \to BG$ and $\phi\colon \mathsf{C} \to BG$ over the twisted n-marked curves C' and C respectively. A morphism between (C', ϕ') and (C, ϕ) is a pair (f, α) such that $\mathsf{f}\colon \mathsf{C}' \to \mathsf{C}$ is a morphism of n-marked twisted curves, and $\alpha\colon \phi' \to \phi \circ \mathsf{f}$ is an isomorphism between objects of $\mathrm{Fun}(\mathsf{C}', BG)$.

Remark 1.44. Given a twisted G-cover $\phi\colon \mathsf{C} \to BG$ and a representable morphism $\mathsf{f}\colon \mathsf{C}' \to \mathsf{C}$, we note $\mathsf{f}^*\phi$ the twisted G-cover $\phi \circ \mathsf{f}\colon \mathsf{C}' \to BG$.

The category $\mathcal{B}_{g,n}(G)$ is equivalently the 2-category of twisted stable n-pointed maps of genus g and degree 0 to the category BG, as introduced in [2]. In the same paper it is observed that the automorphism group of every 1-morphism is trivial, therefore this 2-category is equivalent to the category obtained by replacing 1-morphisms with their 2-isomorphism classes. In [2] this category is noted $\mathcal{K}_{g,n}(BG, 0)$, the notation $\mathcal{B}_{g,n}(G)$ for the case of twisted G-covers appears for example in [1].

Definition 1.45. A balanced twisted G-cover is a twisted G-cover over a twisted balanced curve. We call $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$ the sub-functor of twisted balanced G-covers.

Remark 1.46. Given any twisted balanced S_d-cover $F \to \mathsf{C}$, we consider the natural inclusion $S_{d-1} \subset S_d$. If C is the coarse space of C, and $F' := F/S_{d-1}$, then $F' \to C$ is an admissible cover of degree d, as proven in [1, Lemma 4.2.1]. This gives a morphism of stacks

$$\phi\colon \mathcal{B}^{\mathrm{bal}}_{g,n}(S_d) \to \mathrm{Adm}_{g,n,d},$$

which is a normalization morphism by [1, Proposition 4.2.2].

We recall (see for example [12, p.6]) that given a twisted curve C, a faithful line bundle on it is a line bundle $\mathsf{L} \to \mathsf{C}$ such that the associated morphism $\mathsf{C} \to B\mathbb{C}^*$ is representable. In the following proposition we observe that if the group G is the cyclic group $\boldsymbol{\mu}_\ell$ of order ℓ, the choice of a twisted $\boldsymbol{\mu}_\ell$-cover over a twisted curve C, is equivalent to the choice of an ℓth root of the trivial bundle $\mathcal{O}_{\mathsf{C}}$, *i.e.* a faithful line bundle $\mathsf{L} \to \mathsf{C}$ with the property that $\mathsf{L}^{\otimes \ell} \cong \mathcal{O}_{\mathsf{C}}$ (see [12, §1.2]). This explains part of our work as a generalization of *level* structures on curves, as developed for example in [12] and [11].

Proposition 1.47. *For any twisted curve C, there exists a natural bijection between the set of twisted $\boldsymbol{\mu}_\ell$-covers over C and the set of ℓth roots of $\mathcal{O}_{\mathsf{C}}$.*

Proof. We consider the categories $\boldsymbol{\mu}_\ell - \mathrm{Bun}_{\mathsf{C}}$ and $\mathrm{Root}^\ell_{\mathsf{C}}$ of twisted $\boldsymbol{\mu}_\ell$-covers over C and of ℓth roots of $\mathcal{O}_{\mathsf{C}}$. We will show two functors

$$A\colon \boldsymbol{\mu}_\ell - \mathrm{Bun}_{\mathsf{C}} \to \mathrm{Root}^\ell_{\mathsf{C}}, \qquad B\colon \mathrm{Root}^\ell_{\mathsf{C}} \to \boldsymbol{\mu}_\ell - \mathrm{Bun}_{\mathsf{C}}$$

such that $A \circ B$ and $B \circ A$ are the identity functors of respectively $\mathrm{Root}^{\ell}_{\mathsf{C}}$ and $\boldsymbol{\mu}_\ell - \mathrm{Bun}_{\mathsf{C}}$.

As $\boldsymbol{\mu}_\ell$ is naturally immersed in $\mathbb{G}_m$, there exists a natural inclusion

$$\iota\colon B\boldsymbol{\mu}_\ell \hookrightarrow B\mathbb{G}_m.$$

If we have a twisted $\boldsymbol{\mu}_\ell$-cover $\phi\colon \mathsf{C} \to B\boldsymbol{\mu}_\ell$, then $\iota\circ\phi$ is representable and the induced line bundle is an ℓth root of the trivial line bundle.

Conversely, if L is an ℓth root of $\mathcal{O}_{\mathsf{C}}$, then it corresponds to a representable morphism $\phi'\colon \mathsf{C} \to B\mathbb{G}_m$. If we consider the map $\exp_\ell\colon B\mathbb{G}_m \to B\mathbb{G}_m$ corresponding to the ℓth power, the fact of L being an ℓth root means that the image of ϕ' is in the kernel of $\exp_\ell$, that is $B\boldsymbol{\mu}_\ell$.

We observe that $B \circ A = \mathrm{id}_{\boldsymbol{\mu}_\ell - \mathrm{Bun}_{\mathsf{C}}}$ and $A \circ B = \mathrm{id}_{\mathrm{Root}^{\ell}_{\mathsf{C}}}$, which completes the proof. □

1.3.2 Local structure of a twisted covers

We consider a twisted curve C over a geometric point $S = \mathrm{Spec}(\mathbb{C})$. For any marked point p_i we consider an open neighborhood $U_i \cong \mathbb{A}^1$. As stated above, we know that for any $p_i \in \mathsf{C}$, the local picture of C at p_i is the same as $[U_i/\boldsymbol{\mu}_{r_i}]$ at the origin, with the group $\boldsymbol{\mu}_{r_i}$ naturally acting by multiplication. Any principal G-bundle over C, or equivalently any object of $BG(\mathsf{C})$, is locally isomorphic at p_i to a principal G-bundle on $[U_i/\boldsymbol{\mu}_{r_i}]$.

Remark 1.48. In [1, §2.1.8] is explained how to realize twisted stable maps as twisted objects over scheme theoretic curves. In particular, a principal G-bundle on $[U_i/\boldsymbol{\mu}_{r_i}]$ is a principal G-bundle $f\colon F \to U_i$ plus a $\boldsymbol{\mu}_{r_i}$-action compatible with the $\boldsymbol{\mu}_{r_i}$-action on U_i and G-equivariant. Formally, if $\psi\colon G \times F \to F$ is the G-action on F, then there exists a $\boldsymbol{\mu}_{r_i}$-action $\nu\colon \boldsymbol{\mu}_{r_i} \times F \to F$ such that:

1. $f \circ \nu \circ (\xi \times \mathrm{id}) = \xi \cdot f\colon F \to U_i$, for all $\xi \in \boldsymbol{\mu}_{r_i}$;

2. $\psi \circ (h \times (\nu \circ (\xi \times \mathrm{id}))) = \nu \circ (\xi \times (\psi(h \times \mathrm{id}))\colon F \to F$, for all $h \in G$ and $\xi \in \boldsymbol{\mu}_{r_i}$.

For any $\xi \in \boldsymbol{\mu}_{r_i}$ we define a morphism $\tilde{\alpha}(\xi)\colon F \to F$ such that

$$\tilde{\alpha}(\xi) := \nu \circ (\xi \times \mathrm{id}).$$

By what we saw in the remark above, if p is a marked point and $F \to U$ the local picture of the twisted G-cover at p,then

$$\tilde{\alpha}(\xi_r)(\tilde{p}) = \psi(h_{\tilde{p}}, \tilde{p}),$$

for all preimages $\tilde{p}$ of p, where $h_{\tilde{p}}$ is an element of group G depending on $\tilde{p}$.

Definition 1.49. The element $h_{\tilde{p}}$ of G is called *local index* of the geometric point $\tilde{p}$.

Remark 1.50. By the local description we gave above, the fiber of C at any marked point p is $\mathsf{C}_p = B\boldsymbol{\mu}_r$, for some positive integer r. The twisted G-cover $\phi\colon \mathsf{C} \to BG$ induces then a morphism

$$\phi_p\colon \mathsf{C}_p = B\boldsymbol{\mu}_r \to BG,$$

and a morphism

$$\tilde{\phi}_p\colon \operatorname{Aut}(\mathsf{C}_p) \to \operatorname{Aut}(BG).$$

The group $\operatorname{Aut}(BG)$ is canonically isomorphic to G up to conjugation. This implies that ϕ induces a morphism $\tilde{\phi}_p\colon \boldsymbol{\mu}_r \to G$, and the representability of ϕ means, stack theoretically, that $\tilde{\phi}_p$ is an injection

$$\tilde{\phi}_p\colon \boldsymbol{\mu}_r \hookrightarrow G \ \text{ defined up to conjugation.}$$

Considering the privileged primitive rth root $\xi_r = \exp(2\pi i/r)$, the datum of $\tilde{\phi}_p$ is equivalent to the the conjugacy class of $h_p := \tilde{\phi}_p\left(e^{2\pi i/r}\right)$ in G. We remark that $[\![h_p]\!]$ is also the conjugacy class of the local index of any preimage $\tilde{p}$ of p.

Definition 1.51. The conjugacy class $[\![h_p]\!]$ of $h_p \in G$ is called G-type of ϕ_p, or equivalently G-type of the point p with respect to the twisted G-cover $\phi\colon \mathsf{C} \to BG$.

Remark 1.52. In the next section we will see the equivalence between twisted G-covers and admissible G-covers, this notion of G-type for twisted G-cover is the exact translation of Definition 1.28 for admissible G-covers.

Remark 1.53. By the injectivity of morphism $\tilde{\phi}_p$, any element of the G-type $[\![h_p]\!]$ has the same order $r = r(p)$ of the p stabilizer.

We observe the local description of a twisted G-cover at a node of C. By definition, for any node $q \in \mathsf{C}$, the local picture at q is the same as $[V/\boldsymbol{\mu}_{r_q}]$ at the origin, where

$$V \cong \{x'y' = 0\} \subset \mathbb{A}^2_{x',y'},$$

and the $\boldsymbol{\mu}_{r_q}$-action is given by

$$\xi \cdot (x', y') = (\xi x', \xi^{-1} y')$$

for all $\xi \in \boldsymbol{\mu}_{r_q}$.

The normalization of the node neighborhood V is naturally isomorphic to

$$\mathbb{A}^1_{x'} \sqcup \mathbb{A}^1_{y'} \to V.$$

We consider the normalization $\mathsf{nor}\colon \overline{\mathsf{C}} \to \mathsf{C}$ of the twisted curve C, the local picture of nor morphism at q is

$$[\mathbb{A}^1_{x'}/\boldsymbol{\mu}_{r_q}] \sqcup [\mathbb{A}^1_{y'}/\boldsymbol{\mu}_{r_q}] \to [V/\boldsymbol{\mu}_{r_q}].$$

We note $q_1 \in \mathbb{A}^1_{x'}$ and $q_2 \in \mathbb{A}^1_{y'}$ the two preimages of the node q in V. In general, every node has two preimages on the normalized twisted curve $\overline{\mathsf{C}}$, these are marked points of $\overline{\mathsf{C}}$ and they have the same stabilizer as the node.

As we have seen above, given a twisted G-cover on C, its local picture at a node q is the same as a principal G-bundle over $[V/\boldsymbol{\mu}_{r_q}]$, *i.e.* a principal G-bundle $F \to V$ plus a $\boldsymbol{\mu}_{r_q}$-action compatible with the $\boldsymbol{\mu}_{r_q}$-action on V and G-equivariant. This induces

- two principal G-bundles

$$F' \to \mathbb{A}^1_{x'} \quad \text{and} \quad F'' \to \mathbb{A}^1_{y'}$$

with the naturally associated $\boldsymbol{\mu}_{r_q}$-actions. We note $\nu' \colon \boldsymbol{\mu}_{r_q} \times F' \to F'$ and $\nu'' \colon \boldsymbol{\mu}_{r_q} \times F'' \to F''$ the actions on F' and F'';

- a gluing isomorphism between the central fibers

$$\kappa_q \colon F'_{q_1} \xrightarrow{\cong} F''_{q_2},$$

which is equivariant with respect to the G-actions and the $\boldsymbol{\mu}_{r_q}$-actions, and such that $F = (F' \sqcup F'')/\kappa_q$.

Following Remark 1.48, we define

$$\tilde{\alpha}' \colon F' \to F' \ \text{ such that } \ \tilde{\alpha}'(\xi) := \nu' \circ (\xi \times \mathrm{id}),$$

$$\tilde{\alpha}'' \colon F'' \to F'' \ \text{ such that } \ \tilde{\alpha}''(\xi) := \nu'' \circ (\xi \times \mathrm{id}),$$

for $\xi \in \boldsymbol{\mu}_{r_q}$. By the balancing condition, if we have two points $\tilde{q}_1$ and $\tilde{q}_2$ in F'_{q_1} and F''_{q_2} respectively, such that $\kappa_q(\tilde{q}_1) = \tilde{q}_2$, then the local index at $\tilde{q}_1$ is the inverse of the local index at $\tilde{q}_2$,

$$h_{\tilde{q}_1} = h^{-1}_{\tilde{q}_2}.$$

Therefore, if we note $[\![h_1]\!]$ and $[\![h_2]\!]$ the G-types of q_1 and q_2 with respect to $F' \to \mathbb{A}^1_{x'}$ and $F'' \to \mathbb{A}^1_{y'}$, then

$$[\![h_1]\!] = [\![h_2^{-1}]\!].$$

Remark 1.54. Once we choose a privileged branch of a node, we call G-type of that node the G-type with respect to the restriction of the cover to that branch. For example in the case above, if we choose the branch $\mathbb{A}^1_{x'}$ of node q, the G-type of q is the G-type of q_1 with respect to $F' \to \mathbb{A}^1_{x'}$, that is $[\![h_1]\!]$. Switching the branch changes the G-type into the inverse class $[\![h_2]\!] = [\![h_1^{-1}]\!]$.

Remark 1.55. Given this description, we introduce another notation for a twisted G-cover $\phi \colon \mathsf{C} \to BG$. We consider the universal G-bundle $\mathfrak{U} \to BG$. The pullback

$$\mathsf{F} := \phi^*\mathfrak{U}$$

is a twisted curve with a natural projection $\mathsf{F} \to \mathsf{C}$. Over the generic locus C_{gen}, the restriction $\mathsf{F}_{\mathrm{gen}}$ is isomorphic to the principal scheme theoretic G-bundle F_{gen}, at the nodes the local picture is sketched above.

We consider a twisted curve C and its normalization morphism $\mathsf{nor} \colon \overline{\mathsf{C}} \to \mathsf{C}$. From our description is clear that if $\mathsf{F} \to \mathsf{C}$ is a twisted G-cover over C, then $\mathsf{nor}^*\mathsf{F}$ is a twisted G-cover over $\overline{\mathsf{C}}$, too. Conversely, knowing the twisted G-cover over $\overline{\mathsf{C}}$, and the gluing isomorphisms κ_q at each node of C, we have a twisted G-cover over C. This is resumed in the following proposition, which is a direct consequence of the construction above.

Proposition 1.56. *The datum of a balanced twisted G-cover $\mathsf{F} \to \mathsf{C}$, is equivalent to the data of a twisted G-cover on its normalization $\mathsf{F}' \to \overline{\mathsf{C}}$, and an isomorphism κ_q as above for every node q of C, with the isomorphisms κ_q defined up to automorphism of $\mathsf{F}' \to \overline{\mathsf{C}}$.*

Proof. We denote by $\mathsf{U}_1(\mathsf{C})$ the set of twisted G-covers on C, and by $\mathsf{U}_2(\overline{\mathsf{C}})$ the set of twisted G-covers on $\overline{\mathsf{C}}$ plus a series of automorphisms κ_q at every node, defined up to automorphism of the cover. The construction above gives an injective map $\mathsf{U}_1(\mathsf{C}) \to \mathsf{U}_2(\overline{\mathsf{C}})$, moreover the map is surjective, because for every cover $\mathsf{F}' \to \overline{\mathsf{C}}$ with a series of automorphisms κ_q, it is possible to obtain a cover $\mathsf{F} \to \mathsf{C}$ using the κ_q as gluing automorphisms. □

1.3.3 Local structure of $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$

The local structure of $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$ can be described with a very similar approach of what we did for $\overline{\mathsf{M}}_{g,n}$. We work the case $n = 0$ of unmarked twisted G-covers. Given a twisted G-cover (C, ϕ), its deformation functor is representable and the associated scheme $\mathrm{Def}(\mathsf{C}, \phi)$ is isomorphic to $\mathrm{Def}(\mathsf{C})$ via the forgetful functor $(\mathsf{C}, \phi) \mapsto \mathsf{C}$. The automorphism group $\mathrm{Aut}(\mathsf{C}, \phi)$ naturally acts on $\mathrm{Def}(\mathsf{C}, \phi) = \mathrm{Def}(\mathsf{C})$ and the local picture of $\mathcal{B}^{\mathrm{bal}}_{g}(G)$ at $[\mathsf{C}, \phi]$ is the same of

$$[\mathrm{Def}(\mathsf{C})/\,\mathrm{Aut}(\mathsf{C}, \phi)]$$

at the central point.

Remark 1.57. Consider a twisted curve C whose coarse space is the curve C, we give a description of the scheme $\mathrm{Def}(\mathsf{C})$ as we did in Remark 1.5 for $\mathrm{Def}(C)$. As before we note by $C_1, \dots, C_V$ the irreducible components of C, by $\overline{C}_i$ the normalization of curve C_i and by $\mathsf{nor}\colon \overline{C} \to C$ the normalization morphism. Furthermore, we note by $q_1, \dots, q_\delta$ the C nodes. As C is a twisted curve, every node q_j has an eventually non-trivial stabilizer, which is a cyclic group of order r_j.

The deformation $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C})$ of C alongside with its nodes, is canonically identified with the deformation of C alongside with its nodes.

$$\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C}) = \mathrm{Def}(C; \mathrm{Sing}\,C) = \bigoplus_{i=1}^{V} \mathrm{Def}(\overline{C}_i; D_i) \cong \bigoplus_{i=1}^{V} H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i)). \quad (1.6)$$

As in the previous case, the following quotient has a canonical splitting.

$$\mathrm{Def}(\mathsf{C})/\,\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C}) = \bigoplus_{j=1}^{\delta} R_j. \quad (1.7)$$

In this case $R_j \cong \mathbb{A}^1$ is the deformation scheme of the node q_j together with its stack structure. If we consider the schemes M_j of Equation (1.2) in Remark 1.5, there exists for every j a canonical morphism $R_j \to M_j$ of order r_j ramified in exactly one point.

1.3.4 Equivalence between twisted and admissible covers

We introduced the two categories $\mathcal{B}_g^{\mathrm{bal}}(G)$ and Adm_g^G with the purpose of "well" defining the notion of principal G-bundle over stable non-smooth curves. These two categories are proven isomorphic in [1].

Theorem 1.58 (see [1, Theorem 4.3.2])**.** *There exists a base preserving equivalence between $\mathcal{B}_g^{\mathrm{bal}}(G)$ and Adm_g^G, therefore in particular they are isomorphic Deligne-Mumford stacks.*

The proof of this proposed in [1] can be sketched quickly. Given a twisted G-cover

$$\phi\colon \mathsf{C} \to BG,$$

it is a principal G-bundle $F_{\mathrm{gen}} \to C_{\mathrm{gen}}$ on the generic locus of the coarse space C, and this can be uniquely completed to an admissible G-cover $F \to C$. Conversely, given an admissible G-cover $F \to C$, it induces a quotient stack $\mathsf{C} := [F/G]$ and therefore a representable morphism $\mathsf{C} \to BG$ with balanced action on nodes.

In what follows we will adopt the notation $\overline{\mathsf{R}}_{g,G}$ for the stack $\mathcal{B}_{g,\mathrm{bal}}(G)$ whose objects are twisted G-covers, or equivalently the stack of Adm_g^G whose objects are admissible G-covers. For every class $\mathcal{H}$ in $\mathcal{T}(G)$ we note $\overline{\mathsf{R}}_{g,G}^{\mathcal{H}}$ the full substack of $\overline{\mathsf{R}}_{g,G}$ whose objects are admissible $\mathcal{H}$-covers.

The correspondence of Theorem 1.58 allows the translation of every machinery we developed on twisted G-covers to admissible G-covers, and conversely. For example, the two definitions of G-type we introduced are equivalent, and we can use over twisted G-covers the notion of Hurwitz datum. We will note

$$\overline{\mathsf{R}}^{\mathcal{H}}_{g,[\![h_1]\!],\dots,[\![h_n]\!]}$$

the stack of admissible $\mathcal{H}$-covers of genus g with Hurwitz datum $[\![h_1]\!],\dots,[\![h_n]\!]$.

Furthermore, this correspondence gives a general statement about the automorphism of any twisted G-cover. Suppose C is a non-necessarily connected twisted curve, with coarse space C, and $\phi\colon \mathsf{C} \to BG$ a twisted G-cover on C. We note $\mathrm{Aut}(\mathsf{C},\phi)$ the automorphism group of this twisted G-cover, and we note $F \to C$ the associated admissible G-cover. Therefore by the correspondence we have

$$\mathrm{Aut}(\mathsf{C},\phi) = \mathrm{Aut}(C,F),$$

and we can use all the previous results.

Chapter 2

Spin bundles

In this chapter we consider the case where G is a cyclic group of order ℓ. We already observed that taking a twisted $\boldsymbol{\mu}_\ell$-cover is equivalent to taking a line bundle L and a trivialization $L^{\otimes \ell} \to \mathcal{O}$ over the same twisted curve. We will note $\overline{\mathsf{R}}^0_{g,\ell} = \overline{\mathsf{R}}_{g,\boldsymbol{\mu}_\ell}$ the moduli space of twisted curves equipped with an ℓth root of the trivial bundle. We generalize this by considering line bundles which are ℓth roots of some power of the canonical bundle. In the first section we introduce some notions of graph theory that are necessary to treat the dual graphs of stable curves. In the second section we analyze stable curves with an ℓth root of the canonical bundle, and describe their automorphism group.

2.1 Basic graph theory

Consider a connected graph Γ with vertex set V and edge set E, we call *loop* an edge that starts and ends on the same vertex, we call *separating* an edge e such that the graph with vertex set V and edge set $E\backslash\{e\}$ is disconnected.

We note by E_{sep} the set of separating edges. We note by $\mathbb{E}$ the set of oriented edges: the elements of this set are edges in E equipped with an orientation, in particular for every edge $e \in \mathbb{E}$ we note e_+ the head vertex and e_- the tail, and there is a 2-to-1 projection $\mathbb{E} \to E$. We also introduce a conjugation in $\mathbb{E}$, such that for each $e \in \mathbb{E}$, the conjugated edge $\bar{e}$ is obtained by reversing the orientation, in particular

$$(\bar{e})_+ = e_-.$$

For every graph Γ, when there is no risk of confusion we note by V the cardinality of the vertex set $V(\Gamma)$ and by E the cardinality of the edge set $E(\Gamma)$.

2.1.1 Cochains on graphs

We introduce some simple arithmetics over graphs. We consider a finite group G acting on graph Γ. That is there exist two G actions on the vertex set and on the edge set,

$$G \times V(\Gamma) \to V(\Gamma) \quad \text{and} \quad G \times \mathbb{E}(\Gamma) \to \mathbb{E}(\Gamma).$$

We note these actions by $h \cdot v$ and $h \cdot e$ for every h in G and every vertex v and oriented edge e. These actions must respect the following natural intersection conditions

1. $(h \cdot e)_+ = h \cdot e_+ \ \ \forall h \in G, e \in \mathbb{E}(\Gamma)$;
2. $\overline{h \cdot e} = h \cdot \bar{e} \ \ \forall h \in G, e \in \mathbb{E}(\Gamma)$.

Observe that there are no faithfulness conditions, therefore any vertex or edge may have a non trivial stabilizer. We note H_v and H_e the stabilizers of vertex v and edge e respectively. We remark that

$$H_{s \cdot v} = s \cdot H_v \cdot s^{-1} \ \ \forall v \in V(\Gamma), s \in G,$$

and the same is true for H_e. In general, every orbit of vertices (or oriented edges) is characterized by a conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$, and every element of $\mathcal{H}$ is the stabilizer of some object in the orbit.

We introduce the cochains groups.

Definition 2.1 (Cochains)**.** The group of 0-cochains is the group of G-valued functions on $V(\Gamma)$ compatible with the G-action

$$C^0(\Gamma; G) := \left\{ a\colon V(\Gamma) \to G | \ a(g \cdot v) = g \cdot a(v) \cdot g^{-1} \right\}.$$

The group of 1-cochains is the group of antisymmetric functions on $\mathbb{E}$ with the same compatibility condition

$$C^1(\Gamma; G) := \left\{ b\colon \mathbb{E} \to G | \ b(\bar{e}) = b(e)^{-1}, \ b(g \cdot e) = g \cdot b(e) \cdot g^{-1} \right\}.$$

Clearly, these groups generalize the cochains groups defined by Chiodo and Farkas in [12]. In particular the Chiodo-Farkas groups refer to the case of a trivial G-action on Γ. In this chapter we work in this Chiodo-Farkas case, with $G = \boldsymbol{\mu}_\ell$ a cyclic group. Therefore, $C^0(\Gamma; \boldsymbol{\mu}_\ell)$ is the group of $\boldsymbol{\mu}_\ell$-valued functions on the vertices, and $C^1(\Gamma; \boldsymbol{\mu}_\ell)$ is the group of the $\boldsymbol{\mu}_\ell$-valued functions on the oriented edges of Γ, such that $b(e) = b(\bar{e})^{-1}$.

We observe that via the (non-canonical) choice of the rth root $\xi_r = \exp(2\pi i/r)$, we have the identification

$$C^i(\Gamma; \boldsymbol{\mu}_\ell) = C^i(\Gamma; \mathbb{Z}/\ell) \ \text{ for } i = 0, 1.$$

We use this notation in the following of this chapter. When there is no risk of confusion, we eventually note the groups by $C^0(\Gamma)$ and $C^1(\Gamma)$.

There exists a natural differential

$$\delta\colon C^0(\Gamma; \mathbb{Z}/\ell) \to C^1(\Gamma; \mathbb{Z}/\ell)$$

such that

$$\delta a(e) := a(e_+) - a(e_-), \ \ \forall a \in C^0(\Gamma; \mathbb{Z}/\ell) \ \forall e \in \mathbb{E}.$$

The exterior differential fits into an useful exact sequence of abelian groups.

$$0 \to \mathbb{Z}/\ell \xrightarrow{i} C^0(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\delta} C^1(\Gamma; \mathbb{Z}/\ell). \tag{2.1}$$

Where the injection i sends $h \in \mathbb{Z}/\ell$ on the cochain constatly equal to h.

Given a graph Γ, we have the identification

$$C^0(\Gamma; G) = \bigoplus_{v \in V(\Gamma)} \mathbb{Z}/\ell.$$

Moreover, if we choose a privileged orientation for every edge e in $E(\Gamma)$, then we have a canonical isomorphism

$$C^1(\Gamma; \mathbb{Z}/\ell) \cong \bigoplus_{e \in E(\Gamma)} \mathbb{Z}/\ell.$$

We often refer to a subset $W \subset V$ by referring to its characteristic function $\mathbb{I}_W$, *i.e.* the cochain $V \to \mathbb{Z}$ such that $\mathbb{I}_W(v) = 1$ for all $v \in W$ and $\mathbb{I}_W(v) = 0$ if $v \notin W$.

Definition 2.2. A *path* in a graph Γ is a sequence $e_1, e_2, \ldots, e_k$ of edges in $\mathbb{E}$ overlying k distinct non-oriented edges in E, and such that the head of e_i is the tail of e_{i+1} for all $i = 1, \ldots, k$.

A *circuit* is a closed path, *i.e.* a path $\mathcal{P} = (e_1, \ldots, e_k)$ such that the head of e_k is the tail of e_1. We often refer to a circuit by referring to its characteristic function $\mathbb{I}_{\mathcal{P}}$, *i.e.* the cochain $\mathbb{E} \to \mathbb{Z}$ such that $\mathbb{I}_{\mathcal{P}}(e_i) = 1$ for every i, $\mathbb{I}_{\mathcal{P}}(\bar{e}_i) = -1$ and $\mathbb{I}_{\mathcal{P}}(e) = 0$ if e is not on the circuit.

Proposition 2.3. *A* 1*-cochain b is in* $\operatorname{Im} \delta$ *if and only if, for every circuit $\mathcal{K} = (e_1, \ldots, e_k)$ in $\mathbb{E}$, we have*

$$b(\mathcal{K}) := b(e_1) + b(e_2) + \cdots + b(e_k) \equiv 0 \mod \ell.$$

Proof. If $b \in \operatorname{Im} \delta$, the condition above is easily verified. To complete the proof we will show that if the condition si verified, then there exists a cochain $a \in C^0(\Gamma; Z/\ell)$ such that $\delta a = b$. We choose a vertex $v \in V(\Gamma)$ and pose $a(v) \equiv 0 \mod \ell$, for any other vertex $w \in V(\Gamma)$ we consider a path $\mathcal{P} = (e_1, \ldots, e_m)$ starting in v and ending in w. We pose

$$a(w) := b(\mathcal{P}) = b(e_1) + \cdots + b(e_m).$$

By the condition on circuits, the cochain a is well defined, and by construction we have $b = \delta a$. □

Definition 2.4. A *cut* is a 1-cochain $b \colon \mathbb{E} \to A$ such that there exists a non-empty subset $W \subset V$ and the values of b are the following:

- $b(e) = h_0$ for some $h_0 \in \mathbb{Z}/\ell$, if the head of e is in W and the tail in $V \backslash W$;
- $b(e) = -h_0$ if the head is in $V \backslash W$ and the tail in W;
- $b(e) = 0$ elsewhere.

Remark 2.5. Any cut b is an element of $\operatorname{Im} \delta$. Indeed, if b is the cut associated to subset $W \subset V$ and of value h_0 in $\mathbb{Z}/\ell$, then $b = \delta\,(h_0 \cdot \mathbb{I}_W)$.

Lemma 2.6 (see [6, Lemma 5.1])**.** *If T is a spanning tree for Γ, for every oriented edge $e \in \mathbb{E}_T$ there is a unique cut $b \in C^1(\Gamma; \mathbb{Z}/\ell)$ such that $b(e) = 1$ and the elements of the support of b other than e and $\bar{e}$ are all in $\mathbb{E} \backslash \mathbb{E}_T$.*

Definition 2.7. We call $\mathrm{cut}_\Gamma(e; T)$ the unique cut b resulting from the lemma for all $e \in \mathbb{E}_T$.

Proposition 2.8. *If T is a spanning tree for Γ, then the elements $\mathrm{cut}_\Gamma(e; T)$, with edge e varying on E_T, form a basis of $\mathrm{Im}(\delta)$.*

For a proof of this and a deeper analysis of the image of δ, see [6, Chap.5].

2.1.2 Trees, tree-like graphs and graph contraction

Definition 2.9. A tree is a graph that does not contain any circuit. A *tree-like* graph is a connected graph whose only circuits are loops.

Remark 2.10. For every connected graph Γ, the first Betti number

$$b_1(\Gamma) = E - V + 1$$

is the dimension rank of the homology group $H_1(\Gamma; \mathbb{Z})$. Note that, b_1 being positive,

$$E \geq V - 1.$$

The inequality above is an equality if and only if Γ is a tree.

For every connected graph Γ with vertex set V and edge set E, we can choose a connected subgraph T with the same vertex set and edge set $E_T \subset E$ such that T is a tree.

Definition 2.11. The graph T is a *spanning tree* of Γ.

We call $\mathbb{E}_T$ the set of oriented edges of the spanning tree T. Here we notice that E contains a distinguished subset of edges E_{sep} whose size is *smaller* than $V - 1$.

Lemma 2.12. *If $E_{\mathrm{sep}} \subset E$ is the set of edges in Γ that are separating, then*

$$E_{\mathrm{sep}} \leq V - 1$$

with equality if and only if Γ is tree-like.

Proof. If T is a spanning tree for Γ and E_T its edge set, then $E_{\mathrm{sep}} \subset E_T$. Indeed, an edge $e \in E_{\mathrm{sep}}$ is the only path between its two extremities, therefore, since T is connected, e must be in E_T. Thus $E_{\mathrm{sep}} \leq E_T = V - 1$, with equality if and only if all the edges of Γ are loops or separating edges, *i.e.* if Γ is a tree-like graph. □

Another tool in graph theory is edge contraction, which corresponds to the intuitive operation of collapsing vertices linked to certain edges.

Definition 2.13. Given a graph Γ with vertex set V and edge set E, we choose a subset $D \subset E$. Contracting edges in D means taking the graph Γ_0 such that:

1. the edge set of Γ_0 is $E_0 := E \backslash D$;

2. given the relation in V, $v \sim w$ if v and w are linked by an edge $e \in D$, the vertex set of Γ_0 is $V_0 := V/\sim$.

We have a natural morphism $\Gamma \to \Gamma_0$ called *contraction* of D. Edge contraction will be useful, in particular we will consider the image of the exterior differential δ and its restriction over contractions of a given graph. If Γ_0 is a contraction of Γ, then $E(\Gamma_0)$ is canonically a subset of $E(\Gamma)$. As a consequence, cochains over Γ_0 are cochains over Γ with the additional condition that the values on $E(\Gamma)\backslash E(\Gamma_0)$ are all 0. Then we have a natural immersion

$$C^i(\Gamma_0; \mathbb{Z}/\ell) \hookrightarrow C^i(\Gamma; \mathbb{Z}/\ell).$$

Consider the two exterior differentials

$$\delta\colon C^0(\Gamma; \mathbb{Z}/\ell) \to C^1(\Gamma; \mathbb{Z}/\ell) \quad \text{and} \quad \delta_0\colon C^0(\Gamma_0; \mathbb{Z}/\ell) \to C^1(\Gamma_0; \mathbb{Z}/\ell).$$

Clearly δ_0 is the restriction of δ on $C^0(\Gamma_0; G)$. From this observation we have the following.

Proposition 2.14.

$$\operatorname{Im}\delta_0 = C^1(\Gamma_0; \mathbb{Z}/\ell) \cap \operatorname{Im}\delta.$$

Remark 2.15. We observe that given a graph contraction $\Gamma \to \Gamma_0$, the separating edges who are not contracted remain separating. Moreover, an edge who is not separating cannot become separating.

