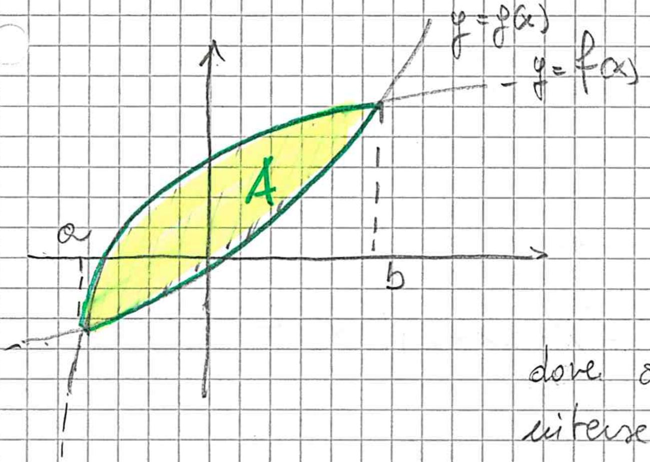
AREA DELLA REGIONE DI PIANO DELIMITATA DA DUE CURVE



con $f(x) \geq g(x)$ in $[a, b]$

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

dove a e b sono le ascisse dei punti di intersezione tra f(x) e g(x), quindi si ottengono risolvendo il sistema tra

$$\begin{cases} y = f(x) \\ y = g(x) \end{cases} \Rightarrow f(x) = g(x)$$

<u>Esempio</u>

Trovare l'area tra le parabole

$f(x) = x^2 - 3x + 2$ e $g(x) = -x^2 + x + 2$

$V_x = -\frac{b}{2a} = +\frac{3}{2}$

$V_y = \frac{9}{4} - 3 \cdot \frac{3}{2} + 2 =$

$= \frac{9}{4} - \frac{9}{2} + 2 = \frac{9 - 18 + 8}{4} =$

$= -\frac{1}{4} \Rightarrow V_1\left(\frac{3}{2}, -\frac{1}{4}\right)$

Intersezione asse y

$x = 0 \Rightarrow f(0) = 2 \Rightarrow (0, 2)$

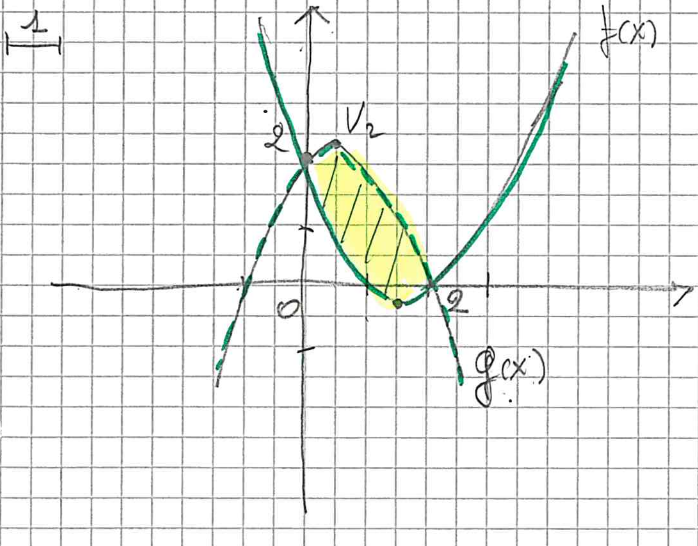
$V_x = -\frac{b}{2a} = \frac{-1}{-2} = \frac{1}{2}$

$V_y = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{-1 + 2 + 8}{4} = \frac{9}{4}$

$\Rightarrow V_2\left(\frac{1}{2}, \frac{9}{4}\right)$ 2,25

Intersezione asse y

$x = 0 \Rightarrow f(0) = 2 \Rightarrow (0, 2)$



Cerco i punti di intersezione tra f(x) e g(x) per trovare l'intervallo dove valutare l'area

$$\begin{cases} y = f(x) \\ y = g(x) \end{cases} \quad \begin{cases} y = x^2 - 3x + 2 \\ y = -x^2 + x + 2 \end{cases} \Rightarrow$$

