INTEGRALE DEFINITO in $[a,b]$ con $[a,b] \subseteq$ Dominio di f

$$\int_a^b f(x)\,dx = \underbrace{[F(x)]_a^b}_{\text{primitive di } f} = F(b) - F(a)$$

con $a < b$

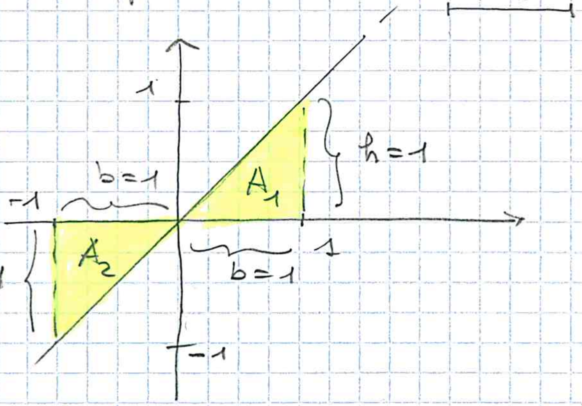
[Calcolando l'integrale indefinito otteniamo l'insieme delle primitive $F(x)$ cioè $\int f(x)\,dx = F(x) + c$]

CALCOLO DELL'AREA DI UNA REGIONE DI PIANO

L'integrale definito può essere usato come strumento per calcolare l'area di una regione di piano vediamo come:

Data la funzione $f(x) = x$ (retta)

calcoliamo in modo classico l'area A_1 e A_2 dei triangoli evidenziati in figura



$$A_1 = \frac{b \cdot h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

$$A_2 = \frac{b \cdot h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

$$A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$$

Ora calcoliamo

- $$\int_0^1 f(x)\,dx = \int_0^1 \underbrace{x}_{\geq 0 \text{ in } [0,1]}\,dx = \left[\frac{x^2}{2}\right]_0^1 = F(1) - F(0) = \left[\frac{(1)^2}{2}\right] - [0] = \frac{1}{2} \Rightarrow A_1 = \frac{1}{2}$$

  questo valore rappresenta un'area perché è $f(x) \geq 0$ in $[0,1]$

- $$\int_{-1}^0 f(x)\,dx = \int_{-1}^0 \underbrace{x}_{\leq 0 \text{ in } [-1,0]}\,dx = \left[\frac{x^2}{2}\right]_{-1}^0 = F(0) - F(-1) = 0 - \frac{(-1)^2}{2} = -\frac{1}{2}$$

  $$\Downarrow \quad A_2 = -\left(-\frac{1}{2}\right) = +\frac{1}{2}$$

  questo valore non può rappresentare un'area perché è < 0 ma è l'opposto del valore dell'area

- $$\int_{-1}^1 f(x) = \int_{-1}^1 \underbrace{x}_{\text{ma} \geq 0 \text{ ma} \leq 0 \text{ in } [-1,1]}\,dx = \left[\frac{x^2}{2}\right]_{-1}^1 = F(1) - F(-1) = \frac{(1)^2}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

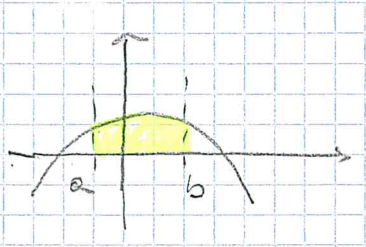
  questo valore non può essere il valore dell'area $A_1 + A_2$

$$A_1 + A_2 = -\int_{-1}^0 \underbrace{f(x)\,dx}_{\leq 0 \text{ in } [-1,0]} + \int_0^1 \underbrace{f(x)\,dx}_{\geq 0} = \underbrace{-\left(-\frac{1}{2}\right)}_{A_2} + \underbrace{\left(\frac{1}{2}\right)}_{A_1} = \frac{1}{2} + \frac{1}{2} = 1$$

