# Analityc estimates for solutions of the Levi equations [1]

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In this paper we introduce a new technique of real analysis for studying the regularity properties of a solution $u$ to the Levi equation in $R^3$. First we study the solution in suitable Sobolev spaces and establish some a priori estimates of analytic type for the "intrinsic" derivatives of the solution. Then we prove that for every $\xi_0$ in the domain there exists a real analytic 2-dimensional manifold $M_{\xi_0}$ passing through $\xi_0$ such that $u_{|M_{\xi_0}}$ is an harmonic function.

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## 1. INTRODUCTION

In this paper we study the regularity properties of solutions of the Levi equation:

$$\mathcal{L}u = 0 \text{ in } \Omega, \tag{1}$$

where $\Omega$ is an open subset of $\mathbb{R}^3$, $\mathcal{L}$ is the second order partial differential operator defined as

$$\mathcal{L} = \partial^2_{xx} + \partial^2_{yy} + 2a\partial^2_{xt} + 2b\partial^2_{yt} + (a^2 + b^2)\partial^2_{tt}, \tag{2}$$

and its coefficients are functions of the gradient of $u$:

$$a = a_u = \frac{u_y - u_x u_t}{1 + u_t^2}, b = b_u = -\frac{u_x + u_y u_t}{1 + u_t^2}. \tag{3}$$

Here we have denoted by $(x, y, t)$ the points of the space $\mathbb{R}^3$, by $u_x$ the partial derivative of $u$ with respect to $x$ and we have used an analogous notation for the other derivatives.

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The Dirichlet problem for the Levi equation naturally arises in complex analysis, when studying flat-hypersurfaces of $\mathbb{C}^2$ with prescribed boundary. After the first qualitative results of Debiard and Gaveau, (see [DG]), an existence result was due to Bedford and Gaveau:

THEOREM. *Let $\Omega$ be a bounded strictly pseudoconvex domain of $\mathbb{C} \times \mathbb{R}$ of class $C^2$ and $\phi \in C^{m+5}(\partial\Omega)$. Let us assume that $\partial\Omega$ is a 2-sphere and there are exactly two complex tangency elliptic points $p$ and $q$ on the graph of $\phi$. Then the Dirichlet problem*

$$\begin{cases} \mathcal{L}u = 0 & \textit{in } \Omega \\ u = \phi & \textit{on } \partial\Omega. \end{cases} \tag{4}$$

*has a solution $u$ in $C^{m+\alpha}(\bar{\Omega} - \{p,q\}) \cap Lip(\bar{\Omega})$, with $0 < \alpha < 1$. Moreover, the graph of $u$ is the envelope of holomorphy of the graph of $\phi$ and it is foliated by complex curves* (see [BG]).

The definition of foliation is given in Definition 5.2.

The conditions on the open set $\Omega$ and on the boundary datum $\phi$ have been subsequently weakened (see [A], [S], [CS], [BK]), always with the same technique, based on Bishop theorem of analytic disks. The most general result of foliation of graphs in this contest is the following

THEOREM. *Let $\Omega$ be a bounded strictly pseudo-convex domain of $\mathbb{C} \times \mathbb{R}$ and let $\phi$ be continuous. Then the polynomial hull of the graph of $\phi$ is foliated by complex analytic curves* (see [CS]).

A real analytic approach was introduced by [ST], for studying equation (1) and the more general equation

$$\mathcal{L}u = q(\cdot, u)\frac{(1 + |\nabla u|^2)^{3/2}}{1 + (\partial_t u)^2} \quad in \;\; \Omega. \tag{5}$$

The operator $\mathcal{L}$ is a quasi linear second order operator, whose characteristic form is non negative defined and has minimum eigenvalue identically zero at every point. However for any $\varepsilon > 0$ it is possible to introduce its elliptic regularization:

$$\mathcal{L}_\varepsilon := \mathcal{L} + \frac{\varepsilon^2}{1 + (\partial_t u)^2}\partial^2_{tt}, \tag{6}$$

and to give the following definition:

Definition 1.1. We say that a function $u \in Lip(\Omega)$ is a strong viscosity solution of the equation (1) if there exist a sequence of positive numbers $\varepsilon_j$ such that $\varepsilon_j \to 0$ when $j \to \infty$, and a sequence $(u_j)$ in $C^\infty(\Omega)$ such that

$$\mathcal{L}_{\varepsilon_j} u_j = q(\cdot, u_j)\frac{(1+|\nabla u_j|^2)^{3/2}}{1+(\partial_t u_j)^2} \quad in \ \ \Omega \ \ \forall j \in N.$$

Besides $(u_j)$ is bounded in $Lip(\Omega)$ and pointwise convergent to $u$ in $\Omega$.

Note that a strong viscosity solution is a viscosity solution in the sense of Crandall, Ichii and Lions [CIL]. In this contest Slodkowsky and Tomassini established the following result:

Theorem. *If $\Omega$ is a bounded strictly pseudoconvex domain of class $C^{2,\alpha}$, then the Dirichlet problem* (4)*, with boundary data $\phi \in C^{2,\alpha}(\partial\Omega)$ has an unique strong viscosity solution $u$ in $Lip(\bar{\Omega})$* (see [ST]).

Moreover in [ST] it is proved that, if the boundary datum is only continuous, then the problem has a continuous solution.

The complex analysis technique applied in [BG] and [S] provides deep regularity results, but can be applied only when the problem is studied in $\mathbb{C} \times \mathbb{R}$ and for $q = 0$, while the real analytic approach provides existence of solutions also in higher dimensions (see [ST2]), but the solutions are proved to be only Lipschitz continuous. Hence a real analysis technique for the regularity seemed necessary, in view of applications to higher dimensional problems. However, because of the strong degeneracy of the operator $\mathcal{L}$, it was pointed out that most of the known techniques could not be applied to the problem. On the other hand in [C] and [CLM] a new approach, based on a Partial Differential Equation Method, was introduced for studying the regularity problem of the equation (5) when $q$ never vanishes. We will see that the situation $q = 0$ is much more degenerate, from the point of view of the differential equations. However we are able to extend the same method to this situation, and we prove the following:

Theorem 1.1. *If $u$ is a Lipschitz continuous strong viscosity solution of the Levi equation, then for every $\xi_0 \in \Omega$ there exists a is a real analytic 2-dimensional manifold $M_{\xi_0}$, such that $u_{|M_{\xi_0}}$ is an harmonic function.*

This result has been announced in [CM2] and we give here a complete proof of it. This kind of regularity is optimal for solutions of equation (1), since the solutions of the equation are in general not regular as functions of three variables. For example any function of the variable $t$ is a solution.

As a corollary of Theorem 1.1 we obtain a foliation result, already contained in the above cited theorems of [S] and [CS]. We explicitly note that in those papers only the regularity of $u_{|M_{\xi_0}}$ is investigated. On the contrary with our method we obtain some regularity results also in the orthogonal direction. Indeed, for every multiindex $I$ we will introduce in (12) the notion of tangential derivative $D_0^I u$ (see for the precise definition of $D_0^I u$), and we obtain the follows:

PROPOSITION 1.1. *If $u$ is a strong solution of (1), then for every multiindex $I$ the derivative $D_0^I u(\xi)$ is Lipschitz continuous, with respect to the euclidean distance of $\mathbb{R}^3$.*

Our approach to the problem is based on a representation of the operator as a sum of squares of vector fields, whose coefficients depend on the gradient of a solution $u$ of the Levi equation. Formally we can call

$$X_u = \partial_x + a_u \partial_t, \quad Y_u = \partial_y + b_u \partial_t, \tag{7}$$

with $a_u$ and $b_u$ defined as in (3) and we can formally represent the operator $\mathcal{L}$ in (2) as

$$\mathcal{L} = (1 + (\partial_t u)^2)(X_u^2 + Y_u^2) \tag{8}$$

see [C]. We recall that the vector fields are "nonlinear" since their coefficients depend on the gradient of $u$. For this reason the regularity of the solution can not be studied with the well known methods introduced by [RS] and [FS], for linear operators associated to $C^\infty$ vector fields. On the other hand the representation in (8) has been used in a previous note by the authors, who proved the following

THEOREM. *If $u \in Lip(\Omega)$ is a strong viscosity solution of* (1), *then $X_u u$ and $Y_u u$ belong to the ordinary Sobolev space $H^1_{loc}(\Omega)$. Moreover the following relations hold:*

$$a_u = Y_u u, \quad b_u = -X_u u. \tag{9}$$

*and equation (1) is satisfied pointwise a.e.* (see [CM1]).