$x^2 - 3x + \cancel{2} = -x^2 + x + \cancel{2}$

$2x^2 - 4x = 0$

$x(2x-4) = 0 \Rightarrow x = 0 \quad \vee \quad 2x - 4 = 0$

$x = \frac{4}{2} = 2$

$\Rightarrow a = 0 \quad b = 2$

Quindi in questo caso è $g(x) \geq f(x) \Rightarrow$

$$A = \int_a^b [g(x) - f(x)]\, dx = \int_0^2 [(-x^2 + x + 2) - (x^2 - 3x + 2)]\, dx =$$

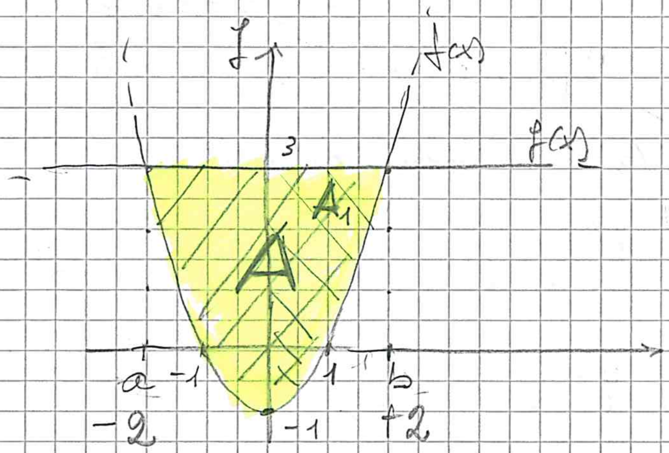
$$= \int_0^2 (-x^2 + x + \cancel{2} - x^2 + 3x - \cancel{2})\, dx = \int_0^2 (-2x^2 + 4x)\, dx =$$

$$\left[\int (-2x^2 + 4x)\, dx = -2 \cdot \frac{x^3}{3} + \cancel{4}^{2} \cdot \frac{x^2}{\cancel{2}} + c = -\frac{2}{3} x^3 + 2x^2 + c \right]$$

$$= \left[-\frac{2}{3} x^3 + 2x^2 \right]_0^2 = F(2) - F(0) = \left(-\frac{2}{3} \cdot 8 + 2 \cdot 4 \right) - (0 + 0) =$$

$$= -\frac{16}{3} + 8 = \frac{-16 + 24}{3} = \frac{8}{3} > 0 \quad \underline{\text{ok}}$$

ES Trovare l'area tra la parabola $f(x)=x^2-1$ e la retta $g(x)=3$



Sol

Cerco gli estremi di integrazione a e b

$$\begin{cases} y=f(x) \\ y=g(x) \end{cases}$$

$$\begin{cases} y=x^2-1 \\ y=3 \end{cases} \Rightarrow \quad x^2-1=3$$

$$x^2=4$$

$$x=\pm 2 \Rightarrow a=-2 \quad e \quad b=2$$

Vale $g(x) \geq f(x)$

$$A=\int_{-2}^{2}\left[g(x)-f(x)\right]dx=\int_{-2}^{2}\left[3-(x^2-1)\right]dx=\int_{-2}^{2}\left(3-x^2+1\right)dx$$

$$=\int_{-2}^{2}\left(-x^2+4\right)dx=\left[-\frac{x^3}{3}+4x\right]_{-2}^{2}=F(2)-F(-2)=\left(-\frac{8}{3}+8\right)-\left(+\frac{8}{3}-8\right)$$

$$\left[\int(-x^2+4)dx=-\frac{x^3}{3}+4x+c\right]$$

$$=-\frac{8}{3}+8-\frac{8}{3}+8$$

$$=-\frac{16}{3}+16=\frac{-16+48}{3}$$

$$=\frac{32}{3}>0 \quad \text{ok}$$

Oppure sfrutto la simmetria della situazione geometrica e calcolo A_1 cioè l'area in $[0;2]$ e poi $A=2A_1$

$$A_1=\int_{0}^{2}\left(-x^2+4\right)dx=\left[-\frac{x^3}{3}+4x\right]_{0}^{2}=F(2)-F(0)=\left(-\frac{8}{3}+8\right)-0=$$