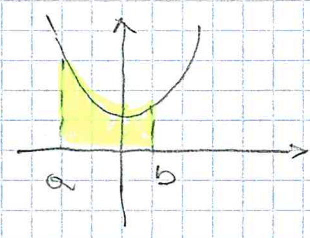
Ok

Quindi

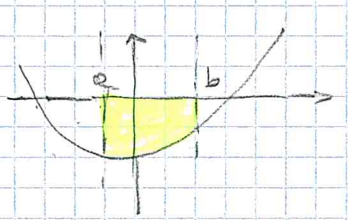
Se $f(x) \geq 0$ in $[a,b]$



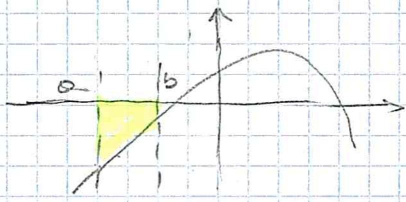
$\Rightarrow A = \int_a^b f(x)\,dx$



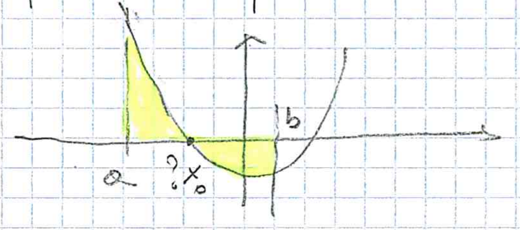
Se $f(x) \leq 0$ in $[a,b]$



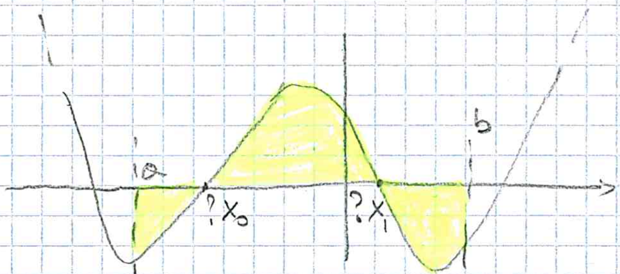
$\Rightarrow A = -\int_a^b f(x)\,dx$



Se $f(x)$ è sia positiva sia negativa in $[a,b]$ $\Rightarrow$



$\Rightarrow A = \int_a^{x_0} f(x)\,dx - \int_{x_0}^b f(x)\,dx$

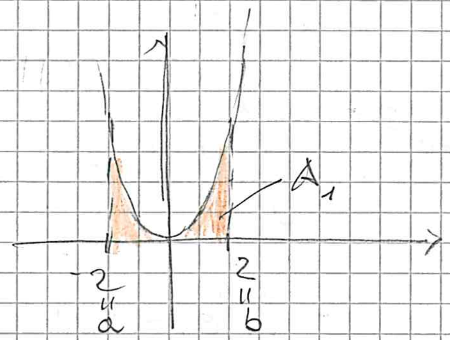


$\Rightarrow A = -\int_a^{x_0} f(x)dx + \int_{x_0}^{x_1} f(x)dx - \int_{x_1}^{b} f(x)dx$

NB x_0, x_1 si calcolano cercando l'intersezione di $f(x)$ con l'asse x cioè risolvendo il sistema $\begin{cases} y = f(x) \\ y = 0 \quad (\text{asse } x) \end{cases}$

ES Trovare l'area compresa tra la curva $f(x)=x^2$, l'asse x e le rette $x=-2$ e $x=2$

Soluzione



$$A=\int_a^b f(x)\,dx=\int_{-2}^{2}x^2\,dx=\left[\frac{x^3}{3}\right]_{-2}^{2}=$$

$$=F(2)-F(-2)$$

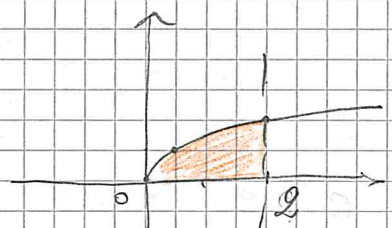
$$=\frac{8}{3}-\left(-\frac{8}{3}\right)=\frac{8}{3}+\frac{8}{3}=\frac{16}{3}$$