Since the coefficients of the vectors fields belong to $H^1_{loc}(\Omega)$, it is possible to define in $L^2_{loc}(\Omega)$ a bracket

$$[X_u, Y_u] = X_u Y_u - Y_u X_u,$$

and the following relation holds

$$[X_u, Y_u] = -\frac{\mathcal{L}u}{1 + (\partial_t u)^2}\partial_t \quad \text{in } L^2_{loc}(\Omega). \tag{10}$$

If $\mathcal{L}u \neq 0$ for every $\xi \in \Omega$, then $X_u, Y_u$ and their commutator $[X_u, Y_u]$ are linear independent almost everywhere. This condition, clearly reminiscent of the Hörmander condition for hypoellipticity for sum of squares of $C^\infty$ vector fields (see [H]), has been recently used in [CLM] for studying the regularity properties of solutions to the equation (5) in the case that $q$ never vanishes.

Here $\mathcal{L}u$ is identically zero and the Lie algebra generated by $X_u$ and $Y_u$ has dimension 2. Hence, even if the coefficients of the vector fields were $C^\infty$, the Hörmander condition would be violated at every point of the space, and the solution would not be regular as a function of three variables.

Assume for simplicity that $X$ and $Y$ are linear vector fields with $C^\infty$ coefficients such that $[X, Y] = 0$, and that $u$ is a solution of $X^2u + Y^2u = 0$. For every fixed $\xi_0$ let us consider the following Cauchy problem

$$\begin{cases} \gamma' = (x - x_0)X(\gamma) + (y - y_0)Y(\gamma) \\ \gamma(0) = \xi_0 = (x_0, y_0, t_0) \in \Omega \end{cases} \tag{11}$$

and let us denote its solution $\gamma_{\xi_0,x,y}$. Then the Lie group associated to $X$ and $Y$ is

$$M_{\xi_0} := \{\gamma_{\xi_0,x,y}(1) : (x, y) \in B\},$$

for a suitable neighborhood $B$ of $(x_0, y_0)$. By construction $X$ and $Y$ are then vector fields on $M = M_{|\xi_0}$ which span the tangent space at every point and the restriction $u_{|M}$ satisfies $X^2u + Y^2u = 0$. Then the hypotheses of [RS] are satisfied, and $u_{|M}$ is $C^\infty$.

In our situation the coefficients $a_u$ and $b_u$ of the operator are only $H^1_{loc}(\Omega)$. The main difficulty of the paper is then to find an unique solution of the Cauchy problem (11) with $X = X_u$, and $Y = Y_u$. In order to do so, we introduce a new technique, completely reversing the classical approach of [RS]. The first step (which is the main part of this work) is the proof that the "intrinsic" derivatives of any order of a solution $u$ are continuous and satisfy an analytic estimate. The second step will be the proof of the uniqueness of the solution of the Cauchy problem (11), and the definition of the manifold $M_{\xi_0}$ on which $u$ is harmonic. Note that at this point we have not yet established the Lipschitz continuity of the right hand side of the Cauchy problem. Only at the end of our regularization procedure we will be able to prove the Lipschitz continuity of the coefficients $a$ and $b$.

Let us give some more details.

For any multi-index $I = (i_1, \cdots, i_k) \in \{1,2\}^k$, we say that $I$ has length $k$ and we define

$$D_0^{(1)} = X_u, \quad D_0^{(2)} = Y_u, \quad D_0^I = D_0^{(i_1)} \cdots D_0^{(i_k)}. \tag{12}$$

Hence $D_0^I$ can be considered as a derivative of order $k$ in the direction of the vector fields.

In Section 2 we recall some a priori estimate for the solutions, already proved in [CLM]. In Section 3 we show that

THEOREM 1.2. *If $u$ is a strong viscosity solution of the Levi equation, then*

$$D_0^I u \in W^{1,p}_{loc}(\Omega)$$

*for every $p > 1$, and for every multi-index $I$. Here $W^{1,p}_{loc}(\Omega)$ denotes the classical Sobolev Space.*

Since $u$ is a strong viscosity solution, this result is proved by mean of a priori estimates for each element of the approximating sequence, and then passing to the limit.

Refining the previous estimates, we then prove

THEOREM 1.3. *For any open set $U \subset\subset \Omega$ there exists a constant $C = C(U) > 0$ such that for every multi-index $I$ of length $k$*

$$||X_u D_0^I u||_{L^6(U)} + ||Y_u D_0^I u||_{L^6(U)} + ||\partial_t D_0^I u||_{L^6(U)} \leq k! C^k. \tag{13}$$

In Section 4 we prove a Sobolev type inequality in terms of the vector fields. The results of this type known in literature only refer to homogeneous Sobolev spaces (see [FL], [J], [ML]), while here we need to work in non-homogeneous ones, in order to apply (13). Besides the coefficients of our vector fields are only hölder continuous, while the standard hypothesis is the Lipschitz continuity.

From this Sobolev-type theorem, and Theorem 1.3 we deduce the following analytic estimate in the intrinsic directions of the solution of (1):

THEOREM 1.4. *For any open set $U \subset\subset \Omega$ there exists a constant $C = C(U) > 0$ such that for every multi-index $I$ of length $k$*

$$||D_0^I u||_{L^\infty(U)} \leq k! C^k. \tag{14}$$

In Section 5 we deduce from here the uniqueness of the solution of Cauchy problem (11). Then the set $M_{\xi_0}$ is well defined, and we can prove Theorem 1.1. Finally we prove that $a$ and $b$ are Lipschitz. As a consequence we recover the previously stated results of [S], with a completely different approach:

COROLLARY 1.1. *Let us identify $\Omega \times i\mathbb{R}$ with a subset of $\mathbb{C} \times \mathbb{R} \times i\mathbb{R}$ and let $u \in Lip(\Omega)$ be a strong viscosity solution of the Levi equation (1). Then the graph of $u$, $\Gamma(u) = \{(\xi, u(\xi)), \xi \in \Omega\}$, is foliated by analytic complex curves.*

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## 2. NOTATIONS AND KNOWN RESULTS

In this section we define a natural setting for studying equations (1) and its regularization

$$\mathcal{L}_\varepsilon u = 0 \; in \; \Omega \subset \mathbb{R}^3, \tag{15}$$

where $\mathcal{L}_\varepsilon$ has been defined in (6). For simplicity we assume that $\varepsilon \geq 0$ and, when $\varepsilon = 0$, we obtain equation (1). We fix a Lipschitz continuous strong viscosity solution $u$ of equation (15). We denote $a = a_u$ and $b = b_u$ the coefficients introduced in (3) and we call $X$ and $Y$ instead of $X_u$ and $Y_u$ the vector fields defined in terms of them in (7). We also assume that

$$a, b, \varepsilon\partial_t u \in H^1_{loc}(\Omega). \tag{16}$$

Under these assumptions the following ones are the main properties of the vector fields and their coefficients (see [CM1] for the proof):

$$Yu = a \quad and \quad Xu = -b, \tag{17}$$

$$\partial_t a = \frac{Y\omega - \omega X\omega}{1+\omega^2} \quad and \quad \partial_t b = -\frac{X\omega + \omega Y\omega}{1+\omega^2}, \tag{18}$$

where, here and in the sequel, we denote by

$$\omega = \partial_t u. \tag{19}$$

In the study of regularity of solutions of equations (1) and (15), the vector fields $X$ and $Y$ play a role analogous to the derivatives in the ordinary setting, so that there is a natural definition of derivatives and Sobolev spaces in terms of them.