2.2 Moduli of rooted curves

Given a non-negative integer k and a positive integer ℓ, we consider a triple $(\mathsf{C}, \mathsf{L}, \theta)$, that is the data of a twisted curve C, plus a faithful line bundle L on it and an isomorphism

$$\theta\colon \mathsf{L}^{\otimes\ell} \to \omega_{\mathsf{C}}^{\otimes k},$$

where ω_{C} is the canonical bundle on C. If we note C the coarse space of C, we observe that ω_{C} is the pullback of the canonical bundle ω_C of C via the coarsening. This allows to call L an *ℓth root* of $\omega_C^{\otimes k}$, and the triple $(\mathsf{C}, \mathsf{L}, \theta)$ a *rooted curve.*

Definition 2.16. We define the category $\overline{\mathsf{R}}^k_{g,\ell}$ whose objects are rooted curves

$$(\mathsf{C} \to S, \mathsf{L}, \theta)$$

such that $\mathsf{C} \to S$ is an S-twisted curve of genus g whose coarse space is C and such that L is an ℓth root of $\omega_C^{\otimes k}$. A morphism between two rooted curves $(\mathsf{C}', \mathsf{L}', \theta')$ and $(\mathsf{C}, \mathsf{L}, \theta)$ is a pair $(\mathsf{f}, \tilde{\mathsf{f}})$ such that $\mathsf{f}\colon \mathsf{C}' \to \mathsf{C}$ is a morphism of twisted curves and the diagram

$$\begin{tikzcd} \mathsf{L}' \arrow[r, "\tilde{\mathsf{f}}"] \arrow[d] & \mathsf{L} \arrow[d] \\ \mathsf{C}' \arrow[r, "\mathsf{f}"] & \mathsf{C}, \end{tikzcd}$$

is a cartesian square such that $\theta \circ \tilde{\mathsf{f}}^{\otimes \ell} = \theta'$.

Remark 2.17. The stack $\overline{\mathsf{R}}^k_{g,\ell}$ is non-empty if and only if ℓ divides $\deg \omega^{\otimes k} = k \cdot (2g-2)$. If this is the case, for every curve C there exists exactly ℓ^{2g} ℓth rooted curves equipped with an ℓth root of $\omega_C^{\otimes k}$.

2.2.1 Local structure of rooted curves

If a curve is nodal, it has several irreducible components. The information about the relative crossing of this components is encoded in the dual graph.

Definition 2.18 (Dual graph)**.** Given a nodal curve C, the dual graph $\Gamma(C)$ has the set of irreducible components of C as vertex set, and the set of nodes of C as edge set E. The edge associated to the node q links the vertices associated to the components where the branches of q lie. If there is a G-action on the curve, with G finite group, this naturally induces a G-action on the dual graph.

Remark 2.19. Given a twisted curve C whose coarse space is C, the dual graph $\Gamma(\mathsf{C})$ is the dual graph $\Gamma(C)$ of curve C.

Remark 2.20. In dual graph setting, where the vertices are the components of a curve C and the edges are the nodes, the oriented edge set $\mathbb{E}$ could be seen as the set of branches at the curve nodes. Indeed, every edge e, equipped with an orientation, is bijectively associated to the branch it is pointing at.

The category $\overline{\mathsf{R}}^k_{g,\ell}$ is a Deligne-Mumford stack generalizing the stack of level curves or spin curves as treated for example in [19], [26] or [12]. Moreover, this generalizes also the stack of twisted $\boldsymbol{\mu}_\ell$-covers. Indeed in the case $k=0$ we have proved in Proposition 1.47 that $\overline{\mathsf{R}}_{g,\boldsymbol{\mu}_\ell} = \overline{\mathsf{R}}^0_{g,\ell}$.

Given a node q of the twisted curve C, the local picture of C at q is $[V/\boldsymbol{\mu}_r]$ where $V = \{x'y' = 0\} \subset \mathbb{A}^2$ and r is a positive integer dividing ℓ. The local picture of a line bundle $\mathsf{L} \to \mathsf{C}$ at q is the following line bundle on V,

$$\begin{array}{ccc} V \times \mathbb{A}^1_s & \hookrightarrow & \mathbb{A}^2_{x',y'} \times \mathbb{A}^1_s \\ \downarrow & & \\ V, & & \end{array}$$

with a $\boldsymbol{\mu}_r$-action compatible with the stack action on V, that is

$$\xi \cdot (x', y', s) = (\xi x', \xi^{-1} y', \xi^m s),$$

for all ξ in $\boldsymbol{\mu}_r$ and all points (x', y', s) of $V \times \mathbb{A}^1_s$. The index m is an element of $\mathbb{Z}/r$, it is called *local multiplicity* of the node q and, by the faithfulness of L, it has order exactly r. We observe that ξ^m is the local index at the node by Definition 1.28, and as $\boldsymbol{\mu}_\ell$ is abelian, this coincides with the $\boldsymbol{\mu}_\ell$-type of the node. The local multiplicity depends on the choice of a privileged branch of node q. In particular in the case above m is the local index with respect to the $\mathbb{A}^1_{x'}$ branch.

Remark 2.21. Following [12, §1.3.4], it is possible to define a multiplicity index

$$M \in C^1(\Gamma(\mathsf{C}); \mathbb{Z}/\ell),$$

a cochain encoding this datum for every node of C. If e is the edge associated to node q and with orientation pointing at the branch of $\mathbb{A}^1_{y'}$, then $M(e) := m$.

2.2.2 Automorphisms of rooted curves

We denote by $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta)$ the group of automorphism of any rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$. We denote by $[\mathsf{C}, \mathsf{L}, \theta]$ the point of $\overline{\mathbf{R}}^k_{g,\ell}$ associated to the rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$, then the local picture of the stack $\overline{\mathbf{R}}^k_{g,\ell}$ at $[\mathsf{C}, \mathsf{L}, \theta]$ is

$$[\mathrm{Def}(\mathsf{C}, \mathsf{L}, \theta)/\,\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta)]\,,$$

where the universal deformation $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \theta)$ is a smooth scheme of dimension $3g-3$, and

$$\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta) = \Big\{(\mathsf{s}, \rho) |\ \mathsf{s} \in \mathrm{Aut}(\mathsf{C}) \text{ and } \rho\colon \mathsf{s}^*\mathsf{L} \xrightarrow{\cong} \mathsf{L} \text{ such that } \phi \circ \rho^{\otimes \ell} = \mathsf{s}^*\phi\Big\}$$

is the automorphism group of $(\mathsf{C}, \mathsf{L}, \theta)$. This implies that the local picture of the moduli space $\overline{\mathcal{R}}^k_{g,\ell}$ is the classical quotient $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \theta)/\,\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta)$. The deformation space $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \theta)$ is canonically isomorphic to $\mathrm{Def}(\mathsf{C})$ via the étale forgetful functor $(\mathsf{C}, \mathsf{L}, \theta) \mapsto \mathsf{C}$. Also we see that the action of $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta)$ on $\mathrm{Def}(\mathsf{C})$ is not faithful. In particular the quasi-trivial automorphisms $(\mathrm{id}_{\mathsf{C}}, \zeta)$ with $\zeta \in \boldsymbol{\mu}_\ell$, whose action scale the fibers, have trivial action. Thus it becomes natural to consider the group

$$\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta) := \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta)/\{(\mathrm{id}_{\mathsf{C}}; \zeta) |\ \zeta \in \boldsymbol{\mu}_\ell\} = \{\mathsf{s} \in \mathrm{Aut}(\mathsf{C}) |\ \mathsf{s}^*\mathsf{L} \cong \mathsf{L}\}\,.$$

Remark 2.22. After what we said, the local picture of $\overline{\mathcal{R}}^k_{g,\ell}$ at $(\mathsf{C}, \mathsf{L}, \theta)$ can be rewritten as

$$\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta).$$

The coarsening $\mathsf{C} \to C$ induces a group homomorphism

$$\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta) \to \mathrm{Aut}(C).$$

We note the kernel and the image of this morphism by $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ and $\mathrm{Aut}'(C)$ (see also [12, chap. 2]). They fit into the following short exact sequence,

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta) \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta) \to \mathrm{Aut}'(C) \to 1. \tag{2.2}$$

Definition 2.23. The group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ is the group of *ghost* automorphisms.

Given a rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$, we come back to the local multiplicity on a node q whose local picture is $[\{xy = 0\}/\boldsymbol{\mu}_r]$ with r positive integer. As we have already seen, once we choose a privileged branch, the action on the bundle fiber near the node is $\xi_r(t, x, y) = (\xi_r^m t, \xi_r x, \xi_r^{-1} y)$. We observe that the canonical line bundle ω_{C} is the pullback of the canonical line bundle over the coarse space C, and this, together with the isomorphism $\mathsf{L}^{\otimes \ell} \cong \omega_{\mathsf{C}}$, implies that $(\xi_r^m)^\ell = 1$. So ℓm is a multiple of r. As a consequence of the faithfulness of L, the order r equals 1 or ℓ.

Given a stable curve C we consider its irreducible components $C_1, \dots, C_V$, as seen before there exists a normalization $\overline{C} = \sqcup \overline{C}_i$ where $\overline{C}_i$ is the normalization of curve C_i. For any i, we call g_i the genus of $\overline{C}_i$, n_i the number of node preimages on $\overline{C}_i$ via the normalization morphism, and v_i the associated vertex in the dual graph $\Gamma = \Gamma(C)$.

Proposition 2.24. *Consider a stable curve C with dual graph Γ and consider a cochain M in $C^1(\Gamma; \mathbb{Z}/\ell)$. Also consider the differential*

$$\partial \colon C^1(\Gamma; \mathbb{Z}/\ell) \to C^0(\Gamma; \mathbb{Z}/\ell).$$

There exists an ℓth root of the canonical bundle $\omega_C^{\otimes k}$ with multiplicity index M, if and only if

$$\partial M(v_i) \equiv k \cdot (2g_i - 2 + n_i) \quad \forall v_i \in V(\Gamma).$$

Proof. If $(\mathsf{C}, \mathsf{L}, \theta)$ is any rooted curve, then we obtain the result simply by verifying a degree condition on every irreducible component of C. We have

$$\deg\left(\mathsf{L}|_{v_i}\right)^{\otimes \ell} \equiv \sum_{e_+ = v_i} M(e) \mod \ell.$$

We call $\tilde{q}_1, \dots \tilde{q}_{n_i}$ the preimages of the C nodes on the normalized curve $\overline{C}_i$. Then,

$$\omega|_{v_i} = \omega_{\overline{C}_i}\left(\tilde{q}_1 + \dots \tilde{q}_{n_i}\right).$$

Knowing that $\left(\mathsf{L}|_{v_i}\right)^{\otimes \ell} \cong \left(\omega|_{v_i}\right)^{\otimes k}$ we obtain, as we wanted,

$$\deg\left(\omega|_{v_i}\right)^{\otimes k} \equiv \sum_{e_+ = v_i} M(e) \mod \ell.$$

To prove the other implication we will show that the multidegree condition implies $\deg \omega_C^{\otimes k} \equiv 0 \mod \ell$, and conclude by Remark 2.17. Indeed,

$$\deg \omega_C^{\otimes k} = \sum_{v_i \in V(\Gamma)} \deg F|_{v_i}.$$

Therefore, using the condition above we obtain

$$\deg \omega_C^{\otimes k} \equiv \sum_{v_i \in V(\Gamma)} \sum_{e_+ = v_i} M(e) \equiv \sum_{e \in \mathbb{E}(\Gamma)} M(e) \equiv 0 \mod \ell.$$

□

Consider a rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$ such that the coarse space of C is C. Starting from the dual graph $\Gamma(\mathsf{C})$ and the multiplicity index M of $(\mathsf{C}, \mathsf{L}, \theta)$, consider the new contracted graph $\Gamma_0(\mathsf{C})$ defined by

1. the vertex set $V_0 = V(C)/\sim$, defined by modding out the relation
$$(e_+ \sim e_- \text{ if } M(e) \equiv 0);$$
2. the edge set $E_0 = \{e \in E(C) | \; M(e) \not\equiv 0\}$.

Remark 2.25. The graph Γ_0 is obtained by contracting the edges of Γ where the function M vanishes.

Definition 2.26. The pair $(\Gamma_0(\mathsf{C}), M)$, where M is the restriction of the multiplicity index on the contracted edge set, is called *decorated graph* of the curve $(\mathsf{C}, \mathsf{L}, \theta)$. If the cochain M is clear from context, we will refer also to $\Gamma_0(\mathsf{C})$ or Γ_0 alone as the decorated graph.

To study $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ we start from a bigger group, the group $\mathrm{Aut}_C(\mathsf{C})$ containing automorphisms of C fixing the coarse space C. Consider a node q of C whose local picture is $[\{xy = 0\}/\boldsymbol{\mu}_r]$. Consider an automorphism $\eta \in \mathrm{Aut}_C(\mathsf{C})$. The local action of η at q is $(x, y) \mapsto (\xi x, y) = (x, \xi y)$, with $\xi \in \boldsymbol{\mu}_r$. As a consequence of the definition of $\mathrm{Aut}_C(\mathsf{C})$, the action of η outside the C nodes is trivial. Then the whole group $\mathrm{Aut}_C(\mathsf{C})$ is generated by automorphisms of the form $(x, y) \mapsto (\xi x, y)$ on a node and trivial elsewhere.

We are interested in representing $\mathrm{Aut}_C(\mathsf{C})$ as acting on the edges of the dual graph, thus we introduce the group of functions $\mathbb{E} \to \mathbb{Z}/\ell$ that are even with respect to conjugation
$$S(\Gamma; \mathbb{Z}/\ell) := \{b\colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\bar{e}) \equiv b(e) \mod \ell\}.$$
We have a canonical identification sending the function $b \in S(\Gamma_0(\mathsf{C}); \mathbb{Z}/\ell)$ to the automorphism η with local action $(x, y) \mapsto (\xi_\ell^{b(e)} x, y)$ on the node associated to the edge e if $M(e) \not\equiv 0$. Therefore the decorated graph encodes the automorphisms acting trivially on the coarse space. We can write
$$\mathrm{Aut}_C(\mathsf{C}) \cong S(\Gamma_0(\mathsf{C}); \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{E_0(\Gamma)}. \tag{2.3}$$
We already saw that elements of $C^1(\Gamma; \mathbb{Z}/\ell)$ are odd functions $\mathbb{E} \to \mathbb{Z}/\ell$. Given $g \in S(\Gamma; \mathbb{Z}/\ell)$ and $N \in C^1(\Gamma; \mathbb{Z}/\ell)$, their natural product $N \cdot g$ is still an odd function, thus an element of $C^1(\Gamma; \mathbb{Z}/\ell)$.

Given the normalization morphism $\mathsf{nor}\colon \overline{C} \to C$, consider the short exact sequence of sheaves over C
$$1 \to \mathbb{Z}/\ell \to \mathsf{nor}_*\mathsf{nor}^*\mathbb{Z}/\ell \xrightarrow{t} \mathbb{Z}/\ell|_{\mathrm{Sing}\, C} \to 1.$$
The sections of the central sheaf are $\mathbb{Z}/\ell$-valued functions over $\overline{C}$. Moreover, the image of a section s by t is the function that assigns to each node the difference between the two values of s on the preimages. The cohomology of this sequence gives the following long exact sequence
$$1 \to \mathbb{Z}/\ell \to C^0(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\delta} C^1(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\tau} \mathrm{Pic}(C)[\ell] \xrightarrow{\mathsf{nor}^*} \mathrm{Pic}(\overline{C})[\ell] \to 1. \tag{2.4}$$
Here, $\mathrm{Pic}(C)[\ell] = H^1(C, \mathbb{Z}/\ell)$ is the subgroup of $\mathrm{Pic}(C)$ of elements of order dividing ℓ, *i.e.* of ℓth roots of the trivial bundle.

As showed in [9], we have the following result

Proposition 2.27. *If* $(\mathsf{C}, \mathsf{L}, \theta)$ *is a rooted curve and* a *an automorphism in* $\mathrm{Aut}_C(\mathsf{C})$, *then*

$$\mathsf{a}^*\mathsf{L} = \mathsf{L} \otimes \tau\left(\mathsf{a}M\right).$$

Proposition 2.27 means that, if a is an automorphism in $\mathrm{Aut}_C(\mathsf{C})$, the pullback $\mathsf{a}^*\mathsf{L}$ is totally determined by the product $\mathsf{a} \cdot M \in \mathbb{C}^1(\Gamma_0; \mathbb{Z}/\ell)$, where a is seen as an element of $S(\Gamma_0; \mathbb{Z}/\ell)$, and M is the multiplicity index of $(\mathsf{C}, \mathsf{L}, \phi)$.

As a consequence we have the following theorem.

Theorem 2.28. *An element* $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$, *lifts to* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ *if and only if*

$$\mathsf{a}M \in \mathrm{Ker}(\tau) = \mathrm{Im}\,\delta.$$

In the following we will state the more general Proposition 4.19 for the case of twisted G-covers.

We recall the subcomplex $C^i(\Gamma_0(\mathsf{C}); \mathbb{Z}/\ell) \subset C^i(\Gamma(\mathsf{C}); \mathbb{Z}/\ell)$. Moreover, if δ_0 is the δ operator on $C^i(\Gamma_0)$, *i.e.* the restriction of the δ operator to this space, from Proposition 2.14 we know that $\mathrm{Im}(\delta_0) = C^1(\Gamma_0) \cap \mathrm{Im}\,\delta$. Given the multiplicity index $M \in \mathbb{C}^1(\Gamma; \mathbb{Z}/\ell)$, we define the subgroup $C^1_M(\Gamma; \mathbb{Z}/\ell) \subset C^1(\Gamma; \mathbb{Z}/\ell)$ of cochains N such that for every oriented edge e there exists a positive integer n such that $N(e) = M(e) \cdot n$.

Remark 2.29. As a consequence of Proposition 2.28, we get the canonical identification

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta) = \mathrm{Im}(\delta_0) \cap C^1_M(\Gamma_0; \mathbb{Z}/\ell)$$

via the multiplication $\mathsf{a} \mapsto M \cdot \mathsf{a}$.

Chapter 3

Singularities of moduli of rooted curves

For any moduli space of rooted curves $\overline{\mathcal{R}}^k_{g,\ell}$, the local picture at any point is a quotient $\mathbb{C}^{3g-3}/\mathfrak{G}$ where $\mathfrak{G}$ is a finite subgroup of $\mathrm{GL}(\mathbb{C}^{3g-3})$. Quotient singularities are a deeply studied kind of singularities, and in particular in this chapter we classify any point $[\mathsf{C}, \mathsf{L}, \phi]$ of $\overline{\mathcal{R}}^k_{g,\ell}$ as smooth point, canonical singular point or non-canonical singular points, by using some characteristics of the decorated dual graph of $(\mathsf{C}, \mathsf{L}, \phi)$. In the first section we introduce the notions of quasireflection automorphism and age invariant, which are central in analyzing quotient singularities. In the second section we completely describe the singular locus $\mathrm{Sing}\,\overline{\mathcal{R}}^k_{g,\ell}$ and in the third section we describe also the non-canonical locus $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}^k_{g,\ell}$ via a new stratification of the moduli space.

The results of this chapter already appeared in the author's paper [22].

3.1 Techniques to treat quotient singularities

3.1.1 Quasireflections

We consider a quotient of the form $\mathbb{C}^n/\mathfrak{G}$ where $\mathfrak{G}$ is a finite subgroup of $\mathrm{GL}(\mathbb{C}^n)$. In this setting we introduce some automorphisms called quasireflections.

Definition 3.1 (Quasireflection)**.** Any finite order complex automorphism $\mathsf{h} \in \mathrm{GL}(\mathbb{C}^n)$ is called a quasireflection if its fixed locus has dimension exactly $n-1$. Equivalently, h is a quasireflection if, for an opportune choice of the basis, we can diagonalize it as

$$\mathsf{h} = \mathrm{Diag}(\xi, 1, 1, \ldots, 1),$$

where ξ is a primitive root of the unit of order equal to the order of h. Given a finite group $\mathfrak{G} \subset \mathrm{GL}(\mathbb{C}^n)$, we note $\mathrm{QR}(\mathfrak{G})$ the subgroup generated by quasireflections.

Quasireflections have the interesting property that any complex vector space, quotiented by them, keeps being a smooth variety. In particular if $\mathsf{h} \in GL(\mathbb{C}^n)$ is a quasireflection, variety $\mathbb{C}^n/\mathsf{h}$ is isomorphic to $\mathbb{C}^n$.

Proposition 3.2 (see [29]). *Consider any vector space quotient*

$$V' := V/\mathfrak{G},$$

where $V \cong \mathbb{C}^n$ is a complex vector space and $\mathfrak{G} \subset \mathrm{GL}(V)$ is a finite group. The variety V' is smooth if and only if H is generated by quasireflections.

In the following, we will work with quotients of the form

$$\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta),$$

therefore the smoothness of any point $[\mathsf{C}, \mathsf{L}, \theta] \in \overline{\mathcal{R}}^k_{g,\ell}$ will depend on the group of automorphisms being generated by quasireflections.

3.1.2 Age invariant

In the case of the group $\mathfrak{G}$ not being generated by quasireflections, we need another tool to distinguish between canonical singularities and non-canonical singularities. The age is an additive positive function from the representations ring of $\mathfrak{G}$ to rational numbers,

$$\mathrm{age}\colon \mathrm{Rep}(\mathfrak{G}) \to \mathbb{Q}.$$

First consider the group $\mathbb{Z}/r$ for every r positive integer. Given the character $\boldsymbol{m}$ such that $1 \mapsto m \in \mathbb{Z}/r$, we define $\mathrm{age}(\boldsymbol{m}) = m/r$. These characters are a basis for the group $\mathrm{Rep}(\mathbb{Z}/r)$, then we can extend age over the entire representation ring.

Consider a $\mathfrak{G}$-representation $\rho\colon \mathfrak{G} \to \mathrm{GL}(V)$, the age function could be defined on any injection $i\colon \mathbb{Z}/r \hookrightarrow \mathfrak{G}$ simply composing the injection with ρ,

$$\mathrm{age}_V\colon i \mapsto \mathrm{age}(\rho \circ i).$$

Definition 3.3 (Age). On any finite group $\mathfrak{G}$, age is defined by

$$\mathfrak{G} \xrightarrow{f} \bigsqcup_{r \geq 1} \{i |\ i\colon \boldsymbol{\mu}_r \hookrightarrow \mathfrak{G}\} \xrightarrow{\mathrm{age}_V} \mathbb{Q},$$

where f is the set bijection sending $\mathsf{h} \in \mathfrak{G}$, element of order r, to the injection obtained by mapping $1 \in \mathbb{Z}/r$ to h.

Remark 3.4. The identification f is non-canonical, it depends on the choice of the primitive root $\xi_r = \exp(2\pi i/r)$.

Remark 3.5. To see all this explicitly, consider any automorphism $\mathsf{h} \in GL(\mathbb{C}^n)$ of finite order r, consider any basis such that

$$\mathsf{h} = \mathrm{Diag}(\xi_r^{a_1}, \xi_r^{a_2}, \ldots, \xi_r^{a_n}).$$

In this setting,

$$\mathrm{age}(\mathsf{h}) = \frac{1}{r}\sum_{i=1}^{n} a_i.$$

Given a quasireflection $\mathsf{h} \in \mathrm{GL}(\mathbb{C}^n)$ and the canonical projection

$$\varphi\colon \mathbb{C}^m \to (\mathbb{C}^n/\mathsf{h}) \cong \mathbb{C}^n,$$

we can chose the basis of $\mathbb{C}^n$ in such a way that φ is the identity in all the coordinates but one, on the non-trivial coordinate φ is the mapping $z \mapsto z^n$.

Definition 3.6 (Junior group). A finite group $\mathfrak{G} \subset \mathrm{GL}(\mathbb{C}^m)$ that contains no quasireflections is called junior if the image of the age function intersects the open interval $]0,1[$,

$$\operatorname{age}\mathfrak{G} \cap\,]0,1[\, \neq \emptyset.$$

The group $\mathfrak{G}$ is called senior if the intersection is empty.

Remark 3.7. The image of $\operatorname{age}\colon \mathfrak{G} \to \mathbb{Q}$ does not depend on the choice of identification f, therefore junior and senior groups are well defined independently from any choice of a primitive root ξ_n.

The following criterion is the central tool in our work, it sets a correspondence between age and canonical singularities.

Proposition 3.8 (Age criterion, see [30]). *Consider any vector space quotient*

$$V' := V/\mathfrak{G},$$

where $V \cong \mathbb{C}^n$ is a complex vector space and $\mathfrak{G} \subset \mathrm{GL}(V)$ is a finite group containing no quasireflections. Then V' has a non-canonical singularity if and only if $\mathfrak{G}$ is junior.

We will use the Age Criterion 3.8 to find non-canonical singularities by the study of group $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\theta)$ action. We point out that to satisfy the hypothesis of Age Criterion, it is necessary for $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\theta)$ to be quasireflections free. As this is often not the case, the following lemma is necessary to represent the same singularity by a group with no quasireflections.

Proposition 3.9 (see [29]). *Consider a finite subgroup $\mathfrak{G} \subset \mathrm{GL}(\mathbb{C}^n)$. Then, there exists an isomorphism $u\colon \mathbb{C}^n/\operatorname{QR}(\mathfrak{G}) \to \mathbb{C}^n$ and a finite subgroup $\mathfrak{K} \subset \mathrm{GL}(\mathbb{C}^n)$ isomorphic to $\mathfrak{G}/\operatorname{QR}(\mathfrak{G})$, such that the following diagram is commutative.*

$$\begin{array}{ccccc}
\mathbb{C}^n & \longrightarrow & \mathbb{C}^n/\operatorname{QR}(\mathfrak{G}) & \xrightarrow{u} & \mathbb{C}^n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}^n/H & \xrightarrow{\cong} & (\mathbb{C}^n/\operatorname{QR}(\mathfrak{G}))/(\mathfrak{G}/\operatorname{QR}(\mathfrak{G})) & \xrightarrow{\cong} & \mathbb{C}^n/\mathfrak{K}
\end{array}$$

In the following section we approach the characterization of singularities for $\overline{\mathcal{R}}^k_{g,\ell}$.

3.2 Smooth points

3.2.1 Ghost quasireflections

In order to characterize any rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$ whose automorphism group $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \theta)$ is generated by quasireflections, in this section we characterize those rooted curves such that the group of ghost automorphisms, $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$, is generated by quasireflections.

Lemma 3.10. *Any automorphism* a *in* $\mathrm{Aut}_C(\mathsf{C})$ *is a ghost quasireflection in the group* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ *if and only if* $\mathsf{a}(e) \equiv 0 \mod \ell$ *for all edges of* $\Gamma_0(\mathsf{C})$ *but one that is a separating edge.*

Proof. If a is a quasireflection, the action on all but one of the coordinates must be trivial. Therefore $\mathsf{a}(e) \equiv 0$ on all the edges but one, say e_1. If e_1 is in any circuit $(e_1, \ldots, e_k)$ of Γ_0 with $k \geq 1$, we have, by Remark 4.28, that $\sum \mathsf{a}(e_i) \equiv 0$. As $\mathsf{a}(e_1) \not\equiv 0$, there exists $i > 1$ such that $\mathsf{a}(e_i) \not\equiv 0$, contradiction. Thus e_1 is not in any circuit, then it is a separating edge.

Reciprocally, consider an automorphism $\mathsf{a} \in S(\Gamma_0; \mathbb{Z}/\ell)$ such that there exists an oriented separating edge e_1 with the property that $\mathsf{a}(e) \equiv 0$ for all $e \in \mathbb{E}\backslash\{e_1, \bar{e}_1\}$ and $\mathsf{a}(e_1) \not\equiv 0$. Then for every circuit $(e'_1, \ldots, e'_k)$ we have $\sum \mathsf{a}(e'_i) \equiv 0$ and so a is in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$. $\square$

Definition 3.11. We call $\mathrm{QR}(\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta))$ or simply $\mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta)$ the group generated by quasireflection automorphisms of the rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$. We call $\mathrm{QR}(\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta))$ or simply $\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \theta)$ the group generated by ghost quasireflections.

Remark 3.12. If we note $E_{\mathrm{sep}} \subset E_0 = E(\Gamma_0)$ the subset of separating edges of Γ_0, we have a simple description of the group $\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \theta)$.

$$\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \theta) \cong \bigoplus_{e \in E_{\mathrm{sep}}} \mathbb{Z}/r(e) \subset \bigoplus_{e \in E_0} \mathbb{Z}/r(e) \cong S(\Gamma_0; M).$$

To show the next theorem we update our tools following [12, §2.4]. We introduce the p-adic valuation. Let e be an oriented edge in $\mathbb{E}(\Gamma)$,

$$\nu_p(e) := \mathrm{val}_p\, r(e).$$

Given the dual graph $\Gamma(\mathsf{C}) = \Gamma$, consider the factorization of ℓ in prime numbers $\ell = \prod p^{\gamma_p}$, where we note $\gamma_p := \nu_p(\ell)$ the p-adic valuation of ℓ, or equivalently the maximum power of p dividing ℓ. For every prime number p we define $\Gamma(\nu_p^j)$ by contracting every edge e of Γ such that p^j divides $r(e)$. We will note $\Gamma_p(\mathsf{C}) := \Gamma(\nu_p^{\gamma_p})$. We have the chain of contractions already introduced in [12].

$$\Gamma \to \Gamma_p = \Gamma(\nu_p^{\gamma_p}) \to \Gamma(\nu_p^{\gamma_p - 1}) \to \cdots \to \Gamma(\nu_p^1) \to \{\cdot\}.$$

We point out that given two integers r and r' such that r divides r', we look at $\mathbb{Z}/r$ as canonically immersed in $\mathbb{Z}/r'$. In particular for every prime number p, we have the inclusion sequence $\mathbb{Z}/p \subset \mathbb{Z}/p^2 \subset \mathbb{Z}/p^3 \subset \cdots$.

Theorem 3.13. *Consider a rooted curve* $(\mathsf{C},\mathsf{L},\theta)$. *The group of ghost automorphisms* $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta)$ *is generated by quasireflections if and only if the graphs* $\Gamma_p(\mathsf{C})$ *are tree-like for every prime number* p *dividing* ℓ.

Proof. As seen above, we have the canonical identification

$$\mathrm{Aut}_C(\mathsf{C}) = \bigoplus_{e\in E(\Gamma)} \mathbb{Z}/r(e).$$

Given the multiplicity index M of $(\mathsf{C},\mathsf{L},\theta)$ and the exterior differential operator $\delta\colon C^0(\Gamma;\mathbb{Z}/\ell)) \to C^1(\Gamma;\mathbb{Z}/\ell)$, as stated in Remark 2.29 the group of ghost automorphisms $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta)$ is canonically identified with $C^1_M(\Gamma;\mathbb{Z}/\ell))\cap \mathrm{Im}\,\delta$.

We consider the function $\bar\nu_p$ such that $\bar\nu_p(e) := \min(\gamma_p,\nu_p(e))$. We observe that, by definition, for every oriented edge $e\in\mathbb{E}(\Gamma)$ the order $r(e)$ factorizes as

$$r(e) = \prod_{p\mid\ell} p^{\gamma_p-\bar\nu_p(e)}.$$

Furthermore, given the canonical immersion

$$C^i(\Gamma(\nu_p^k);\mathbb{Z}/\ell) \hookrightarrow C^i(\Gamma;\mathbb{Z}/\ell)$$

for $i=0,1$. We define δ_p^j as the restriction of the delta operator on the group $C^0(\Gamma(\nu_p^j);\mathbb{Z}/p^{\gamma_p-j+1})$ for all p in the factorization of ℓ and for all j between 1 and γ_p.

As stated above, an automorphism $\mathsf{a}\in\mathrm{Aut}_C(\mathsf{C})$ is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta)$ if and only if $\mathsf{a}(e)\equiv 0 \mod r(e)$ for all edges but one that is a separating edge. This allows the following decomposition

$$\mathrm{QR}_C(\mathsf{C},\mathsf{L},\theta) = \bigoplus_{e\in E_{\mathrm{sep}}(\Gamma)} \mathbb{Z}/r(e) = \bigoplus_{e\in E_{\mathrm{sep}}}\bigoplus_{p\mid\ell} \mathbb{Z}/p^{\gamma_p-\bar\nu_p(r(e))} = \bigoplus_{p\mid r_{\mathrm{lcm}}}\bigoplus_{j=1}^{\gamma_p}(\mathbb{Z}/p^j)^{\beta_p^j},$$

where we define the integers β as $\beta_p^j := E_{\mathrm{sep}}\left(\Gamma(\nu_p^{\gamma_p-j+1})\right) - E_{\mathrm{sep}}\left(\Gamma(\nu_p^{\gamma_p-j})\right)$ if $j<\gamma_p$ and $\beta_p^{\gamma_p} := E_{\mathrm{sep}}\left(\nu_p^1\right)$. Following [12, Lemma 2.22], we have a similar decomposition for $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta)$:

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta) = C^1_M(\Gamma;\mathbb{Z}/\ell)\cap\mathrm{Im}\,\delta = \bigoplus_{p\mid\ell}\sum_{j=1}^{\gamma_p}\mathrm{Im}\,\delta_p^j = \bigoplus_{p\mid\ell}\bigoplus_{j=1}^{\gamma_p}(\mathbb{Z}/p^j)^{\alpha_p^j},$$

where $\alpha_p^j := V\left(\Gamma(\nu_p^{\gamma_p-j+1})\right) - V\left(\Gamma(\nu_p^{\gamma_p-j})\right)$ for all $j\geq 0$. We observe that $\alpha_p^j\geq\beta_p^j$ for all p dividing ℓ and $k\geq 0$. Moreover, $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta)$ coincides with $\mathrm{QR}_C(\mathsf{C},\mathsf{L},\theta)$ if and only if $\alpha_p^j=\beta_p^j$ for all p and j. Fixing p, this is equivalent to impose $\sum_j\beta_p^j = \sum_j\alpha_p^j$. In the previous expression the left hand side is $E_{\mathrm{sep}}(\Gamma_p)$ and the right hand side is $V(\Gamma_p)-1$, we saw in Lemma 2.12 that the equality is achieved if and only if Γ_p is tree-like. $\square$

Remark 3.14. A *bouquet* is a graph with only one vertex, or equivalently a tree-like graph with no non-loop edges. Theorem 3.13 is a generalization of [12, Theorem 2.28]. In particular Chiodo and Farkas proved that in the case $k = 0$, the moduli of level curves, if $[\mathsf{C}, \mathsf{L}, \theta]$ is smooth then the contracted graphs Γ_p are bouquet. This happens because in the case $k = 0$, every separating node must have trivial stabilizer, *i.e.* the M cochain must be 0 and therefore they disappear after the contraction. In the general case, M could be non-zero on separating edges too, and the theorem is still true using the more general notion of tree-like graph.

3.2.2 Smooth points of $\overline{\mathcal{R}}^k_{g,\ell}$

We recall the quasireflection analysis in the case of a stable curve C.

Definition 3.15. Within a stable curve C, an elliptic tail is an irreducible component of geometric genus 1 that meets the rest of the curve in only one point called an elliptic tail node. Equivalently, T is an elliptic tail if and only if its algebraic genus is 1 and $T \cap \overline{C \backslash T} = \{q\}$.