$$=\frac{-8+24}{3}=\frac{16}{3}$$

da cui

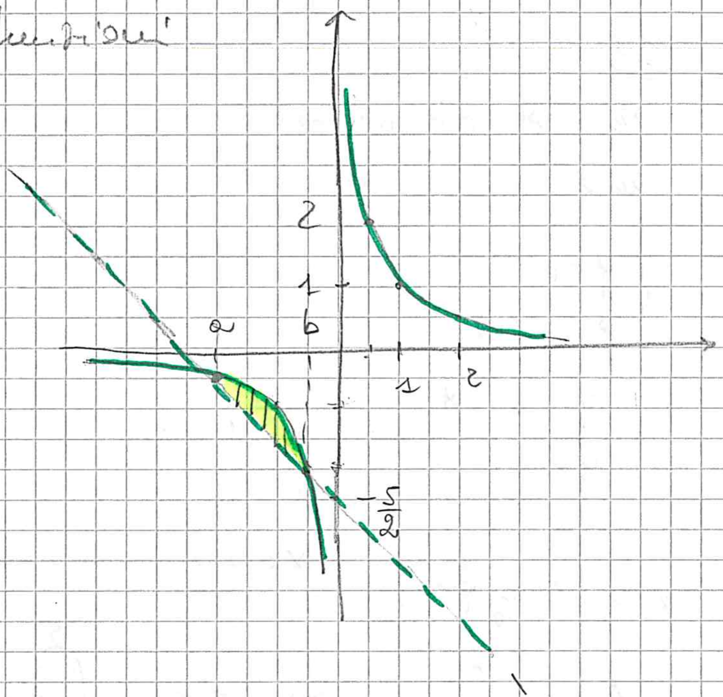
$$A=2\cdot A_1=2\cdot\frac{16}{3}=\frac{32}{3}$$

ES

Trovare l'area tra la curva $f(x)=\frac{1}{x}$
e la retta $g(x)=-x-\frac{5}{2}$.
Rappresentare graficamente le funzioni

SOL

Cerco i valori a e b



$$\begin{cases} y=f(x) \\ y=g(x) \end{cases}$$

$$\begin{cases} y=\frac{1}{x} \\ y=-x-\frac{5}{2} \end{cases} \Rightarrow$$

$$\Rightarrow \frac{1}{x}=-x-\frac{5}{2}$$

$$\frac{2}{\cancel{2x}}=\frac{-2x^2-5x}{\cancel{2x}} \qquad x\neq 0$$

$$2x^2+5x+2=0 \qquad x_{1,2}=\frac{-5\pm\sqrt{25-16}}{4}=\frac{-5\pm 3}{4}=$$

$$= \begin{cases} -\frac{8}{4}=-2 \Rightarrow a=-2 \\ -\frac{2}{4}=-\frac{1}{2} \Rightarrow b=-\frac{1}{2} \end{cases}$$

Vale $f(x)\geq g(x)$

$$A=\int_{a=-2}^{b=-\frac{1}{2}}\left(f(x)-g(x)\right)dx=\int_{-2}^{-\frac{1}{2}}\left[\frac{1}{x}-\left(-x-\frac{5}{2}\right)\right]dx=$$

$$=\int_{-2}^{-\frac{1}{2}}\left(\frac{1}{x}+x+\frac{5}{2}\right)dx=$$

$$=\left[\ln|x|+\frac{x^2}{2}+\frac{5}{2}x\right]_{-2}^{-\frac{1}{2}}=F\left(-\frac{1}{2}\right)-F(-2)$$

$$=\left[\ln\left|-\frac{1}{2}\right|+\frac{\frac{1}{4}}{2}+\frac{5}{2}\left(-\frac{1}{2}\right)\right]-\left[\ln|-2|+\frac{\cancel{4}^{2}}{\cancel{2}}+\frac{5}{\cancel{2}}(-\cancel{2})\right]$$

$$=\ln\left(\frac{1}{2}\right)+\frac{1}{4}\cdot\frac{1}{2}-\frac{5}{4}-\ln 2-2+5=$$