Oppure usiamo la simmetria geometrica

$$A_1=\int_0^2 x^2\,dx=\left[\frac{x^3}{3}\right]_0^2=F(2)-F(0)=\frac{8}{3}-0=\frac{8}{3}$$

$$A_{TOT}=2\cdot A_1=2\cdot\frac{8}{3}=\frac{16}{3}$$

ES Trovare l'area compresa tra la curva $f(x)=\sqrt{x}$, l'asse x, l'asse y e la retta $x=2$

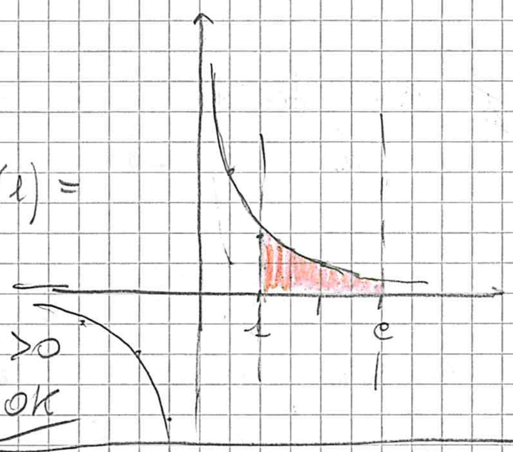


Soluzione

$$A=\int_0^2\sqrt{x}\,dx=\int_0^2 x^{\frac{1}{2}}\,dx=\left[\frac{x^{\frac{1}{2}+1}}{\frac{3}{2}}\right]_0^2=\quad \left(\tfrac{1}{2}+1=\tfrac{3}{2}\right)$$

$$=\left[\frac{2}{3}\sqrt{x^3}\right]_0^2=F(2)-F(0)=\frac{2}{3}\sqrt{8}-0=\frac{2\cdot2\sqrt{2}}{3}=\frac{4\sqrt{2}}{3}$$

ES Trovare l'area compresa tra la curva $f(x) = \frac{1}{x}$, l'asse x e le rette $x = 1$ e $x = e$



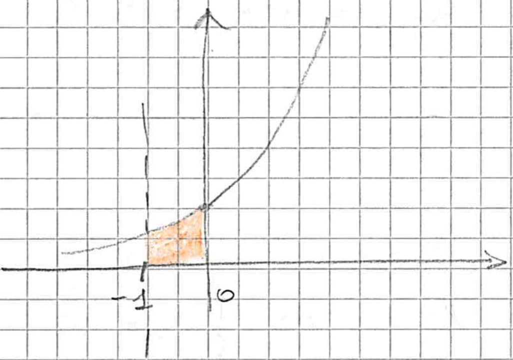
Soluzione

$$A = \int_1^e \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^e = F(e) - F(1) =$$

$$= \ln|e| - \ln 1 = \underbrace{\ln e}_{=1} - \underbrace{\ln 1}_{=0} = 1 > 0$$

OK

ES Trovare l'area compresa tra la curva $f(x) = e^x$, l'asse x, l'asse y e la retta $x = -1$