Definition 3.16. An element $i \in \mathrm{Aut}(C)$ is an elliptic tail automorphism if there exists an elliptic tail T of C such that i fixes T and his restriction to $\overline{C \backslash T}$ is the identity. An elliptic tail automorphism of order 2 is called an elliptic tail quasireflection (ETQR). In the literature ETQRs are called elliptic tail involutions (or ETIs), we changed this convention in order to generalize the notion.

Remark 3.17. Every scheme theoretic curve of algebraic genus 1 with one marked point has exactly one involution i. Then there is a unique ETQR associated to every elliptic tail.

More precisely an elliptic tail E could be of two types. The first type is a smooth curve of geometric genus 1 with one marked point, *i.e.* an elliptic curve: in this case we have $E = \mathbb{C}/\Lambda$, for Λ integral lattice of rank 2, the marked point is the origin, and the only involution is the map induced by $x \mapsto -x$ on $\mathbb{C}$. The second type is the rational line with one marked point and an autointersection point: in this case we can write $E = \mathbb{P}^1/\{1 \equiv -1\}$, the marked point is the origin, and the only involution is the map induced by $x \mapsto -x$ on $\mathbb{P}^1$.

From Remark 1.5 we have a coordinate system on $\mathrm{Def}(C)$. Our notation is again $C_1, \dots, C_V$ for the irreducible components of C, $\overline{C}_i$ for the normalization of curve C_i for every i, and $q_1, \dots, q_\delta$ for the nodes of C. We have a canonical subscheme $\mathrm{Def}(C; \mathrm{Sing}\, C)$ such that

$$\mathrm{Def}(C) \supset \mathrm{Def}(C; \mathrm{Sing}\, C) \cong \bigoplus_{i=1}^{V} H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i)).$$

Furthermore, the quotient of these two schemes has a splitting

$$\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C) \cong \bigoplus_{j=1}^{\delta} \mathbb{A}^1_{t_j},$$

where $\mathbb{A}^1_{t_j} \cong M_j$ as defined in Remark 1.5. These coordinates systems on the space $\mathrm{Def}(C; \mathrm{Sing}\, C)$ and $\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C)$ allow the detection of quasireflections and the age calculations for every automorphism $\mathsf{a} \in \mathrm{Aut}(C)$. Indeed, the diagonalizations of the a-action on the two spaces determines a diagonalization of the a-action on the whole $\mathrm{Def}(C)$. Therefore, a is a quasireflection if it acts non-trivially on exactly one coordinate on $\mathrm{Def}(C; \mathrm{Sing}\, C)$ or $\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C)$. Furthermore, the age of a is the sum of its ages calculated on the two spaces.

The choice of the coordinate t_j associated to the smoothing of node q_j is non-canonical. In particular the scheme M_j is the deformation scheme for node q_j, *i.e.* it comes with a flat morphism $X \to M_j$ isomorphic to

$$\begin{array}{ccc} \{xy = t_j\} & \subset & \mathbb{A}^2_{x,y} \times \mathbb{A}^1_{t_j} \\ \downarrow & & \\ \mathbb{A}^1_{t_j}. & & \end{array}$$

Given any automorphism of the central fiber $\{xy = 0\} \subset \mathbb{A}^2$ of the family, this representation allows to catch the associated action on M_j, *i.e.* on the coordinate smoothing the node.

The following theorem by Harris and Mumford describes the action of the automorphism group $\mathrm{Aut}(C)$ on $\mathrm{Def}(C)$.

Theorem 3.18 (See [24, Theorem 2])**.** *Consider a stable curve C of genus $g \geq 4$. An element of $\mathrm{Aut}(C)$ acts as a quasireflection on $\mathrm{Def}(C)$ if and only if it is an ETQR. In particular, if $\eta \in \mathrm{Aut}(C)$ is an ETQR acting non trivially on the tail T with elliptic tail node q_j, then η acts trivially on $\mathrm{Def}(C; \mathrm{Sing}\, C)$, and its action on $\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C)$ is $t_j \mapsto -t_j$ on the coordinate associated to q_j, and the identity $t \mapsto t$ on the remaining coordinates.*

This theorem allows to conclude that on the moduli space $\overline{\mathcal{M}}_g$ of stable curves, any point $[C]$ is smooth if and only the group $\mathrm{Aut}(C)$ is generated by ETQRs.

We discussed in Remark 2.22 the fact that every point $[\mathsf{C}, \mathsf{L}, \theta] \in \overline{\mathcal{R}}^k_{g,\ell}$ has a local picture isomorphic to

$$\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta).$$

Therefore, to find the smooth points of $\overline{\mathcal{R}}_{g,G}$, by Proposition 3.2 we need to know when $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$ is generated by quasireflections. To do this we generalize the notion of ETQR.

In Remark 1.57 we have seen that the deformation schemes $\mathrm{Def}(C)$ and $\mathrm{Def}(\mathsf{C})$ are very similar. The deformations $\mathrm{Def}(C; \mathrm{Sing}\, C)$ and $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C})$ are canonically identified,

$$\mathrm{Def}(C; \mathrm{Sing}\, C) = \mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C}) = \bigoplus_{i=1}^{V} \mathrm{Def}(\overline{C}_i, D_i).$$

For the deformation of the nodes, the description is slightly different. We note $r_1, \ldots, r_\delta$ the order of the cyclic stabilizers in C of the nodes $q_1, \ldots, q_\delta$ respectively.

We have

$$\mathrm{Def}(\mathsf{C})/\,\mathrm{Def}(\mathsf{C};\mathrm{Sing}\,\mathsf{C}) = \bigoplus_{j=1}^{\delta} \mathbb{A}^1_{\tilde{t}_j},$$

where $\mathbb{A}^1_{\tilde{t}_j} \cong R_j$ as defined in Remark 1.57. The scheme R_j is the deformation scheme of the node q_j with its stack structure, in particular it comes with a flat representable morphism of Deligne-Mumford stacks $X \to R_j$ isomorphic to

$$[\{x'y' = \tilde{t}_j\}/\boldsymbol{\mu}_{r_j}] \to \mathbb{A}^1_{\tilde{t}_j},$$

where the local stabilizer $\boldsymbol{\mu}_{r_j}$ acts by

$$\xi \cdot (x', y', \tilde{t}_j) = (\xi x', \xi^{-1} y', \tilde{t}_j).$$

After this construction we also get the morphism $R_j \to M_j$, which is isomorphic to

$$\mathbb{A}^1_{\tilde{t}_j} \to \mathbb{A}^1_{t_j} \quad \text{such that} \quad (\tilde{t}_j)^{r_j} = t_j.$$

Remark 3.19. Consider a stack-theoretic curve E of genus 1 with one marked point. We call E its coarse space. In the case of an elliptic tail of a curve C, the marked point is the point of intersection between E and $\overline{\mathsf{C}\backslash\mathsf{E}}$.

If E is an elliptic curve, then $\mathsf{E} = E$ and the curve has exactly one involution i_0. In case E is rational, its normalization is the stack $\overline{\mathsf{E}} = [\mathbb{P}^1/\boldsymbol{\mu}_r]$, with $\boldsymbol{\mu}_r$ acting by multiplication, and $\mathsf{E} = \overline{\mathsf{E}}/\{0 \equiv \infty\}$. There exists a canonical involution i_0 in this case too: we consider the pushforward of the involution on $\mathbb{P}^1$ such that $z \mapsto 1/z$. As every involution of E project to the canonical involution of E via the coarsening, we consider the autointersection node of E and its local picture $[\{x'y' = 0\}/\boldsymbol{\mu}_r]$, then we observe that i_0 is the only E involution acting trivially on the product $x'y'$ and therefore acting trivially on the smoothing coordinate $\tilde{t}$ associated to this node.

Given any twisted curve C with an elliptic tail E whose elliptic tail node is called q, the construction above defines a canonical involution i_0 on E up to non-trivial action on q.

Definition 3.20. We generalize the notion of ETQR to rooted curves. An element $\mathsf{i} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ is an ETQR if there exists an elliptic tail E of C with elliptic tail node q, such that the action of i on $\mathsf{C}\backslash\mathsf{E}$ is trivial, and the action on E, up to non-trivial action on q, is the canonical involution i_0.

Lemma 3.21. *Consider an element* h *of* $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$*. It acts as a quasireflection on* $\mathrm{Def}(\mathsf{C})$ *if and only if one of the following is true:*

1. *the automorphism* h *is a ghost quasireflection, i.e. an element of* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ *which moreover operates as a quasireflection;*

2. *the automorphism* h *is an ETQR, using the generalized Definition 3.20.*

Proof. We first prove the "only if" part. If h acts trivially on certain coordinates of $\mathrm{Def}(\mathsf{C})$, *a fortiori* we have that h acts trivially on the corresponding coordinates of $\mathrm{Def}(C)$. Therefore h acts as the identity or as a quasireflection on $\mathrm{Def}(C)$. In the first case, h is a ghost automorphism and we are in case (1). If h acts as a quasireflection, then it is a classical ETQR as we pointed out on Theorem 3.18, and it acts non-trivially on the elliptic tail node q associated to an elliptic tail.

As we know that the action of h is trivial on the components $\mathrm{Def}(\overline{C}_i, D_i)$, so is the action of h on these components. It remains to know the action of h on the nodes with non-trivial stabilizer and other than q. If the elliptic tail where h operates non trivially is a rational component with an autointersection node q_1, by hypothesis h acts trivially on the universal deformation R_1 of this node. Therefore, the h restriction to the elliptic tail has to be the canonical involution i_0 (see Remark 3.19). For every node other than q and q_1, if the local picture is $[\{x'y'=0\}/\boldsymbol{\mu}_{r_j}]$, the action of h must be of the form

$$(x',y') \mapsto (\xi x', y') \equiv (x', \xi y') \quad \text{for some} \quad \xi \in \boldsymbol{\mu}_{r_j}.$$

If $\xi \neq 1$ this gives a non-trivial action on the associated universal deformation M_j, against our hypothesis. By Definition 3.20 this implies that h is an ETQR of (C, ϕ).

For the "if" part, we observe that a ghost quasireflection is automatically a quasireflection. It remains to prove the case of point (2). By definition of ETQR, its action on $\mathrm{Def}(\mathsf{C})$ can be non-trivial only on the components associated to the separating node q of the tail. As a consequence h acts as the identity or as a quasireflection. The local coarse picture at q is $\{xy=0\}$, where $y=0$ is the branch lying on the elliptic tail. Then the action of h on the coarse space is $(x,y) \mapsto (-x,y)$. Therefore the action is *a fortiori* non trivial on the coordinate associated to the stack node q in $\mathrm{Def}(\mathsf{C})$. $\square$

Lemma 3.22. *If* $\mathrm{QR}(\mathrm{Aut}(C))$ *(also called* $\mathrm{QR}(C)$*) is the group generated by ETQRs inside* $\mathrm{Aut}(C)$*, then any element* $h \in \mathrm{QR}(C)$ *which could be lifted to* $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$*, has a lifting in* $\mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta)$*, too.*

Proof. By definition, $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$ is the set of automorphisms $\mathsf{s} \in \mathrm{Aut}(\mathsf{C})$ such that $\mathsf{s}^*\mathsf{L} \cong \mathsf{L}$. Consider $h \in \mathrm{QR}(C)$ such that its decomposition in quasireflections is $h = i_0 i_1 \cdots i_m$, and every i_t is an ETQR acting non-trivially on an elliptic tail E_t. Any lifting of q is in the form $\mathsf{q} = \mathsf{i}_0\mathsf{i}_1 \cdots \mathsf{i}_m \cdot \mathsf{a}$, where i_t is a (generalized) ETQR acting non-trivially on E_t, and a is a ghost acting non-trivially only on nodes other than the elliptic tail nodes of the E_t. We observe that every i_t is a lifting in $\mathrm{Aut}(\mathsf{C})$ of i_t, we are going to prove that moreover $\mathsf{i}_t \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$. By construction, $\mathsf{h}^*\mathsf{L} \cong \mathsf{L}$ if and only if $\mathsf{i}_t^*\mathsf{L} \cong \mathsf{L}$ for all t and $\mathsf{a}^*\mathsf{L} \cong \mathsf{L}$. This implies that every i_t lies in $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$, and therefore $\mathsf{h}\mathsf{a}^{-1}$ is a lifting of h lying in $\mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta)$. $\square$

Remark 3.23. We recall the short exact sequence

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta) \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta) \xrightarrow{\beta} \mathrm{Aut}'(C) \to 1$$

and introduce the group $\mathrm{QR}'(C) \subset \mathrm{Aut}'(C)$, generated by liftable quasireflections, *i.e.* by those quasireflections h in $\mathrm{Aut}'(C)$ such that there exists an automorphism in $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ whose coarsening is h. Knowing that

$$\beta(\mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta)) \subset \mathrm{QR}'(C) \subset \mathrm{Aut}'(C) \cap \mathrm{QR}(C),$$

the previous lemma shows that $\mathrm{QR}'(C) = \beta(\mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta))$. Using also Lemma 4.29, we obtain that the following is a short exact sequence

$$1 \to \mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \theta) \to \mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta) \to \mathrm{QR}'(C) \to 1.$$

Theorem 3.24. *The group $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$ is generated by quasireflections if and only if both $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ and $\mathrm{Aut}'(C)$ are generated by quasireflections.*

Proof. After the previous remark, the following is a short exact sequence

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)/\, \mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \theta) \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)/\, \mathrm{QR}(\mathsf{C}, \mathsf{L}, \theta) \to \mathrm{Aut}'(C)/\, \mathrm{QR}'(C) \to 1.$$

The theorem follows. □

As we know that any point $[\mathsf{C}, \mathsf{L}, \theta] \in \overline{\mathcal{R}}^k_{g,\ell}$ is smooth if and only if the group $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)$ is generated by quasireflections, then the next theorem follows from Theorems 3.13 and 3.24.

Theorem 3.25. *For any $g \geq 4$ and ℓ positive integer. The point $[\mathsf{C}, \mathsf{L}, \phi]$ is smooth if and only if $\mathrm{Aut}'(C)$ is generated by ETQRs of C and the $\Gamma_p(\mathsf{C})$ are tree-like.*

We introduce two closed loci of $\overline{\mathcal{R}}^k_{g,\ell}$,

$$N^k_{g,\ell} := \{[\mathsf{C}, \mathsf{L}, \theta] |\ \mathrm{Aut}'(C) \text{ is not generated by ETQRs}\},$$

$$H^k_{g,\ell} := \{[\mathsf{C}, \mathsf{L}, \theta] |\ \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta) \text{ is not generated by quasireflections}\}.$$

We have by Theorem 3.25 that the singular locus $\mathrm{Sing}\, \overline{\mathcal{R}}^k_{g,\ell}$ is their union

$$\mathrm{Sing}\, \overline{\mathcal{R}}^k_{g,\ell} = N^k_{g,\ell} \cup H^k_{g,\ell}.$$

Remark 3.26. Consider the natural projection $\pi\colon \overline{\mathcal{R}}^k_{g,\ell} \to \overline{\mathcal{M}}_g$, we observe that

$$N^k_{g,\ell} \subset \pi^{-1} \mathrm{Sing}\, \overline{\mathcal{M}}_g.$$

Indeed, after Remark 3.23, $\mathrm{QR}'(C) = \mathrm{Aut}'(C) \cap \mathrm{QR}(C)$ and therefore $\mathrm{Aut}(C) = \mathrm{QR}(C)$ if and only if $\mathrm{Aut}'(C) = \mathrm{QR}'(C)$. This implies that $\left(\pi^{-1} \mathrm{Sing}\, \overline{\mathcal{M}}_g\right)^c \subset (N^k_{g,\ell})^c$, and taking the complementary we obtain the result.

3.2.3 A new stratification of the singular locus

As we saw, the information about the automorphism group of a certain rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$ is encoded by its dual decorated graph $(\Gamma_0(\mathsf{C}), M)$. It is therefore quite natural to introduce a stratification of $\overline{\mathcal{R}}^k_{g,\ell}$ using this notion. For this, we extend the notion of graph contraction: if $\Gamma'_0 \to \Gamma'_1$ is a usual graph contraction, the ring $C^1(\Gamma'_1; \mathbb{Z}/\ell)$ is naturally immersed in $C^1(\Gamma'_0; \mathbb{Z}/\ell)$, then the contraction of a pair (Γ'_0, M'_0) is the pair (Γ'_1, M'_1) where the cochain M'_1 is the restriction of M'_0. If it is clear from the context, we could refer to the decorated graph restriction simply by the graph contraction $\Gamma'_0 \to \Gamma'_1$.

Definition 3.27. Given a decorated graph (Γ, M) with $M \in C^1(\Gamma; \mathbb{Z}/\ell)$, consider this locus in $\overline{\mathcal{R}}^k_{g,\ell}$:

$$\mathcal{S}_{(\Gamma,M)} := \left\{ [\mathsf{C}, \mathsf{L}, \theta] \in \overline{\mathcal{R}}^k_{g,\ell} : \ \Gamma_0(\mathsf{C}) = \Gamma, \text{ and } M \text{ is the multiplicity index of } (\mathsf{C}, \mathsf{L}, \theta) \right\}.$$

These loci makes a new stratification of the space. We can find a first link between the decorated graphs and geometric properties of the strata.

Proposition 3.28. *If E is the cardinality of the edge set $E(\Gamma)$, then the codimension of $\mathcal{S}_{(\Gamma,M)}$ inside $\overline{\mathcal{R}}^k_{g,\ell}$ is*

$$\operatorname{Codim} \mathcal{S}_{(\Gamma,M)} = E.$$

Proof. We take a general point $[\mathsf{C}, \mathsf{L}, \theta]$ of stratum $\mathcal{S}_{(\Gamma,M)}$, where the curve C has V irreducible components $\mathsf{C}_1, \dots, \mathsf{C}_V$. We call g_i the genus of C_i and n_i the number node preimages on the normalized curve $\overline{\mathsf{C}}_i$. Then we have $\sum n_i = 2E$. We obtain that the dimension of $\operatorname{Def}(\mathsf{C}, \mathsf{L}, \theta)$ is

$$\dim \operatorname{Def}(\mathsf{C}, \mathsf{L}, \theta) = \sum_{i=1}^{V} (3g_i - 3 + n_i) = 3 \sum g_i - 3V + 2E.$$

It is known that if g is the C genus, then $g = \sum g_i - V + E + 1$, and therefore $\dim \operatorname{Def}(\mathsf{C}, \mathsf{L}, \theta) = 3g - 3 - E$. The result on the codimension follows. $\square$

Using contraction, we have this description of the closed strata.

$$\overline{\mathcal{S}}_{(\Gamma'_1, M'_1)} = \left\{ [\mathsf{C}, \mathsf{L}, \theta] \in \overline{\mathbf{R}}^k_{g,\ell} : \begin{array}{l} \text{if } (\Gamma_0, M) \text{ is the decorated graph of } (\mathsf{C}, \mathsf{L}, \theta), \\ \text{there exists a contraction } (\Gamma_0, M) \to \\ (\Gamma'_1, M'_1) \end{array} \right\}.$$

After Theorem 3.25 we have seen that the singular locus of $\overline{\mathcal{R}}^k_{g,\ell}$ is easily described as the union of two loci

$$\operatorname{Sing} \overline{\mathcal{R}}^k_{g,\ell} = N^k_{g,\ell} \cup H^k_{g,\ell}.$$

The stratification just introduced is particularly useful in describing the "new" locus $H^k_{g,\ell}$. We recall the definition of *vine* curves.

Definition 3.29. We note $\Gamma_{(2,n)}$ a graph with two vertices linked by n edges. A curve C is an n-vine curve if $\Gamma(\mathsf{C})$ contracts to $\Gamma_{(2,n)}$ for some $n \geq 2$. Equivalently an n-vine curve has coarse space $C = C_1 \cup C_2$ which is the union of two curves intersecting each other n times.

Lemma 3.30. *If* $[\mathsf{C},\mathsf{L},\theta] \in \operatorname{Sing}\overline{\mathcal{R}}^k_{g,\ell}$ *is a point in* $H^k_{g,\ell}$ *whose decorated graph is* (Γ_0, M)*, then there exists* $n \geq 2$ *and a graph contraction* $\Gamma_0 \to \Gamma_{(2,n)}$*. Equivalently* C *is an* n*-vine curve for some* $n \geq 2$*.*

Proof. If $[\mathsf{C},\mathsf{L},\theta] \in H^k_{g,\ell}$, by definition Γ_0 contains a circuit that is not a loop. We will show an edge contraction $\Gamma_0 \to \Gamma_{(2,n)}$ for some $n \geq 2$.

Consider two different vertices v_1 and v_2 that are consecutive on a non-loop circuit $\mathcal{K} \subset \Gamma_0$. Now consider a partition $V = V_1 \sqcup V_2$ of the vertex set such that $v_1 \in V_1$ and $v_2 \in V_2$. This defines an edge contraction $\Gamma_0 \to \Gamma_{(2,n)}$ where $e \in E(\Gamma)$ is contracted if its two extremities lie in the same V_i. As $\mathcal{K}$ is a circuit, necessarily $n \geq 2$ and the theorem is proved. □

We conclude that $H^k_{g,\ell}$ is the union of the closed strata associated to vine graphs.

$$H^k_{g,\ell} \quad = \bigcup_{\substack{n\geq 2\\ M\in C^1(\Gamma_{(2,n)};\mathbb{Z}/\ell)}} \overline{\mathcal{S}}_{(\Gamma_{(2,n)},M)}.$$

3.3 The non-canonical singular locus for ℓ prime number

As suggested by the title, in this section we consider the case of ℓ prime number, and give a classification of the non-canonical singular locus $\operatorname{Sing}^{\mathrm{nc}}\overline{\mathcal{R}}^k_{g,\ell}$. In particular we will see that this locus is the union of two subloci $T^k_{g,\ell}$ and $J^k_{g,\ell}$, we will focus on the second one.

3.3.1 T-curves and J-curves

We introduce two closed loci which are central in our description.

Definition 3.31 (T-curves)**.** A rooted curve $(\mathsf{C},\mathsf{L},\phi)$ is a T-curve if there exists an automorphism $\mathsf{a} \in \underline{\operatorname{Aut}}(\mathsf{C},\mathsf{L},\phi)$ such that its coarsening a is an elliptic tail automorphism of order 6. The locus of T-curves in $\overline{\mathcal{R}}^k_{g,\ell}$ is noted $T^k_{g,\ell}$.

Definition 3.32 (J-curves)**.** A rooted curve $(\mathsf{C},\mathsf{L},\phi)$ is a J-curve if the group

$$\underline{\operatorname{Aut}}_C(\mathsf{C},\mathsf{L},\phi)/\operatorname{QR}_C(\mathsf{C},\mathsf{L},\phi),$$

which is the group of ghosts quotiented by its subgroup of quasireflections, is junior. The locus of J-curve in $\overline{\mathcal{R}}^k_{g,\ell}$ is noted $J^k_{g,\ell}$.

Theorem 3.33. *For* $g \geq 4$*, the non-canonical locus of* $\overline{\mathcal{R}}^k_{g,\ell}$ *is formed by* T*-curves and* J*-curves, i.e. it is the union*

$$\operatorname{Sing}^{\mathrm{nc}}\overline{\mathcal{R}}^k_{g,\ell} = T^k_{g,\ell} \cup J^k_{g,\ell}.$$

About this result, we will prove a more general theorem in the next chapter (see Theorem 4.36). In particular we will show that the same classification is true for the moduli space of twisted G-covers. In [22, Theorem 5.8], the author gives a specific proof for the $\overline{\mathcal{R}}^k_{g,\ell}$ case.

Remark 3.34. We observe that Theorem 2.44 of Chiodo and Farkas [12], affirms exactly that in the case $k = 0$, $\ell \leq 6$ and $\ell \neq 5$, the J-locus $J^0_{g,\ell}$ is empty for every genus g, and therefore $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}^0_{g,\ell}$ coincides with the T-locus for these values of ℓ.

The J-locus is the "new" part of the non-canonical locus which appears passing from $\overline{\mathsf{M}}_g$ to one of its coverings $\overline{\mathsf{R}}^k_{g,\ell}$. For some values of ℓ and k this locus could be empty, however, a consequence of our analysis is that it is actually not empty for any $\ell > 2$ and $k \neq 0$.

We will exhibit an explicit decomposition of $J^k_{g,\ell}$ in terms of the strata $\mathcal{S}_{(\Gamma,M)}$. We point out a significant difference with respect to the description of the singular locus: in the case of $H^k_{g,\ell}$, we showed a decomposition in terms of loci whose generic point represents a two component curve, *i.e.* a vine curve. For the J-locus we do not have such an elegant minimal decomposition: for values of ℓ large enough, there exists strata representing J-curves with an arbitrary high number of components, such that each one of their smoothing is not a J-curve. Equivalently, there are decorated graphs with an arbitrary high number of vertices and admitting junior automorphisms, but such that each one of its contraction does not admit junior automorphisms.

Remark 3.35. We already observed that a ghost automorphism always acts trivially on loop edges of decorated dual graphs, and that quasireflections only act on separating edges. Thus we can ignore these edges in studying

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\theta)/\,\mathrm{QR}_C(\mathsf{C},\mathsf{L},\theta).$$

From this point we will automatically contract loops and separating edges as they appear. This is not a big change in our setting, in fact graphs without loops and separating edges are a subset of the graphs we considered until now. Reducing our analysis to this subset is done for the sole purpose of simplifying the notation.

Age is not well-behaved with respect to graph contraction, but there is another invariant which is better behaved: we will define a number associated to every ghost, which respects a super-additive property in the case of strata intersection (see Theorem 3.41). Here we only work under the condition of ℓ being a prime number.

3.3.2 Evaluating age with dual graphs

Given a graph Γ and a multiplicity index $M \in C^1(\Gamma;\mathbb{Z}/\ell)$ we recall the group $S(\Gamma;M)$ of even functions $f\colon \mathbb{E}(\Gamma) \to \mathbb{Z}/\ell$ such that for all e in $\mathbb{E}$, $f(e)$ is in the group $\mathbb{Z}/r(e) \subset \mathbb{Z}/\ell$, where $r(e)$ is the order of $M(e)$. We consider the multiplicative morphism

$$S(\Gamma;M) \to C^1(\Gamma;\mathbb{Z}/\ell),$$

such that $\mathsf{a} \mapsto M \cdot \mathsf{a}$ and define the subgroup

$$\mathcal{A}(\Gamma; M) \subset S(\Gamma; M)$$

which is the subset of automorphisms a such that $\mathsf{a} \cdot M$ is in $\mathrm{Im}\,\delta$. By Remark 2.29, $\mathcal{A}(\Gamma; M)$ is canonically identified with the group of ghost automorphisms $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)$ of any rooted curve whose multiplicity index is M.

We note (Γ_0, M) the decorated dual graph associated to Γ. As ℓ is a prime number, $S(\Gamma_0; M)$ is canonically identified to $C^1(\Gamma_0; \mathbb{Z}/\ell)$ via multiplication by M. When there is no risk of confusion, from now on we will not repeat the notation of $\mathbb{Z}/\ell$ and M on the label of groups $\mathcal{A}$, S and C^1.

Consider a graph contraction $\Gamma_0 \to \Gamma_1$ and if M is the multiplicity index associated to Γ_0, we denote by M_1 its restriction to $C^1(\Gamma_1; \mathbb{Z}/\ell)$. Because of contraction, the edge set $E(\Gamma_1)$ is a subset of $E(\Gamma_0)$, so that the group $S(\Gamma_1)$ is naturally immersed in $S(\Gamma_0)$, and $C^1(\Gamma_1)$ in $C^1(\Gamma_0)$. These immersions are compatible with multiplication by M, thus $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_0) \cap S(\Gamma_1)$ and we have a natural immersion of the $\mathcal{A}$ groups too.

Proposition 3.36. *Given a contraction* $(\Gamma_0, M) \to (\Gamma_1, M_1)$, *the elements of* $\mathcal{A}(\Gamma_1)$ *are those cochains of* $\mathcal{A}(\Gamma_0)$ *whose support is contained in* $E(\Gamma_1)$.

This gives an interesting correspondence between curve specialization, decorated graph contraction, strata inclusion and canonical immersions between the associated ghost automorphism groups.

We note V_0 and E_0 the cardinalities of $V(\Gamma_0)$ and $E(\Gamma_0)$ respectively. From the definition of $\mathcal{A}(\Gamma_0; \mathbb{Z}/\ell)$ and Proposition 2.8, we know that $\mathcal{A}(\Gamma_0) \cong \mathbb{Z}/\ell^{V_0 - 1}$, and we have an explicit basis for it. We consider a spanning tree T for Γ_0, we call $e_1, e_2, \ldots, e_{V_0-1}$ the edges of T, each one with an orientation. Then the cuts $\mathrm{cut}_{\Gamma_0}(e_i; T)$ form a basis of $\mathcal{A}(\Gamma_0)$. We can also write

$$\mathcal{A}(\Gamma_0) = \bigoplus_{i=1}^{V_0-1} \left(\mathrm{cut}_{\Gamma_0}(e_i; T) \cdot \mathbb{Z}/\ell\right).$$

Therefore, for an n-vine decorated graph $(\Gamma_{(2,n)}, M)$, the $\mathcal{A}(\Gamma_{(2,n)})$ is cyclic.

Remark 3.37. An element $\mathsf{a} \in \mathcal{A}(\Gamma_0)$ could be seen as living on stratum $\mathcal{S}_{(\Gamma_0, M)}$. We observe that if $(\Gamma_0, M) \to (\Gamma_1, M_1)$ is a contraction, then a lies in $\mathcal{A}(\Gamma_1)$ by the natural injection. Thus every automorphism living on $\mathcal{S}_{(\Gamma_0, M)}$ lives on the closure $\overline{\mathcal{S}}_{(\Gamma_0, M)}$.

If a ghost automorphism is junior, it carries a non-canonical singularity that spreads all over the closure of the stratum where the automorphism lives. This informal statement justifies the following definition.

Definition 3.38. The age of a stratum $\mathcal{S}_{(\Gamma_0, M)}$ is the minimum age of a ghost automorphism a in $\mathcal{A}(\Gamma_0)$. As in the case of group age, the age of an automorphism depends on the primitive root chosen, but the stratum age does not.

With this new notation, the J-locus could be written as follows,

$$J^k_{g,\ell} = \bigcup_{\operatorname{age} \mathcal{S}_{(\Gamma_0,M)}<1} \overline{\mathcal{S}}_{(\Gamma_0,M)}.$$

Indeed, if $[\mathsf{C},\mathsf{L},\theta] \in \overline{\mathcal{R}}^k_{g,\ell}$ has a junior ghost group, then this point lies on the closure of a junior stratum. Conversely, every point in the closure of a junior stratum has a junior ghost group.

Definition 3.39. We say that a set of contractions $\{(\Gamma_0, M) \to (\Gamma_i, M_i)\}$, for $i = 1, \dots, k$, *covers* Γ_0 if $E(\Gamma_0) = \bigcup_i E(\Gamma_i)$. We observe that if this set is a covering, then $\mathcal{A}(\Gamma_i, M_i) \subset \mathcal{A}(\Gamma_0, M)$ for every i and

$$\overline{\mathcal{S}}_{(\Gamma_0,M)} \subset \bigcap_{i=1}^k \overline{\mathcal{S}}_{(\Gamma_i,M_i)}.$$

Definition 3.40. An automorphism $\mathsf{a} \in \mathcal{A}(\Gamma_0)$ is *supported* on stratum $\mathcal{S}_{(\Gamma_0,M)}$ if its support is the whole $E(\Gamma_0)$ set. We observe that this property has an immediate moduli interpretation: an automorphism supported on $\mathcal{S}_{(\Gamma_0,M)}$ appears in the ghost group of every curve in $\overline{\mathcal{S}}_{(\Gamma_0,M)}$ but it does not appear in any other stratum whose closure contains $\mathcal{S}_{(\Gamma_0,M)}$.

The age of a strata intersection is not bounded by the sum of the ages of strata intersecting. However, there exists another invariant which has a superadditive property with respect to strata intersection. We will pay attention to the new automorphisms that appear at the intersection, *i.e.* those automorphisms supported on the intersection stratum, using the notion just introduced. We note by E the cardinality of $E(\Gamma)$, by V the cardinality of $V(\Gamma)$, by E_i the cardinalities of the edge sets $E(\Gamma_i)$ and by V_i the cardinalities of the edge sets $V(\Gamma_i)$.

Theorem 3.41. *Consider a covering* $(\Gamma_0, M) \to (\Gamma_i, M_i)$*, with* $i = 1, \dots, m$*, such that*

$$\mathcal{A}(\Gamma_0) = \sum_{i=1}^m \mathcal{A}(\Gamma_i).$$

Then for every a *supported on* $\mathcal{S}_{(\Gamma_0,M)}$ *we have*

$$\operatorname{age} \mathsf{a} - E_0 \geq \sum_{i=1}^m \left(\operatorname{age} \mathcal{S}_{(\Gamma_i,M_i)} - E_i \right).$$

To prove the theorem we need the following lemma

Lemma 3.42. *If* a *is supported on* $\mathcal{S}_{(\Gamma_0,M)}$*, then*

$$\operatorname{age} \mathsf{a} + \operatorname{age} \mathsf{a}^{-1} = E_0.$$

Proof. Given an edge $e \in E(\Gamma_0)$, by definition, $\mathsf{a}^{-1}(e), \mathsf{a}(e)$ are elements of $\mathbb{Z}/\ell$. As a is supported on $\mathcal{S}_{(\Gamma,M)}$, $\mathsf{a}(e) \neq 0$ for all e in $E(\Gamma_0)$, and then this component brings to $\operatorname{age} \mathsf{a}$ and $\operatorname{age} \mathsf{a}^{-1}$ respectively a value of $\mathsf{a}(e)/\ell$ and $(1 - \mathsf{a}(e)/\ell)$. As a consequence we obtain $\mathrm{age}\,\mathsf{a} + \mathrm{age}\,\mathsf{a}^{-1} = E = \mathrm{Codim}\,\mathcal{S}_{(\Gamma_0,M)}$. $\square$

As a direct consequence of the previous lemma, we have $\mathrm{age}\,\mathsf{a}^{-1} = E_0 - \mathrm{age}\,\mathsf{a}$. By hypothesis we can write,

$$\mathsf{a}^{-1} = \mathsf{a}_1 + \mathsf{a}_2 + \cdots + \mathsf{a}_m,$$

where $\mathsf{a}_i \in \mathcal{A}(\Gamma_i, M_i)$ for every i, and we call c_i the cardinality of a_i support. By subadditivity of age, we have $\mathrm{age}\,\mathsf{a}^{-1} \leq \sum \mathrm{age}\,\mathsf{a}_i$, then using Lemma 3.42 we obtain

$$\mathsf{a} - E \geq \sum_{i=1}^{m} \left(\mathrm{age}\,\mathsf{a}_i^{-1} - c_i\right).$$

By the fact that $c_i \leq E_i$ for every i, and by the definition of age for the strata, the Theorem is proved. $\square$

We observe that $E_i = \mathrm{Codim}\,\mathcal{S}_{(\Gamma_i,M_i)}$ by Proposition 3.28, we found an inequality about age involving geometric data. There is another formulation of the statement. We already observed that for every graph Γ, the first Betti number is

$$b_1(\Gamma) = E - (V - 1).$$

If the sum in the hypothesis is a direct sum, the rank condition is equivalent to $V_0 - 1 = \sum V_i - 1$. Therefore we have the following.