If $f$ is a $L^1_{loc}(\Omega)$ function, we say that it is weakly differentiable with respect to $X$ if there exists a function $g \in L^1_{loc}(\Omega)$ such that

$$\int f X^* \phi = \int g \phi \quad \forall \phi \in C_0^\infty(\Omega) \tag{20}$$

where $X^*$ is the formal adjoint of $X$. The weak derivative with respect to $Y$ is analogously defined.

Definition 2.1. If $\varepsilon \geq 0$ is fixed, we will denote

$$D_\varepsilon^{(1)} = X, \quad D_\varepsilon^{(2)} = Y, \quad D_\varepsilon^{(3)} = T_\varepsilon = \frac{\varepsilon}{\sqrt{1+\omega^2}}\partial_t,$$

where $\omega$ is defined in (19). When $\varepsilon > 0$ we call intrinsic gradient

$$D_\varepsilon = (X, Y, T_\varepsilon),$$

and, when $\varepsilon = 0$, we will simply call

$$D_0 = (X, Y).$$

For every moltiindex $I = (I_1, \cdots, I_k) \in \{1,2,3\}^k$ we set $|I| = k$ and

$$D_\varepsilon^I = D_\varepsilon^{(I_1)} \cdots D_\varepsilon^{(I_k)}.$$

Then, for any $U \subset \Omega$ we will call

$$W_\varepsilon^{k,p}(U) = \{f : D_\varepsilon^I f \in L^p(U), \forall I : |I| \leq k\}$$

$$||f||_{W_\varepsilon^{k,p}(U)} = \sum_{|I|\leq k} ||D_\varepsilon^I f||_{L^p(U)},$$

and we will say that $f \in W^{k,p}_{\varepsilon,loc}(\Omega)$ if $f \in W_\varepsilon^{k,p}(U)$ for every $U \subset\subset \Omega$. In particular

$$||f||_{W_\varepsilon^{0,p}(U)} = ||f||_{L^p(U)}.$$

Finally we will denote by $W^{k,p}(\Omega) = W_1^{1,p}(\Omega)$ the usual Sobolev space, and by $\nabla = (\partial_x, \partial_y, \partial_t)$ the euclidean gradient in $\mathbb{R}^3$.

In these Sobolev spaces we will make use of the following interpolation inequalities, already proved in [CLM].

PROPOSITION 2.1. *For every $p \geq 3$, and for every $\phi \in C_0^\infty(\Omega)$, there exists a constant $C_p$, depending on $p$ such that for every function $z \in C^\infty(\Omega)$*

$$\int |Xz|^p \phi^{2p} \leq C_p \Big( \int |z|^{2p}\phi^{2p} + \int |D_\varepsilon(|Xz|^{(p-1)/2})|^2\phi^{2p} + \int |D_\varepsilon \phi|^{2p} \Big).$$

*Besides*

$$\int |Xz|^p \phi^{2p} \leq C_p \Big( ||z||_\infty^2 \int |D_\varepsilon(|Xz|^{(p-2)/2})|^2\phi^{2p} + ||z||_\infty^4 \int \phi^p |D_\varepsilon \phi|^p \Big).$$

*Analogous relations hold if we replace $X$ with $Y$ or $\varepsilon\partial_t$.*

### 2.1. A linear Levi type equation

In this subsection we recall some properties of a linear equation associated to equation (1), already proved in [CLM]. We call linear operator associated to $\mathcal{L}_\varepsilon$

$$L_\varepsilon z = X^2 z + Y^2 z + T_\varepsilon^2 z, \tag{21}$$

and consider the solutions of the following linear equation:

$$L_\varepsilon z = f_0 + f_1 Xz + f_2 Yz + f_3 T_\varepsilon z + X f_4 + Y f_5 + T_\varepsilon f_6 \ \ in\ \Omega \tag{22}$$

for suitable functions $f_i, i = 1, \cdots, 6$.

Let us note that, if we define

$$v = arctg(u_t), \tag{23}$$

by relations (18) and (19) we get

$$Yv = \frac{Y\omega}{1+\omega^2} = \frac{\partial_t a - \partial_t u \partial_t b}{1 + (\partial_t u)^2}, \quad Xv = -\frac{\partial_t b + \partial_t u \partial_t a}{1 + (\partial_t u)^2}, \quad T_\varepsilon v = \frac{\partial_t(\varepsilon \partial_t u)}{(1 + (\partial_t u)^2)^{3/2}}.$$

Since $u$ is Lipschitz continuous, and condition (16) holds, we deduce that $v \in W^{1,2}_{\varepsilon, loc}(\Omega)$. Hence for every fixed $R > 0$, there exists a constant $C > 0$ such that

$$||D_\varepsilon u||_{L^\infty(B(R))} + ||\partial_t u||_{L^\infty(B(R))} + ||v||_{W^{1,2}_\varepsilon(B(R))} \leq C, \tag{24}$$

where $B(R)$ denotes a sphere in the euclidean metrics compactly imbedded in $\Omega$.

The following Lemma ensures that the derivatives of solutions of equation (22) are solutions of the same equation

Lemma 2.1. *If $z$ is a solution of equation*

$$L_\varepsilon z = 0,$$

*then $s_1 = Xz$ is a solution of the equation*

$$L_\varepsilon s_1 = 2XvT_\varepsilon vT_\varepsilon z - Y\Big(2T_\varepsilon vT_\varepsilon z\Big) + 2T_\varepsilon\Big(YvT_\varepsilon z\Big), \tag{25}$$

*$s_2 = Yz$ is a solution of the equation*

$$L_\varepsilon s_2 = 2YvT_\varepsilon vT_\varepsilon z + X\Big(2T_\varepsilon vT_\varepsilon z\Big) - 2T_\varepsilon\Big(XvT_\varepsilon z\Big),$$

*$s_3 = \varepsilon\partial_t z$ and $s_3 = \partial_t z$ are solutions of the equation*

$$L_\varepsilon s_3 = -2\partial_t aXs_3 - 2\partial_t bYs_3 + 2\omega T_\varepsilon vT_\varepsilon s_3.$$

In particular the function $v$ previously defined satisfies

$$L_\varepsilon v = 0 \ \ in\ \Omega. \tag{26}$$

Under the a priori bound (24) the following estimate, of Caccioppoli type, holds.