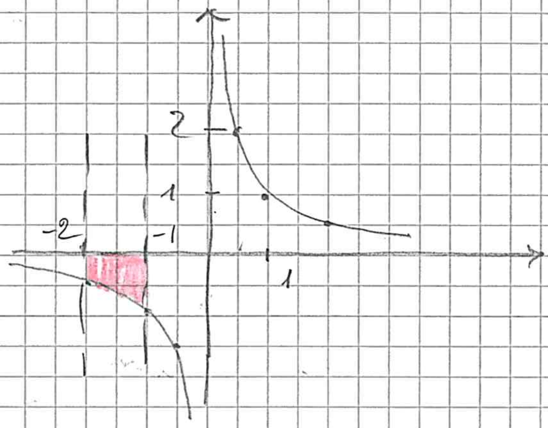
Soluzione

$$A = \int_{-1}^{0} e^x\,dx = \Big[e^x\Big]_{-1}^{0} = F(0) - F(-1)$$

$$= e^0 - e^{-1} = 1 - \frac{1}{e} > 0$$

OK

ES Trovare l'area compresa tra la curva $f(x)=\frac{1}{x}$, l'asse x e le rette $x=-2$ e $x=-1$



Soluzione

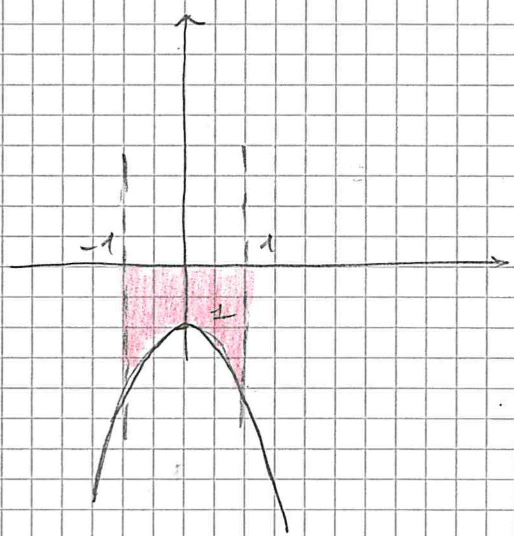
$$A=-\int_{-2}^{-1}\frac{1}{x}\,dx=-\Big[\ln|x|\Big]_{-2}^{-1}$$

$$=-\Big[\ln|-1|-\ln|-2|\Big]=$$

$$=-\Big[\underbrace{\ln 1}_{=0}-\ln 2\Big]=+\ln 2\geq 0 \quad \text{ok}$$

ES trovare l'area compresa tra la curva $f(x)=-x^2-1$, l'asse x e le rette $x=-1$ e $x=1$

Soluzione



$$A=-\int_{-1}^{1}\left(-x^2-1\right)dx$$

$$=-\left[-\frac{x^3}{3}-x\right]_{-1}^{1}=$$

$$=-\left[\left(-\frac{1}{3}-1\right)-\left(\frac{1}{3}+1\right)\right]=$$

$$=-\left[-\frac{1}{3}-1-\frac{1}{3}-1\right]=$$

$$=\frac{1}{3}+1+\frac{1}{3}+1=\frac{2}{3}+2=\frac{2+6}{3}=\frac{8}{3}\geq 0 \quad \text{ok}$$

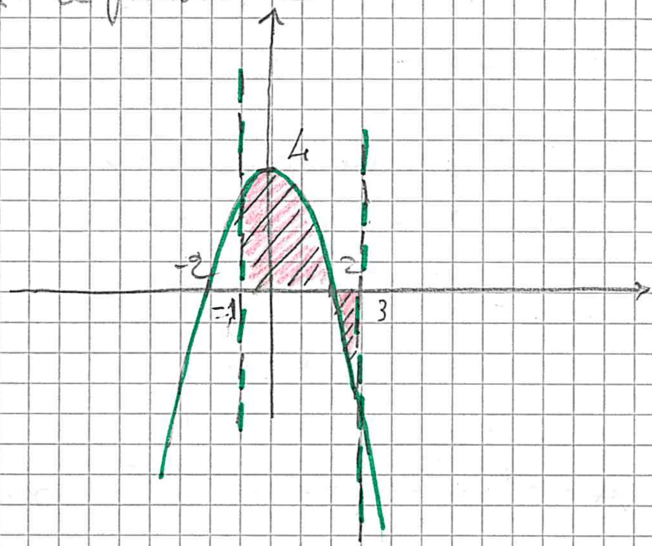
oppure sfrutto la simmetria

$$A=-2\cdot\int_{0}^{1}\left(-x^2-1\right)dx=-2\left[-\frac{x^3}{3}-x\right]_{0}^{1}=$$

$$=-2\left[\left(-\frac{1}{3}-1\right)-0\right]=+\frac{2}{3}+2=\frac{2+6}{3}=\frac{8}{3} \quad \text{ok}$$