Corollary 3.43. *If the decorated graphs (Γ_i, M_i) cover (Γ_0, M) and*

$$\mathcal{A}(\Gamma_0) = \bigoplus_{i=1}^{m} \mathcal{A}(\Gamma_i),$$

then for every automorphism a supported on $\mathcal{A}(\Gamma_0)$ we have

$$\mathrm{age}\,\mathsf{a} - b_1(\Gamma_0) \geq \sum_{i=1}^{m} \left(\mathrm{age}\,\mathcal{S}_{(\Gamma_i,M_i)} - b_1(\Gamma_i)\right).$$

In the case of two strata intersecting, a rank condition implies the splitting of $\mathcal{A}(\Gamma_0, M)$ in a direct sum.

Lemma 3.44. *Consider the contractions of decorated graphs $(\Gamma_0, M) \to (\Gamma_i, M_i)$ for $i = 1, 2$. If Γ_1 and Γ_2 cover Γ_0, and moreover*

$$V_0 - 1 = (V_1 - 1) + (V_2 - 1),$$

then we have

$$\mathcal{A}(\Gamma_0) = \mathcal{A}(\Gamma_1) \oplus \mathcal{A}(\Gamma_2).$$

Proof. Before proving it, we point out a useful fact: given any contraction $\Gamma_0 \to \Gamma_i$, the natural injection $\mathcal{A}(\Gamma_i) \hookrightarrow \mathcal{A}(\Gamma_0)$ sends cuts on cuts. We suppose, without loss of generalities, that $V_1 \leq V_2$ and we prove the lemma by induction on V_1.

The base case $V_1 = 1$ is empty, $\Gamma_0 = \Gamma_2$ and the thesis follows obviously. Now suppose $V_1 = q > 1$, then $V_2 < V_0$ and so there exists two vertices of Γ_0 connected by edges who lie in $E(\Gamma_1)$ but not in $E(\Gamma_2)$. We call e_1 one of these edges in $E(\Gamma_1)$ and T_1 a spanning tree of Γ_1 containing e_1. If $\mathrm{cut}_{\Gamma_1}(e_1; T_1)$ is the corresponding cut, it is also an element of $\mathcal{A}(\Gamma_0)$. By rank conditions it suffices to prove that $\mathcal{A}(\Gamma_0) = \mathcal{A}(\Gamma_1) + \mathcal{A}(\Gamma_2)$. Consider $\mathsf{a} \in \mathcal{A}(\Gamma_0)$ which is not a sum of elements in $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$. Now we define

$$\mathsf{a}' := \mathsf{a} - u \cdot \mathrm{cut}_{\Gamma_1}(e_1; T_1),$$

where u is the necessary integer such that $\mathsf{a}'(e_1) \equiv 0$. Consider the graphs (Γ_0', M') and (Γ_1', M_1') obtained contracting the edge e_1 in Γ_0 and Γ_1 respectively. By construction the contractions $\Gamma_0' \to \Gamma_1'$ and $\Gamma_0' \to \Gamma_2$ still respect the hypothesis, and $V(\Gamma_1') = V_1 - 1$. Therefore by induction the automorphism a', which is an element of $\mathcal{A}(\Gamma_0', M')$, is a sum

$$\mathsf{a}' = \mathsf{a}_1' + \mathsf{a}_2,$$

with $\mathsf{a}_1' \in \mathcal{A}(\Gamma_1') \subset \mathcal{A}(\Gamma_1)$ and $\mathsf{a}_2 \in \mathcal{A}(\Gamma_2)$. Finally $\mathsf{a} = u \cdot \mathrm{cut}_{\Gamma_1}(e; T_1) + \mathsf{a}_1' + \mathsf{a}_2$, then it is in $\mathcal{A}(\Gamma_1) + \mathcal{A}(\Gamma_2)$. This is a contradiction and so the lemma is proved. $\square$

3.3.3 The locus $J^k_{g,\ell}$ for $\ell = 2$

In what follows we will find, for some small prime values of ℓ, a description of $J^k_{g,\ell}$ via our stratification. Before starting we point out that our analysis will focus on the cases $J^0_{g,\ell}$ and $J^1_{g,\ell}$.

Proposition 3.45. *If ℓ is prime, we have a natural stack isomorphism $\mathsf{R}^1_{g,\ell} \cong \mathsf{R}^k_{g,\ell}$ for every k between 1 and $\ell - 1$.*

Proof. Consider a scheme S and a triple $(\mathsf{C} \to S, \mathsf{L}, \phi)$ in $\mathsf{R}^1_{g,\ell}(S)$. Consider the map sending it to

$$(\mathsf{C} \to S, \mathsf{L}^{\otimes k}, \phi^{\otimes k}) \in \mathsf{R}^k_{g,\ell}(S).$$

As ℓ is prime and $k \not\equiv 0 \mod \ell$, this morphism has a canonical inverse, so we obtained an isomorphism of categories. $\square$

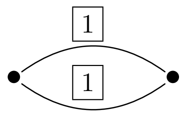
In this case the J-locus is always empty. Indeed, every automorphism in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)/\mathrm{QR}$ must have a support of cardinality at least 2, but for every edge in the support, a ghost a has a contribution of $1/2$ to its age. Hence there are no junior automorphisms in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \theta)/\mathrm{QR}$. This result was already obtained by Farkas and Ludwig for the Prym space $\overline{\mathcal{R}}^0_{g,2}$ in [19], and by Ludwig for $\overline{\mathcal{R}}^1_{g,2}$ in [26].

3.3.4 The locus $J^k_{g,\ell}$ for $\ell = 3$

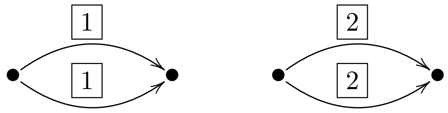
The process of finding the $J^k_{g,\ell}$ decomposition, for specific values of ℓ and k, will always follows three steps.

Step 1. We identify at first the graphs which can support a junior automorphism in $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \theta)/\mathrm{QR}$, *i.e.* those graphs with $E < \ell$ and no separating edges. If Γ is one of these graphs, we identify the $\mathbb{Z}/\ell$-valued automorphisms supported on Γ, *i.e.* the elements of $\bigoplus_{e \in E(\Gamma)} \mathbb{Z}/\ell$ which are non-trivial on every edge and are junior. These automorphisms are the junior elements in $\mathrm{Aut}_C(\mathsf{C})$ for an ℓ-twisted curve whose dual graph is Γ.

If $\ell = 3$ there is only one junior automorphism which can be supported on a $\mathbb{Z}/3$-valued decorated graph, the one represented in the image below and supported on $\Gamma_{(2,2)}$.



Step 2. For each one of these junior automorphisms, we search for multiplicity cochains that respect the lift condition of Theorem 2.28 on the automorphisms above. In this case the only possibilities are the following cochains



M_1 M_2

In fact, the two decorated graphs $(\Gamma_{(2,2)}, M_1)$ and $(\Gamma_{(2,2)}, M_2)$ are isomorphic by the isomorphism inverting the two vertices. Thus for $\ell = 3$, there is only one class of decorated graphs admitting junior automorphisms.

Step 3. By Proposition 2.24, there is an additional multidegree condition that the decorated graph must satisfy.

$$\sum_{e_+ = v_i} M_1(e) \equiv \deg \omega|_{v_i}^{\otimes k} \equiv k \cdot (2g_i - 2 + n_i) \mod \ell \quad \forall v_i \in V(\Gamma). \tag{3.1}$$

By Proposition 3.45 we can focus on the cases $k = 0$ and 1. If $k = 1$ the condition of (3.1) is empty, because 2 and 3 are coprime and there always exists a sequence of g_i satisfying the equality. Then we have

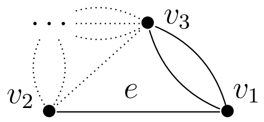
$$J^1_{g,3} = \overline{\mathcal{S}}_{(\Gamma_{(2,2)}, M_1)}.$$

In case $k = 0$, by (3.1) we have $\sum_{e_+ = v} M_1(e) \equiv 0 \mod 3$ for both vertices, but this condition is not satisfied by $\big(\Gamma_{(2,2)}, M_1\big)$, then

$$J^0_{g,3} = \varnothing.$$

3.3.5 The locus $J^k_{g,\ell}$ for $\ell = 5$

Step 1 and 2. For $\ell = 5$, every graph (Γ, M) such that there exists a junior automorphism $\mathsf{a} \in \mathcal{A}(\Gamma, M)$, contracts on a vine stratum. This is a consequence of the following lemma.



Lemma 3.46. *If ℓ is a prime number, consider a decorated graph (Γ, M) such that there exists a vertex $v_1 \in V(\Gamma)$ connected with exactly two vertices v_2, v_3 and such that between v_1 and v_2 there is only one edge called e.*

If $\mathcal{S}_{(\Gamma,M)}$ is a junior stratum, then there exists a non-trivial decorated graph contraction $(\Gamma, M) \to (\Gamma_1, M_1)$ such that $\mathcal{S}_{(\Gamma_1,M_1)}$ is also junior.

Remark 3.47. This lemma allows one to simplify the analysis of junior strata. Every stratum labeled with a graph containing the configuration above, can be ignored in the analysis. Indeed, if $\mathcal{S}_{(\Gamma,M)}$ is a subset of the J-locus, there exists a decorated graph (Γ_1, M_1) with less vertices such that $\mathcal{S}_{(\Gamma,M)} \subset \overline{\mathcal{S}}_{(\Gamma_1,M_1)} \subset J^k_{g,\ell}$.

Proof of the Lemma. Consider $\mathsf{a} \in \mathcal{A}(\Gamma, M)$ such that $\operatorname{age} \mathsf{a} < 1$. If a is not supported on $\mathcal{S}_{(\Gamma,M)}$, we contract one edge where a acts trivially and the lemma is proved. Thus we suppose a supported on $\mathcal{S}_{(\Gamma,M)}$. We call e' one of the edges connecting v_1 and v_3. We consider a spanning tree T of Γ passing though e' and not passing through e. Then we call Γ_1 and Γ_2 the two contractions of Γ obtained by contracting respectively e' and $E_T \backslash \{e'\}$. By Lemma 3.44, we have

$$\mathcal{A}(\Gamma, M) = \mathcal{A}(\Gamma_1, M_1) \oplus \mathcal{A}(\Gamma_2, M_2).$$

Therefore we use Theorem 3.41 to obtain

$$\operatorname{age} \mathsf{a} - E \geq \left(\operatorname{age} \mathcal{S}_{(\Gamma_1,M_1)} - E(\Gamma_1)\right) + \left(\operatorname{age} \mathcal{S}_{(\Gamma_2,M_2)} - E(\Gamma_2)\right).$$

As $E = E(\Gamma_1) + E(\Gamma_2) - 1$ by construction, the fact that a is junior implies that at least one between $\mathcal{S}_{(\Gamma_1,M_1)}$ and $\mathcal{S}_{(\Gamma_2,M_2)}$ is junior. $\square$

The configuration of Lemma 3.46 appears in every non-vine graph with less than 5 edges. As a consequence the reduction to vine strata follows.

To summarize these vine strata we introduce a new notation. Consider a n-vine graph, with vertices v_1 and v_2 and edges $e_1, \ldots, e_n$ all taken oriented from v_1 to v_2. If the multiplicity index M on the graph takes values $M(e_i)$ in $\mathbb{Z}/\ell$, we note this decorated graph by $(M(e_1), M(e_2), \ldots, M(e_n))$. For example the graphs appeared in the precedent paragraph are noted $(1,1)$ and $(2,2)$. We can now state the ten classes of vine graphs which support a junior ghost for $\ell = 5$,

$$(1,1),\ (2,2),\ (1,2),\ (1,3),\ (1,1,1),\ (2,2,2),\ (1,1,3),\ (2,2,1),\ (1,1,1,1),\ (2,2,2,2).$$

Step 3. For $k = 1$, equation (3.1) is always respected for some genus labellings of the graph. Therefore we have

$$J^1_{g,5} = \overline{\mathcal{S}}_{(1,1)} \cup \overline{\mathcal{S}}_{(2,2)} \cup \overline{\mathcal{S}}_{(1,2)} \cup \overline{\mathcal{S}}_{(1,3)} \cup \overline{\mathcal{S}}_{(1,1,1)} \cup \overline{\mathcal{S}}_{(2,2,2)} \cup \overline{\mathcal{S}}_{(1,1,3)} \cup \overline{\mathcal{S}}_{(2,2,1)} \cup \overline{\mathcal{S}}_{(1,1,1,1)} \cup \overline{\mathcal{S}}_{(2,2,2,2)}.$$

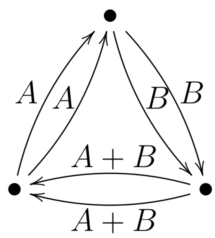
If $k = 0$ the equation is satisfied by two vine graphs, and we obtain the following

$$J^0_{g,5} = \overline{\mathcal{S}}_{(1,1,3)} \cup \overline{\mathcal{S}}_{(2,2,1)}.$$

In particular, this result fills the gap in Chiodo and Farkas analysis in [12]. They proved that the J-locus in the space $\overline{\mathcal{R}}^0_{g,\ell}$ is empty for $\ell \leq 6$ and $\ell \neq 5$.

3.3.6 The locus $J^k_{g,\ell}$ for $\ell = 7$

Step 1 and 2. Using Lemma 3.46, we observe that for $\ell = 7$ there are two kinds of graphs admitting junior automorphisms. The first kind is the usual vine graph, but we can also have a 3-cycle graph such that every pair of vertices is connected by two edges. In this second case, the only possible automorphism with age lower than 1 takes value 1 on every edge. As a consequence the possible decorations are like in figure below.



We define two sets of decorated graphs

$$V_7 := \{\text{vine decorated graphs admitting junior automorphism}\}/\cong$$

$$C_7 := \{\text{graphs decorated as in the figure}\}/\cong .$$

Step 3. If $k = 1$, condition (3.1) is always verified by some genus labeling. Therefore, we have

$$J^1_{g,7} = \bigcup_{(\Gamma,M)\in V_7} \overline{\mathcal{S}}_{(\Gamma,M)} \cup \bigcup_{(\Gamma,M)\in C_7} \overline{\mathcal{S}}_{(\Gamma,M)}.$$

If $k = 0$, we call V'_7 the subset of V_7 of decorated graphs respecting equation (3.1). Every graph in C_7 does not respect the equation. Indeed, we must have $4A + 2B \equiv 0 \mod 7$ and $2B - 2A \equiv 0 \mod 7$, therefore $A \equiv B \equiv 0$ which is not allowed. Finally, we have

$$J^0_{g,7} = \bigcup_{(\Gamma,M)\in V'_7} \overline{\mathcal{S}}_{(\Gamma,M)}.$$

In other words, the J-locus of $\overline{\mathcal{R}}^0_{g,7}$ is a union of vine strata.

Chapter 4

Moduli of twisted G-covers

In this chapter we consider the moduli space of twisted G-curves $\overline{\mathcal{R}}_{g,G}$ for any finite group G, and we develop an analysis similar to the one that we did for the moduli spaces of rooted curves $\overline{\mathcal{R}}^k_{g,\ell}$. In particular, in the first section we update the dual graph tools. In the second section we show a similar description of the singular locus, and moreover the fact that the non-canonical locus $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G}$ is the union of the T-locus and the J-locus also in this case. Finally, we show the pluricanonical forms extension result 4.43 for the case G abelian group, and state a similar conjecture in the case $G = S_3$ the symmetric group of order 3.

4.1 Decorated dual graphs and G-covers

4.1.1 Graph G-covers

In Section 2.1 we introduced graphs with a G-action, and the associated groups of 0-cochains and 1-cochains. To treat twisted G-covers and their associated dual graphs, we need to focus on the notion of graph G-cover. Given a graph Γ, we start by recalling the natural differential $\delta\colon C^0(\Gamma) \to C^1(\Gamma)$, this can be defined also in the case of a general finite group G.

$$\delta\colon C^0(\Gamma; G) \to C^1(\Gamma; G)$$

such that

$$\delta a(e) := a(e_+) \cdot a(e_-)^{-1}, \quad \forall a \in C^0(\Gamma; G)\ \forall e \in \mathbb{E}.$$

Given a not necessarily connected graph Γ, we note $\mathcal{T}(\Gamma)$ the set of connected components of the graph Γ, and we observe that the G-action on Γ induces naturally a G-action on $\mathcal{T}(\Gamma)$ too. The exterior differential fits into an useful exact sequence of groups

$$0 \to \operatorname{Hom}^G(\mathcal{T}(\Gamma), G) \xrightarrow{i} C^0(\Gamma; G) \xrightarrow{\delta} C^1(\Gamma; G). \tag{4.1}$$

Where the injection i sends $f \in \operatorname{Hom}^G(\mathcal{T}(\Gamma), G)$ on the cochain a such that for every component $\gamma \in \mathcal{T}(\Gamma)$, a is constantly equal to $f(\gamma)$ on γ.

Remark 4.1. We recall that for any group we can define a (non-associative) $\mathbb{Z}$-action via $h \cdot n := h^n$ for all h in G and $n \in \mathbb{Z}$.

We generalize Proposition 2.3 to the case of any finite group G.

Proposition 4.2. *A* 1*-cochain b is in* $\operatorname{Im}\delta$ *if and only if, for every circuit* $\mathcal{K} = (e_1, \ldots, e_k)$ *in* $\mathbb{E}$*, we have*

$$b(\mathcal{K}) := b(e_1) \cdot b(e_2) \cdots b(e_k) = 1.$$

Proof. If $b \in \operatorname{Im}\delta$, the condition above is easily verified. To complete the proof we will show that if the condition si verified, then there exists a cochain $a \in C^0(\Gamma; G)$ such that $\delta a = b$. We choose a vertex $v \in V(\Gamma)$ and pose $a(v) = 1$, for any other vertex $w \in V(\Gamma)$ we consider a path $\mathcal{P} = (e_1, \ldots, e_m)$ starting in v and ending in w. We pose

$$a(w) := b(\mathcal{P}) = b(e_1) \cdots b(e_m).$$

By the condition on circuits, the cochain a is well defined, and by construction we have $b = \delta a$. $\square$

We can now introduce edge contraction for a graph with a non-trivial G-action.

Definition 4.3. Given a graph Γ with a G-action, a vertex set V and edge set E, we choose a subset $D \subset E$ which is stable by the G-action. Contracting edges in D means taking the graph Γ_0 such that:

1. the edge set of Γ_0 is $E_0 := E \backslash D$;
2. given the relation in V, $v \sim w$ if v and w are linked by an edge $e \in D$, the vertex set of Γ_0 is $V_0 := V/\sim$.

The G-action on Γ induces naturally a G-action on Γ_0.

All the properties of graph contraction proved in Section 2.1.2 are still verified, because they do not depend on the G-action.

Given any graph $\tilde{\Gamma}$ with a G-action, we define its G-quotient

$$\Gamma := \tilde{\Gamma}/G$$

by $V(\Gamma) := V(\tilde{\Gamma})/G$ and $E(\Gamma) := E(\tilde{\Gamma})/G$. The conditions on the G-action stated at the beginning of section 2.1.1 assure that Γ is well defined. Moreover, the edge contraction of $D \subset E(\tilde{\Gamma})$ is compatible with the quotient, so that if $\tilde{\Gamma} \to \tilde{\Gamma}_0$ is the contraction, then

$$\tilde{\Gamma}/G \to \tilde{\Gamma}_0/G$$

is the contraction of D/G.

We call the natural morphism $\tilde{\Gamma} \to \Gamma$ a graph G-cover. For any vertex $\tilde{v}$ of $\tilde{\Gamma}$, we note $H_{\tilde{v}}$ its stabilizer in G. For every vertex v of Γ, its preimages in $V(\tilde{\Gamma})$ all have a stabilizer in the same conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$, *i.e.* for all $\tilde{v}$ in $f^{-1}(v)$ we have $H_{\tilde{v}} \in \mathcal{H}$. Moreover, for every subgroup H in the class $\mathcal{H}$, there exists a vertex preimage $\tilde{v}$ of v with stabilizer exactly H. In particular the cardinality of the v fiber is $|G|/|H|$ where $|H|$ is the cardinality of any subgroup in $\mathcal{H}$. The same is true for any edge e in $E(\Gamma)$.

This behavior suggests to introduce a labelling on the vertices of graph Γ.

We observe that it is possible to give another description of the cochain groups of $\tilde{\Gamma}$ by considering the graph G-cover $\tilde{\Gamma} \to \Gamma$.

Proposition 4.4. *If $F \to C$ is an admissible G-cover of a stable curve, and $f\colon \tilde{\Gamma} \to \Gamma$ is its associated graph G-cover, then we have the identification*

$$C^0(\tilde{\Gamma}; G) = \prod_{v \in V(\Gamma)} \mathrm{Hom}^G(f^{-1}(v), G).$$

Moreover, if we choose a privileged orientation for every edge e in $E(\Gamma)$, then we have a canonical isomorphism

$$C^1(\tilde{\Gamma}; G) \cong \prod_{e \in E(\Gamma)} \mathrm{Hom}^G(f^{-1}(e), G).$$

Definition 4.5. Given a graph $\tilde{\Gamma}$ with a G-action on it, its *subgroup function* is the map $V(\tilde{\Gamma}) \to \mathrm{Sub}(G)$ sending any vertex $\tilde{v}$ on its stabilizer $H_{\tilde{v}}$.

Given the associated graph G-cover $\tilde{\Gamma} \to \Gamma$, the associated *class function* of graph Γ is a map $V(\Gamma) \to \mathcal{T}(G)$ sending any vertex v on the conjugacy class $\mathcal{H}$ of the stabilizer $H_{\tilde{v}}$ for any vertex $\tilde{v}$ in the set $f^{-1}(v) \subset V(\tilde{\Gamma})$.

Consider a 1-cochain b in $C^1(\tilde{\Gamma}; G)$. For any oriented edge e of Γ, and for every preimage $\tilde{e}$ in $f^{-1}(e)$, the conjugacy class $[\![h_e]\!]$ of $b(\tilde{e})$ in $[\![G]\!]$ only depends on edge e. Moreover, for every element in $\tilde{h}$ in $[\![h_e]\!]$, there exists an edge preimage $\tilde{e}'$ such that $b(\tilde{e}') = \tilde{h}$.

Definition 4.6. Given a graph G-cover $f\colon \tilde{\Gamma} \to \Gamma$, and a 1-cochain b on $\tilde{\Gamma}$, the associated *type function* of b is a map $M_b\colon \mathbb{E}(\Gamma) \to [\![G]\!]$ such that for any e in $\mathbb{E}(\Gamma)$, $M_b(e)$ is the conjugacy class of any $b(\tilde{e})$ with $\tilde{e}$ in the set $f^{-1}(e)$.

Given a type function on Γ and an oriented edge $e \in \mathbb{E}(\Gamma)$, the order of $M(e)$ is well defined as the order of any element in the conjugacy class, therefore we say that $r(e)$ is the order of e with respect to M. We observe that $r(e) = r(\bar{e})$ for all e, therefore the order is well defined as a function on the edge set (without orientation), $r\colon E(\Gamma) \to \mathbb{Z}_{>0}$. In the following it will be useful to consider every graph with a type function, as it was equipped with the cyclic group $\mathbb{Z}/r(e)$ at the edge e.

Remark 4.7. If we have an admissible G-cover $F \to C$, we consider $\tilde{\Gamma}$ and Γ the dual graphs associated respectively to F and C. The first with the naturally associated G-action. Therefore $\Gamma = \tilde{\Gamma}/G$ and $\tilde{\Gamma} \to \Gamma$ is a graph G-cover.

Remark 4.8. We recall the correspondence between admissible G-covers over a stable curve C, and twisted G-covers over C, treated in section 1.3.4. By this, the dual graphs $\tilde{\Gamma}$ and Γ introduced for any admissible G-cover, are well defined for the associated twisted G-cover, too.

For any admissible G-cover $F \to C$, consider the dual graphs $\tilde{\Gamma} = \Gamma(F)$ and $\Gamma = \Gamma(C)$, consider the cochain $b_F \in C^1(\tilde{\Gamma}; G)$ such that for any oriented edge $\tilde{e}$ in $\mathbb{E}(\tilde{\Gamma})$, $b_F(\tilde{e})$ is the local index of the associated node with the orientation inducing the privileged branch (see Definition 1.30 and Remark 1.54). The correspondent type function $M_{b_F}\colon \mathbb{E}(\Gamma) \to [\![G]\!]$, is the function sending any oriented edge e in the G-type of the associated node.

Definition 4.9. We call *indices cochain* of the admissible G-cover $F \to C$, the cochain b_F in $C^1(\tilde{\Gamma}; G)$. Moreover, we call type function of $F \to C$, the type function M_{b_F} associated to the indices cochain b_F. When there is no risk of confusion, we note the type function of $F \to C$ simply by M.

Remark 4.10. In the case of an admissible G-cover with G abelian group, the type function uniquely determines the indices cochain. In the case of $G = \boldsymbol{\mu}_\ell$ a cyclic group, we already introduced the type function, see Remark 2.21, by using the multiplicity index notation of Chiodo and Farkas.

We can now update the definition of decorated dual graph to G-covers. Consider an admissible G-cover $F \to C$ and the associated graph G-cover $\tilde{\Gamma} \to \Gamma$. We define the contracted graphs $\tilde{\Gamma}_0$ and Γ_0. Let $D \subset \mathbb{E}(\tilde{\Gamma})$ bet the subset of edges where the cochain b_F of local indices is trivial, that is

$$D := \{\tilde{e} \in \mathbb{E}(\tilde{\Gamma}) |\ b_F(\tilde{e}) = 1\}.$$

The graph $\tilde{\Gamma}_0$ is the result of D contraction on $\tilde{\Gamma}$. The graph Γ_0 is the quotient $\tilde{\Gamma}_0/G$, *i.e.* it is the graph Γ after conctraction of the edges where the type function M has value $[\![1]\!]$.

Definition 4.11. The pair $(\Gamma_0(C), M)$, where M is the restriction of the type function on the contracted edge set, is called *decorated graph* of the admissible G-cover $F \to C$ (or equivalently of the associated twisted G-cover (C, ϕ)). If the function M is clear from context, we will refer also to $\Gamma_0(C)$ or Γ_0 alone as the decorated graph.

4.1.2 Basic theory of sheaves in groups and torsors

To have a good description of the ghost automorphism group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$, we introduce some basic notions in sheaf and torsor theory. We refer in particular to Calmès and Fasel paper [7] for notations and definitions.

Consider a scheme S and a site $\mathbb{T}$ over the category Sch/S of S-schemes. An S-sheaf is a sheaf over the category Sch/S with the $\mathbb{T}$ topology. Consider $\mathcal{G}$ an S-sheaf in groups, and P an S-sheaf in sets with a left $\mathcal{G}$-action.

Definition 4.12 (torsor)**.** The sheaf P is a torsor under $\mathcal{G}$ (or a $\mathcal{G}$-torsor) if

1. the application
$$\mathcal{G} \times P \to P \times P,$$
where the components are the action and the identity, is an isomorphism;

2. for every covering $\{S_i\}$ of S, $P(S_i)$ is non-empty for every i.

For example, if G is a finite group, a principal G-bundle over a scheme S, is a $\mathcal{G}$-torsor, where $\mathcal{G}$ is the S-sheaf in groups defined by

$$\mathcal{G}(S') := S' \times G \quad \forall S' \text{ scheme in } Sch/S.$$

When we consider any S-sheaf in groups $\mathcal{G}$ as acting on itself, we get a $\mathcal{G}$-torsor called *trivial $\mathcal{G}$-torsor*.

Proposition 4.13 (see [7, Proposition 2.2.2.4])**.** *An S-sheaf P with a left $\mathcal{G}$-action is a torsor if and only if it is $\mathbb{T}$-locally isomorphic to the trivial torsor $\mathcal{G}$.*

Consider two S-sheaves P and P' with $\mathcal{G}$-action respectively on the left and on the right.

Definition 4.14. We note $P' \wedge^{\mathcal{G}} P$ the cokernel sheaf of the two morphisms

$$\mathcal{G} \times P' \times P \rightrightarrows P' \times P$$

given by the $\mathcal{G}$-action on P and P' respectively. This is called *contracted product.*

Equivalently, $P' \wedge^{\mathcal{G}} P$ is the sheafification of the presheaf of the orbits of $\mathcal{G}$ acting on $P' \times P$ by

$$(h, (z', z)) \mapsto (z'h^{-1}, hz).$$

Remark 4.15. If $\mathcal{G}$ is the sheaf in groups constantly equal to $\mathbb{C}^*$ and P, P' are two line bundles, then the contracted product is simply the usual tensor product $P \otimes P'$.

If another S-sheaf in groups $\mathcal{G}'$ acts on the left on P', then the contracted product $P' \wedge^{\mathcal{G}} P$ has a $\mathcal{G}'$-action on the left, too. The same is true for a $\mathcal{G}'$-action on the right on P.

Lemma 4.16 (see [7, Lemma 2.2.2.10])**.** *The $\wedge$ construction is associative. Consider $\mathcal{G}$ and $\mathcal{G}'$ two S-sheaves in groups, P and P' two S-sheaves with respectively left $\mathcal{G}$-action and right $\mathcal{G}'$-action, finally P'' an S-sheaf with $\mathcal{G}'$-action on the left and $\mathcal{G}$-action on the right, and the actions commute. Then there exists a canonical isomorphism*

$$(P' \wedge^{\mathcal{G}'} P'') \wedge^{\mathcal{G}} P \cong P' \wedge^{\mathcal{G}'} (P'' \wedge^{\mathcal{G}} P).$$

Moreover, we have $\mathcal{G} \wedge^{\mathcal{G}} P \cong P$ for every $\mathcal{G}$-torsor P.

Proposition 4.17 (see [7, Proposition 2.2.2.12])**.** *Consider a morphism $\mathcal{G} \to \mathcal{G}'$ of S-sheaves in groups. Therefore $\mathcal{G}$ acts on the right on $\mathcal{G}'$. We obtain the map*

$$P \mapsto \mathcal{G}' \wedge^{\mathcal{G}} P,$$

from the category of $\mathcal{G}$-torsors to $\mathcal{G}'$-torsors. This is a functor.

Definition 4.18. Given an S-scheme S' and a site $\mathbb{T}$ on Sch/S, we note $H^1_{\mathbb{T}}(S', \mathcal{G})$ the pointed set of $\mathcal{G}$-torsors (on the left) over S' with respect to the $\mathbb{T}$ topology. The base point of the set is the torsor $\mathcal{G}$ itself.

This cohomology type notation fits with the cohomology type behavior we are going to describe. We refer for the following results for example to [7, §2.2.5] or [23, Chap.3]. Consider three S-sheaves in groups fitting in a short exact sequence

$$1 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 1. \tag{4.2}$$

This gives a long exact sequence in cohomology

$$1 \to \mathcal{G}_1(S) \to \mathcal{G}_2(S) \xrightarrow{\delta} \mathcal{G}_3(S) \xrightarrow{\tau} H^1_{\mathbb{T}}(S, \mathcal{G}_1) \to H^1_{\mathbb{T}}(S, \mathcal{G}_2) \to H^1_{\mathbb{T}}(S, \mathcal{G}_3). \tag{4.3}$$

This is an exact sequence of pointed sets, and it is exact in $\mathcal{G}_1(S)$ and $\mathcal{G}_2(S)$ as a sequence of groups.

To describe the map τ, observe that $\mathcal{G}_3 = \mathcal{G}_2/\mathcal{G}_1$. By [23, Proposition 3.1.2], the set $\mathcal{G}_3(S)$ is in bijection with the set of sub-$\mathcal{G}_1$-torsors of $\mathcal{G}_2$. Any object Q in $\mathcal{G}_3(S)$ in sent by τ on the $\mathcal{G}_1$-torsor induced by the pullback along $\mathcal{G}_2 \to \mathcal{G}_2/\mathcal{G}_1$. As a consequence $\tau(Q)$ is a $\mathcal{G}_1$-bitorsor (on the left and on the right).

Via the τ map we also have a $\mathcal{G}_3(S)$-action on $H^1_{\mathbb{T}}(S, \mathcal{G}_1)$. Indeed, for every Q in $\mathcal{G}_3(S)$ and for every $\mathcal{G}_1$-torsor P, we obtain by contracted product the $\mathcal{G}_1$-torsor

$$\tau(Q) \wedge^{\mathcal{G}_1} P.$$

To state the next proposition, we observe that $\mathcal{G}_1$ acts trivially on the right on $\mathcal{G}_3$, therefore given any $\mathcal{G}_1$-torsor P (on the left), we have the identification of sheaves

$$\mathcal{G}_3 \wedge^{\mathcal{G}_1} P = \mathcal{G}_3.$$

Consider the map $\mathcal{G}_2 \to \mathcal{G}_3$ in the short exact sequence (4.2), and its image via the contracted product functor of Proposition 4.17,

$$\mathcal{G}_2 \wedge^{\mathcal{G}_1} P \xrightarrow{\bullet \wedge P} \mathcal{G}_3 \wedge^{\mathcal{G}_1} P = \mathcal{G}_3.$$

We note $\mathcal{G}_2^P := \mathcal{G}_2 \wedge^{\mathcal{G}_1} P$, and $\delta^P \colon \mathcal{G}_2^P(S) \to \mathcal{G}_3(S)$.