$$=\ln 2^{-1}+\frac{1}{8}-\frac{5}{4}-\ln 2+3=$$

$$=-\ln 2-\ln 2+\frac{1-10+24}{8}=-2\ln 2+\frac{15}{8}$$

$\approx 0{,}5>0$ Ok

E5 | Calcolare l'area tra le curve $f(x) = x^2-4$ e $g(x) = 3x$

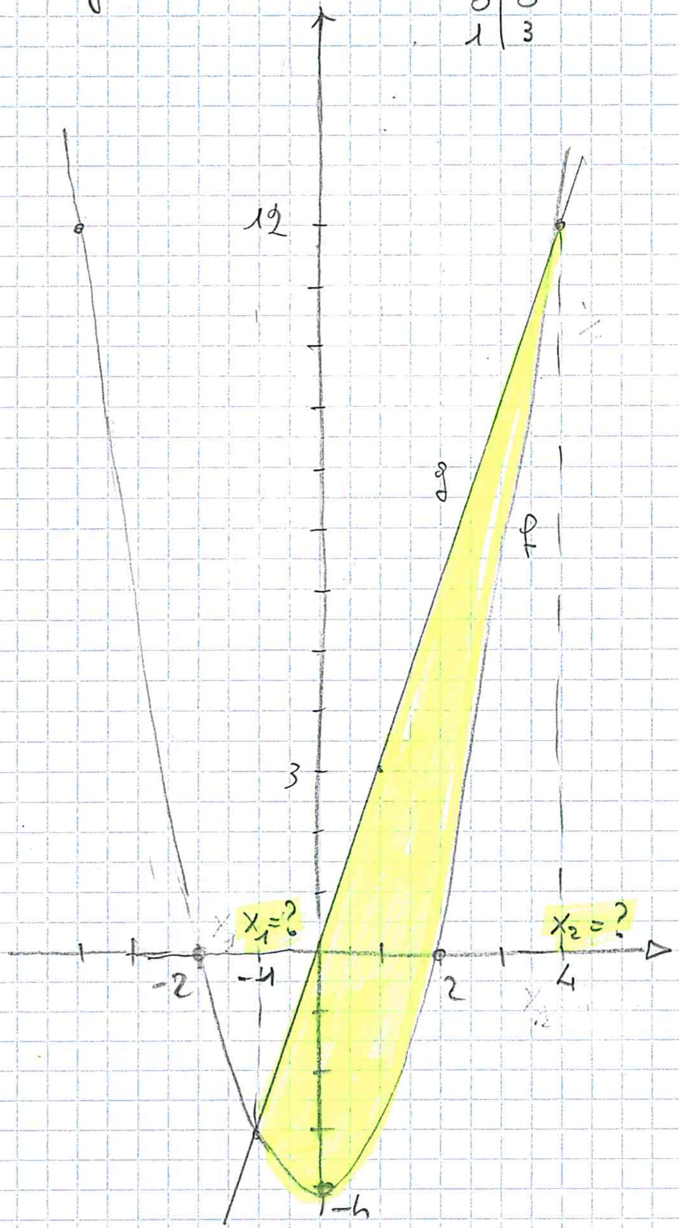
Soluzione

$f(x) = x^2-4$ parabola $x_V = 0$, $y_V = 0^2-4 = -4$ $\Rightarrow V(0,-4)$

Intersezione asse x $\begin{cases} y = x^2-4 \\ y = 0 \end{cases}$ $x^2-4=0$, $x^2=4$, $x = \pm 2$

$g(x) = 3x$ retta

x	y
0	0
1	3



Cerco le intersezioni tra le 2 funzioni

$\begin{cases} y = f(x) \\ y = g(x) \end{cases}$ $\begin{cases} y = x^2-4 \\ y = 3x \end{cases}$

$x^2-4 = 3x$

$x^2-3x-4 = 0$

$x_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} \begin{matrix} 4 \\ -1 \end{matrix}$