Lemma 2.2. *If $f_0, \cdots, f_6 \in L^q_{loc}(\Omega)$, for $q > 2$, $z \in W^{2,2}_{\varepsilon,loc}(\Omega) \cap W^{1,3}_{loc}(\Omega)$ is a solution of equation (22) then there exist constants $C_1, C_2, C_3$ depending only on $\beta$, $\gamma$, the constant $C$ in (24) and independent of $\varepsilon$ such that*

$$\int |D_\varepsilon(|z|^\beta)|^2\phi^\gamma \leq$$

$$\leq C_1\int |z|^{2\beta}\big(\phi^2+|D_\varepsilon\phi|^2\big)\phi^{\gamma-2}+\int |f_0||z|^{2\beta-1}\phi^\gamma+C_2\int |z|^{2\beta}\big(|f_1|^2+|f_2|^2+|f_3|^2\big)\phi^\gamma$$

$$+C_3\int |z|^{2\beta-2}\big(|f_4|^2+|f_5|^2+|f_6|^2\big)\phi^\gamma+C_3\int |z|^{2\beta-1}\big(|f_4|+|f_5|+|f_6|\big)|D_\varepsilon v|\phi^\gamma.$$

*If $f_1 = -2\partial_t a$, $f_2 = -2\partial_t b$, $f_3 = 2\omega T_\varepsilon v$, we can choose $C_2 = 0$.*

Note that the right hand side is bounded. Indeed, by the hypothesis on $z$ and the classical Sobolev imbedding theorem, $z \in L^p_{loc}(\Omega)$ for every $p > 1$.

A similar estimate holds for second derivatives of $z$ containing a derivative with respect to $t$.

PROPOSITION 2.2. *If $z \in W^{2,2}_{\varepsilon,loc}(\Omega) \cap W^{1,2}_{loc}(\Omega)$ is a solution of equation*

$$L_\varepsilon z = 0 \ \ in\ \Omega,$$

*If $e = 0$ also assume that $D_\varepsilon z \in W^{1,2}_{loc}(\Omega)$. Then there exists a constant $\widetilde{C} > 0$, only dependent on $C$ in (24) such that for every $\phi \in C^\infty_0(\Omega)$,*

$$\int \Big(|\partial_t D_\varepsilon z|^2\Big)\phi^2 \leq \widetilde{C} \int (\partial_t z)^2 (|D_\varepsilon \phi|^2 + |\phi|^2).$$

More generally an analogous estimate is satisfied by the intrinsic derivatives of $z$ of any order.

THEOREM 2.1. *Let $\varepsilon > 0$, $p \geq 2$ be fixed, let $z$ solution of equation $L_\varepsilon z = 0$ in $\Omega$. Let $B(R)$ be a sphere such that $B(R) \subset\subset \Omega$, and let $k \in N$, $k \geq 1$. Then for every $\theta < 1$ there exists $\widetilde{C} = \widetilde{C}_{p,R,\theta}$, which depends on the constant $C$ in (24) on $p$, $R$, and on $\theta$, but not on $\varepsilon$, such that the solution $z$ satisfies the following estimate*

$$||z||^p_{W^{k+1,p}_\varepsilon(B(\theta R))} + \sum_{|i|=k+1} |||D^i z|^{(p-1)/2}||^2_{W^{1,2}_\varepsilon(B(\theta R))} \leq$$

$$\leq \widetilde{C}\Big(||v||^{2p}_{W^{k,2p}_\varepsilon(B(R))} + ||z||^{2p}_{W^{k,2p}_\varepsilon(B(R))}\Big),$$

*where $v$ is defined in (23).*

## 3. $L^P$ A PRIORI ESTIMATES

In this section we give an a priori estimate in the natural Sobolev spaces for the solution of equation (1). Since this is defined as a limit of solutions of problems (15), we start with studying these last problems.

### 3.1. Estimates for the regularized equation

Let $\varepsilon > 0$ be fixed and $u$ be a solution of equation (15), satisfying (24). Here we estimate its derivatives, using the results stated in Section 2.

PROPOSITION 3.1. *For every ball $B(R)$ contained in $\Omega$, for every $k \in \mathbb{N}$, $0 < \theta < 1$, $p \geq 1$ there exists a constant $C_{pR\theta k} > 0$, which depends on the constant $C$ in (24), $p$, $\theta$, $R$ and $k$, but is independent of $\varepsilon$ such that*

$$||v||_{W^{k,p}_\varepsilon(B(\theta R))} \leq C_{pR\theta k}.$$

*Since* $v = \arctan(u_t)$*, then the same estimate also holds for* $u_t$*.*

**Proof** Let us start with the proof for $k = 1$. Let us fix a function $\phi \in C_0^\infty(B(R))$ such that $\phi = 1$ on $B(\theta R)$, and $|D\phi| \leq 4/R(1-\theta)$. By Proposition 2.1 with $p = 2\beta + 1$, we have

$$\int |\varepsilon \partial_t v|^{2\beta+1} \phi^{2(2\beta+1)} \leq$$

$$\leq \widetilde{C}\Big( \int |v|^{2(2\beta+1)} |\phi|^{2(2\beta+1)} + \int |D_\varepsilon (\varepsilon \partial_t z)^\beta|^2 \phi^{2(2\beta+1)} + \int |D_\varepsilon \phi|^{2(2\beta+1)} \Big).$$

Here $\widetilde{C}$ denotes a constant independent of $\varepsilon$, which will not always be the same. By (26) and Lemma 2.1 $s_3 = \varepsilon \partial_t v$ is a solution of

$$L_\varepsilon s_3 = -\partial_t a X s_3 - \partial_t b Y s_3 + 2\omega T_\varepsilon v T_\varepsilon s_e.$$

Then we can apply Lemma 2.2 with $C_2 = 0$, and we deduce

$$\int |\varepsilon \partial_t v|^{2\beta+1} |\phi|^{2(2\beta+1)} \leq$$

$$\leq \widetilde{C}\Big( \int |v|^{2(2\beta+1)} \phi^{2(2\beta+1)} + \int |\varepsilon \partial_t v|^{2\beta} (|D_\varepsilon \phi|^2 + \phi^2) \phi^{4\beta} + \int |D_\varepsilon \phi|^{2(2\beta+1)} \Big) \leq$$

(by Schwartz inequality and the boundeness of $v$)

$$\leq \widetilde{C} + \delta \int |\varepsilon \partial_t v|^{2\beta+1} |\phi|^{2(2\beta+1)},$$

for a suitable constant $\widetilde{C}$. If we choose $\delta$ sufficiently small, this implies that there exists a new constant $\widetilde{C} = \widetilde{C}_{\beta R \theta}$ such that

$$\int_{B(\theta R)} |T_\varepsilon v|^{2\beta+1} \leq \int |\varepsilon \partial_t v|^{2\beta+1} |\phi|^{2(2\beta+1)} \leq \widetilde{C}. \tag{27}$$

On the other hand by the second assertion of Proposition 2.1 we also have

$$\int |Xv|^{2\beta+2} \phi^{4\beta+4} \leq \widetilde{C} \int |D_\varepsilon (|Xv|^\beta)|^2 \phi^{4\beta+4} + \widetilde{C}. \tag{28}$$

By Lemma 2.1 $Xv$ is a solution of equation

$$L_\varepsilon (Xv) = -2Xv(T_\varepsilon v)^2 - 2Y\big((T_\varepsilon v)^2\big) + 2T_\varepsilon (Y v T_\varepsilon v)$$

and by Lemma 2.2 we have

$$\int |D_\varepsilon(|Xv|^\beta)|^2\phi^{4\beta+4} \le C_1 \int |Xv|^{2\beta}\big(|D_\varepsilon\phi|^2+\phi^2\big)\phi^{4\beta+2} + \tag{29}$$

$$+2\int |T_\varepsilon v|^2|Xv|^{2\beta}\phi^{4\beta+4} + C_3\int (|T_\varepsilon v|^4+|T_\varepsilon v|^2|Yv|^2)|Xv|^{2\beta-2}\phi^{4\beta+4}$$

$$+C_3\int |Xv|^{2\beta-1}|T_\varepsilon v|(|T_\varepsilon v|+|Yv|)|D_e v|\phi^{4\beta+4}.$$

By (27), (28), (29) there exists a constant $\widetilde{C}=\widetilde{C}_{\beta R\theta}$ such that

$$\int |Xv|^{2\beta+2}\phi^{4\beta+4} \le \delta\int_{B(R)}\Big(|Xv|^{2\beta+2}+|Yv|^{2\beta+2}\Big)\phi^{4\beta+4}+\widetilde{C}.$$

Arguing in the same way with $Yv$ we deduce the thesis for $k=1$. The general case $k>1$ follows by induction from Theorem 2.1, choosing $z=v$.