Proposition 4.19 (see [23, Proposition 3.3.3])**.** *For every P in $H^1_{\mathbb{T}}(S, \mathcal{G}_1)$, the stabilizer of P by the $\mathcal{G}_3(S)$-action induced by τ, is the image of $\delta^P \colon \mathcal{G}_2^P(S) \to \mathcal{G}_3(S)$.*

4.2 Singularities of $\overline{\mathcal{R}}_{g,G}$

4.2.1 Ghost automorphisms of a twisted curve

We generalize the notion of ghost automorphisms for twisted G-covers. We start by considering the automorphism group of (C, ϕ), where C is a twisted curve and $\phi \colon \mathsf{C} \to BG$ is a twisted G-cover.

$$\mathrm{Aut}(\mathsf{C}, \phi) := \{(\mathsf{f}, \rho) |\ \mathsf{f} \in \mathrm{Aut}(\mathsf{C}),\ \rho \colon \phi \xrightarrow{\cong} \mathsf{f}^*\phi\}.$$

We observe that this group does not act faithfully on the universal deformation $\mathrm{Def}(\mathsf{C}, \phi)$. Indeed, Proposition 1.38 and Remark 1.39 describe the group $\mathrm{Aut}_{\mathsf{C}}(\mathsf{C}, \phi)$ of automorphisms of (C, ϕ) acting trivially on C, and these automorphisms act trivially on $\mathrm{Def}(\mathsf{C}, \phi)$, too. Thus it becomes natural to consider the group

$$\underline{\mathrm{Aut}}(\mathsf{C}, \phi) = \mathrm{Aut}(\mathsf{C}, \phi) / \mathrm{Aut}_{\mathsf{C}}(\mathsf{C}, \phi) = \{\mathsf{f} \in \mathrm{Aut}(\mathsf{C}) |\ \mathsf{f}^*\phi \cong \phi \text{ as twisted } G\text{-covers}\}.$$

Remark 4.20. The local description of $\overline{\mathcal{R}}_{g,G}$ at $[\mathsf{C}, \phi]$ could be rewritten

$$\mathrm{Def}(\mathsf{C}) / \underline{\mathrm{Aut}}(\mathsf{C}, \phi).$$

The coarsening $\mathsf{C} \to C$ induces moreover a group morphism

$$\underline{\mathrm{Aut}}(\mathsf{C}, \phi) \to \mathrm{Aut}(C).$$

We note the kernel and the image of this morphism by $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)$ and $\mathrm{Aut}'(C)$ (see also [12, chap. 2]). They fit into the following short exact sequence,

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C},\phi) \to \underline{\mathrm{Aut}}(\mathsf{C},\phi) \to \mathrm{Aut}'(C) \to 1. \tag{4.4}$$

As in the case of rooted curves, the group $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)$ is called the group of ghost automorphisms of the twisted G-cover $\phi\colon \mathsf{C} \to BG$.

As for rooted curves, we start from the group $\mathrm{Aut}_C(\mathsf{C})$ of the automorphisms of C whose coarsening is the identity on C. Consider a node q of C whose local picture is $[\{x'y'=0\}/\boldsymbol{\mu}_r]$. Consider an automorphism $\eta \in \mathrm{Aut}_C(\mathsf{C})$, then the local action of η at q can be represented by the automorphism of $V=\{x'y'=0\}\subset \mathbb{A}^2$ such that

$$(x',y') \mapsto (\xi x', y'),$$

with ξ a primitive root in $\boldsymbol{\mu}_r$. We observe moreover that $(\xi x', y') \equiv (\xi^{u+1}x', \xi^{-u}y')$ by the $\boldsymbol{\mu}_r$-action on V. Anyway, when it is not specified otherwise, we will use the lifting acting trivially on the y' coordinate. Consider the dual graph $\Gamma = \Gamma(\mathsf{C})$ and the type function associated to the twisted G-cover (C,ϕ). Consider a node q of C and its associated edge e in $E(\Gamma)$. By Remark 1.53, the order r of the q stabilizer, is the same as the order of e with respect to the type function M.

As a consequence of the definition of $\mathrm{Aut}_C(\mathsf{C})$, the action of η outside the C nodes is trivial. Then the whole group $\mathrm{Aut}_C(\mathsf{C})$ is generated by automorphisms of the form $(x',y') \mapsto (\xi x', y')$ on a node, and trivial elsewhere. We are interested in representing $\mathrm{Aut}_C(\mathsf{C})$ as acting on the edges of the dual graph, thus we introduce the following group.

Definition 4.21. Given a graph Γ (in this case with no G-action) and a type function M on it, we note r_{lcm} as the least common multiple of all the orders $r(e)$ of the edges of Γ. We define the group

$$S(\Gamma; M) := \{f\colon \mathbb{E}(\Gamma) \to \mathbb{Z}/r_{\mathrm{lcm}} |\ f(e) = f(\bar{e}) \in \mathbb{Z}/r(e) \subset \mathbb{Z}/r_{\mathrm{lcm}}\}.$$

Indeed, $\mathbb{Z}/r$ could be seen as canonically embedded in $\mathbb{Z}/r'$ if r divides r'.

If $(\Gamma_0(\mathsf{C}), M)$ is the decorated dual graph of the twisted G-cover (C,ϕ), we define a morphism $S(\Gamma_0(\mathsf{C}); M) \to \mathrm{Aut}_C(\mathsf{C})$, sending any function a on the automorphism whose action at the node associated to $e \in E(\Gamma_0)$ is

$$(x',y') \mapsto (\xi_{r(e)}^{\mathsf{a}(e)} \cdot x', y').$$

We observe that once again we have chosen a privileged $r(e)$th root of the unit, *i.e.* $\xi_{r(e)} = \exp(2\pi i/r(e))$. The morphism above only depends on this non-canonical choice, and it is an isomorphism, thus proving

$$\mathrm{Aut}_C(\mathsf{C}) \cong S(\Gamma_0(\mathsf{C}); M) \cong \bigoplus_{e\in E(\Gamma_0)} \mathbb{Z}/r(e).$$

Consider a function $\mathsf{a} \in S(\Gamma; M)$ and a cochain $b \in C^1(\tilde{\Gamma}; G)$. We can define a cochain

$$b \cdot \mathsf{a} \in C^1(\tilde{\Gamma}; G).$$

Consider an oriented edge $\tilde{e}$ in $\mathbb{E}(\tilde{\Gamma})$ and its projection e in $\mathbb{E}(\Gamma)$. We pose

$$(b \cdot \mathsf{a})(\tilde{e}) := b(\tilde{e}) \cdot \mathsf{a}(e).$$

This is again a 1-cochain compatible with the G-action.

Clearly, the group of ghost automorphisms

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi) = \{\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C}) |\ \mathsf{a}^*\phi \cong \phi\}$$

is a subset of $\mathrm{Aut}_C(\mathsf{C})$. To describe $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ we try to characterize the automorphisms in $\mathrm{Aut}_C(\mathsf{C})$ lifting to the twisted G-cover ϕ. If C is the coarse space of C, we consider the admissible G-cover $F \to C$ associated to (C, ϕ), and the normalization morphism $\mathsf{nor}\colon \overline{C} \to C$.

If $C_1, \ldots, C_V$ are the irreducible components of C, and $\overline{C}_1, \ldots, \overline{C}_V$ their normalizations, then

$$\overline{C} = \bigsqcup_{i=1}^{V} \overline{C}_i.$$

We call $F_i := F|_{C_i}$ the restriction of F to C_i for every i. Given the category Sch/C and the usual Zarisky topology $\mathbb{T}$, for every C-scheme U, we consider the restriction $F|_U \to U$ of F, which is an admissible G-cover over U. Moreover, we note $\overline{F} := \mathsf{nor}^*F$ the pullback cover, and in general for every C-scheme U, we note $\overline{U} := \mathsf{nor}^*U$ the $\overline{C}$-scheme obtained by pulling back U.

Definition 4.22. We define the C-sheaf in groups $\mathfrak{H}^F$ such that,

$$\mathfrak{H}^F(U) := \mathrm{Aut}_{\mathrm{Adm}}(U, F|_U).$$

We observe that F can be seen as a C-sheaf with a left $\mathfrak{H}^F$-action. We have a short exact sequence of C-sheaves in groups,

$$1 \to \mathfrak{H}^F \to \mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F \xrightarrow{t} \mathfrak{H}^F|_{\mathrm{Sing}\, C} \to 1. \tag{4.5}$$

The central sheaf is defined over any C-scheme U as

$$\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(U) = \mathrm{Aut}_{\mathrm{Adm}}(\overline{U}, \overline{F}|_{\overline{U}}).$$

There exists a $2:1$ cover $\overline{F}|_{\mathrm{Sing}\, C} \to F|_{\mathrm{Sing}\, C}$. If ε is a section of $\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(U)$, its image via t is obtained on every point p of $F|_{\mathrm{Sing}}(U)$ by taking the difference between the actions of ε on the two preimages, and therefore $t(\varepsilon)$ is well defined up to ordering the branches of every node.

We pass to the associated long exact sequence. We observe that $\mathfrak{H}^F(C) = \mathrm{Hom}^G(\mathcal{T}(F), G) = \mathrm{Hom}^G(\mathcal{T}(\tilde{\Gamma}), G)$ by Proposition 1.16. Moreover,

$$\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(C) = \mathrm{Aut}_{\mathrm{Adm}}(\overline{C}, \overline{F}) = \prod_i \mathrm{Hom}^G(\mathcal{T}(F_i), G),$$

by the result of Remark 1.40. If we note $f\colon \tilde{\Gamma} \to \Gamma$ the graph G-cover associated to $F \to C$, by the result of Proposition 4.4 we have

$$\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(C) = C^0(\tilde{\Gamma}; G).$$

Finally, if $q_1, \dots, q_\delta$ are the nodes of C, then

$$\mathfrak{H}^F|_{\operatorname{Sing} C}(C) = \prod_j \operatorname{Aut}_{\mathrm{Adm}}(q_j, F_{q_j}) = \prod_j \operatorname{Hom}^G(F_{q_j}, G).$$

By the definition of the dual graph $\tilde{\Gamma}$, the right hand side of the equality above is identified with $\prod_{e \in E(\Gamma)} \operatorname{Hom}^G(f^{-1}(e), G)$, up to fixing an orientation to every edge e in $E(\Gamma)$. Therefore,

$$\mathfrak{H}^F|_{\operatorname{Sing} C}(C) \cong C^1(\tilde{\Gamma}; G).$$

Theorem 4.23. *Taking the long exact sequence associated to (4.5), we get*

$$1 \to \operatorname{Hom}^G(\mathcal{T}(\tilde{\Gamma}), G) \xrightarrow{i} C^0(\tilde{\Gamma}; G) \xrightarrow{\delta} C^1(\tilde{\Gamma}; G) \xrightarrow{\tau} H^1_{\mathbb{T}}(C; \mathfrak{H}^F). \tag{4.6}$$

The first part of this sequence is exactly the sequence (4.1) defined on the dual graph $\tilde{\Gamma} = \Gamma(F)$ and treated in Section 2.1.1.

We come back to the twisted G-cover (C, ϕ), and we denote by $\overline{\mathsf{C}}$ the normalization of C. The cover $\phi\colon \mathsf{C} \to BG$ induces another cover $\overline{\phi}\colon \overline{\mathsf{C}} \to BG$ on the normalized curve, and for every node q of C an isomorphism κ_q as described precisely in Section 1.3.2 and Proposition 1.56. Given an automorphism a in $\operatorname{Aut}_C(\mathsf{C})$, we consider the induced cover $\mathsf{a}^*\phi$, the ghost automorphism a lifts to ϕ if $\mathsf{a}^*\phi \cong \phi$ as a twisted G-cover. We note a^*F the admissible G-cover associated to $(\mathsf{C}, \mathsf{a}^*\phi)$.

Remark 4.24. Given any cochain $b \in C^1(\tilde{\Gamma}; G)$, we consider the admissible G-cover obtained by contracted product $F' := \tau(b) \wedge^{\mathfrak{H}^F} F$. We observe that, by definition, the twisted G-cover associated to F' corresponds (see Proposition 1.56) to the same cover $\overline{\phi}$ on the normalization $\overline{\mathsf{C}}$, and for every node q a new node isomorphisms κ'_q such that

$$\kappa'_q(\tilde{q}) := \psi \circ (b(\tilde{e}) \times \kappa_q)(\tilde{q}).$$

Here ψ is the G-action, $\tilde{q}$ is a point over q on the domain branch of the cover, and $\tilde{e}$ is the oriented edge in $\mathbb{E}(\tilde{\Gamma})$ associated to node q and pointing to the codomain branch of κ_q.

Proposition 4.25. *Given a finite abelian group G and a twisted G-cover (C, ϕ) whose associated admissible G-cover is $F \to C$, consider the cochain of local indices $b_F \in C^1(\tilde{\Gamma}; G)$ and its associated type function M. If $\mathsf{a} \in \operatorname{Aut}_C(\mathsf{C})$ is a ghost automorphism of C, seen as an element of the group $S(\Gamma; M)$, then*

$$\mathsf{a}^*F \cong \tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F.$$

Proof. We consider a node q of C. If $V = \{x'y' = 0\} \subset \mathbb{A}^2$, then the local picture of the cover ϕ at q is a principal G-bundle $F \to V$ with a $\boldsymbol{\mu}_r$-action on it, compatible with the stack action on V, that is $\xi \cdot (x', y') = (\xi x', \xi^{-1} y')$ forall $\xi \in \boldsymbol{\mu}_r$. By definition of automorphism a around the node q, we can note its action on V as

$$(x', y') \mapsto (\xi_r^{\mathsf{a}(e)} x', y').$$

Noting $F' \to \mathbb{A}^1_{x'}$ and $F'' \to \mathbb{A}^1_{y'}$ the restrictions of F to the components of V, the bundle F can be rewritten as

$$F = (F' \sqcup F'')/\kappa_q.$$

We note e the oriented edge in $\mathbb{E}(\Gamma)$ corresponding to node q and pointing to the branch of q_2, and we note $\psi'\colon A \times F' \to F'$ and $\psi''\colon A \times F'' \to F''$ the A-actions on F' and F''.

Moreover, by Remark 1.48 there exists a natural map $\tilde{\alpha}'(\xi)\colon F' \to F'$ for every $\xi \in \boldsymbol{\mu}_r$, induced by the $\boldsymbol{\mu}_r$-action on F'. We have a cartesian square

$$\begin{tikzcd} F' \arrow[r, "\tilde{\alpha}'(\xi_r^{\mathsf{a}(e)})"] \arrow[d] & F' \arrow[d] \\ \mathbb{A}^1_{x'} \arrow[r, "\mathsf{a}"] & \mathbb{A}^1_{x'}. \end{tikzcd}$$

Our lifting of a to $\mathbb{A}^1_{y'}$ is trivial, and by also using the cartesian diagram above, we have

$$\mathsf{a}^*F = (F' \sqcup F'')/(\kappa_q \circ \tilde{\alpha}'(\xi_r^{\mathsf{a}(e)})).$$

In particular, by construction if we consider a point $\tilde{q}_1$ over q_1 and its associated edge $\tilde{e}_1$ over e, $\alpha'(\xi_r^{\mathsf{a}(e)})(\tilde{q}_1) = \psi((b_F \cdot \mathsf{a})(\tilde{e}_1), \tilde{q}_1)$. As

$$\kappa_q \circ \psi \circ (b(\tilde{e}) \times \mathrm{id}) = \psi \circ (b(\tilde{e}) \times \kappa_q),$$

for every 1-cochain b, then by Remark 4.24, the proof is completed. □

Theorem 4.26. *Given a twisted G-cover (C, ϕ) with associated admissible G-cover $F \to C$, any ghost automorphism $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ lifts to a ghost automorphism of (C, ϕ) if and only if the 1-cochain $b_F \cdot \mathsf{a}$ is in $\operatorname{Ker}\tau = \operatorname{Im}\delta$ of sequence (4.6).*

Proof. After the proposition above, we have that $\phi \cong \mathsf{a}^*\phi$ if and only if $\tau(b_F \cdot \mathsf{a})$ acts trivially via the contracted product on F. We consider the restriction $F_{\mathrm{gen}} \to C_{\mathrm{gen}}$ over the generic locus. We observe that F_{gen} is an $\mathfrak{H}^F$-torsor on C_{gen}, then we apply Proposition 4.19 to obtain that $\tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F_{\mathrm{gen}} = F_{\mathrm{gen}}$ if and only if

$$b_F \cdot \mathsf{a} \in \operatorname{Im}\delta^F.$$

This is a necessary condition to have $\tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F = F$, but it is also sufficient because F_{gen} completes uniquely to F.

It remains to prove that $\operatorname{Im}\delta^F = \operatorname{Im}\delta$. In particular we observe that $\delta^F = \delta$, *i.e.* the contracted product does not act on the δ morphism, and so the proof is concluded. □

Remark 4.27. Given the dual graph G-cover $\tilde{\Gamma} \to \Gamma$ associated to $F \to C$, and the decorated graph $(\tilde{\Gamma}_0, M)$, we recall the subcomplexes inclusion

$$C^i(\tilde{\Gamma}_0; G) \subset C^i(\tilde{\Gamma}; G)$$

for $i = 0, 1$. We also consider the exterior differential δ_0 on $C^0(\tilde{\Gamma}; G)$, *i.e.* the restriction of the δ operator to this group. Because of Proposition 2.14, we have $\mathrm{Im}(\delta_0) = C^1(\tilde{\Gamma}_0; G) \cap \operatorname{Im}\delta$.

Remark 4.28. Previously we obtained a characterization of the cochains in $\mathrm{Im}(\delta)$ that we could restate in our new setting. Indeed, because of Proposition 4.2, an automorphism $\mathsf{a} \in S(\tilde{\Gamma}_0; M)$ is an element of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ if and only if for every circuit $(\tilde{e}_1, \ldots, \tilde{e}_k)$ in $\tilde{\Gamma}_0$ we have $\prod_{i=1}^k (b_F \cdot \mathsf{a})(\tilde{e}_i) = 1$.

4.2.2 Smooth points

In Remark 4.20 we discussed the fact that every point $[\mathsf{C}, \phi] \in \overline{\mathcal{R}}_{g,G}$ has a local picture isomorphic to

$$\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \phi).$$

Therefore, to find the smooth points of $\overline{\mathcal{R}}_{g,G}$, by Proposition 3.2 we need to know when $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ is generated by quasireflections.

We recall the generalized definition of elliptic tail quasireflection (ETQR) 3.20: given a twisted G-cover (C, ϕ) an ETQR is an element $\mathsf{i} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ such that there exists an elliptic tail E of C with elliptic tail node q, the action of i on $\mathsf{C}\backslash\mathsf{E}$ is trivial, and the action on E is the canonical involution i_0, up to non-trivial action on q.

Moreover, the analogous of Lemmata 4.29 and 3.22 are still verified.

Lemma 4.29. *Consider an element* h *of* $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$*. It acts as a quasireflection on* $\mathrm{Def}(\mathsf{C})$ *if and only if one of the following is true:*

1. *the automorphism* h *is a ghost quasireflection, i.e. an element of* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ *which moreover operates as a quasireflection;*

2. *the automorphism* h *is an ETQR, using the generalized Definition 3.20.*

Lemma 4.30. *If* $\mathrm{QR}(\mathrm{Aut}(C))$ *(also called* $\mathrm{QR}(C)$*) is the group generated by ETQRs inside* $\mathrm{Aut}(C)$*, then any element* $h \in \mathrm{QR}(C)$ *which could be lifted to* $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$*, has a lifting in* $\mathrm{QR}(\mathsf{C}, \phi)$*, too.*

The proofs of these lemmata follow easily from the ones in Section 3.2.2. Indeed, there exists a canonical identification

$$\mathrm{Def}(\mathsf{C}, \phi) = \mathrm{Def}(\mathsf{C}) = \mathrm{Def}(\mathsf{C}, \mathsf{L}, \theta)$$

for any twisted G-cover (C, ϕ) and for any rooted curve $(\mathsf{C}, \mathsf{L}, \theta)$, and it is sufficient to look at the action of $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ on $\mathrm{Def}(\mathsf{C})$, with minor adjustments from one case to the other.

Remark 4.31. We recall the short exact sequence

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \phi) \to \underline{\mathrm{Aut}}(\mathsf{C}, \phi) \xrightarrow{\beta} \mathrm{Aut}'(C) \to 1$$

and introduce the group $\mathrm{QR}'(C) \subset \mathrm{Aut}'(C)$, generated by liftable quasireflections, *i.e.* by those quasireflections h in $\mathrm{Aut}'(C)$ such that there exists an automorphism in $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ whose coarsening is h. Similarly to the case of rooted curves, $\mathrm{QR}'(C) = \beta(\mathrm{QR}(\mathsf{C}, \phi))$ and we have the short exact sequence

$$1 \to \mathrm{QR}_C(\mathsf{C}, \phi) \to \mathrm{QR}(\mathsf{C}, \phi) \to \mathrm{QR}'(C) \to 1.$$

Theorem 4.32. *The group* $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ *is generated by quasireflections if and only if both* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ *and* $\mathrm{Aut}'(C)$ *are generated by quasireflections.*

Proof. After the previous remark, the following is a short exact sequence

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C},\phi)/\,\mathrm{QR}_C(\mathsf{C},\phi) \to \underline{\mathrm{Aut}}(\mathsf{C},\phi)/\,\mathrm{QR}(\mathsf{C},\phi) \to \mathrm{Aut}'(C)/\,\mathrm{QR}'(C) \to 1.$$

The theorem follows. □

This gives a first important result for the moduli space of twisted G-covers $\overline{\mathcal{R}}_{g,G}$. As we know that any point $[\mathsf{C},\phi] \in \overline{\mathcal{R}}_{g,G}$ is smooth if and only if the group $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ is generated by quasireflections, then the following theorem is straightforward.

Theorem 4.33. *Given a twisted G-cover $\phi\colon \mathsf{C} \to BG$ over a twisted curve C of genus $g \geq 4$ whose coarse space is C, the point $[\mathsf{C},\phi]$ of the moduli space $\overline{\mathcal{R}}_{g,G}$ is smooth if and only if the group $\mathrm{Aut}'(C)$ is generated by ETQRs and the group of ghost automorphisms $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)$ is generated by quasireflections.*

We introduce two closed loci of $\overline{\mathcal{R}}_{g,G}$,

$$N_{g,G} := \{[\mathsf{C},\phi]|\ \mathrm{Aut}'(C) \text{ is not generated by ETQRs}\},$$

$$H_{g,G} := \{[\mathsf{C},\phi]|\ \underline{\mathrm{Aut}}_C(\mathsf{C},\phi) \text{ is not generated by quasireflections}\}.$$

We have by Theorem 4.33 that the singular locus $\mathrm{Sing}\,\overline{\mathcal{R}}_{g,G}$ is their union

$$\mathrm{Sing}\,\overline{\mathcal{R}}_{g,G} = N_{g,G} \cup H_{g,G}.$$

Remark 4.34. Consider the natural projection $\pi\colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$, we observe that

$$N_{g,G} \subset \pi^{-1}\,\mathrm{Sing}\,\overline{\mathcal{M}}_g.$$

Indeed, after Remark 3.23, $\mathrm{QR}'(C) = \mathrm{Aut}'(C) \cap \mathrm{QR}(C)$ and therefore $\mathrm{Aut}(C) = \mathrm{QR}(C)$ if and only if $\mathrm{Aut}'(C) = \mathrm{QR}'(C)$. This implies that $\left(\pi^{-1}\,\mathrm{Sing}\,\overline{\mathcal{M}}_g\right)^c \subset (N_{g,G})^c$, and taking the complementary we obtain the result.

4.2.3 T-curves and J-curves

The definition of T-curves and J-curves we stated in Section 3.3.1 generalizes easily to rooted curves. Indeed, a T-curve is a twisted G-cover (C,ϕ) such that there exists a in $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ whose coarsening a is an elliptic tail automorphism of order 6. A J-curve is a rooted curve such that the group $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)/\,\mathrm{QR}_C(\mathsf{C},\phi)$ is junior. The loci of T-curves and J-curves inside $\overline{\mathcal{R}}_{g,G}$ are noted $T_{g,G}$ and $J_{g,G}$ respectively.

To better analyze the locus of J-curves, we need the following lemma on quasireflections generalizing Lemma 3.10.

Lemma 4.35. *Consider a ghost automorphism a in $\mathrm{Aut}_C(\mathsf{C})$, we look at it as an element of $S(\Gamma_0;M)$. If a is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)$ then $\mathsf{a}(e) = 0$ for all edges but one that is a separating edge of $\Gamma_0(\mathsf{C})$.*

Proof. If a is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)$, the value on all but one of the coordinates must be 0. Therefore $\mathsf{a}(e) = 0 \in \mathbb{Z}/r(e)$ on all the edges but one, say e_1. If there exists a preimage $\tilde{e}_1$ in $\mathbb{E}(\tilde{\Gamma}_0)$ that is in any circuit $(\tilde{e}_1,\ldots,\tilde{e}_k)$ of $\tilde{\Gamma}_0$ with $k \geq 1$, then we have, by Remark 4.28, that $\prod(b_F\mathsf{a})(\tilde{e}_i) = 1$. As $\mathsf{a}(e_1) \neq 0$, then

$(b_F\mathsf{a})(\tilde{e}_1) \neq 1$ and therefore there exists $i > 1$ such that $(b_F \cdot \mathsf{a})(\tilde{e}_i) \neq 1$ too. This would imply that, if e_i is the image in Γ_0 of $\tilde{e}_i$, then $\mathsf{a}(e_i) \neq 0$, contradiction. Thus $\tilde{e}_1$ is not in any circuit, then it is a separating edge and so is e_1.

Reciprocally, consider an automorphism $\mathsf{a} \in S(\Gamma_0; M)$ such that there exists an oriented separating edge e_1 with the property that $\mathsf{a}(e) = 0$ for all e in $\mathbb{E}\backslash\{e_1, \bar{e}_1\}$ and $\mathsf{a}(e_1)$ is a non-zero element of $\mathbb{Z}/r(e_1)$. Then for every circuit $(\tilde{e}'_1, \dots, \tilde{e}'_k)$ of $\mathbb{E}(\tilde{\Gamma}_0)$, we have $\prod(b_F \cdot \mathsf{a})(\tilde{e}'_i) = 1$ and so a is in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ by Theorem 4.26. ☐

Theorem 4.36. *For $g \geq 4$, the non-canonical locus of $\overline{\mathcal{R}}_{g,G}$ is formed by T-curves and J-curves, i.e. it is the union*

$$\mathrm{Sing}^{\mathrm{nc}}\, \overline{\mathcal{R}}_{g,G} = T_{g,G} \cup J_{g,G}.$$

Remark 4.37. We observe that Theorem 2.44 of Chiodo and Farkas in [12], affirms exactly that in the case $G = \boldsymbol{\mu}_\ell$ with $\ell \leq 6$ and $\ell \neq 5$, the J-locus $J_{g,\boldsymbol{\mu}_\ell}$ is empty for every genus g, and therefore $\mathrm{Sing}^{\mathrm{nc}}\, \overline{\mathcal{R}}_{g,\boldsymbol{\mu}_\ell}$ coincides with the T-locus for these values of ℓ.

To show Theorem 4.36 we will prove a stronger proposition.

Proposition 4.38. *Given a twisted G-cover (C, ϕ) of genus $g \geq 4$ which is not a J-curve, if $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)/\,\mathrm{QR}(\mathsf{C}, \phi)$ is a junior automorphism, then its coarsening a is an elliptic tail automorphism of order 3 or 6.*

Proof. We introduce the notion of $\star$-smoothing, following [24] and [26].

Definition 4.39. Consider a twisted G-cover (C, ϕ), we say that it is $\star$-smoothable if there is a junior automorphism $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)/\,\mathrm{QR}(\mathsf{C}, \phi)$ such that

- there exists a cycle of m non-separating nodes $q_0, \dots, q_{m-1}$, *i.e.* we have $\mathsf{a}(q_i) = q_{i+1}$ for all $i = 0, 1 \dots, m-2$ and $\mathsf{a}(q_{m-1}) = q_0$;
- the action of a^m over the coordinate associated to every node is trivial. This is equivalent to ask $\mathsf{a}^m(\tilde{t}_{q_i}) = \tilde{t}_{q_i}$ for all $i = 0, 1, \dots, m-2$, where $\tilde{t}_{q_i}$ is the coordinate associated to node q_i.

If (C, ϕ) is $\star$-smoothable, there exists a deformation $(\mathsf{C}', \phi', \mathsf{a}')$ of the triple $(\mathsf{C}, \phi, \mathsf{a})$ that smoothens the m nodes and with $\mathsf{a}' \in \underline{\mathrm{Aut}}(\mathsf{C}', \phi')$. Moreover, this deformation preserves the age of the a-action on $\mathrm{Def}(\mathsf{C}, \phi)/\,\mathrm{QR}$. Indeed, the eigenvalues of a are a discrete and locally constant set, thus constant by deformation. The T-locus and the J-locus are closed by $\star$-smoothing, *i.e.* if the deformation (C', ϕ') above is a T-curve or a J-curve, then (C, ϕ) is a T-curve or a J-curve respectively. Therefore we can suppose as an additional hypothesis of Theorem 4.36, that our curves are $\star$-*rigid*, *i.e.* non-$\star$-smoothable. From this point we suppose that (C, ϕ) is $\star$-rigid.

We will show in eight steps that if the group

$$\underline{\mathrm{Aut}}(\mathsf{C}, \phi)/\,\mathrm{QR}\,(\mathsf{C}, \phi)$$

is junior, and (C, ϕ) is not a J-curve, then it is a T-curve. After the Age Criterion 3.8 and Proposition 3.9, this will prove Theorem 4.36. From now on we work under the hypothesis that $\mathsf{a} \in \underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)/\,\mathrm{QR}$ is a non-trivial automorphism aged less than 1, and (C, ϕ) is not a J-curve.

In steps 1 and 2 we fix the setting and prove two useful lemmata. In step 3 we prove that all the nodes of C are fixed by a except at most 2 of them which are exchanged. In step 4 we show that every irreducible component $\mathsf{Z} \subset \mathsf{C}$ is fixed by a. In step 5 we can therefore conclude that there are no couple of exchanging nodes. In step 6 and 7 we study the action of a on the irreducible components of C and the contributions to age a. Finally we prove the result in step 8.

Step 1. Consider the decorated graph (Γ_0, M) of (C, ϕ). As before, we call E_{sep} the set of separating edges of Γ_0. Then, following Section 3.1, we can estimate the age by a splitting of the form

$$\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C}) \cong \bigoplus_{i=1}^{V} H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i)),$$

$$\mathrm{Def}(\mathsf{C})/\,\mathrm{Def}(\mathsf{C}; \sin \mathsf{C}) \cong \bigoplus_{e \in E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_e} \oplus \bigoplus_{e' \in E \setminus E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_{e'}},$$

where $\tilde{t}_e$ is a coordinate parametrizing the smoothing of the node associated to the edge e, and the curves $\overline{C}_i$ are the normalizations of the irreducible components of the coarse space C of C. In particular for every vector subspace $V \subset \mathrm{Def}(\mathsf{C})$ or $V \subset \mathrm{Def}(\mathsf{C})/\,\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C})$ and every automorphism a of (C, ϕ), we will note $\mathrm{age}(\mathsf{a}|V)$ the age of the restriction $\mathsf{a}|_V$. If Z is a subcurve of C, then there exists a canonical inclusion $\mathrm{Def}(\mathsf{Z}) \subset \mathrm{Def}(\mathsf{C})$, and we note $\mathrm{age}(\mathsf{a}|\mathsf{Z}) := \mathrm{age}(\mathsf{a}|\,\mathrm{Def}(\mathsf{Z}))$.

Every ghost automorphism in $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ fixes $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C})$ and the two summands of $\mathrm{Def}(\mathsf{C})/\,\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C})$. Moreover, every quasireflection acts only on the summand $\bigoplus_{e \in E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_e}$ by Lemmata 4.29 and 4.35. As a consequence, by Propostion 3.9, the group $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)/\,\mathrm{QR}$ acts on

$$\frac{\mathrm{Def}(\mathsf{C})/\,\mathrm{QR}}{\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C})} = \left(\frac{\bigoplus_{e \in E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_e}}{\mathrm{QR}(\mathsf{C}, \phi)} \right) \oplus \bigoplus_{e' \in E \setminus E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_{e'}}. \tag{4.7}$$

Every quasireflection acts on exactly one coordinate $\tilde{t}_e$ with $e \in E_{\mathrm{sep}}$. We rescale all the coordinates $\tilde{t}_e$ by the action of $\mathrm{QR}(\mathsf{C}, \phi)$. We call τ_e, for $e \in E(\Gamma_0)$, the new set of coordinates. Obviously $\tau_{e'} = \tilde{t}_{e'}$ if $e' \in E(\Gamma_0) \setminus E_{\mathrm{sep}}$.

Step 2. We show two lemmata about the age contribution of the a-action on nodes, which we will call *aging* on nodes.

Definition 4.40 (coarsening order)**.** If $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ and a is its coarsening, then we define

$$\text{c-ord}\,\mathsf{a} := \mathrm{ord}\, a.$$

The coarsening order is the least integer n for which a^n is a ghost automorphism.

Lemma 4.41. *Suppose that $\mathsf{Z} \subset \mathsf{C}$ is a subcurve of C such that $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$ and $q_0, \dots q_{m-1}$ is a cycle, by a, of nodes in Z. Then we have the following inequalities:*

1. $\mathrm{age}(\mathsf{a}|\mathsf{Z}) \geq \frac{m-1}{2}$;
2. *if the nodes $q_0, \dots, q_{m-1}$ are non-separating,* $\mathrm{age}(\mathsf{a}) \geq \frac{m}{\mathrm{ord}(\mathsf{a}|\mathsf{Z})} + \frac{m-1}{2}$;
3. *if $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is a senior ghost, we have* $\mathrm{age}(\mathsf{a}) \geq \frac{1}{\text{c-ord}(\mathsf{a})} + \frac{m-1}{2}$.

Proof. We call $\tau_0, \tau_1, \dots, \tau_{m-1}$ the coordinates associated to nodes $q_0, \dots, q_{m-1}$ respectively. By hypothesis, $\mathsf{a}(\tau_0) = c_1 \cdot \tau_1$ and $\mathsf{a}^i(\tau_0) = c_i \cdot \tau_i$ for all $i = 2, \dots, m-1$, where c_i are complex numbers. If $n' = \mathrm{ord}(\mathsf{a}|_\mathsf{Z})$, we have

$$\mathsf{a}^m(\tau_0) = \xi_{n'}^{um} \cdot \tau_0$$

where $\xi_{n'}$ is a primitive n'th root of the unit and u is an integer such that

$$0 \leq u < n'/m.$$

The integer u is called *exponent* of the cycle $(q_0, \dots, q_{m-1})$ with respect to the curve Z. Observe that $\mathsf{a}(\tau_{i-1}) = (c_i/c_{i-1}) \cdot \tau_i$ and $\mathsf{a}^m(\tau_i) = \xi_{n'}^{um} \cdot \tau_i$ for every i.

We can explicitly write the eigenvectors for the action of a on the coordinates $\tau_0, \dots, \tau_{m-1}$. Set $d := n'/m$ and $b := sd + u$ with $0 \leq s < m$, and consider the vector

$$\mathfrak{v}_b := (\tau_0 = 1,\ \tau_1 = c_1 \cdot \xi_{n'}^{-b}, \dots,\ \tau_i = c_i \cdot \xi_{n'}^{-ib}, \dots).$$

Then $\mathsf{a}(\mathfrak{v}_b) = \xi_{n'}^b \cdot \mathfrak{v}_b$. The contribution to the age of the eigenvalue $\xi_{n'}^b$ is b/n', thus we have

$$\mathrm{age}\,\mathsf{a} \geq \sum_{s=0}^{m-1} \frac{sd+u}{n'} = \frac{mu}{n'} + \frac{m-1}{2},$$

proving point (1).

If the nodes are non-separating, as we are working on a $\star$-rigid curve, we have $u \geq 1$ and the point (2) is proved.