B) Trovare l'area delimitata dalla parabola $f(x) = -x^2+4$, dall'asse x e dalle rette $x=-1$ e $x=3$. Rappresentare graficamente la funzione

Sol $f(x) = -x^2+4$



Intersezione con asse x

$f(x)=0 \Rightarrow -x^2+4=0$

$x^2=4$

$x=\pm 2$

$$A = \int_{-1}^{2}\left(-x^2+4\right)dx - \int_{2}^{3}\left(-x^2+4\right)dx =$$

$$= \left[-\frac{x^3}{3}+4x\right]_{-1}^{2} - \left[-\frac{x^3}{3}+4x\right]_{2}^{3} =$$

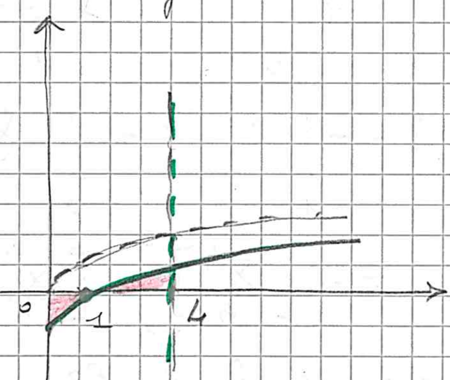
$$= \left(-\frac{8}{3}+8\right)-\left(+\frac{1}{3}-4\right)-\left[\left(-\frac{\cancel{27}^{9}}{\cancel{3}}+12\right)-\left(-\frac{8}{3}+8\right)\right] =$$

$$= -\frac{8}{3}+\cancel{8}-\frac{1}{3}+\cancel{4}+9-\cancel{12}-\frac{8}{3}+8 = -\frac{17}{3}+17 = \frac{-17+51}{3} = \frac{34}{3} \geq 0$$

ok

B) Trovare l'area della curva $f(x)=\sqrt{x}-1$, l'asse x, l'asse y e la retta $x=4$ e rappresentare graficamente la funzione.

Sol $f(x)=\sqrt{x}-1$ D: $x\geq 0$



Intersezione con asse x

$f(x)=0 \Rightarrow \sqrt{x}-1=0$

$(\sqrt{x})^2=(1)^2$

$x=1$

$$A = -\int_{0}^{1}\left(\sqrt{x}-1\right)dx + \int_{1}^{4}\left(\sqrt{x}-1\right)dx = -\left[\frac{x^{\frac{1}{2}+1=\frac{3}{2}}}{\frac{3}{2}}-x\right]_{0}^{1} + \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-x\right]_{1}^{4}$$

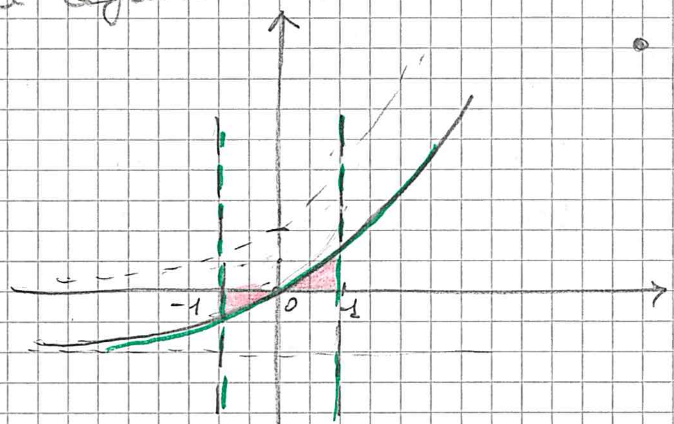
$$= -\left[\frac{2}{3}\sqrt{x^3}-x\right]_{0}^{1} + \left[\frac{2}{3}\sqrt{x^3}-x\right]_{1}^{4} = -\left[\left(\frac{2}{3}-1\right)-0\right]+\left[\left(\frac{2}{3}\sqrt{64}-4\right)-\left(\frac{2}{3}-1\right)\right]$$