Proposition 3.2. *Let $u$ be a solution of (15), satisfying (24). Then for every ball $B(R)$ contained in $\Omega$, for every $k\in\mathbb{N}$, $\theta<1$, for every $p\ge 1$ there exists a constant $C_{p\theta kR}>0$, independent of $\varepsilon$ such that*

$$||D^I_\varepsilon u||_{L^p(B(\theta R))}+||\partial_t(D^I_\varepsilon u)||_{L^p(B(\theta R))}\le C_{pR\theta k}$$

*for every $I$ such that $|I|=k$.*

**Proof** By the assumption (24), and Theorem 2.1 we immediately deduce that there exists $C_{pR\theta k}$ such that

$$||D^I_\varepsilon u||_{L^p(B(\theta R))}\le C_{pR\theta k} \tag{30}$$

for every $I$ such that $|I|=k$. Let us now show by induction on $k$ that, for every multi-index $I\in\{1,2,3\}^k$, we have

$$\partial_t(D^I_\varepsilon u)=P(\omega,D_\varepsilon\omega,\cdots,D^k_\varepsilon\omega) \tag{31}$$

where $\omega$ is defined in (19), $D^k_\varepsilon$ denotes all the derivatives of order $k$ and $P$ is a smooth function. Indeed, by definition of $X$,

$$\partial_t Xu = (by\ (17))\ = -\partial_t b = (by\ \ (18)) = \frac{X\omega+\omega Y\omega}{1+\omega^2}.$$

If $|I|=k$, we can denote $I=(i_1,I')$ where $|I'|=k-1$. Assume for simplicity that $i_1=1$. Then

$$\partial_t(D^I_\varepsilon u)=\partial_t(XD^{I'}_\varepsilon u)=\partial_t a\partial_t D^{I'}_\varepsilon u + X\partial_t D^{I'}_\varepsilon u =$$

(by the inductive hypothesis there exists a smooth function $P$ such that the following relation is true)

$$= \partial_t a P(\omega, D_\varepsilon \omega, \cdots, D_\varepsilon^{k-1}\omega) + X\Big(P(\omega, D_\varepsilon \omega, \cdots, D_\varepsilon^{k-1}\omega)\Big) =$$

(by (18))

$$= \frac{Y\omega - \omega X\omega}{1+\omega^2} P(\omega, D_\varepsilon \omega, \cdots, D_\varepsilon^{k-1}\omega) + \sum_{j=0}^{k-1} \partial_j P\big(\omega, D_\varepsilon \omega, \cdots, D_\varepsilon^{k-1}\omega\big) X D_\varepsilon^j \omega.$$

From the last assertion of Proposition (3.1), it immediately follows that there exists $C = C_{p,I}$ independent of $\epsilon$ such that

$$||\partial_t (D_\varepsilon^I u)||_{L^p(B(\theta R))} \le C_{p,I}.$$

### 3.2. Estimates for the Levi equation

Now we conclude the estimate of the strong viscosity solution $u$ of equation (1), in the Spaces $W_0^{k,p}(\Omega)$, letting $\varepsilon$ go to 0 in the estimates just obtained.

Let $u$ be a strong viscosity solution, and $(u_j)$ its approximating sequence, as defined in Definition 1.1. For each function $u_j$ we will denote $a_j = a_{u_j}$ and $b_j = b_{u_j}$, the coefficients introduced in (3), $X_j$, $Y_j$ the corresponding vector fields, defined in (7), $D_{\varepsilon_j}$ and $W_{\varepsilon_j}^{k,p}(\Omega)$ the natural gradient and Sobolev spaces, introduced in Definition 2.1. Besides $a$ and $b$, $X$, $Y$, $D_0$ will be the coefficients and vector fields associated to the limit equation $u$, while $W_0^{k,p}(\Omega)$ will be the associated Sobolev space. Note that $\nabla$ and $W^{k,p}(\Omega)$ are the usual gradient and Sobolev space. It has been proved in [CM1] that

THEOREM 3.1. *If $u \in Lip(\Omega)$ is a strong viscosity solution of* (1), *and $v_j = arctg(\partial_t u_j)$, then for every ball $B(R) \subset\subset \Omega$ there exists a constant $C > 0$ such that*

$$||D_{\varepsilon_j} u_j||_{L^\infty(B(R))} + ||\partial_t u_j||_{L^\infty(B(R))} + ||v_j||_{W_{e_j}^{1,2}(B(R))} \le C \qquad (32)$$

*and*

$$X_j u_j \to Xu, \quad Y_j u_j \to Yu, \quad T_{\varepsilon_j} u_j \to 0 \qquad (33)$$

*as $j \to +\infty$ weakly in $H^1_{loc}(\Omega)$, and equation (1), represented as*

$$X^2 u + Y^2 u = 0$$

*is satisfied pointwise a.e. in* $\Omega$.

An analogous result holds for the derivatives of any order:

PROPOSITION 3.3. *For every* $k \in N$ *for every* $p > 1$ *and for every multiindex* $I$ *of length* $k$, *the sequence* $(D^I_{\varepsilon_j} u_j)$ *is bounded in* $W^{1,p}_{loc}(\Omega)$. *Moreover*

$$D^I_{\varepsilon_j} u_j \to D^I_0 u \ \ as \ \ j \to +\infty, \ \ weakly \ \ in \ \ W^{1,p}(\Omega),$$

*and* $[X, Y] = 0$.

**Proof** By (32) and Proposition 3.2 we deduce that for every ball $B(R) \subset\subset \Omega$ there exists $\widetilde{C}$ such that

$$||\partial_t D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \leq \widetilde{C},$$

$$||\partial_x D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \leq ||X D^I_{\varepsilon_j} u_j||_{L^p(B(R))} + ||a_j \partial_t D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \leq \widetilde{C},$$

$$||\partial_y D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \leq ||Y D^I_{\varepsilon_j} u_j||_{L^p(B(R))} + ||b_j \partial_t D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \leq \widetilde{C}$$

so that $(D^I_{\varepsilon_j} u_j)$ is bounded in $W^{1,p}_{loc}(\Omega)$ for every $p > 1$, and every multi index $I$. Let us prove that the weak limit is $D^I_0 u$. If $|I| = 1$ the assertion is true by (33). If $I$ is a multiindex such that $|I| = k$, we can assume by simplicity that $I = (1, I')$, where $|I'| = k - 1$. We can also assume by inductive hypothesis that

$$\partial_t a_j \to \partial_t a \ \ as \ \ j \to \infty \ \ weakly \ \ in \ \ L^p(\Omega)$$

$$a_j \to a \ \ as \ \ j \to \infty \ \ in \ \ L^p_{loc}(\Omega)$$

$$D^{I'}_{\varepsilon_j} u_j \to D^{I'}_0 u \ \ as \ \ j \to \infty \ \ in \ \ L^p_{loc}(\Omega).$$