Suppose that a has order $n = \mathrm{ord}\,\mathsf{a}$ and its action on C has j nodes cycles of order $m_1, m_2, \dots, m_j$ and exponents respectively $u_1, \dots, u_j$ with respect to C. If $k = \text{c-ord}\,\mathsf{a}$, then a^k fixes every node, then we consider the coordinate τ_i of a node of the first cycle and we have

$$\mathsf{a}^k(\tau_i) = \xi_n^{w \cdot k} \cdot \tau_i,$$

where w is an integer such that $0 \leq w < n/k$. Repeating the same operation for every cycle we obtain another series of integers $w_1, w_2, \dots, w_j$. Clearly the age of a^k is

$$\mathrm{age}\,\mathsf{a}^k = \sum_{i=1}^{j} \frac{m_i w_i k}{n},$$

and it is greater or equal to 1 by hypothesis.

We observe that m_i divides k for all $i = 1, \dots, j$, and

$$u_i \cdot m_i \cdot \frac{k}{m_i} \equiv w_i \cdot k \mod n.$$

This implies that $u_i \geq w_i$ for every i.

By the point (2), the age of a on the ith cycle is bounded from below by

$$m_i u_i/n + (m_i - 1)/2.$$

As a consequence

$$\operatorname{age}\mathsf{a} \geq \sum_{i=1}^{j}\left(\frac{m_i u_i}{n} + \frac{m_i - 1}{2}\right) \geq \sum_{i=1}^{j}\left(\frac{m_i w_i}{n} + \frac{m_i - 1}{2}\right) \geq \frac{1}{k} + \frac{m_1 - 1}{2}.$$

$\square$

Step 3. Because of Lemma 4.41, if the automorphism a induces a cycle of m nodes, then this cycles contributes by at least $\frac{m-1}{2}$ to the aging of a. Therefore, as a is junior, all the nodes of C are fixed except at most two of them, that are exchanged. Moreover, if a pair of non-fixed nodes exists, they contribute by at least 1/2.

Step 4. Consider an irreducible component $\mathsf{Z} \subset \mathsf{C}$, then $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$. To show this, we suppose there exists a cycle of irreducible components $\mathsf{C}_1, \ldots, \mathsf{C}_m$ with $m \geq 2$ such that $\mathsf{a}(\mathsf{C}_i) = \mathsf{C}_{i+1}$ for $i = 1, \ldots, m-1$, and $\mathsf{a}(\mathsf{C}_m) = \mathsf{C}_1$. We call $\overline{\mathsf{C}}_i$ the normalizations of these components, and D_i the preimages of C nodes on $\overline{\mathsf{C}}_i$. We point out that this construction implies that $(\overline{\mathsf{C}}_i, D_i) \cong (\overline{\mathsf{C}}_j, D_j)$ for all i, j. Then, an argument of [24, p.34] shows that the action of a on $\operatorname{Def}(\mathsf{C}; \operatorname{Sing}\mathsf{C})$ gives a contribution of at least $k \cdot (m-1)/2$ to $\operatorname{age}\mathsf{a}$, where

$$k = \dim H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i)) = 3g_i - 3 + |D_i|.$$

This gives us two cases for which m could be greater than 1 with still a junior age: $k = 1$ and $m = 2$ or $k = 0$.

If $k = 1$ and $m = 2$, we have $g_i = 0$ or 1 for $i = 1, 2$. Moreover, the aging of at least 1/2 sums to another aging of 1/2 if there is a pair of non-fixed nodes. As a is junior, we conclude that $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1)$ but this implies $g(\mathsf{C}) \leq 3$, contradiction.

If $k = 0$, we have $g_i = 1$ or $g_i = 0$, the first is excluded because it implies $|D_i| = 0$ but the component must intersect the curve somewhere. Thus, for every component in the cycle, the normalization $\overline{\mathsf{C}}_i$ is the projective line $\mathbb{P}^1$ with 3 marked points. We have two cases: the component C_i intersects $\overline{\mathsf{C}\backslash\mathsf{C}_i}$ in 3 points or in 1 point, in the second case C_i has an autointersection node and $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1)$, which is a contradiction because $g(\mathsf{C}) < 4$. It remains the case in the image below.

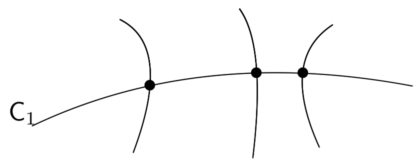


Figure 4.1: Case with $\mathsf{C}_1 \cong \mathbb{P}^1$ and 3 marked points

As $\mathsf{C}_1, \mathsf{C}_2, \ldots, \mathsf{C}_m$ are moved by a, every node on C_1 is transposed with another one or is fixed with its branches interchanged. If at least two nodes are transposed we have an age contribution bigger or equal to 1 by Lemma 4.41. If only one node is transposed we have two cases. In the first case $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1) \cup \mathsf{C}_2 \cup \mathsf{a}(\mathsf{C}_2)$, where C_2 intersects only the component C_1 and in exactly one point. If $g(\mathsf{C}_2) \geq 2$, then the age is bigger than 1, if $g(\mathsf{C}_2) < 2$, then $g(\mathsf{C}) \leq 3$, contradiction.

In the second case, $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1) \cup \mathsf{C}_2$ where C_2 intersects C_1 and $\mathsf{a}(\mathsf{C}_1)$, both in exactly one point. If $g(\mathsf{C}_2) < 2$ we have another genus contradiction. By the results of [24, p.28], to have $\operatorname{age}(\mathsf{a}|\mathsf{C}_2) < 1$ we must have $g(\mathsf{C}_2) = 2$ and the coarsening of a has order 2. Therefore by Lemma 4.41 point (3), a has age bigger or equal to 1.

Step 5. We prove that every node is fixed by a. Consider the normalization $\mathsf{nor}\colon \bigsqcup_i \overline{\mathsf{C}}_i \to \mathsf{C}$ already introduced. If the age of a is lower than 1, *a fortiori* we have $\operatorname{age}(\mathsf{a}|\overline{\mathsf{C}}_i) < 1$ for every i. In [24, p.28] there is a list of those smooth stable curves for which there exists a non-trivial junior action.

i. The projective line $\mathbb{P}^1$ with $\mathsf{a}\colon z \mapsto (-z)$ or $(\xi_4 z)$;

ii. an elliptic curve with a of order 2, 3, 4 or 6;

iii. an hyperelliptic curve of genus 2 or 3 with a the hyperelliptic involution;

iv. a bielliptic curve of genus 2 with a the canonical involution.

We observe that the order of the a-action on these components is always 2, 3, 4 or 6. As a consequence, if a is junior, then $n = \text{c-ord}\,\mathsf{a} = 2, 3, 4, 6$ or 12, as it is the greatest common divisor between the c-ord $\left(\mathsf{a}|\overline{\mathsf{C}}_i\right)$.

First we suppose $\operatorname{ord}\mathsf{a} > \text{c-ord}\,\mathsf{a}$, thus $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is a ghost and it must be senior. Indeed, if $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is aged less than 1, then (C, ϕ) admits junior ghosts, contradicting our assumption. By point (3) of Lemma 4.41, if there exists a pair of non-fixed nodes, we obtain an aging of $1/n + 1/2$ on node coordinates. If $\operatorname{ord}\mathsf{a} = \text{c-ord}\,\mathsf{a}$ the bound is even greater. As every component is fixed by a, the two nodes are non-separating, and by point (2) of Lemma 4.41 we obtain an aging of $2/n + 1/2$.

If $\overline{\mathsf{C}}_i$ admits an automorphism of order 3, 4 or 6, by a previous analysis of Harris and Mumford (see [24] again), this yields an aging of, respectively, 1/3, 1/2 and 1/3 on $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$.

These results combined, show that a non-fixed pair of nodes gives an age greater than 1. Thus, if a is junior, every node is fixed.

Step 6. We study the action of a separately on every irreducible component. The a-action is non-trivial on at least one component C_i, and this component must lie in the list above.

In case (i), $\overline{\mathsf{C}}_i$ has at least 3 marked points because of the stability condition. Actions of type $x \mapsto \xi x$ have two fixed points on $\mathbb{P}^1$, thus at least one of the marked points is non-fixed. A non-fixed preimage of a node has order 2, thus the coarsening a of a is the involution $z \mapsto -z$. Moreover, C_i is the autointersection of the projective line and a exchanges the branches of the node. Therefore $\mathsf{a}^2|_{\overline{\mathsf{C}}_i}$ is a ghost automorphism of $\overline{\mathsf{C}}_i$. As a direct consequence of Theorem 4.26 and Remark 4.28, the action of a^2 on the coordinate associated to the autointersection node, is trivial. Therefore the action of a^2 on the same coordinate gives an aging of 0 or 1/2, by $\star$-rigidity it is 1/2.

The analysis for cases (iii) and (iv) is identical to that developed in [24]: the only possibility of a junior action is the case of an hyperelliptic curve E of genus 2 intersecting $\overline{\mathsf{C}\backslash\mathsf{E}}$ in exactly one point, whose hyperinvolution gives an aging of 1/2 on $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$.

Finally, in case (ii), we use again the analysis of [24]. The elliptic component E has 1 or 2 point of intersection with $\overline{\mathsf{C}\backslash\mathsf{E}}$. If there is 1 point of intersection, elliptic tail case, for a good choice of coordinates the coarsening a acts as $z \mapsto \xi_n z$, where n is $2, 3, 4$ or 6. The aging is, respectively, 0, 1/3, 1/2, 1/3. If there are 2 points of intersection, elliptic ladder case, the order of a on E must be 2 or 4 and the aging respectively 1/2 or 3/4.

Step 7. Resuming what we saw until now, if a is a junior automorphism of (C, ϕ), a its coarsening and C_1 an irreducible component of C, then we have one of the following:

A. component C_1 is an hyperelliptic tail, crossing the curve in one point, with a acting as the hyperelliptic involution and aging 1/2 on $H^1(\overline{C}_1, T_{\overline{\mathsf{C}}_i}(-D_1))$;

B. component C_1 is a projective line $\mathbb{P}^1$ autointersecting itself, crossing the curve in one point, with a the involution which fixes the nodes, and aging 1/2;

C. component C_1 is an elliptic ladder, crossing the curve in two points, with a of order 2 or 4 and aging respectively 1/2 or 3/4;

D. component C_1 is an elliptic tail, crossing the curve in one point, with a of order $2, 3, 4$ or 6 and aging $0, 1/3, 1/2$ or $1/3$;

E. automorphism a acts trivially on C_1 with no aging.

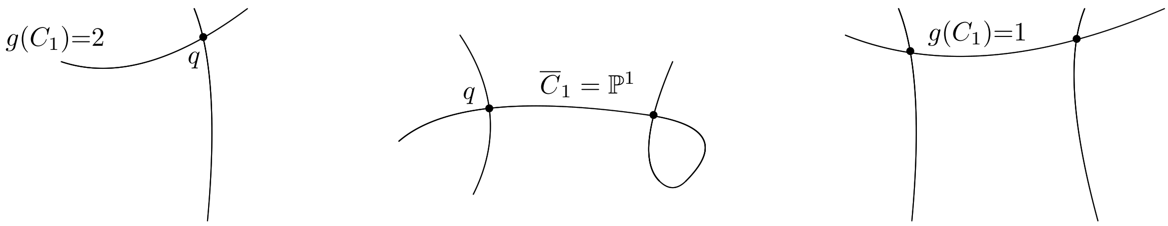


Figure 4.2: Components of type A, B and C.

We rule out cases (A), (B) and (C). At first we suppose there is a component of type (A) or (B). For genus reasons, the component intersected in both cases must be of type (E). We study the local action on the separating node q. The local picture of q is $[\{x'y' = 0\}/\boldsymbol{\mu}_r]$. The smoothing of the node is given by the stack $\left[\{xy = \tilde{t}_q\}/\boldsymbol{\mu}_r\right]$. Consider the action of the automorphism a at the node, as the coarsening of a has order 2, then $\mathsf{a}\colon \tilde{t}_q \mapsto \varsigma \cdot \tilde{t}_q$ and $\varsigma^2 \in \boldsymbol{\mu}_r$. Therefore a^2 acts as the identity or as a quasireflection of factor ς^2. Thus $\tau_q = \tilde{t}_q^{r'}$ where $r'|r$ is the order of ς^2. Therefore the action of a on $\mathbb{A}^1_{\tilde{t}_q} / \operatorname{QR} = \mathbb{A}^1_{\tau_q}$ is $\tau_q \mapsto \varsigma^{r'} \cdot \tau_q = -\tau_q$. The additional age contribution is 1/2, ruling out this case.

In case there is a component of type (C), if its nodes are separating, then one of them must intersect a component of type (E) and we use the previous idea. In case nodes are non-separating, we use Lemma 4.41. If $\operatorname{ord}\mathsf{a} > \operatorname{c-ord}\mathsf{a}$, then $\mathsf{a}^{\operatorname{c-ord}\mathsf{a}}$ is a senior ghost because (C, ϕ) is not a J-curve, thus by point (3) of the lemma there is an aging of $1/\operatorname{c-ord}\mathsf{a}$ on the node coordinates. If $\operatorname{ord}\mathsf{a} = \operatorname{c-ord}\mathsf{a}$, the bound is even greater, as by point (2) we have an aging of $2/\operatorname{c-ord}\mathsf{a}$. We observe that $\operatorname{c-ord}\mathsf{a} = 2, 4$ or 6, and in case $\operatorname{c-ord}\mathsf{a} = 6$ there must be a component of type (E). Using additional contributions listed above we rule out the case (C).

Step 8. We proved that C contains components of type (D) or (E), *i.e.* the automorphism a acts non-trivially only on elliptic tails. If q is the elliptic tail node, there are two quasireflections acting on the coordinate $\tilde{t}_q$: a ghost automorphism associated to this node and the elliptic tail quasireflection. If the order of the local stabilizer is r, then $\tau_q = \tilde{t}_q^{2r}$.

If $\operatorname{ord}\mathsf{a} = 2$ we are in the ETQR case, this action is a quasireflection and it contributes to rescaling the coordinate $\tilde{t}_q$.

If $\operatorname{ord}\mathsf{a} = 4$, the action on the (coarse) elliptic tail is $z \mapsto \xi_4 z$. The space $H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i))$ is the space of 2-forms $H^0(\overline{C}_i, \omega_{\overline{C}_i}^{\otimes 2})$: this space is generated by $dz^{\otimes 2}$ and the action of a is $dz^{\otimes 2} \mapsto \xi_4^2 dz^{\otimes 2}$. Moreover, if the local picture of the elliptic tail node is $[\{x'y' = 0\}/\boldsymbol{\mu}_r]$, then $\mathsf{a}\colon (x', y') \mapsto (\zeta x', \varrho y')$ such that $\zeta^r = \xi_4$ and $\varrho^r = 1$. As a consequence $\mathsf{a}\colon \tilde{t}_q \mapsto \zeta \cdot \varrho \cdot \tilde{t}_q$ and therefore $\tau_q \mapsto \xi_2 \tau_q$. Then, $\operatorname{age}\mathsf{a} = 1/2 + 1/2$, proving the seniority of a.

If E admits an automorphism a of order 6, the action on the (coarse) elliptic tail is $\mathsf{a}\colon z \mapsto \xi_6^k z$. Then $dz^{\otimes 2} \mapsto \xi_3^k dz^{\otimes 2}$ and $\tau_q \mapsto \xi_3^k \tau_q$. For $k = 1, 4$ we have age lower than 1.

If (C, ϕ) is not a J-curve, we have shown that the only case where an automorphism a in $\underline{\operatorname{Aut}}(\mathsf{C}, \phi)/\operatorname{QR}$ is junior, is when its coarsening a is an elliptic tail automorphism of order 6. $\square$

4.3 Extension of pluricanonical forms

In order to evaluate the Kodaira dimension of any moduli space $\overline{\mathcal{R}}_{g,G}$, it is fundamental to prove an extension result of pluricanonical forms, as done for example by Harris and Mumford for $\overline{\mathcal{M}}_g$ (see [24]) and by Chiodo and Farkas for $\overline{\mathcal{R}}^0_{g,\ell}$ with $\ell < 5$ and $\ell = 6$. In particular given a desingularization $\widehat{\mathcal{R}}_{g,G} \to \overline{\mathcal{R}}_{g,G}$, and denoting by $\overline{\mathcal{R}}^{\mathrm{reg}}_{g,G}$ the sublocus of regular points, we would like to prove

$$H^0\left(\widehat{\mathcal{R}}_{g,G}, nK_{\widehat{\mathcal{R}}_{g,G}}\right) = H^0\left(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,G}, nK_{\overline{\mathcal{R}}_{g,G}}\right)$$

for n sufficiently big and divisible. This condition is verified locally for smooth points and canonical singularities. It remains to treat the non-canonical locus $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G}$. As this locus is the union of the T-locus and the J-locus, we consider the cases where the second one is empty, and therefore we treat the T-locus by generalizing the Harris-Mumford technique for $\overline{\mathcal{M}}_g$. We focus in the case of G abelian group and in the case of $G = S_3$ the symmetric group of order 3.

4.3.1 The case G abelian group

In the case of an abelian group G it is possible to be more specific on the description of the T-curves.

Lemma 4.42. *Consider a twisted G-cover (C,ϕ) with G finite abelian group. If (C,ϕ) is a T-curve, then the restriction of the cover to the elliptic tail is trivial.*

Proof. By Proposition 1.33, the set of admissible G-covers over the elliptic curve E is in bijection with the set of maps $\varpi\colon \pi_1(E_{\mathrm{gen}},p_*)\to G$, where p_* is the E base point. We recall that if E is an elliptic curve admitting an order 6 automorphism a, then

$$E\cong \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\cdot\Omega),$$

where Ω is a primitive 6th root of the unit and p_* is the origin. Therefore a acts on E as multiplication by Ω^2. The fundamental group $\pi_1(E,p_*)\subset\pi_1(E_{\mathrm{gen}},p_*)$ is generated by a_1 and b_1 which are the classes of the two laces γ_a and γ_b such that

$$\gamma_a\colon[0,1]\to\mathbb{C}\ :\quad \gamma_a(t)=t;$$

$$\gamma_b\colon[0,1]\to\mathbb{C}\ :\quad \gamma_b(t)=t\cdot\omega.$$

We have as a consequence

$$\mathsf{a}(a_1)=b_1a_1^{-1},\quad \mathsf{a}(b_1)=a_1^{-1}.$$

Therefore if we call ϖ' the map $\mathsf{a}^*\varpi$, we have

$$\varpi'(a_1)=\varpi(b_1)\varpi(a_1)^{-1},\quad \varpi'(b_1)=\varpi(a_1)^{-1}.$$

By Proposition 1.33, a lifts to the cover if and only if $\varpi'=\varpi$. This is true if and only if $\varpi\equiv 1$, *i.e.* the restriction of (C,ϕ) to the elliptic tail must be trivial. □

Theorem 4.43. *In the case of a moduli space $\overline{\mathcal{R}}_{g,G}$ of twisted G-covers with G finite abelian group, we consider a desingularization $\widehat{\mathcal{R}}_{g,G}\to\overline{\mathcal{R}}_{g,G}$. If the locus $J_{g,G}\subset\overline{\mathcal{R}}_{g,G}$ is empty, then*

$$H^0(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,G},nK_{\overline{\mathcal{R}}^{\mathrm{reg}}_{g,G}})=H^0(\widehat{\mathcal{R}}_{g,G},nK_{\widehat{\mathcal{R}}_{g,G}}),$$

for n sufficiently big and divisibile.

In [24] the same is proved for the moduli space $\overline{\mathcal{M}}_g$. The idea is the following. Consider a general non-canonical singularity of $\overline{\mathcal{M}}_g$, that is a point $[C]$ where $C=C_1\cup C_2$ is a genus g stable curve with two irreducible components: C_1 ha genus $g-1$ and C_2 is an elliptic tail. We note q the intersection node of C_1 and C_2. Consider the operation of gluing any elliptic tail C_2' at C_1 along the same node q. This gives an immersion

$$\Psi\colon\overline{\mathcal{M}}_{1,1}\to\overline{\mathcal{M}}_g,\tag{4.8}$$

and the image of Ψ passes through $[C]$.

There exists a neighborhood $S([C])$ of $\operatorname{Im}\Psi$ in $\overline{\mathcal{M}}_g$ with the following properties:

1. there exists a smooth variety B of dimension $3g-3$ and a birational morphism $g\colon S \to B$;

2. there exists à subvariety $Z \subset B$ of codimension 2 such that $g^{-1}(B\backslash Z)$ is isomorphic to $B\backslash Z$;

3. as $B\backslash Z \subset S \subset \overline{\mathcal{M}}_g$, we have $B\backslash Z \subset \overline{\mathcal{M}}_g^0 \subset \overline{\mathcal{M}}_g^{\mathrm{reg}}$, where $\overline{\mathcal{M}}_g^0$ is the subspace of stable curves with trivial automorphism group.

$$
\begin{tikzcd}
 & \operatorname{Im}\Psi \arrow[d, hook] \arrow[dr] & \\
 & S([C]) \arrow[r, hook] \arrow[dl, "g"'] & \overline{\mathcal{M}}_g \\
B & B\backslash Z \arrow[l, hook'] \arrow[u, hook] \arrow[r, hook] & \overline{\mathcal{M}}_g^0 \arrow[u, hook]
\end{tikzcd}
$$

This allows to conclude, because for every pluricanonical form ω on $S([C])^{\mathrm{reg}}$, we consider its restriction to $B\backslash Z$, this extends to the smooth variety B and pullback to a desingularization $\widehat{S}([C]) \to S([C]) \to B$.

We will generalize this technique, by using the same approach of Ludwig in [26] for the case of the moduli space $\overline{\mathcal{R}}_{g,\boldsymbol{\mu}_2}$. In order to complete this, we need a generalization of the age criterion 3.8.

Proposition 4.44 (see Appendix 1 to §1 of [24])**.** *Consider a complex vector space $V \cong \mathbb{C}^n$, $\mathfrak{G} \subset \mathrm{GL}(V)$ finite subgroup, a desingularization $\widehat{V/\mathfrak{G}} \to V/\mathfrak{G}$ and a $\mathfrak{G}$-invariant pluricanonical form ω on V. Consider an element* h *in $\mathfrak{G}$, and note $V^0 \subset V$ the subset where $\mathfrak{G}$ acts freely and* $\mathrm{Fix}(\mathsf{h}) \subset V$ *the fixed point set of* h*. With abuse of notation we note* $\mathrm{Fix}(\mathsf{h})$ *also the image of the fixed point set in $V/\mathfrak{G}$. Let $U \subset V/\mathfrak{G}$ an open subset such that $V^0/\mathfrak{G} \subset U$ and such that for every* h *with* $\operatorname{age}\mathsf{h} < 1$ *(with respect to some primitive root of the unit), the intersection* $U \cap \mathrm{Fix}(\mathsf{h})$ *is non-empty. We note $\widehat{U} \subset \widehat{V/\mathfrak{G}}$ the preimage of U under the desingularization.*

If ω, as a meromorphic form on $\widehat{V/\mathfrak{G}}$, is holomorphic on $\widehat{U}$, then it is holomorphic on $\widehat{V/\mathfrak{G}}$.

Proof of Theorem 4.43. Consider a pluricanonical form ω on $\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}$, We show that ω lifts to a desingularization of an open neighborhood of every point $[\mathsf{C},\phi]$ of $\overline{\mathcal{R}}_{g,G}$.

If $[\mathsf{C},\phi]$ is a canonical singularity this is obvious by definition.

If $[\mathsf{C},\phi]$ is a non-canonical singularity, at first we consider the case of a general non-canonical singularity. As $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G} = T_{g,G}$, then by Lemma 4.42 a general point $[\mathsf{C},\phi] \in T_{g,G}$ is a T-curve C whose coarse space C has two components, $C = C_1 \cup C_2$ with C_1 a stable curve of genus $g-1$ and C_2 an elliptic tail, we note q the intersection node. Moreover, if (C_1,ϕ_1) and (C_2,ϕ_2) are the restrictions of the twisted S_3-cover on C_1 and C_2, then (C_2,ϕ_2) is the trivial cover of C_2, $\mathsf{C}_1 = C_1$ and $\underline{\mathrm{Aut}}(\mathsf{C}_1,\phi_1)$ is trivial.

Once we fixed the twisted G-cover (C_1, ϕ_1) with one marked point on C_1, we consider the morphism

$$\Psi' \colon \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{R}}_{g,G}$$

sending any point $[C_2]$ of $\overline{\mathcal{M}}_{1,1}$ to the point $[C, \phi]$ obtained by joining C_1 and C_2 along their marked points, and considering the G-cover obtained by joining ϕ_1 to the trivial cover on C_2. By following [26], we see that the projection $\pi \colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ sends $\operatorname{Im}\Psi'$ isomorphically on $\operatorname{Im}\Psi$, which implies that $\pi|_{\operatorname{Im}\Psi}$ is a local isomorphism. Indeed, $\operatorname{Def}(C; \phi) = \operatorname{Def}(C)$ and for every point of $\operatorname{Im}\Psi'$ the automorphism group $\underline{\operatorname{Aut}}(C; \phi_1)$ is isomorphic to $\operatorname{Aut}(C)$, which implies π being a local isomorphism. Therefore if we consider the neighborhood $S([C])$ of $[C] \in \operatorname{Im}\Psi$ introduced by Harris and Mumford, then up to shrinking $\pi^* S([C]) \cong S([C])$ we have a neighborhood of $[C, \phi]$ with the same properties.

It remains to consider the case of any non-canonical singularity $[\mathsf{C}, \phi]$. Here C is a twisted curve such that

$$\mathsf{C} = \mathsf{C}_1 \cup \bigcup \mathsf{C}_2^{(i)},$$

where any C_2 is an elliptic tail admitting an elliptic tail automorphism of order 6. Again we follow the last part of the Ludwig's demonstration of [26, Theorem 4.1]. We consider for each i a small deformation $(\mathsf{C}^{(i)}, \phi^{(i)})$ of (C, ϕ) which fixes the ith elliptic tail. That is, $\mathsf{C}^{(i)} = \mathsf{C}_1^{(i)} \cup \mathsf{C}_2^{(i)}$ is the union of two irreducible components. Moreover, the twisted G-cover admits no non-trivial automorphism over $\mathsf{C}_1^{(i)}$ and it is unchanged over $\mathsf{C}_2^{(i)}$. By the previous point we consider $S^{(i)} := S([\mathsf{C}^{(i)}, \phi^{(i)}])$. Up to shrinking the open subsets $S^{(i)}$, they are all disjoint.

Given the point $[\mathsf{C}, \phi]$ of $\overline{\mathcal{R}}_{g,G}$, we consider the local picture of its universal deformation

$$V := \operatorname{Def}(\mathsf{C}, \phi) \cong \mathbb{C}^{3g-3},$$

and recall that the local picture at $[\mathsf{C}, \phi]$ is the same of

$$V / \underline{\operatorname{Aut}}(\mathsf{C}, \phi)$$

at the origin. We define

$$S([\mathsf{C}, \phi]) := (V / \underline{\operatorname{Aut}}(\mathsf{C}, \phi)) \cup \left(\bigcup S^{(i)} \right).$$

If V^0 is the V subset where $\underline{\operatorname{Aut}}(\mathsf{C}, \phi)$ acts freely, let $U \subset S([\mathsf{C}, \phi])$ be the set

$$U := (V^0 / \underline{\operatorname{Aut}}(\mathsf{C}, \phi)) \cup \left(\frac{V}{\underline{\operatorname{Aut}}(\mathsf{C}, \phi)} \cap \bigcup S^{(i)} \right).$$

Then U is exactly the open set that allows to apply Proposition 4.44. This completes the proof. □

Remark 4.45. We focus on the case $G = \boldsymbol{\mu}_\ell$ a cyclic group. The proof above applies with minor adjustments also to the case of the moduli space $\overline{\mathcal{R}}^k_{g,\ell}$ for $k \neq 0$.

Therefore we have that the extension result is true in this case, too. If the J-locus $J^k_{g,\ell}$ of $\overline{\mathcal{R}}^k_{g,\ell}$ is empty, and $\widehat{\mathcal{R}}^k_{g,\ell} \to \overline{\mathcal{R}}^k_{g,\ell}$ is a desingularization, then

$$H^0\left(\widehat{\mathcal{R}}^k_{g,\ell}, nK_{\widehat{\mathcal{R}}^k_{g,\ell}}\right) = H^0\left(\overline{\mathcal{R}}^{k,\mathrm{reg}}_{g,\ell}, nK_{\overline{\mathcal{R}}^k_{g,\ell}}\right)$$

for n sufficiently big and divisible.

4.3.2 The case $G = S_3$

An important consequence of Theorem 4.36 comes when considering the moduli space of G-twisted covers with $G = S_3$. In this section we prove the following

Theorem 4.46. *If G is the symmetric group S_3, then the non-canonical locus coincides with the T-locus,*

$$\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G} = T_{g,G}.$$

In particular, a point $[\mathsf{C}, \phi]$ is a non-canonical singular points if and only if there exists an automorphism $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ whose coarsening is an elliptic tail automorphism of order 6.

In order to prove this, we look deeper the global structure of an admissible G'-cover $F \to C$ over a 2-marked stable curve $(C; p_1, p_2)$, where G' is an abelian group. We observe that by definition, in the case of an abelian group G', every G'-type is an element of G'. Moreover, for if $\tilde{p}$ is a preimage on F of the marked point p on C, then the local index at $\tilde{p}$ equals the G'-type at p.

Lemma 4.47. *If G' is an abelian group, $(C; p_1, p_2)$ a 2-marked stable curve, and $F \to C$ and admissible G'-cover over $(C; p_1, p_2)$, then the G'-types h_1 and h_2 at p_1 and p_2 respectively, are inverses,*

$$h_1 = h_2^{-1}.$$

Proof. We consider at first the case of a smooth 2-marked curve $(C; p_1, p_2)$. As a consequence of Remark 1.21, the product $h_1 h_2$ is in the commutators subgroup of G', which is trivial. Therefore $h_1 h_2 = 1$.

In the case of a general stable curve C, we call $C_1, \dots, C_V$ its connected components, and $\overline{C}_1, \dots, \overline{C}_V$ their normalizations. Furthermore, we note $p_1^{(i)}, \dots, p_{m_i}^{(i)}$ the marked points on $\overline{C}_i$, which are the preimages of p_1, p_2 or the C nodes. By the previous point, if $h_j^{(i)}$ is the G'-type of F at the marked point $p_j^{(i)}$, then

$$\prod_{j=1}^{m_i} h_j^{(i)} = 1,$$

for every i. By the balancing condition, for every G'-type $h_j^{(i)}$ coming from a C-node, there exists another marked point with G'-type $h_{j'}^{(i')} = (h_j^{(i)})^{-1}$. Therefore

$$1 = \prod_{j,i} h_j^{(i)} = h_1 \cdot h_2.$$

□

Lemma 4.48. *If* (C, ϕ) *is a twisted* S_3*-cover and* a *is a ghost automorphism in* $\mathrm{Aut}_C(\mathsf{C}, \phi)/\,\mathrm{QR}(\mathsf{C}, \phi)$*, then* $\mathrm{age}(\mathsf{a}) \geq 1$.

Proof. We note $\tilde{\Gamma} \to \Gamma$ the graph G-cover associated to (C, ϕ) We call $F \to C$ the admissible S_3-cover associated to (C, ϕ) and we note b_F the local indices cochain of F.

We prove that if a is a ghost automorphism in $\mathrm{Aut}_C(\mathsf{C}, \phi)$ such that $\mathsf{a}(e) = 0$ for every separating edge of Γ, then $\mathrm{age}\,\mathsf{a} \geq 1$. By Lemma 4.35 this implies the thesis.

By Remark 4.28, for any cycle $(\tilde{e}_1, \dots, \tilde{e}_k)$ of $\tilde{\Gamma}$, we must have $\prod (b_F \cdot \mathsf{a})(\tilde{e}_i) = 1$. As any $\mathsf{a}(e)$ has order 2 or 3, and thus gives an aging of $1/2$ or $1/3$ respectively, the only case where $\mathrm{age}\,\mathsf{a} < 1$ is if there exist two edges $e_1, e_2 \in \mathbb{E}(\Gamma)$ such that $\mathsf{a}(e) = 0$ if $e \notin \{e_1, e_2\}$ $\mathsf{a}(e_1) = \mathsf{a}(e_2) = 1 \in \mathbb{Z}/3$. In order to respect the cycles condition, we have a dual graph Γ of the type

$$\begin{tikzcd} \Gamma' \arrow[r, bend left, no head, "e_1"] \arrow[r, bend right, no head, "e_2"'] & \Gamma'', \end{tikzcd}$$

where Γ' and Γ'' are two subgraphs of Γ such that $\mathsf{a}(e) = 0$ for every edge in $E(\Gamma')$ or $E(\Gamma'')$. These two subgraphs are associated to two components C', C'' of C such that $C = C' \cup C''$ and they intersect in exactly two nodes q_1, q_2, corresponding to edges e_1, e_2.

We note $\tilde{\Gamma}'$ and $\tilde{\Gamma}''$ the restrictions of $\tilde{\Gamma}$ over Γ' and Γ'' respectively. As $b_F \cdot \mathsf{a}$ is in $\mathrm{Im}\,\delta$ by Theorem 4.26, this means that $C^0(\tilde{\Gamma}; S_3)$ is non-empty. By the structure of a, we have that $\tilde{\Gamma}'$ or $\tilde{\Gamma}''$, say the first, has two non-connected components $\tilde{\Gamma}'_1, \tilde{\Gamma}'_2$ with stabilizer the normal subgroup $N \subset S_3$ of order 3. The other component $\tilde{\Gamma}''$ has one or two connected components. In fact, the restriction $F|_{C_1}$ is the union of two admissible $\boldsymbol{\mu}_3$-covers over the 2-marked curve $(C_1; q_1, q_2)$. We note $\tilde{e}_1, \tilde{e}_2$ the two oriented edges over e_1 and e_2 in $\mathbb{E}(\tilde{\Gamma})$ such that the tail of $\tilde{e}_1$ and the head of $\tilde{e}_2$ are in $\tilde{\Gamma}'_1$. By Lemma 4.47, $b_F(\tilde{e}_1) = b_F(\tilde{e}_2)$ and as $\mathsf{a}(e_1)$ has order 3, then $b_F(\tilde{e}_1)$ and $b_F(\tilde{e}_2)$ have order 3. By local indices considerations, $\tilde{e}_1$ and $\tilde{e}_2$ touch the same connected components of $\tilde{\Gamma}''$.

$$\begin{tikzcd} \tilde{\Gamma}'_1 \arrow[r, bend left, "\tilde{e}_1"] & \tilde{\Gamma}'' \arrow[l, bend left, "\tilde{e}_2"'] \arrow[dl, bend left] \arrow[dd, rightsquigarrow] \\ \tilde{\Gamma}'_2 \arrow[ur] \arrow[d, rightsquigarrow] & \\ \Gamma' \arrow[r, bend left, no head, "e_1"] \arrow[r, bend right, no head, "e_2"'] & \Gamma''. \end{tikzcd}$$

Finally, a consequence of $b_F \cdot \mathsf{a} \in \mathrm{Im}\,\delta$ is that

$$(b_F \cdot \mathsf{a})(\tilde{e}_1) \cdot (b_F \cdot \mathsf{a})(\tilde{e}_2) = b_F(\tilde{e}_1)^2 = 1,$$

but this is a contradiction because $b_F(e_1)$ has order 3. $\square$

Therefore, as in the case of G abelian group, also for $G = S_3$ the non-canonical locus $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G}$ coincides with the T-locus. In this case, anyway, the Harris-Mumford proof of the extension of pluricanonical forms does not adapt. Indeed, if we look at the proof of Lemma 4.42, and consider the elliptic curve E admitting an automorphism a of order 6, then for $G = S_3$ there exists a non-trivial S_3-cover on E such that a lifts to this cover. That is the S_3-cover induced by the map $\varpi\colon \pi_1(E, p_*) \to S_3$ such that

$$\varpi(a_1) = (123),\ \varpi(b_1) = (132), \quad \text{or a conjugated morphism.}$$

We observe as a consequence that in $\overline{\mathcal{R}}_{g,S_3}$ there exists a component of the T-locus sent in $\operatorname{Im}\Psi$ by $\pi\colon \overline{\mathcal{R}}_{g,S_3} \to \overline{\mathcal{M}}_g$ but non-isomorphic to $\operatorname{Im}\Psi$, and therefore the Ludwig approach does not generalize to this case.