$x_1 = -1$

$x_2 = 4$

Quindi

$$A = \int_{-1}^{4} \left(g(x) - f(x)\right) dx$$

$$= \int_{-1}^{4} \left[(3x) - (x^2-4)\right] dx =$$

$$= \int_{-1}^{4} \left(3x - x^2 + 4\right) dx =$$

$$= \left[3\frac{x^2}{2} - \frac{x^3}{3} + 4x\right]_{-1}^{4} = F(4) - F(3)$$

$$= \left[3\frac{(4)^2}{2} - \frac{(4)^3}{3} + 4\cdot(4)\right] - \left[3\frac{(-1)^2}{2} - \frac{(-1)^3}{3} + 4(-1)\right]$$

$$= \left[3\cdot\frac{\cancel{16}^{8}}{\cancel{2}} - \frac{64}{3} + 16\right] - \left[\frac{3}{2} + \frac{1}{3} - 4\right] =$$

$$= 24 - \frac{64}{3} + 16 - \frac{3}{2} - \frac{1}{3} + 4 = 44 - \frac{65}{3} - \frac{3}{2} = \frac{264-130-9}{6} = \frac{125}{6} > 0 \quad \text{Ok}$$

DS | Calcolare l'area tra le curve $f(x) = -x^2+2x$ e $g(x) = -2x$ e le rette $x=-1$ e $x=2$

Soluzione

$f(x) = -x^2+2x$ PARABOLA $\cap$

$V_x = \frac{-b}{2a} = \frac{-2}{2(-1)} = 1 \implies V(1,1)$

$V_y = -(1)^2+2(1) = 1$

Intersezione con asse x

$\begin{cases} y = -x^2+2x \\ y = 0 \text{ (asse x)} \end{cases}$

$-x^2+2x=0$

$x(-x+2)=0$

$x=0$

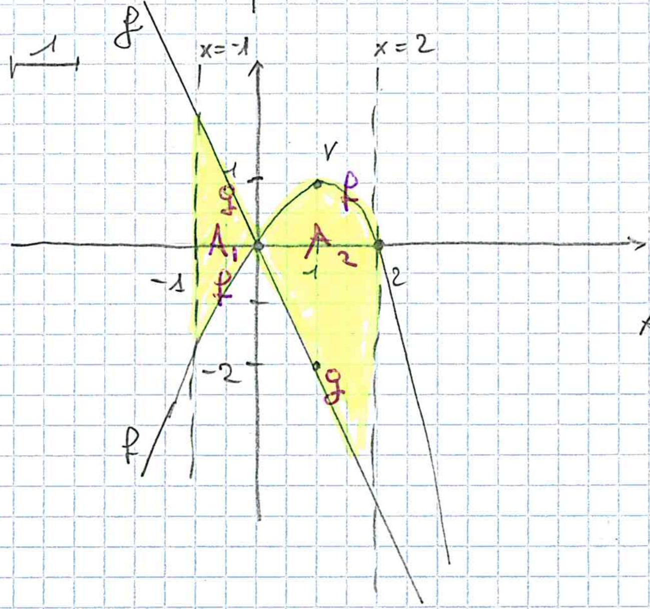
$-x+2=0 \quad -x=-2$

$x=2$

$(0,0) \quad (2,0)$

$g(x) = -2x$ RETTA

x	y
0	0
1	-2



$$A = \underbrace{\int_{-1}^{0} (g(x)-f(x))\,dx}_{A_1} + \underbrace{\int_{0}^{2} (f(x)-g(x))\,dx}_{A_2} = (*)$$

$$A_1 = \int_{-1}^{0} (g(x)-f(x))\,dx = \int_{-1}^{0} [-2x-(-x^2+2x)]\,dx$$

$$= \int_{-1}^{0} (-2x+x^2-2x)\,dx$$

$$= \int_{-1}^{0} (x^2-4x)\,dx = \left[\frac{x^3}{3} - \cancel{4}^{2}\frac{x^2}{\cancel{2}}\right]_{-1}^{0} = \left[\frac{x^3}{3} - 2x^2\right]_{-1}^{0} = F(0)-F(-1)$$

$$= 0 - \left[\frac{(-1)^3}{3} - 2(-1)^2\right] = -\left[-\frac{1}{3} - 2\right] = \frac{1}{3} + 2 = \frac{7}{3}$$

$$A_2 = \int_{0}^{2} (f(x)-g(x))\,dx = \int_{0}^{2} [-x^2+2x-(-2x)]\,dx =$$

$$= \int_{0}^{2} (-x^2+2x+2x)\,dx = \int_{0}^{2} (-x^2+4x)\,dx = \left[-\frac{x^3}{3} + \cancel{4}^{2}\frac{x^2}{\cancel{2}}\right]_{0}^{2} = F(2)-F(0)$$

$$= \left[-\frac{(2)^3}{3} + 2\,(2)^2\right] - [0] = -\frac{8}{3} + 8 = \frac{-8+24}{3} = \frac{16}{3}$$