Then integrating by parts

$$lim_{j\to\infty} \int D^I_{\varepsilon_j} u_j \phi = -lim_{j\to\infty} \int D^{I'}_{\varepsilon_j} u_j X_j \phi - \int \partial_t a_j D^{I'}_{\varepsilon_j} u_j \phi =$$

$$= -\int D^{I'}_0 u X\phi - \int \partial_t a D^{I'}_0 u \phi,$$

and this ensures the weak convergence of $(D^I_{\varepsilon_j}u_j)$ to $D^I_0u$. Finally

$$[X,Y] = XY - YX = (Xb - Ya)\partial_t = 0.$$

We can now prove the main regularity properties of the limit function $u$:

PROPOSITION 3.4. *For every $k$, for every $p > 1$ and for every multiindex $I$, such that $|I| = k$, the function $z = D^I_0u$ belongs to $W^{1,p}_{loc}(\Omega) \cap W^{2,p}_{0,loc}(\Omega)$ and it is a solution of*

$$X^2z + Y^2z = 0 \quad in \;\; \Omega. \tag{34}$$

*In particular*

$$D^I_0u \in C^\alpha_{loc}(\Omega) \tag{35}$$

*for every $\alpha < 1$.*

**Proof** Since $u$ is a solution of $X^2u + Y^2u = 0$ in $\Omega$ and $X$ and $Y$ commute, then we deduce that also the derivatives are solutions of the same equation. From Proposition 3.3 we immediately obtain that $D^I_0u$ belongs to $W^{1,p}_{loc}(\Omega) \cap W^{2,p}_{0,loc}(\Omega)$ for every $I$, and for every $p > 1$, so that (35) is satisfied by the classical Sobolev imbedding theorem.

The following estimate will be the main tool for our analytic estimate.

PROPOSITION 3.5. *If $B(2R) \subset\subset \Omega$ there exist constants $C_1 > 0$ and $C_2 > 0$ such that for every multiindex $I$ of length $k$,*

$$||X(D^I_0u)||_{L^6(B(R))} + ||Y(D^I_0u)||_{L^6(B(R))} + ||\partial_t(D^I_0u)||_{L^2(B(R))} \le C_2k!C_1^k.$$

**Proof** Let $I \in \{1,2\}^{k+2}$. By definition $I = (I_1, I')$, where $I' \in \{1,2\}^{k+1}$. By Proposition 3.4 the function $z = D^{I'}_0u$ is a solution of (34). Let us choose $\psi$ such that $\psi = 1$ on $B(R)$, $\psi \in C^\infty_0(B(2R))$, $0 \le \psi \le 1$. Then we apply Lemma 2.2 with $\varepsilon = 0$, $\beta = 1$, $\phi = \psi^k$, $\gamma = 2$ and we deduce that there exists a constant $\widetilde{C}$ independent of $k$ such that

$$\int_{B(R)} |D^I_0u|^2 \le \int |D^I_0u|^2\psi^{2k}$$

$$\le \widetilde{C}^2 \int |D^{I'}_0u|^2\Big(|\psi|^{2k} + k^2|D_0\psi|^2\psi^{2(k-1)}\Big) \le \widetilde{C}^2k^2 \int |D^{I'}_0u|^2\psi^{2(k-1)}.$$

Iterating this inequality we get

$$\int_{B(R)} |D_0^I u|^2 \le \widetilde{C}^{2k}(k!)^2 \int |u|^2\psi^2 \le C_1\widetilde{C}^{2k}(k!)^2.$$

Using Lemma 2.1 we get in the same way

$$\int_{B(R)} |\partial_t D_0^{I'} u|^2 \le C^k(k!)^2 \int |\partial_t u|^2\phi^2 \le C_1\widetilde{C}^{2k}(k!)^2.$$

It follows that

$$||\partial_x D_0^{I'} u||_{L^2(B(R))} \le ||XD_0^{I'} u||_{L^2(B(R))} + ||a\partial_t D_0^{I'} u||_{L^2(B(R))} \le C_1\widetilde{C}^k k!$$

$$||\partial_y D_0^{I'} u||_{L^2(B(R))} \le ||YD_0^{I'} u||_{L^2(B(R))} + ||b\partial_t D_0^{I'} u||_{L^2(B(R))} \le C_1\widetilde{C}^k k!$$

By the classical Sobolev imbedding theorem the thesis follows.

## 4. SOBOLEV IMBEDDING THEOREMS

In this section we will always assume that $u$ is a fixed function defined on an open set $\Omega$ of $\mathbb{R}^3$, such that the coefficients $a$ and $b$ satisfy the following assumptions:

$$a, b \in C^\beta_{loc}(\Omega)\ Xa, Xb, Ya, Yb \in C^\beta_{loc}(\Omega)\ for\ \beta \in ]0,1[, \quad Xb = Ya, \tag{36}$$

$$\partial_t a, \partial_t b \in L^s_{loc}(\Omega) \quad s > N/\beta. \tag{37}$$

Then we prove a nonomogeneous Morrey type inequality, in terms of the vector fields. Since we will not assume that the coefficients are Lipschitz continuous, we can not apply any standard result, but we will use a freezing method first introduced in [C] in a situation similar to the present one, and also applied in [CLM].

For a function $f \in C^\beta_{loc}(\Omega)$ such that $Xf, Yf \in C^\beta_{loc}(\Omega)$ we can introduce the first order polynomial

$$P^1_{\xi_0} f(\xi) = f(\xi_0) + Xf(\xi_0)(x - x_0) + Yf(\xi_0)(y - y_0). \tag{38}$$

In particular, by (36), we can define

$$X_{\xi_0} = \partial_x + P^1_{\xi_0} a(\xi)\partial_t, \quad Y_{\xi_0} = \partial_x + P^1_{\xi_0} b(\xi)\partial_t.$$

These are $C^\infty$ vector fields, and the associated metrics can be defined as follows. For every $\xi \in \Omega$ the exponential mapping

$$\phi_\xi(\alpha) = exp\big(\alpha_1 X_{\xi_0} + \alpha_2 Y_{\xi_0} + \alpha_3\partial_t\big)(\xi)$$

is a local diffeomorphism.

On the domain of $\phi_\xi$ we define a group of dilations and norm as follows:

$$||\alpha|| = \Big(\big(\alpha_1^2 + \alpha_2^2\big)^3 + \alpha_3^2\Big)^{\frac{1}{6}} \tag{39}$$

$$\delta_s(\alpha) = (s\alpha_1, s\alpha_2, s^3\alpha_3).$$

By the hypothesis (36), and the results in NSW], a distance and a family of dilations are also induced on the image of $\phi_\xi$, and for every $\xi, \xi_1$ we can set

$$d_{\xi_0}(\xi, \xi_1) = ||\phi_\xi^{-1}(\xi_1)||, \quad \widetilde{\delta}_s(\xi_1) = \phi_\xi(\delta_s(\phi_\xi^{-1}(\xi_1)))). \tag{40}$$

In the sequel we will need an explicit expression of the functions $\phi_{\xi_0}$ and $\phi_{\xi_0}^{-1}$.