Our conjecture is that the extension result for pluricanonical forms is true for $G = S_3$. In the following we will show how to evaluate the Kodaira dimension of $\overline{\mathcal{R}}_{g,S_3}$ in the hypothesis that

$$H^0\left(\widehat{\mathcal{R}}_{g,S_3}, nK_{\widehat{\mathcal{R}}_{g,S_3}}\right) = H^0\left(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}, nK_{\overline{\mathcal{R}}_{g,S_3}}\right)$$

for n sufficiently big and divisible.

Chapter 5

Evaluation of the Kodaira dimension of $\mathcal{R}_{g,S_3}$

5.1 Moduli boundary

In the previous chapter we introduced the moduli space $\mathcal{R}_{g,G}$ for any genus $g \geq 2$ and any finite group G, and its compactification $\overline{\mathcal{R}}_{g,G}$. In this chapter we study the boundary of this compactification, *i.e.* the locus $\overline{\mathcal{R}}_{g,G} \backslash \mathcal{R}_{g,G}$. This locus is particularly important, because in order to prove $\overline{\mathcal{R}}_{g,S_3}$ being of general type, we will describe the canonical divisor of this space as a combination of the Hodge divisor $\lambda \in \operatorname{Pic}(\overline{\mathcal{R}}_{g,G})$ and boundary divisors.

5.1.1 Components of the moduli space

We recall, for every class $\mathcal{H}$ in $\mathcal{T}(G)$, the moduli stack $\overline{\mathbf{R}}^{\mathcal{H}}_{g,G}$ of admissible $\mathcal{H}$-covers over stable curves of genus g, *i.e.* admissible G-covers such that the stabilizer of every connected component of a cover is in the class $\mathcal{H}$. These are all pairwise disjoint substacks of $\overline{\mathbf{R}}_{g,G}$.

We focus on the case $G = S_3$. We observe that S_3 has 6 subgroups and 4 subgroup classes.

- The trivial subgroup $(1) \subset S_3$. In this case $\overline{\mathcal{R}}^{1}_{g,S_3}$ is isomorphic to $\overline{\mathcal{M}}_g$.
- The three subgroups T_1, T_2, T_3 of order 2, generated respectively by transpositions $(23), (13), (12)$. These all lie in the same conjugacy class $T = \{T_1, T_2, T_3\}$, and the stack $\overline{\mathbf{R}}^{T}_{g,S_3}$ is isomorphic to $\overline{\mathbf{R}}^{\boldsymbol{\mu}_2}_{g,\boldsymbol{\mu}_2} \subset \overline{\mathbf{R}}^{0}_{g,2}$, the moduli stack of twisted curves equipped with a non-trivial square root of the trivial bundle. Indeed, given an admissible T-cover $F \to C$, consider any connected component $E \subset F$, and map it to the admissible connected $\boldsymbol{\mu}_2$-cover $E \to C$. This is a well defined isomorphism $\overline{\mathbf{R}}^{T}_{g,S_3} \to \overline{\mathbf{R}}^{\boldsymbol{\mu}_2}_{g,\boldsymbol{\mu}_2}$. It is known that $\overline{\mathbf{R}}^{T}_{g,S_3} = \overline{\mathbf{R}}^{\boldsymbol{\mu}_2}_{g,\boldsymbol{\mu}_2}$ is connected.
- The normal subgroup $N \subset S_3$, a cyclic group generated by the 3-cycle (123). For a matter of simplicity, with a little abuse of notation we call N the class $\{N\} \in \mathcal{T}(S_3)$. Consider the moduli stack $\overline{\mathbf{R}}^{\boldsymbol{\mu}_3}_{g,\boldsymbol{\mu}_3} \subset \overline{\mathbf{R}}^{0}_{g,\boldsymbol{\mu}_3}$ of twisted curves

equipped with a non-trivial third root of the trivial bundle. There exists a $2:1$ map $\overline{\mathsf{R}}^{\boldsymbol{\mu}_3}_{g,\boldsymbol{\mu}_3} \to \overline{\mathsf{R}}^{N}_{g,S_3}$. Consider a third root $(\mathsf{C},\mathsf{L},\theta)$, and the map $\exp_3\colon \mathsf{L} \to \mathsf{L}^3 = \mathcal{O}_{\mathsf{C}}$ such that $z \mapsto z^3$. We denote by R the preimage via $\exp_3$ of the unit section of $\mathcal{O}_{\mathsf{C}}$. Therefore $(\mathsf{C},\mathsf{L},\theta)$ is mapped to $(R \sqcup R) \to C$ where the S_3-action is given by the two natural non-trivial actions of $\boldsymbol{\mu}_3$ on the two copies of R. The moduli space $\overline{\mathcal{R}}^{\boldsymbol{\mu}_3}_{g,\boldsymbol{\mu}_3}$ being connected, this also implies the connectedness of $\overline{\mathcal{R}}^{N}_{g,S_3}$.

- The group S_3 itself. In this case the stack $\overline{\mathcal{R}}^{S_3}_{g,S_3}$ is the moduli of curves equipped with a connected admissible S_3-cover. This is the "really new" component of moduli space $\overline{\mathcal{R}}_{g,S_3}$, and our analysis will focus on it.

Furthermore, we observe that there exists a canonical map $\overline{\mathsf{R}}_{g,S_3} \to \overline{\mathsf{R}}_{g,\boldsymbol{\mu}_2}$. Given any admissible G-cover $F \to C$, we send it to the $\boldsymbol{\mu}_2$-cover $(F/N) \to C$ obtained via the quotient by the normal subgroup $N \subset S_3$.

Remark 5.1. In general, if $\overline{\mathcal{R}}^{G}_{g,G}$ is the moduli space of connected admissible G-covers, then by [5, Proposition 2.7] we can classify its connected components. In particular, we call π_1 the canonical presentation of the fundamental group $\pi_1(C)$ of any smooth curve of genus g, that is

$$\pi_1 = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \;\middle|\; \prod_i (\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}) = 1 \right\rangle.$$

We note $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ the group of outer automorphisms, that is the quotient of $\mathrm{Aut}(G)$ by $\mathrm{Inn}(G)$ the group of conjugations. Then the connected components of $\overline{\mathcal{R}}^{G}_{g,G}$ are in bijection with the classes

$$\mathrm{Out}(G)\backslash(\mathrm{Surj}(\pi_1, G)/G),$$

where $\mathrm{Surj}(\pi_1, G) \subset \mathrm{Hom}(\pi_1, G)$ is the subgroup of surjections, and G acts on it by conjugation.

Via the action of Dehn twists on π_1, it is possible to prove that the action of $\mathrm{Out}(S_3)$ on $\mathrm{Surj}(\pi_1, S_3)/S_3$, is transitive and therefore $\overline{\mathcal{R}}^{S_3}_{g,S_3}$ is a connected component of $\overline{\mathcal{R}}_{g,S_3}$. Anyway, we will use this property of the moduli space without stating a complete proof.

5.1.2 The boundary divisors

To classify the irreducible divisors on $\overline{\mathcal{R}}_{g,G}\backslash\mathcal{R}_{g,G}$ we start by recalling the boundary divisors of $\overline{\mathcal{M}}_g$.

Definition 5.2. A node q of a nodal curve C is a disconnecting node if the partial normalization of C at q is a disconnected scheme. This disconnected scheme has two connected components noted C_1 and C_2.

Consider C of genus g and q a disconnecting node, q is of type i, with i between 1 and $\lfloor g/2 \rfloor$, if C_1 and C_2 have respectively genus i and $g-i$.

If q is a non-disconnecting node, then it is a node of type 0.

Definition 5.3. For every i between 1 and $\lfloor g/2 \rfloor$, the divisor $\Delta_i \subset \overline{\mathcal{M}}_g$ is the locus of curves with a disconnecting node of type i.

For $i = 0$, the divisor $\Delta_0 \subset \overline{\mathcal{M}}_g$ is the locus of curves with a node of type 0, or equivalently the closure of the locus of nodal irreducible curves.

For every i we will note δ_i the class of Δ_i in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$. We consider the natural morphism $\pi \colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ and look at the preimages $\pi^{-1}(\Delta_i)$ for every i. The intersections of this preimages with the connected components of $\overline{\mathcal{R}}_{g,G}$ are the components of $\overline{\mathcal{R}}_{g,G}$ boundary that we are going to consider.

We start by focusing on the loci of curves with a disconnecting node, *i.e.* the preimages of Δ_i with $i \neq 0$. In what follows for any curve C in Δ_i we note q the node who separates components C_1, of genus i, and C_2, of genus $g - i$. As a consequence

$$C_1 \sqcup C_2 \to C$$

is the partial normalization of C at q, and we note q_1 and q_2 the preimages of q respectively on C_1 and C_2.

There exists, at the level of the moduli space $\overline{\mathcal{M}}_g$, a natural gluing map

$$\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g,$$

defined in such a way that a pair of points $([C_1, q_1], [C_2, q_2])$, where C_1 and C_2 have genus i and $g - i$ respectively, is sent on the point associated to the nodal curve

$$C := (C_1 \sqcup C_2)/(q_1 \sim q_2).$$

We recall that any admissible G-cover on C induces two admissible G-covers F_1 on $(C_1; q_1)$ and F_2 on $(C_2; q_2)$ such that the G-types $[\![h_1]\!]$ and $[\![h_2]\!]$ on q_1 and q_2 are one the inverse of the other, $[\![h_1]\!] = [\![h_2^{-1}]\!]$.

Definition 5.4. Consider two subgroup conjugacy classes $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{T}(G)$, a conjugacy class $[\![h]\!]$ in $[\![G]\!]$ and a positive integer i between 1 and $\lfloor g/2 \rfloor$. We note

$$\Delta_{i,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$$

the locus in $\overline{\mathcal{R}}_{g,G}$ of curves C with a node q of type i and with an admissible G-cover $F \to C$ such that if we note $F_1 \to (C_1; q_1)$ and $F_2 \to (C_2; q_2)$ the restrictions of F, then these are an admissible $\mathcal{H}_1$-cover of $(C_1; q_1)$ and an admissible $\mathcal{H}_2$-cover of $(C_2; q_2)$ respectively, and the G-type at q, with respect to the branch of C_1, is $[\![h]\!]$. In order to simplify the notation we will omit to note the G-type when it is trivial.

Remark 5.5. The loci $\Delta_{i,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ are (not necessarily connected) divisors of the moduli space $\overline{\mathcal{R}}_{g,G}$.

For example in the case of $\overline{\mathcal{R}}_{g,S_3}$ we have defined the following classes:

- Δ_i^{1,S_3}, $\Delta_i^{S_3,1}$, $\Delta_i^{1,T}$, $\Delta_i^{T,1}$, $\Delta_i^{1,N}$, $\Delta_i^{N,1}$ and $\Delta_i^{1,1} \cong \Delta_i$ are the cases of admissible G-covers which are trivial over C_1 or C_2;
- $\Delta_i^{T,T}$, $\Delta_i^{T,N}$, $\Delta_i^{N,T}$, Δ_i^{T,S_3}, $\Delta_i^{N,N}$, Δ_i^{N,S_3}, $\Delta_i^{S_3,N}$ and $\Delta_i^{S_3,S_3}$ are the other cases with trivial G-type at the node q;

- $\Delta_{i,c_3}^{S_3,S_3}$ is the only case with non-trivial stabilizer at q. Here we noted by c_3 the conjugacy class $[\![123]\!] = \{(123),(132)\}$.

Remark 5.6. We observe that by Remark 1.37, it is possible to have non-trivial stabilizer at q only if both $F_1 \to C_1$ and $F_2 \to C_2$ are connected admissible S_3-covers.

Finally we consider the preimage of Δ_0, *i.e.* the locus of curves with a node of type 0. We start by working on a covering stack of $\pi^{-1}(\Delta_0)$.

Definition 5.7. Category D_0 has for objects curves with a node q of type 0 plus an admissible G-cover and a privileged branch at q.

Remark 5.8. Category D_0 is a Deligne-Mumford stack. Its coarse space $\mathcal{D}_0$ comes with a natural $2:1$ morphism $\gamma\colon \mathcal{D}_0 \to \pi^{-1}(\Delta_0)$.

In what follows we consider a curve C with a node q of type 0, we call

$$\mathsf{nor}\colon \overline{C} \to C$$

the partial normalization of C at q. Given any admissible G-cover $F \to C$, the pullback nor^*F over $\overline{C}$ is still an admissible G-cover.

Definition 5.9. Consider two conjugacy classes $\mathcal{H}_1, \mathcal{H}_2$ in $\mathcal{T}(G)$ such that $\mathcal{H}_2 \leq \mathcal{H}_1$. The category $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ is a full subcategory of D_0. Its objects are stable curves C with a node q of type 0, a privileged branch at q and an admissible $\mathcal{H}_1$-cover $F \to C$ such that the pullback $\mathsf{nor}^*F \to \overline{C}$ is an admissible $\mathcal{H}_2$-cover of $\overline{C}$, and the G-type of F at q with respect to the privileged branch is $[\![h]\!]$. In case $h = 1$ we will omit its notation and the divisor will simply be noted $\mathsf{D}_0^{\mathcal{H}_1,\mathcal{H}_2}$.

Category $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ is again a Deligne-Mumford stack and we note its coarse space as $\mathcal{D}_{0,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$.

Remark 5.10. By Remark 1.37, there exists a compatibility condition to have the stack $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ being non-empty. There must exist an element h in the class $[\![h]\!]$ and a subgroup H_2 in the class $\mathcal{H}_2$ such that h lies in the subgroup of commutators of H_2.

Remark 5.11. There exists a natural automorphism

$$\mathrm{inv}\colon \mathsf{D}_0 \to \mathsf{D}_0,$$

which sends any curve with an admissibile G-cover to the same curve and G-cover but changing the privileged branch at the node. This sends isomorphically $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ in $\mathsf{D}_{0,[\![h^{-1}]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$.

Definition 5.12. For any group G we consider the *inverse* relation in the set of conjugacy classes $[\![G]\!]$,

$$[\![h]\!] \sim [\![h']\!]$$

if and only if $[\![h]\!] = [\![h']\!]$ or $[\![h]\!] = [\![h'^{-1}]\!]$. Then we define the set $\underline{[\![G]\!]}$ as

$$\underline{[\![G]\!]} := [\![G]\!]/\sim .$$

When there is no risk of confusion, we note $\underline{[\![h]\!]}$ the class in $\underline{[\![G]\!]}$ of any element $h \in G$.

We observe that any point $[C,F]$ of $\mathcal{D}_0$ and its image $\mathrm{inv}([C,F])$ are sent by $\gamma\colon \mathcal{D}_0 \to \pi^{-1}(\Delta_0)$ to the same point. This means that the image of $\mathcal{D}^{\mathcal{H}}_{0,\underline{[\![h]\!]}}$ via γ depends only on $\mathcal{H}$ and on the class $[\![h]\!] \in \underline{[\![G]\!]}$. Equivalently, for all $\underline{[\![h]\!]} \in \underline{[\![G]\!]}$, $\mathcal{D}^{\mathcal{H}}_{0,[\![h]\!]}$ and $\mathcal{D}^{\mathcal{H}}_{0,[\![h^{-1}]\!]}$ have the same image.

Definition 5.13. For every class $\mathcal{H}$ in $\mathcal{T}(G)$ and $[\![h]\!]$ in $\underline{[\![G]\!]}$, the locus $\Delta^{\mathcal{H}}_{0,\underline{[\![h]\!]}}$ is the image of $\mathcal{D}^{\mathcal{H}}_{0,\underline{[\![h]\!]}}$ via γ for any G-type $[\![h]\!]$. As before we will omit to note $\underline{[\![h]\!]}$ in the case of the trivial class $\underline{[\![1]\!]}$.

As before, we list the divisors in $\pi^{-1}(\Delta_0)$ we just defined:

- $\Delta_0^{N,1}$, $\Delta_0^{T,1}$ and $\Delta_0^{1,1} \cong \Delta_0$ are the divisor of admissible G-covers $F \to C$ such that $\mathsf{nor}^*F \to \overline{C}$ is trivial;
- $\Delta_0^{T,T}$, $\Delta_0^{S_3,T}$, $\Delta_0^{N,N}$, $\Delta_0^{S_3,N}$ and $\Delta_0^{S_3,S_3}$ are the other cases with trivial associated G-type at node q;
- with another small abuse of notation, we note $c_2 := \underline{[\![12]\!]} = \{(12),(13),(23)\}$ and $c_3 := \underline{[\![123]\!]} = \{(123),(132)\}$. Then, $\Delta^{T,T}_{0,c_2}$, Δ^{S_3,S_3}_{0,c_2}, $\Delta^{N,N}_{0,c_3}$ and Δ^{S_3,S_3}_{0,c_3} are the cases with non-trivial stabilizer at q.

Remark 5.14. The divisors $\Delta^{S_3,T}_{0,c_2}$ and $\Delta^{S_3,N}_{0,c_3}$ are both empty, because of glueing reasons.

5.2 The canonical divisor

5.2.1 Evaluation of the divisor

To evaluate the canonical divisor we start by evaluating the pullbacks of the $\overline{\mathcal{M}}_g$ boundary divisors. We note $\delta^{\mathcal{H}_1,\mathcal{H}_2}_{i,[\![h]\!]}$ the class of divisor $\Delta^{\mathcal{H}_1,\mathcal{H}_2}_{i,[\![h]\!]}$ in the ring $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3})$. Furthermore, we introduce some other notation to simplify the formulas that will follow.

- For any $i \geq 1$ we note
$$\delta'_i = \sum \delta_i^{\mathcal{H}_1,\mathcal{H}_2},$$
where the sum is over al the divisors with trivial S_3-type at the node.
- For $i = 0$ we note
$$\delta'_0 = \sum_{\mathcal{H}_2 \leq \mathcal{H}_1} \delta_0^{\mathcal{H}_1,\mathcal{H}_2},$$
$$\delta_{0,c_2} = \delta^{T,T}_{0,c_2} + \delta^{S_3,S_3}_{0,c_2},$$
and
$$\delta_{0,c_3} = \delta^{N,N}_{0,c_3} + \delta^{S_3,S_3}_{0,c_3}.$$

Given the classes δ_i in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$, and the natural morphism $\pi\colon \overline{\mathcal{R}}_{g,S_3} \to \overline{\mathcal{M}}_g$, we consider the pullbacks $\pi^*\delta_i$ to the ring $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3})$.

Lemma 5.15. *If $i > 1$, then*

$$\pi^*\delta_i = \delta_i' + 3\delta_{i,c_3}^{S_3,S_3}.$$

Proof. This relation is true also in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathsf{R}}_{g,S_3})$. We consider a general S_3-cover (C, ϕ) in the divisor $\Delta_{i,c_3}^{S_3,S_3}$, and we note q its node and C the coarse space of C. By construction we have

$$\frac{\mathrm{Aut}(\mathsf{C}, \phi)}{\mathrm{Aut}(C)} = \mathrm{Aut}(\mathsf{C}_q) = \boldsymbol{\mu}_3.$$

This implies that the morphism $\overline{\mathsf{R}}_{g,S_3} \to \overline{\mathsf{M}}_g$ is 3-ramified over $\Delta_{i,c_3}^{S_3,S_3}$. Over Δ_i but outside $\Delta_{i,c_3}^{S_3,S_3}$ the morphism is étale, and the thesis follows. □

Remark 5.16. In the case $i = 1$, we observe that the divisor $\delta_1^{S_3,\mathcal{H}_2}$ is empty for every class $\mathcal{H}_2$. Indeed, by Proposition 1.33 an admissible cover on an elliptic curve (E, p_*) is either cyclic or it has non-trivial stabilizer at the marked point.

Lemma 5.17. *If $i = 1$, then*

$$\pi^*\delta_1 = \delta_1' + 6\delta_{1,c_3}^{S_3,S_3}.$$

Proof. In the case of a curve C with an elliptic tail, there exists an elliptic tail involution on C, and it lifts to the admissible S_3-cover if and only if the restriction of the cover to the elliptic tail is cyclic. Therefore

$$\pi^*\Delta_1 = \Delta_1' + 2\Delta_{1,c_3}^{S_3,S_3}.$$

To obtain the result we observe that, as in the case of Lemma 5.15, the morphism $\overline{\mathsf{R}}_{g,S_3} \to \overline{\mathsf{M}}_g$ is 3-ramified over $\Delta_{1,c_3}^{S_3,S_3}$. □

Lemma 5.18. *If $i = 0$, then*

$$\pi^*\delta_0 = \delta_0' + 2\delta_{0,c_2} + 3\delta_{0,c_3}.$$

Proof. Similarly to what observed in the case of Lemma 5.15, the morphism $\overline{\mathsf{R}}_{g,S_3} \to \overline{\mathsf{M}}_g$ is 2-ramified over Δ_{0,c_2}, 3-ramified over Δ_{0,c_3} and étale elsewhere over Δ_0. □

From the Harris and Mumford work [24], we know the evaluation of the canonical divisor $K_{\overline{\mathcal{M}}_g}$ of the moduli space $\overline{\mathcal{M}}_g$. Knowing that the morphism $\pi\colon \overline{\mathcal{R}}_{g,S_3}^{S_3} \to \overline{\mathcal{M}}_g$ is étale outside the divisors Δ_i , we can now evaluate the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}^{S_3}}$.

Lemma 5.19. *On the smooth variety $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$, the sublocus of regular points, we have the following evaluation of the canonical divisor,*

$$K_{\overline{\mathcal{R}}_{g,S_3}} = 13\lambda - (2\delta_0' + 3\delta_{0,c_2} + 4\delta_{0,c_3}) - (3\delta_1' + 13\delta_{1,c_3}^{S_3,S_3}) - \sum_{i=2}^{\lfloor g/2 \rfloor} (2\delta_i' + 4\delta_{i,c_3}^{S_3,S_3}).$$

Proof. As proved in [24, Theorem 2 bis, p.52], on $\overline{\mathcal{M}}_g^{\mathrm{reg}}$ we have

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor g/2 \rfloor}.$$

We know, from the description above, that the ramification divisor of the morphism $\pi\colon \overline{\mathcal{R}}_{g,S_3}^{S_3} \to \overline{\mathcal{M}}_g$ is

$$R = \delta_{0,c_2} + 2\delta_{0,c_3} + 5\delta_{1,c_3}^{S_3,S_3} + 2\sum_{i=2}^{\lfloor g/2 \rfloor} \delta_{i,c_3}^{S_3,S_3}.$$

By Hurwitz formula we have

$$K_{\overline{\mathcal{R}}_{g,S_3}} = \pi^* K_{\overline{\mathcal{M}}_g} + R,$$

this is true on $\pi^{-1}(\overline{\mathcal{M}}_g^{\mathrm{reg}})$, but this locus and $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$ differ by a codimension 2 locus, therefore the evaluation is unchanged on $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$. The thesis follows by the evaluations of the divisors $\pi^*\delta_i$ of Lemmata 5.15, 5.17 and 5.18. $\square$

5.2.2 The subspace of covers on irreducible curves

We start by recalling the definition of the Kodaira dimension of a normal compact scheme.

Definition 5.20 (Kodaira dimension)**.** For every normal compact scheme S there exists an integer $\kappa \leq \dim S$ such that

$$h^0(S, nK_S) \sim c \cdot n^\kappa$$

where c is a positive constant. The integer $\kappa(S)$ is called Kodaira dimension of the scheme. If $h^0(nK_S) = 0$ for all $n > 0$, we set $\kappa(S) := -\infty$.

It is well known that $\kappa(S) \leq \dim(S)$, and in the case of equality we say that the scheme S is of *general type*.

If S is any scheme, its Kodaira dimension is calculated by the normalization of a compactification $\overline{S}^\nu$ of S.

When $\overline{S}$ is a singular scheme, let $\overline{S}^{\mathrm{reg}}$ be the sublocus of smooth points. It is easier to work with the space of global sections $H^0(\overline{S}^{\mathrm{reg}}, nK)$ over $\overline{S}^{\mathrm{reg}}$ than over $\overline{S}^\nu$. We observe that since $\overline{S}^{\mathrm{reg}} \subset \overline{S}^\nu$, then the restriction morphism induces an inclusion $H^0(\overline{S}^\nu, nK) \hookrightarrow H^0(\overline{S}^{\mathrm{reg}}, nK)$ for all n.

In the following we note $\widehat{\mathcal{R}}_{g,S_3}$ a normalization of $\overline{\mathcal{R}}_{g,S_3}$. Moreover, we note $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$ the open and dense locus of smooth points of $\overline{\mathcal{R}}_{g,S_3}$.

As already said, there exists a natural inclusion

$$H^0\left(\widehat{\mathcal{R}}_{g,S_3},\ nK_{\widehat{\mathcal{R}}_{g,S_3}}\right) \subset H^0\left(\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}},\ nK_{\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}}\right),$$

for any n non-negative integer, via the natural restriction morphism. In the following we will work under the hypothesis that

$$H^0\left(\widehat{\mathcal{R}}_{g,S_3},\ nK_{\widehat{\mathcal{R}}_{g,S_3}}\right) = H^0\left(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3},\ nK_{\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}}\right) \tag{5.1}$$

for n sufficiently big and divisible. This result have been proved in Theorem 4.43 for $\overline{\mathcal{R}}_{g,G}$ when G is an abelian group and the J-locus is empty. We will show the bigness of the canonical divisor under the conjectured condition that Equation (5.1) is verified.

As a consequence, it suffices to prove that $K_{\overline{\mathcal{R}}_{g,S_3}}$ is big over $\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}$ to conclude that $\widehat{\mathcal{R}}_{g,S_3}$, and thus $\overline{\mathcal{R}}_{g,S_3}$, is of general type. Indeed, if the canonical divisor is big over $\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}$, then

$$h^0(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}, nK) \sim c \cdot n^{\dim(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3})},$$

and as $\dim(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}) = \dim(\widehat{\mathcal{R}}_{g,S_3}) = 3g-3$, the same asymptotic relation is true for $\widehat{\mathcal{R}}_{g,S_3}$, too.

We consider the full substack $\widetilde{\mathbf{R}}_{g,S_3} \subset \overline{\mathbf{R}}_{g,S_3}$ of irreducible stable curves of genus g with an admissible S_3-cover. We note $\widetilde{\mathcal{R}}_{g,S_3}$ its coarse space. We prove that the bigness of the canonical divisor over $\widetilde{\mathcal{R}}_{g,S_3}$ implies the bigness over $\overline{\mathcal{R}}_{g,S_3}$. As a consequence it would suffice to prove that $K_{\overline{\mathcal{R}}_{g,S_3}}$ is big over $\widetilde{\mathcal{R}}_{g,S_3}$, to prove $\overline{\mathcal{R}}_{g,S_3}$ being of general type.

We build some pencils filling up specific divisors of $\overline{\mathcal{R}}^{S_3}_{g,S_3}$. To do this we follow [19] and [20]. Given a general K3 surface X of degree $2i-2$ in $\mathbb{P}^i$, the map $\mathrm{Bl}_{i^2}(X) \to \mathbb{P}^1$, where $\mathrm{Bl}_{i^2}(X)$ is the blow-up of X in i^2 points, is a family of genus i stable curves lying on X. This induces a pencil $B \subset \overline{\mathcal{M}}_i$. Moreover, there exists at least one section σ on B, therefore for every genus $g > i$ we can glue along σ a fixed 1-marked curve (C_2, p) of genus $g-i$, thus inducing a pencil $B_i \subset \overline{\mathcal{M}}_g$. The pencils B_i fill up the divisors Δ_i except for $i = 10$, if $i = 10$ then the B_{10} fill up a divisor $\mathcal{Z}$ in Δ_{10} which is the locus of smooth curves of genus 10 lying on a K3 surface and attached to a curve of genus $g-10$. We note $\mathcal{Z}'$ the preimage of $\mathcal{Z}$ in $\overline{\mathcal{R}}^{S_3}_{g,S_3}$.

Lemma 5.21. *Consider an effective divisor E in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{R}^{S_3}_{g,S_3})$ such that*

$$[\overline{E}] = a\cdot\lambda - b_0'\cdot\delta_0' - b_{0,c_2}\cdot\delta_{0,c_2} - b_{0,c_3}\cdot\delta_{0,c_3} - b_1'\cdot\delta_1' - b^{S_3,S_3}_{1,c_3}\cdot\delta^{S_3,S_3}_{1,c_3} - \sum_{i=2}^{\lfloor g/2\rfloor}(b_i'\cdot\delta_i' + b^{S_3,S_3}_{i,c_3}\cdot\delta^{S_3,S_3}_{i,c_3})$$

in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}^{S_3}_{g,S_3})$, where the a and b are rational coefficients.

If $a \leq 13$, $b_0' \geq 2$, $b_{0,c_2} \geq 3$ and $b_{0,c_3} \geq 4$, then $b_i' \geq 3$ and $b^{S_3,S_3}_{i,c_3} \geq 13$ for all $i \geq 1$ and $i \neq 10$. The same is true for $i = 10$ if $\overline{E}$ does not contain $\mathcal{Z}'$.

Proof. By [20, Lemma 2.4] we have

$$B_i\cdot\lambda = i+1,\quad B_i\cdot\delta_0 = 6i+18,\quad B_i\cdot\delta_i = -1,\quad B_i\cdot\delta_j = 0\ \forall j\neq 0, i.$$

We introduce some pencils lying in the preimage of B_i with respect to the natural projection $\pi\colon \overline{\mathcal{R}}^{S_3}_{g,S_3} \to \overline{\mathcal{M}}_g$.

The pencil $A_i^{T,N}$ is the preimage of B_i in $\Delta_i^{T,N}$. It is obtained by taking any 1-marked stable curve C_1 of genus i with an admissible T-cover, and gluing it to a fixed general 1-marked curve C_2 of genus $g-i$ with an admissible N-cover.

The pencil $A_{i,c_3}^{S_3,S_3}$ is a connected component of the preimage of B_i in $\Delta_{i,c_3}^{S_3,S_3}$. It is obtained by taking any 1-marked stable curve C_1 of genus i with an admissible connected S_3-cover, and gluing it to a fixed general 1-marked curve C_2 of genus $g-i$ with an admissible connected S_3 cover, and moreover the gluing node must have a cyclic stabilizer of order 3.

We can write down some intersection numbers for $A_i^{T,N}$:

- $A_i^{T,N}\cdot\lambda=(2^{2i}-1)(i+1)$, which is true because $\pi^*\lambda_{\overline{\mathcal{M}}_g}=\lambda_{\overline{\mathcal{R}}_{g,S_3}}$ and $\pi_*A_i^{T,N}=B_i\cdot\deg(A_i^{T,N}/B_i)$;
- $A_i^{T,N}\cdot\delta_0'=(2^{2i-1}+1)(6i+18)$, $A_i^{T,N}\cdot\delta_{0,c_2}=(2^{2i-1}-2)(6i+18)$, $A_i^{T,N}\cdot\delta_{0,c_3}=0$. The third equality is clear. The second one is obtained by counting the admissible T-cover over a curve of genus $i-1$ and by multiplying for the 2 possible gluing factors at the node. The first equality is obtained by difference;
- $A_i^{T,N}\cdot\delta_{j,c_3}^{S_3,S_3}=0$ for all j and $A_i^{T,N}\cdot\delta_j'=0$ for all $j\neq i$ and $A_i^{T,N}\cdot\delta_i'=-(2^{2i}-1)$ because it is the same of $A_i^{T,N}\cdot\pi^*\delta_i$.

Consider an effective divisor E of $\mathcal{R}_{g,S_3}^{S_3}$, as the pencils $A_i^{T,N}$ fill up the boundary divisor $\Delta_i^{T,N}$ for $i\neq 10$, we have $A_i^{T,N}\cdot\overline{E}\geq 0$ that is, by the relations above,

$$(i+1)\cdot a-\frac{2^{2i-1}+1}{2^{2i}-1}\cdot(6i+18)\cdot b_0'-\frac{2^{2i-1}-2}{2^{2i}-1}\cdot(6i+18)b_{0,c_2}+b_i'\geq 0,$$

which implies $b_i'\geq 3$ for all $i\geq 1$. The same is true for $i=10$ if $\overline{E}$ does not contain the locus $\mathscr{Z}'$.

We note $d:=\deg(A_{i,c_3}^{S_3,S_3})$. As in the previous case, also for $A_{i,c_3}^{S_3,S_3}$ we have the equalities

- $A_{i,c_3}^{S_3,S_3}\cdot\lambda=d\cdot(i+1)$;
- $A_{i,c_3}^{S_3,S_3}\cdot\delta_{i,c_3}^{S_3,S_3}=-d$;
- $A_{i,c_3}^{S_3,S_3}\cdot\delta_j'=0$ for all j and $A_{i,c_3}^{S_3,S_3}\delta_{j,c_3}^{S_3,S_3}=0$ for all $j\neq i$.