$$= -\frac{2}{3}+1+\frac{16}{3}-4-\frac{2}{3}+1 = \frac{\cancel{12}^{4}}{\cancel{3}}-2 = 2 \geq 0$$

ok

ES Trovare l'area compresa tra la curva $f(x)=e^{x}-1$, l'asse x e le rette $x=-1$ e $x=1$
Rappresentare graficamente la funzione.

Sol
$$f(x)=e^{x}-1$$



Intersezione con asse x

$$f(x)=0 \Rightarrow e^{x}-1=0$$
$$e^{x}=1$$
$$x=0$$

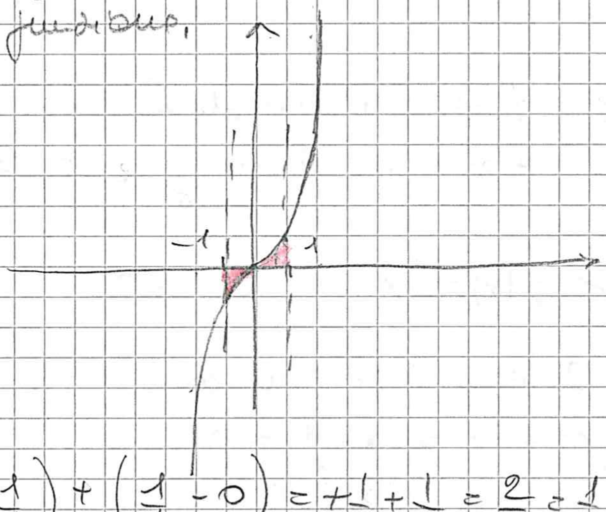
$$A=-\int_{-1}^{0}(e^{x}-1)\,dx+\int_{0}^{1}(e^{x}-1)\,dx=-\left[e^{x}-x\right]_{-1}^{0}+\left[e^{x}-x\right]_{0}^{1}$$

$$=-\left[(e^{0}-0)-(e^{-1}+1)\right]+\left[(e^{1}-1)-(e^{0}-0)\right]=$$

$$=-\left[\cancel{1}-\frac{1}{e}-\cancel{1}\right]+e^{1}-1-1=\frac{1}{e}+e-2$$

ES Trovare l'area compresa tra la curva $f(x)=x^{3}$ l'asse x e le rette $x=-1$ e $x=1$
Rappresentare graficamente la funzione.

Sol $f(x)=x^{3}$



intersezione con asse x

$$f(x)=0 \Rightarrow x^{3}=0 \Rightarrow x=0$$

$$A=-\int_{-1}^{0}x^{3}\,dx+\int_{0}^{1}x^{3}\,dx=$$

$$=-\left[\frac{x^{4}}{4}\right]_{-1}^{0}+\left[\frac{x^{4}}{4}\right]_{0}^{1}=-\left(0-\frac{1}{4}\right)+\left(\frac{1}{4}-0\right)=+\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

oppure sfrutto la simmetria cioè le due parti di aree sono equivalenti quindi

$$A_{tot}=2\int_{0}^{1}x^{3}\,dx=2\left[\frac{x^{4}}{4}\right]_{0}^{1}=2\cdot\left[\frac{1}{4}-0\right]=\frac{\cancel{2}}{\cancel{4}_{2}}=\frac{1}{2}$$