*Remark 4. 1.* If $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a solution of

$$\gamma' = \alpha_1 X_{\xi_0} + \alpha_2 Y_{\xi_0} + \alpha_3 \partial_t, \quad \gamma(0) = \xi_0 = (x_0, y_0, t_0),$$

then

$$\gamma_1(s) = \alpha_1 s + x_0, \quad \gamma_2(s) = \alpha_2 s + y_0,$$

$$\gamma_3'(s) = \alpha_1 P^1_{\xi_0} a(\gamma(s)) + \alpha_2 P^1_{\xi_0} b(\gamma(s)) + \alpha_3.$$

It follows that

$$\phi_{\xi_0}(\alpha) = \gamma(1) = \Big(\alpha_1 + x_0, \alpha_2 + y_0, \alpha_3 + t_0 +$$

$$+\alpha_1 \int_0^1 P^1_{\xi_0} a(\gamma(\sigma)) d\sigma + \alpha_2 \int_0^1 P^1_{\xi_0} b(\gamma(\sigma)) d\sigma\Big) =$$

$$= \Big(\alpha_1 + x_0, \alpha_2 + y_0, \alpha_3 + t_0 + \alpha_1 \big(a(\xi_0) + Xa(\xi_0)\frac{\alpha_1}{2} + Ya(\xi_0)\frac{\alpha_2}{2}\big) +$$

$$+\alpha_2 \big(b(\xi_0) + Xb(\xi_0)\frac{\alpha_1}{2} + Yb(\xi_0)\frac{\alpha_2}{2}\big)\Big).$$

Inverting this expression we get

$$\phi_{\xi_0}^{-1}(\xi) = \Big(x - x_0, y - y_0, t - t_0 - (x - x_0)\big(a(\xi_0) + Xa(\xi_0)\frac{x - x_0}{2} + Ya(\xi_0)\frac{y - y_0}{2}\big) - \tag{41}$$

$$-(y-y_0)\big(b(\xi_0)+Xb(\xi_0)\frac{x-x_0}{2}+Yb(\xi_0)\frac{y-y_0}{2})\Big).$$

*Remark 4. 2.* Since the homogeneous dimension of $R^3$ with respect to the group of dilatation defined in (39) is $N=5$, this is also the dimension of the space with the distance (40). In particular if $B_{\xi_0}(\xi_1,r)$ denotes a sphere of the metrics $d_{\xi_0}$ with center $\xi_0$ and radius $r$, then the Lebesgue measure of $B_{\xi_0}(\xi_1,r)$ is $\widetilde{C}r^5$, for a suitable dimensional constant $\widetilde{C}$.

We can now state a Morrey-type inequality, in terms of the vector fields $X$ and $Y$:

PROPOSITION 4.1. *Let $\xi_0\in\Omega$, $R>0$ such that $B_{\xi_0}(\xi_0,4R)\subset\Omega$. Let*

$$z\in C(B_{\xi_0}(\xi_0,2R))\cap W^{1,p}(B_{\xi_0}(\xi,2R))$$

*and let $z_{B,\xi_0}$ denote its mean on the sphere $B_{\xi_0}(\xi_0,d_{\xi_0}(\xi,\xi_0))$. Then for every $\xi\in B_{\xi_0}(\xi_0,R)$, the following inequality holds,*

$$|z(\xi)-z_{B,\xi_0}|\leq$$

$$\leq\widetilde{C}\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))}d_{\xi_0}^{-N+1}(\xi,\zeta)\big(|Xz(\zeta)|+|Yz(\zeta)|\big)d\zeta+$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))}d_{\xi_0}^{-N+1}(\xi,\zeta)\Big(|a(\zeta)-P^1_{\xi_0}a(\zeta)|+|b(\zeta)-P^1_{\xi_0}b(\zeta)|\Big)|\partial_t z(\zeta)|d\zeta$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))}d_{\xi_0}^{-N+3}(\xi,\zeta)|\partial_t z(\zeta)|d\zeta,$$

*where $\widetilde{C}$ is a dimensional constant.*

**Proof** We can obviously assume that $z\in C^\infty$, and obtain the general result with a simple density argument. For fixed $\xi$, $\xi_0$, we call $r=d_{\xi_0}(\xi,\xi_0)$ and we have, by (40)

$$z(\xi)-z(\xi_1)=z(\widetilde{\delta}_0(\xi_1))-z(\widetilde{\delta}_1(\xi_1))=-\int_0^1\frac{d}{ds}(z(\widetilde{\delta}_s(\xi_1)))ds. \quad (42)$$

In order to evaluate this integral, we first consider the function

$$Z(\sigma,s)=z\Big(exp\big(\sigma\alpha_1X_{\xi_0}+\sigma\alpha_2Y_{\xi_0}+s^3\alpha_3\partial_t\big)(\xi)\Big),$$

defined in such a way that, if we denote $\phi_\xi^{-1}(\xi_1)=\alpha$, then

$$Z(s,s)=z(\phi_\xi(\delta_s(\alpha)))=z(\widetilde{\delta}_s(\xi_1)).$$

Since $\alpha_1 X_{\xi_0}+\alpha_2 Y_{\xi_0}$ and $\partial_t$ commute we have

$$\frac{\partial}{\partial\sigma}Z(\sigma,s)=\frac{\partial}{\partial\sigma}z\Big(exp\big(\sigma\alpha_1 X_{\xi_0}+\sigma\alpha_2 Y_{\xi_0}\big)\big(exp(s^3\alpha_3\partial_t)(\xi)\big)\Big)=$$

(by definition of exponential mapping)

$$=\big(\alpha_1 X_{\xi_0}z+\alpha_2 Y_{\xi_0}z\big)\Big(exp\big(\sigma\alpha_1 X_{\xi_0}+\sigma\alpha_2 Y_{\xi_0}+s^3\alpha_3\partial_t\big)(\xi)\Big).$$

Analogously

$$\frac{\partial}{\partial s}Z(\sigma,s)=\frac{\partial}{\partial s}z\Big(exp(s^3\alpha_3\partial_t)\big(exp\big(\sigma\alpha_1 X_{\xi_0}+\sigma\alpha_2 Y_{\xi_0}\big)(\xi)\big)\Big)=$$

$$=3s^2\alpha_3\partial_t z\Big(exp\big(\sigma\alpha_1 X_{\xi_0}+\sigma\alpha_2 Y_{\xi_0}+s^3\alpha_3\partial_t\big)(\xi)\Big).$$

By the chain rule we immediately get

$$\frac{d}{ds}(z(\widetilde{\delta}_s(\xi_1)))=\frac{\partial}{\partial\sigma}Z(\sigma,s)_{|\sigma=s}+\frac{\partial}{\partial s}Z(\sigma,s)_{|\sigma=s}=$$

$$=\alpha_1 X_{\xi_0}z(\widetilde{\delta}_s(\xi_1))+\alpha_2 Y_{\xi_0}z(\widetilde{\delta}_s(\xi_1))+3s^2\alpha_3\partial_t z(\widetilde{\delta}_s(\xi_1)).$$

Inserting in (42) we get

$$z(\xi)-z(\xi_1)=-\int_0^1\Big(s\alpha_1 X_{\xi_0}z(\widetilde{\delta}_s(\xi_1))+s\alpha_2 Y_{\xi_0}z(\widetilde{\delta}_s(\xi_1))+3s^3\alpha_3\partial_t z(\widetilde{\delta}_s(\xi_1))\Big)\frac{ds}{s}$$

(by (39) we have $|s\alpha_1|+|s\alpha_2|\le||\delta_s\alpha||=$ (by the choice of $\alpha$) $=||\delta_s(\phi_\xi^{-1}(\xi_1)||=$ (by (40)) $=||\phi_\xi^{-1}(\widetilde{\delta}_s(\xi_1)||=d_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))$, and analogously $|s^3\alpha_3|\le d^3_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))$)

$$\le\int_0^1 d_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))\Big(|X_{\xi_0}z(\widetilde{\delta}_s(\xi_1))|+|Y_{\xi_0}z(\widetilde{\delta}_s(\xi_1))|\Big)\frac{ds}{s}+$$

$$+3\int_0^1 d^3_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))|\partial_t z(\widetilde{\delta}_s(\xi_1))|\frac{ds}{s}.$$