Furthermore we have

$$A_{i,c_3}^{S_3,S_3}\cdot(\delta_0'+\delta_{0,c_2}+\delta_{0,c_3})=A_{i,c_3}^{S_3,S_3}\cdot\pi^*\delta_0=d\cdot(6i+18).$$

Moreover, every curve in the intersection $A_{i,c_3}^{S_3,S_3}\cap\Delta_{0,c_2}$ induces a curve in the intersection $A_{i,c_3}^{S_3,S_3}\cap\Delta_0'$ by deleting the non-trivial S_3-type at the auto-intersection node. The same is true for $A_{i,c_3}^{S_3,S_3}\cap\Delta_{0,c_3}$, and it is possible to obtain all the points of of $A_{i,c_3}^{S_3,S_3}\cap\Delta_0'$ via these operations. This implies

$$A_{i,c_3}^{S_3,S_3}\cdot(\delta_{0,c_2}+\delta_{0,c_3})\geq A_{i,c_3}^{S_3,S_3}\cdot\delta_0'.$$

Consider an effective divisor E of $\mathcal{R}_{g,S_3}^{S_3}$, as the pencils $A_{i,c_3}^{S_3,S_3}$ fill up a connected component of the boundary divisor $\Delta_{i,c_3}^{S_3,S_3}$ for $i \neq 10$, we have $A_{i,c_3}^{S_3,S_3} \cdot \overline{E} \geq 0$ that is, by the relations above,

$$d \cdot b_{i,c_3}^{S_3,S_3} \geq A_{i,c_3}^{S_3,S_3} \cdot (2\delta_0' + 3\delta_{0,c_2} + 4\delta_{0,c_3}) - d \cdot (i+1) \cdot a \geq$$

$$\frac{5}{2} \cdot A_{i,c_3}^{S_3,S_3} \cdot (\delta_0' + \delta_{0,c_2} + \delta_{0,c_3}) - d \cdot (i+1) \cdot a \geq d \cdot 13.$$

The same is true if $i = 10$ and $\overline{E}$ does not contain the locus $\mathcal{Z}'$. □

The previous lemma allows to prove the following.

Proposition 5.22. *We work under the hypothesis that Equation (5.1) is true. If we can write down*

$$K_{\overline{\mathcal{R}}_{g,S_3}^{S_3}} = a' \cdot \lambda + E'$$

on $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, where a' is a positive coefficient and E' an effective divisor not containing $\mathcal{Z}'$, then the canonical divisor is big over the whole space $\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}}$ and therefore $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is of general type.

Proof. The equation in the hypothesis implies that the canonical divisor is big on $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, because λ is a big divisor. If we consider cloture $\overline{E}'$ of E' on the whole space $\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}}$, by Lemma 5.21

$$K_{\overline{\mathcal{R}}_{g,S_3}^{S_3}} - \overline{E}'$$

is big on $\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}}$, and the proof is completed. □

Chapter 6

Moduli spaces of general type

To calculate the Kodaira dimension of $\overline{\mathcal{R}}_{g,S_3}$, and in particular to find the genus g such that $\overline{\mathcal{R}}_{g,S_3}$ is of general type, we need to develop some calculations in the tautological ring of the moduli space. Consider $\mathcal{C}_g \to \overline{\mathcal{R}}_{g,S_3}$ the universal curve over the moduli stack, in Section 6.1 we develop Grothendieck Riemann-Roch type calculations for vector bundles over $\mathcal{C}_g$. In Section 6.2 we apply it to evaluate the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}}$ and prove its bigness over $\overline{\mathcal{R}}_{g,S_3}^{S_3}$, the connected component of $\overline{\mathcal{R}}_{g,S_3}$ of admissible connected S_3-covers.

6.1 Adapted Grothendieck Riemann-Roch

6.1.1 Tautological classes

In this section we recall some well known *tautological* classes in the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ of the moduli space of curves, and their generalizations to $\overline{\mathcal{R}}_{g,G}$. For a wider survey of the tautological relations and the tautological rings structure see [4, §17], [28] and [15].

There exists two natural morphisms on the moduli spaces of the type $\overline{\mathcal{M}}_{g,n}$ that "come from geometry": with this we mean that we can define them using the modular interpretation of the space.

- The *forgetful morphism* is a morphism

$$\mu\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$$

  sending any geometric point $[C; p_1, \ldots, p_n]$ to the same marked stable curve without the last point, $[C; p_1, \ldots, p_{n-1}]$.

- The *gluing morphisms* are of two types

$$\iota\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

  and

$$\iota\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}.$$

In the first case $[C; p_1, \dots, p_{n_1+1}] \times [C'; p'_1, \dots, p'_{n_2+1}]$ is sent two the curve

$$(C \cup C')/(p_{n_1+1} \sim p'_{n_2+1}),$$

the junction of C and C' along the marked points p_{n_1+1} and p'_{n_2+1}. The new curve maintains all the other marked points. In the second case $[C; p_1, \dots, p_{n+2}]$ is sent to the curve $C/(p_{n+1} \sim p_{n+2})$ and with the same other marked points.

Definition 6.1. The system of tautological rings $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$ with g, n varying on the non-negative integers, is the smallest system of $\mathbb{Q}$-algebras closed under the pushforwards of the forgetful morphisms and the gluing morphisms.

We define n tautological ψ-classes inside the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$. We consider the universal family

$$u \colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n},$$

where $\mathcal{C}_{g,n}$ is a Deligne-Mumford stack such that every geometric fiber of u is isomorphic to the associated n-marked stable curve, and there exist n sections

$$\sigma_1, \dots, \sigma_n \colon \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}.$$

For every $i = 1, \dots, n$ the line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$ is the ith cotangent line bundle,

$$\mathcal{L}_i := \sigma_i^* \left(T_u^\vee \right).$$

Then we define

$$\psi_i := c_1(\mathcal{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n}).$$

Remark 6.2. The ψ-classes are in the tautological ring $R^1(\overline{\mathcal{M}}_{g,n})$. Indeed, as showed for example in [15], they can be obtained by pushforwarding some fundamental classes of the type $[\overline{\mathcal{M}}_{g,n}]$. In particular,

$$\psi_i = -\mu_* \left(\left(\iota_* \left([\overline{\mathcal{M}}_{g,n}] \times [\overline{\mathcal{M}}_{0,3}] \right) \right)^2 \right) \in R^1(\overline{\mathcal{M}}_{g,n}).$$

There are two other type of classes, very important for our analysis and belonging to the tautological ring. To introduce κ-classes we consider the log-canonical line bundle on $\mathcal{C}_{g,n}$,

$$\omega_u^{\log} := \omega_u(\sigma_1 + \cdots + \sigma_n).$$

Therefore we have

$$\kappa_d := u_* \left(c_1(\omega_u^{\log})^{d+1} \right).$$

As before the class κ_d is well defined in the Chow ring $A^d(\overline{\mathcal{M}}_{g,n})$. We state without proof the known fact that

$$\kappa_d := \mu_*(\psi_{n+1}^{d+1}) \in A^d(\overline{\mathcal{M}}_{g,n}),$$

and being ψ_{n+1} in the tautological ring, the κ-classes is contained in the tautological ring too.

The Hodge bundle over $\overline{\mathbf{M}}_{g,n}$ is the rank g vector bundle

$$\mathcal{E} := u_* \omega_u,$$

i.e. the vector bundle whose fiber at $[C; p_1, \dots, p_n]$ is $H^0(C; \omega_C)$. The *Hodge class* is

$$\lambda := c_1(\mathcal{E}) \in A^1(\overline{\mathcal{M}}_{g,n}).$$

The Hodge class is proved to be in the tautological ring $R^1(\overline{\mathcal{M}}_{g,n})$ in [27].

The line bundles $\mathcal{L}_i$ and $\mathcal{E}$ can be easily generalized to $\overline{\mathcal{R}}_{g,G}$ for every finite group G. Therefore we can define in $A^*(\overline{\mathcal{R}}_{g,G})$ the ψ and κ classes and the Hodge class λ.

6.1.2 Using Grothendieck Riemann-Roch in the Chow ring of $\mathcal{R}_{g,G}$

We call $u\colon \mathcal{C}_{g,G} \to \overline{\mathcal{R}}_{g,G}$ the universal family of curves over $\overline{\mathcal{R}}_{g,G}$. Moreover, we note $\Phi\colon \mathcal{C}_{g,G} \to BG$ the universal twisted G-cover over $\mathcal{C}_{g,G}$. In particular for every geometric point $[\mathsf{C}, \phi]$ of $\overline{\mathcal{R}}_{g,G}$, the restriction of Φ to the associated geometric fiber is isomorphic to the twisted G-cover $\phi\colon \mathsf{C} \to BG$.

We consider the singular locus $\mathcal{N} \subset \mathcal{C}_{g,G}$ of the universal family, whose points are the nodes of $\mathcal{C}_{g,G}$. Furthermore, we consider the stack $\mathcal{N}'$ whose points are nodes equipped with the choice of a privileged branch. There exists a natural étale double cover $\mathcal{N}' \to \mathcal{N}$, and an involution $\varepsilon\colon \mathcal{N}' \to \mathcal{N}'$.

There exists also a natural decomposition of $\mathcal{N}'$. Given a conjugacy class $[\![h]\!]$ in $[\![G]\!]$, we note $\mathcal{N}'_{i,[\![h]\!]} \subset \mathcal{N}'$ the substack of nodes such that the associated privileged branch lies in a component of genus i, and it has $[\![h]\!]$ as G-type (see Remark 1.54). In the case of a node of type 0, the component is $\mathcal{N}'_{0,[\![h]\!]}$. Therefore

$$\mathcal{N}' = \bigsqcup_{\substack{0 \le i \le g-1, \\ [\![h]\!] \in [\![G]\!]}} \mathcal{N}'_{i,[\![h]\!]}.$$

We call j the projection from $\mathcal{N}'$ to $\overline{\mathcal{R}}_{g,G}$,

$$\begin{tikzcd} \mathcal{N}' \arrow[r] \arrow[drr, "j"'] & \mathcal{N} \arrow[r, hook] & \mathcal{C}_{g,G} \arrow[d, "u"] \\ & & \overline{\mathcal{R}}_{g,G}. \end{tikzcd}$$

Furthermore, we note $j_{i,[\![h]\!]}$ the restriction of map j to the component $\mathcal{N}'_{i,[\![h]\!]}$.

We finally define the classes ψ and ψ' on $\mathcal{N}'$:

$$\psi := c_1(T^\vee_{\mathcal{N}'}); \quad \psi' := c_1(\varepsilon^* T^\vee_{\mathcal{N}'}).$$

We state a generalization of the Grothendieck Riemann-Roch formula, following the approach of Chiodo in [9] to evaluate the Chern character of some line bundle pushforwards. In the case treated by Chiodo, this vector bundle is the universal root of the trivial (or canonical) line bundle. Here we generalize to the case of any vector bundle coming from a representation of the group G.

Consider W, a dimension w representation of group G, then W can be regarded as a vector bundle on BG. We consider the universal cover $\Phi\colon \mathcal{C}_{g,G} \to BG$, then the pullback

$$W_{\mathcal{C}_{g,G}} := \Phi^* W$$

is a vector bundle of rank w on the universal family $\mathcal{C}_{g,G}$. From now on we will use the more compact notation $W_{\mathcal{C}}$.

We observe that the projection $j_{i,[\![h]\!]}\colon \mathcal{N}'_{i,[\![h]\!]} \to \overline{\mathcal{R}}_{g,G}$ is locally isomorphic to $B\boldsymbol{\mu}_r \to \operatorname{Spec}\mathbb{C}$ at every point of $\overline{\mathcal{R}}_{g,G}$, where $r = r(h)$ is the order of class $[\![h]\!]$ in G. We follow the approach of [32, §2.2] to decompose the restriction of $W_{\mathcal{C}}$ to any locus $\mathcal{N}'_{i,[\![h]\!]}$. This locus is locally a quotient of an affine scheme by the finite group $\boldsymbol{\mu}_r$, therefore $W_{\mathcal{C}}$ over it is locally a $\boldsymbol{\mu}_r$-equivariant vector bundle, similarly to what we saw analyzing the local structure of twisted G-covers in Section 1.3.2.

Therefore there exists a decomposition in subbundles,

$$W_{\mathcal{C}}|_{\mathcal{N}'_{i,[\![h]\!]}} = W_0 \oplus W_1 \oplus \cdots \oplus W_{r-1},$$

where W_k is the eigen-subbundle with eigenvalue ξ_r^k for all $k = 0, \ldots, r-1$. We note $w_{i,[\![h]\!]}(k)$ the rank of W_k, and clearly these integers satisfy the equation

$$\sum_k w_{i,[\![h]\!]}(k) = w.$$

We also recall the Bernoulli polynomials $B_d(x)$ defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{d=0}^{\infty} B_d(x) \frac{t^d}{d!}.$$

The Bernoulli numbers $B_d := B_d(0)$ are the evaluations of the Bernoulli polynomials at 0. With this setting we can state the following

Proposition 6.3. *On $\overline{\mathcal{R}}_{g,G}$ we have the following evaluation for the degree d component of the Chern character of $Ru_* W_{\mathcal{C}}$.*

$$\mathrm{ch}_d(Ru_* W_{\mathcal{C}}) = \frac{w \cdot B_{d+1}}{(d+1)!}\kappa_d +$$

$$+\frac{1}{2} \cdot \sum_{\substack{0 \le i \le g-1,\\ [\![h]\!] \in [\![G]\!]}} \left(\sum_{0 \le k < r(h)} \left(\frac{r(h)^2 \cdot w_{i,[\![h]\!]}(k) \cdot B_{d+1}\left(k/r(h)\right)}{(d+1)!} \right) \cdot (j_{i,[\![h]\!]})_* \left(\sum_{a+a'=d-1} \psi^a (-\psi')^{a'} \right) \right).$$

This formula is in fact Tseng formula (7.3.6.1) in [32]. In Tseng notation the morphism ev_{n+1} is the u morphism, the representation W is noted F and moreover $(u_*(\mathrm{ch}(ev^* W) Td^\wedge(L_{n+1})))_d$ is the first term in our formula, the one with κ classes. The ψ_i-classes terms are associated to marked points and therefore are absent in our formula. Finally, the terms A_m (see [32, Definition 4.1.2]) give the last term of our formula.

6.2 Bigness of the canonical divisor

We consider the moduli space $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ of genus g curves equipped with a connected admissible S_3-cover. This is a connected component of $\overline{\mathcal{R}}_{g,S_3}$, and $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$ is the subspace of covers on irreducible curves.

The goal of this section is to prove that the canonical divisor of $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is big over the moduli space $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$ for every odd genus $g > 11$. Considering Proposition 5.22 this will imply that $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is of general type for every odd genus $g \geq 13$ under the pluricanonical form extension hypothesis given by Equation (5.1).

The approach follows the strategy of [11] for $\overline{\mathcal{R}}_{g,\boldsymbol{\mu}_3}^{\boldsymbol{\mu}_3}$. We write down the canonical divisor as a sum

$$K = \alpha \cdot \mathcal{U} + \beta \cdot \mathcal{M} + E + \gamma \cdot \lambda \in \operatorname{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}_{g,S_3}^{S_3}). \tag{6.1}$$

Here $\mathcal{U}, \mathsf{M}, E$ are effective divisors, λ is the hodge class, α, β positive coefficients and γ a strictly positive coefficient. This would imply that K is big.

6.2.1 Basic notions of syzygies theory

The divisor $\mathcal{U}$ in equation (6.1) will be defined following the approach for example of Chiodo-Eisenbud-Farkas-Schreyer paper [11]. It is the locus of curves with "extra" syzygies with respect to a particular vector bundle, and it has a determinantal structure over an open subset of $\overline{\mathcal{R}}_{g,S_3}^{S_3}$. To properly define this, we recall some fundamental notions of syzygies theory over stable curves, we will follow the notations of Aprodu-Farkas paper [3].

Consider a finitely generated graded module N over the polynomial ring $S = \mathbb{C}[x_0, \dots, x_n]$. The module has a minimal free resolution

$$0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$$

where

$$F_i = \sum_j S(-i-j)^{b_{i,j}}.$$

The numbers $b_{i,j}$ are well defined and are called the Betti numbers of N, moreover we have

$$b_{i,j} = \dim(\operatorname{Tor}_i^S(N; \mathbb{C}))_{i+j}.$$

Remark 6.4. In an irreducible flat family of modules N_t, the Betti numbers $b_{i,j}(N_t)$ are semicontinuous, and simultaneously take minimal values on an open set. The jumping locus for their values is where one of the values $b_{i,j}(N_t)$ is bigger than this minimum.

For every stable curve we consider a line bundle $L \in \operatorname{Pic}(C)$, a sheaf $\mathcal{F}$ on C, the polynomial ring $S := \operatorname{Sym} H^0(C; L)$ and a the graded S-module

$$\Gamma_C(\mathcal{F}; L) := \bigoplus_{n \in \mathbb{Z}} H^0(C; \mathcal{F} \otimes L^{\otimes n}).$$

From now on we will use the Green notation by calling

$$K_{i,j}(C;\mathcal{F},L) := (\mathrm{Tor}_i^S(\Gamma_C(\mathcal{F},L);\mathbb{C}))_{i+j}.$$

One fundamental idea of Syzygies theory is that the vector space $K_{i,j}(C;\mathcal{F},L)$ can be evaluated via a minimal S resolution of $\mathbb{C}$, where the latter is seen as a graded S-module. To do this we introduce the vector bundle M_L of rank $H^0(L)-1$ over C, via the following short exact sequence

$$1 \to M_L \to H^0(C;L)\otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}} L \to 1. \tag{6.2}$$

Using this vector bundle we can state the following lemma

Lemma 6.5 (see [3, Theorem 2.6]). *Consider C stable curve, L line bundle on it, $\mathcal{F}$ coherent sheaf on it and m positive integer. Then,*

$$K_{m,1}(C;\mathcal{F},L) = H^0\left(\bigwedge^m M_L\otimes\mathcal{F}\otimes L\right).$$

6.2.2 Representation theory of S_3

The symmetric group S_3 has 3 irreducible representations.

1. The trivial representation $\mathbb{I}\colon S_3\to \mathrm{GL}(1;\mathbb{C}) = \mathbb{C}^*$;

2. the parity representation $\epsilon\colon S_3\to\mathbb{C}^*$, which sends even elements to 1 and odd elements to -1;

3. given a vector space $R\cong\mathbb{C}^2$, the representation

$$\rho\colon S_3\to \mathrm{GL}(R)\cong\mathrm{GL}(2;\mathbb{C})$$

such that

$$\rho((12)) = \begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix} \quad \rho((23)) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \rho((13)) = \begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}.$$

Remark 6.6. In particular, if we consider the tautological representation over the vector space

$$P := \langle v_1, v_2, v_3\rangle_{\mathbb{C}}$$

such that S_3 acts naturally by permutation, then P is the direct sum of the trivial representation and R,

$$P = \mathbb{C}_{\mathbb{I}}\oplus R.$$

At the same time, if we consider the regular representation $\mathbb{C}[S_3]$ of dimension $|S_3| = 6$, we have

$$\mathbb{C}[S_3] = \mathbb{C}_{\mathbb{I}}\oplus\mathbb{C}_\epsilon\oplus R^{\oplus 2}.$$

As explained above, we consider R as a line bundle over the stack BS_3. Given a twisted G-cover $[\mathsf{C}, \phi]$, the pullback $R_{\mathsf{C}} := \phi^* R$ is a rank 2 vector bundle over C. In particular if $\mathsf{C} = C$ is a scheme theoretic curve, $\phi^* R_C$ is a usual scheme theoretic vector bundle.

We consider the case of odd genus $g = 2i + 1$, and we focus on the Koszul cohomology $K_{i,1}$. We introduce the locus $\mathcal{U}_g$ as the locus with non-zero cohomology $K_{i,1}$. Supposing that the minimal value assumed by $\dim K_{i,1}$ on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is 0, this is therefore the jumping locus for $K_{i,1}$. In particular we will show that $\mathcal{U}_g$ is a virtual divisor, *i.e.* $\mathcal{U}_g$ is an effective divisor if and only if $\dim K_{i,1}$ takes value 0 on a general curve, or equivalently on at least one curve.

Definition 6.7. Given an odd genus $g = 2i + 1$,

$$\mathcal{U}_g := \left\{ [C, \phi] \in \mathcal{R}_{g,S_3}^{S_3} \mid K_{i,1}(C; R_C, \omega_C) \neq 0 \right\} \subset \overline{\mathcal{R}}_{g,S_3}^{S_3}.$$

We want to show that $\mathcal{U}_g$ is a virtual divisor, *i.e.* it is the degeneration locus of a morphism between two vector bundles of the same rank. This implies that $\mathcal{U}_g$ is either an effective divisor or the whole space $\mathcal{R}_{g,S_3}^{S_3}$. For any stable curve C, by Lemma 6.5 applied in the case $L = \omega_C$ with the vector bundle R_C as the sheaf $\mathcal{F}$, we have

$$K_{i,1}(C; R_C, \omega_C) = H^0\left(\bigwedge^i M_\omega \otimes \omega_C \otimes R_C \right).$$

We can reformulate the definition of $\mathcal{U}_g$ with another scheme theoretic condition. Consider the Equation (6.2) in the case of the canonical line bundle $L = \omega_C$,

$$0 \to M_\omega \to H^0(\omega_C) \otimes \mathcal{O} \to \omega_C \to 0.$$

As ω_C is a line bundle, by a well known property we have the short exact sequence

$$0 \to \bigwedge^i M_\omega \otimes \omega_C \otimes R_C \to \bigwedge^i H^0(\omega_C) \otimes \omega_C \otimes R_C \to \bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C \to 0.$$

Passing to the long exact sequence we have

$$0 \to H^0\left(\bigwedge^i M_\omega \otimes \omega_C \otimes R_C \right) \to \bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C) \xrightarrow{\Lambda} H^0\left(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C \right)$$

Proposition 6.8. *The two vector spaces*

$$\bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C) \quad \text{and} \quad H^0\left(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C \right)$$

have the same dimension. As a consequence any point $[C; \phi]$ of $\mathcal{R}_{g,S_3}^{S_3}$ is in $\mathcal{U}_g$ if and only if the associated Λ morphism is not an isomorphism.

Proof. We start by proving that

$$\bigwedge^i H^0(\omega_C) \otimes H^1(\omega_C \otimes R_C) = H^1(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C) = 0. \tag{6.3}$$

We observe that this two terms fit in the long exact sequence

$$\cdots \to \bigwedge^{i} H^0(\omega_C)\otimes H^1(\omega_C\otimes R_C) \to H^1(\bigwedge^{i-1} M_\omega\otimes\omega_C^{\otimes 2}\otimes R_C) \to H^2(\bigwedge^{i} M_\omega\otimes\omega_C\otimes R_C) \to \cdots$$

and because the last term is 0, it suffices to prove that $H^1(\omega_C \otimes R_C) = H^0(R^\vee) = 0$. Given the admissible S_3-cover $F \to C$ associated to, (C, ϕ), we consider the quotient $F/N =: F' \xrightarrow{\varphi} C$, where $N \subset S_3$ is the normal subgroup of order 3. The cover $\varphi\colon F' \to C$ in an admissible $\boldsymbol{\mu}_2$-cover and by definition of R_C we have $\varphi^* R_C^\vee = \eta \oplus \eta^2$ where η is a non-trivial line bundle such that $\eta^3 \cong \mathcal{O}_{F'}$. Therefore $H^0(F'; \eta) = H^0(F'; \eta^2) = 0$ and *a fortiori* $H^0(C; R_C^\vee) = 0$.

Since the first cohomology group being trivial, we have that the Euler characteristic of both vector bundles coincides with the dimension of their spaces of global sections. In particular, if $\mathcal{E}$ is one of these vector bundles,

$$H^0(\mathcal{E}) = \chi(\mathcal{E}) = \deg(\mathcal{E}) + \mathrm{rk}(\mathcal{E})(1-g).$$

- If $\mathcal{E} = \bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C)$: we have

$$\dim(\mathcal{E}) = \dim\left(\bigwedge^{i} H^0(\omega_C)\right) \cdot \dim\left(H^0(\omega_C \otimes R_C)\right).$$

  The first term is simply $\binom{g}{i}$ because $h^0(\omega_C) = g$. For the second term we have

$$h^0(\omega_C \otimes R_C) = \chi(\omega_C \otimes R_C) = \deg(\omega_C \otimes \mathcal{R}_C) + 2 - 2g = 2g - 2.$$

  Therefore

$$\dim\left(\bigwedge^{i} H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C)\right) = 4i\binom{2i+1}{i}.$$

- If $\mathcal{E} = H^0(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C)$:

$$\deg(\mathcal{E}) = 2 \cdot \deg\left(\bigwedge^{i-1} M_\omega\right) + \binom{g-1}{i-1} \cdot \deg(\omega_C^{\otimes 2} \otimes R_C) = 4(3i+1)\binom{2i}{i-1}.$$

  As a consequence, knowing that $\mathrm{rk}(\mathcal{E}) = 2 \cdot \binom{g-1}{i-1}$, we have

$$\dim(\mathcal{E}) = 4\binom{2i}{i-1}(2i+1) = 4i\binom{2i+1}{i},$$

  that completes the proof.

□

6.2.3 Evaluation of the virtual Koszul divisor

We use the result of Proposition 6.8 to evaluate the class of $\mathcal{U}_g$ in the Chow ring, and also the class of its closure on the space $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$.

Consider the universal family $u\colon \mathcal{C}_{g,S_3} \to \overline{\mathcal{R}}_{g,S_3}^{S_3}$ and the universal rank 2 vector bundle $R_{\mathcal{C}}$ associated to the representation R. Introduce the vector bundle $\mathfrak{M}_u$ on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ defined by the short exact sequence

$$0 \to \mathfrak{M}_u \to u^*(u_*\omega_u) \to \omega_u \to 0.$$

The geometric fiber of $\mathfrak{M}_u$ over any point $[\mathsf{C},\phi]$ of $\mathcal{R}_{g,S_3}^{S_3}$ is the previously defined vector bundle M_ω.

Definition 6.9. We introduce a series of sheaves on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$,

$$\mathbb{E}_{j,b} := u_*\left(\bigwedge^j \mathfrak{M}_u \otimes \omega_u^{\otimes b} \otimes R_{\mathcal{C}}\right),$$

with $j \geq 0$ and $b \geq 1$ integers.

Remark 6.10. To prove these sheaves to be locally free on $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, by Grauert's Theorem it suffices to prove that $h^1(M_\omega \otimes \omega_{\mathsf{C}}^{\otimes b} \otimes R_{\mathsf{C}}) = 0$ for every twisted S_3-cover (C,ϕ) in $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$. As showed in the proof of Proposition 6.8, this reduces to prove $h^0((\omega_{\mathsf{C}}^{\vee})^{\otimes(b-1)} \otimes R_{\mathsf{C}}^{\vee}) = 0$ for all $b \geq 1$. In the proof of Proposition 6.8 we showed this equality for $b = 1$ and a scheme theoretic curve $\mathsf{C} = C$, but the same proof works for every $b \geq 1$ and for every twisted curve C such that its coarse space is irreducible.

By Proposition 6.8, on $\mathcal{R}_{g,S_3}^{S_3}$ the locus $\mathcal{U}_g$ is the degeneration locus of a morphism between the vector bundles

$$\mathbb{E}_{i-1,2} \quad \text{and} \quad \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1},$$

where $\mathbb{E}$ is the Hodge bundle $u_*\omega_u$. In the following we will evaluate this degeneration locus on the space $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$.

We recall that $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$ is the moduli space of admissible connected S_3-covers over irreducible stable curves. We note $\overline{\mathcal{U}}_g$ the closure of the locus $\mathcal{U}_g$ on the space $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$.

Lemma 6.11. *Given an odd genus $g = 2i+1$, the class of $\overline{\mathcal{U}}_g$ in $\mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}_{g,S_3}^{S_3})$ is*

$$[\overline{\mathcal{U}}_g] = c_1\left(\mathrm{Hom}\left(\mathbb{E}_{i-1,2}, \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1}\right)\right) = \mathrm{rk}(\mathbb{E}_{i-1,2})\cdot\left(\sum_{b=0}^{i}(-1)^{b+1}c_1\left(\bigwedge^{i-b}\mathbb{E} \otimes \mathbb{E}_{0,b+1}\right)\right).$$

Proof. Given two vector bundles $\mathcal{A}$ and $\mathcal{B}$ over $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, the morphism vector bundle $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ is isomorphic to $\mathcal{B} \otimes \mathcal{A}^{\vee}$ and therefore

$$c_1(\mathrm{Hom}(\mathcal{A},\mathcal{B})) = \mathrm{rk}(\mathcal{A})c_1(\mathcal{B}) - \mathrm{rk}(\mathcal{B})c_1(\mathcal{A}).$$

To conclude we observe that by the definition of the vector bundles $\mathbb{E}_{j,b}$, they fit in the short exact sequences

$$0 \to \mathbb{E}_{j,b+1-j} \to \bigwedge^j \mathbb{E} \otimes \mathbb{E}_{0,b+1-j} \to \mathbb{E}_{j-1,b+2-j} \to 0,$$

for all $j \geq 0$ and $b \geq j$. □

Lemma 6.12. *The first Chern class of* $\mathbb{E}_{0,b}$ *is*

$$c_1(\mathbb{E}_{0,b}) = 2\lambda + 2\binom{b}{2}\kappa_1 - \frac{1}{4}\delta_0^T - \frac{2}{3}\delta_0^N \in \mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}_{g,S_3}^{S_3}).$$

Proof. This is a direct application of Proposition 6.3 in the evaluation of $\mathrm{ch}_1(Ru_*W_{\mathcal{C}})$. By [10] we have

$$\lambda = \mathrm{ch}_1(u_*\omega_u) = \frac{B_2}{2}\kappa_1 + \frac{1}{2} \cdot \sum_{\substack{0 \leq i \leq g-1, \\ [\![h]\!] \in [\![G]\!]}} r(h) \cdot (j_{i,[\![h]\!]})_* \left(\frac{B_2}{2} \cdot \sum_{a+a'=d-1} \psi^a(-\psi')^{a'} \right).$$

To complete by the formula of Proposition 6.3, we only need the eigenvalues decomposition of $R((12))$ and $R((123))$, where $R\colon S_3 \to \mathrm{GL}(\mathbb{C}, 2)$ is the irreducible S_3 representation of dimension 2. □

With these lemmata we can develop the calculations to evaluate $[\overline{\mathcal{U}}_g]$ in terms of the Hodge class and boundary classes.

Proposition 6.13. *In the Picard group* $\mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}_{g,S_3}^{S_3})$ *we have,*

$$[\overline{\mathcal{U}}_g] = \mathrm{rk}(\mathbb{E}_{i-1,2}) \cdot 2 \cdot \binom{2i-2}{i-1} \left(\frac{2(3i+1)}{i}\lambda - \delta_0' - \left(\frac{6i+1}{4i}\right)\delta_0^T - \left(\frac{5i+2}{3i}\right)\delta_0^N \right).$$

Proof. From the result of Lemma 6.12, we have that

$$c_1\left(\bigwedge^{i-b} \mathbb{E} \otimes \mathbb{E}_{0,b+1}\right) = \mathrm{rk}(\bigwedge^{i-b}\mathbb{E}) \cdot c_1(\mathbb{E}_{0,b+1}) + \mathrm{rk}(\mathbb{E}_{0,b+1}) \cdot c_1(\bigwedge^{i-b}\mathbb{E}) =$$

$$= \binom{g}{i-b} \cdot \left(2\lambda + 2\binom{b+1}{2}\kappa_1 - \frac{1}{4}\delta_0^T - \frac{2}{3}\delta_0^N\right) + 2(2b+1)(g-1) \cdot \binom{g-1}{i-b-1}\lambda.$$

This, thanks to Lemma 6.11, allows to conclude the evaluation. □

We are ready to introduce the divisor $\mathcal{M}$ of Equation 6.1 over $\overline{\mathcal{R}}_{g,S_3}^{S_3}$, with $g = 2i+1$ odd genus. Harris and Mumford introduced in [24] the divisor

$$\mathcal{M}_{g,i+1}^i := \{[C] \in \mathcal{M}_g \mid W_{i+1}^1(C) \neq \emptyset\} \subset \mathcal{M}_g,$$

where $W_d^r(C)$ is the set of complete linear series over C of degree d and dimension at least r. They proved that

$$\mathcal{M}_{g,i+1}^i = c' \cdot \left(\frac{6(i+2)}{i+1}\lambda - \delta_0\right) \in \mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{M}}_g),$$

where $\widetilde{\mathcal{M}}_g$ is the locus of irreducible stable curves inside $\overline{\mathcal{M}}_g$, and c' is a positive coefficient. This gives

$$[\pi^*\mathcal{M}^1_{g,i+1}] = c' \cdot \left(\frac{6(i+2)}{i+1}\lambda - \delta_0' - 2\delta_0^T - 3\delta_0^N\right) \in \mathrm{Pic}_Q(\widetilde{\mathcal{R}}^{S_3}_{g,S_3}),$$

where π is the natural projection $\widetilde{\mathcal{R}}^{S_3}_{g,S_3} \to \widetilde{\mathcal{M}}_g$.

We can now write down

$$[K_{\widetilde{\mathcal{R}}_{g,S_3}}] = \alpha \cdot [\overline{\mathcal{U}}_g] + \beta \cdot [\pi^*\mathcal{M}^1_{g,i+1}] + E + \gamma \cdot \lambda, \tag{6.4}$$

for every odd genus $g = 2i+1$, with α and β positive coefficients and E an effective sum of boundary divisors.

Proposition 6.14. *In Equation (6.4) the coefficient γ can be chosen strictly positive for $i > 5$.*

Proof. By scaling appropriately every coefficient, the equation is equivalent to choosing a real number $s \in [0,1]$ such that

$$\begin{aligned} & s \cdot \left(\frac{2(3i+1)}{i}\lambda - \delta_0' - \left(\frac{6i+1}{4i}\right)\delta_0^T - \left(\frac{5i+2}{3i}\right)\delta_0^N\right) + \\ & +(1-s)\cdot\left(\frac{6(i+2)}{i+1}\lambda - \delta_0' - 2\delta_0^T - 3\delta_0^N\right) + E + \gamma\cdot\lambda = \\ & = \frac{13}{2}\lambda - \delta_0' - \frac{3}{2}\delta_0^T - 2\delta_0^N. \end{aligned}$$

For E to be an effective divisor we must have

$$s \cdot \left(\frac{6i+1}{4i}\right) + (1-s)\cdot 2 \geq \frac{3}{2}$$

and

$$s \cdot \left(\frac{5i+2}{3i}\right) + (1-s)\cdot 3 \geq 2,$$

and therefore $s \leq \frac{3i}{4i-2}$ is a necessary and sufficient condition for E to be an effective boundary divisor.

To complete the proof we evaluate the γ coefficient,

$$s \cdot \left(\frac{6i+2}{i}\right) + (1-s)\cdot\left(\frac{6i+12}{i+1}\right) + \gamma = \frac{13}{2}.$$

After calculations this gives

$$\gamma = \frac{i-11}{2(i+1)} + s\cdot\frac{4i-2}{i(i+1)},$$

which means a maximal possible value of

$$\gamma = \frac{i-5}{2(i+1)},$$

which is positive if and only if $i > 5$. $\square$

After this proposition, the effectiveness of virtual divisor $\mathcal{U}_g$ would imply that the canonical divisor over $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$ is big for every odd genus $g \geq 13$. Then, thanks to Proposition 5.22, we would have $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ being of general type for every odd genus $g \geq 13$.

Therefore we have two main steps to prove this general type conjecture on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$: obtain an extension result as the one stated in Theorem 4.43 but for $\overline{\mathcal{R}}_{g,S_3}$; show the effectiveness of the virtual divisor $\mathcal{U}_g$, that is equivalent to prove the existence of a twisted S_3-cover (C, ϕ) outside the virtual divisor.

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