Quindi

$(*) \implies A = A_1 + A_2 = \frac{7}{3} + \frac{16}{3} = \frac{23}{3}$

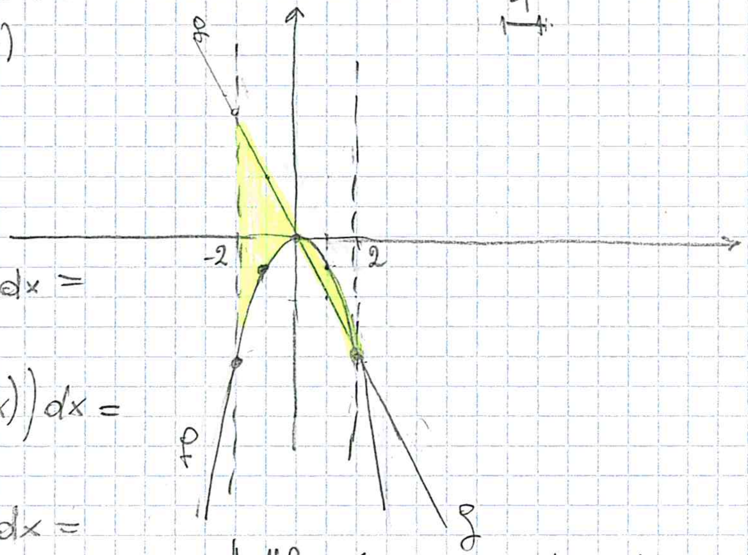
35 Determinare l'area della regione di piano compresa tra le curve di equazione $f(x) = -x^2$ e $g(x) = -2x$ e le rette di equazione $x=2$ e $x=-2$

<u>Soluzione</u> Rappresento le funzioni

$f(x) = -x^2$ parabola $V(0,0)$

$g(x) = -2x$ retta

x	y
0	0
1	-2



$$A_T = \int_{-2}^{0} \left(g(x) - f(x)\right) dx + \int_{0}^{2} \left(f(x) - g(x)\right) dx =$$

$$= \int_{-2}^{0} \left(-2x - (-x^2)\right) dx + \int_{0}^{2} \left(-x^2 - (-2x)\right) dx =$$

$$= \int_{-2}^{0} \left(-2x + x^2\right) dx + \int_{0}^{2} \left(-x^2 + 2x\right) dx =$$

$$= \left[-\cancel{2}\cdot\frac{x^2}{\cancel{2}} + \frac{x^3}{3}\right]_{-2}^{0} + \left[-\frac{x^3}{3} + \cancel{2}\cdot\frac{x^2}{\cancel{2}}\right]_{0}^{2} =$$

$$= \left(F(0) - F(-2)\right) + \left(F(2) - F(0)\right) =$$

$$= \left(0 - \left[-(-2)^2 + \frac{(-2)^3}{3}\right]\right) + \left(\left[-\frac{(2)^3}{3} + (2)^2\right] - [0]\right) =$$

$$= -\left[-4 - \frac{8}{3}\right] + \left[-\frac{8}{3} + 4\right]$$

$$= +4 + \cancel{\frac{8}{3}} - \cancel{\frac{8}{3}} + 4 = 8$$

<u>NB</u> Cerco i punti in cui le due funzioni si intersecano:

$\begin{cases} y = f(x) \\ y = g(x) \end{cases}$ $\begin{cases} y = -x^2 \\ y = -2x \end{cases}$ $\Rightarrow$

$-x^2 = -2x \Rightarrow -x^2 + 2x = 0$

$x(-x+2) = 0$

$x = 0$

$-x + 2 = 0$

$-x = -2$

$x = 2$

Quindi

$x_1 = 0$

$x_2 = 2$