Integrating over $B_{\xi_0}(\xi_0,r)$ with respect to $\xi_1$, and dividing by its measure, we get

$$\Big|z(\xi)-\frac{1}{r^5}\int_{B_{\xi_0}(\xi_0,r)}z(\xi_1)d\xi_1\Big|\le$$

$$\leq \frac{1}{r^5}\int_{B_{\xi_0}(\xi_0,r)}\int_0^1 d_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))\Big(|X_{\xi_0}z(\widetilde{\delta}_s(\xi_1))|+|Y_{\xi_0}z(\widetilde{\delta}_s(\xi_1))|\Big)\frac{ds}{s}d\xi_1+$$

$$+\frac{3}{r^5}\int_{B_{\xi_0}(\xi_0,r)}\int_0^1 d^3_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))|\partial_t z(\widetilde{\delta}_s(\xi_1))|\frac{ds}{s}d\xi_1 \leq$$

(since $r = d_{\xi_0}(\xi,\xi_0)$, then $B_{\xi_0}(\xi_0,r)\subseteq B_{\xi_0}(\xi,2r)$)

$$\leq \frac{1}{r^5}\int_{B_{\xi_0}(\xi,2r)}\int_0^1 d_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))\Big(|X_{\xi_0}z(\widetilde{\delta}_s(\xi_1))|+|Y_{\xi_0}z(\widetilde{\delta}_s(\xi_1))|\Big)\frac{ds}{s}d\xi_1+$$

$$+\frac{3}{r^5}\int_{B_{\xi_0}(\xi,2r)}\int_0^1 d^3_{\xi_0}(\xi,\widetilde{\delta}_s(\xi_1))|\partial_t z(\widetilde{\delta}_s(\xi_1))|\frac{ds}{s}d\xi_1$$

(with the change of variable $\eta = \widetilde{\delta}_s(\xi_1)$)

$$\leq \frac{1}{r^5}\int_0^1\int_{B_{\xi_0}(\xi,2rs)} d_{\xi_0}(\xi,\eta)\Big(|X_{\xi_0}z(\eta)|+|Y_{\xi_0}z(\eta)|\Big)d\eta\frac{ds}{s^6}+$$

$$+\frac{3}{r^5}\int_0^1\int_{B_{\xi_0}(\xi,2rs)} d^3_{\xi_0}(\xi,\eta)|\partial_t z(\eta)|d\eta\frac{ds}{s^6} =$$

(by Coarea formula)

$$\leq \frac{1}{r^5}\int_0^1\int_0^{2rs}\int_{d_{\xi_0}(\xi,\eta)=\rho} d_{\xi_0}(\xi,\eta)\Big(|X_{\xi_0}z(\eta)|+|Y_{\xi_0}z(\eta)|\Big)\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}d\rho\frac{ds}{s^6}+$$

$$+\frac{3}{r^5}\int_0^1\int_0^{2rs}\int_{d_{\xi_0}(\xi,\eta)=\rho} d^3_{\xi_0}(\xi,\eta)|\partial_t z(\eta)|\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}d\rho\frac{ds}{s^6} =$$

(exchanging the first two integrals)

$$= \frac{1}{r^5}\int_0^{2r}\int_{\frac{\rho}{2r}}^1\int_{d_{\xi_0}(\xi,\eta)=\rho}(\cdots)\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\frac{ds}{s^6}d\rho =$$

(since $d_{\xi_0}(\xi,\eta)=\rho$)

$$= \frac{1}{r^5}\int_0^{2r}\Big(\int_{\frac{\rho}{2r}}^1\Big(\int_{d_{\xi_0}(\xi,\eta)=\rho}\frac{|X_{\xi_0}z(\eta)|+|Y_{\xi_0}z(\eta)|}{d^4_{\xi_0}(\xi,\eta)}\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\Big)\frac{ds}{s^6}\Big)\rho^5 d\rho+$$

$$+\frac{3}{r^5}\int_0^{2r}\Big(\int_{\frac{\rho}{2r}}^1\Big(\int_{d_{\xi_0}(\xi,\eta)=\rho}\frac{|\partial_t z(\eta)|}{d^2_{\xi_0}(\xi,\eta)}\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\Big)\frac{ds}{s^6}\Big)\rho^5 d\rho$$

(by the change of variable $\tau=\frac{\rho}{s}$)

$$=\frac{1}{r^5}\int_0^{2r}\Big(\int_r^{2r}\Big(\int_{d_{\xi_0}(\xi,\eta)=\rho}\frac{|X_{\xi_0}z(\eta)|+|Y_{\xi_0}z(\eta)|}{d^4_{\xi_0}(\xi,\eta)}\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\Big)\tau^4 d\tau\Big)d\rho+$$

$$+\frac{3}{r^5}\int_0^{2r}\Big(\int_r^{2r}\Big(\int_{d_{\xi_0}(\xi,\eta)=\rho}\frac{|\partial_t z(\eta)|}{d^2_{\xi_0}(\xi,\eta)}\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\Big)\tau^4 d\tau\Big)d\rho$$

(if we compute the second integral)

$$=\widetilde{C}\int_0^{2r}\Big(\int_{d_{\xi_0}(\xi,\eta)=\rho}\frac{|X_{\xi_0}z(\eta)|+|Y_{\xi_0}z(\eta)|}{d^4_{\xi_0}(\xi,\eta)}\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\Big)d\rho+$$

$$+\widetilde{C}\int_0^{2r}\Big(\int_{d_{\xi_0}(\xi,\eta)=\rho}\frac{|\partial_t z(\eta)|}{d^2_{\xi_0}(\xi,\eta)}\frac{dH_{n-1}(\eta)}{|\nabla d_{\xi_0}(\xi,\eta)|}\Big)d\rho$$

(by Coarea formula)

$$=\widetilde{C}\int_{B_{\xi_0}(\xi,2r)}d_{\xi_0}(\xi,\eta)^{-4}\big(|X_{\xi_0}z(\eta)|+|Y_{\xi_0}z(\eta)|\big)d\eta+\widetilde{C}\int_{B_{\xi_0}(\xi,2r)}d_{\xi_0}(\xi,\eta)^{-2}|\partial_t z(\eta)|d\eta.$$

Now the thesis immediately follows, since $X-X_{\xi_0}=(a-P^1_{\xi_0}a)\partial_t$ and $Y-Y_{\xi_0}=(b-P^1_{\xi_0}b)\partial_t$.

From the previous result it follows that

Corollary 4.1. *Let $z$ be a continuous function in $\Omega$, belonging to $W^{1,p}_{loc}(\Omega)$. If $B_{\xi_0}(\xi_0,4R)\subset\subset\Omega$, and $\xi\in B_{\xi_0}(\xi_0,R)$, then*

$$|z(\xi)-z(\xi_0)|\le\widetilde{C}\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))}d_{\xi_0}^{-N+1}(\xi,\zeta)\big(|Xz(\zeta)|+|Yz(\zeta)|\big)+ \tag{43}$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))}d_{\xi_0}^{-N+1}(\xi,\zeta)\Big(|a(\zeta)-P^1_{\xi_0}a(\zeta)|+|b(\zeta)-P^1_{\xi_0}b(\zeta)|\Big)||\partial_t z(\zeta)|$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))}d_{\xi_0}^{-N+3}(\xi,\zeta)|\partial_t z(\zeta)|,$$

*where $\widetilde{C}$ is a dimensional constant.*

We can now show that, if $z$ is sufficiently regular, the first order polynomial $P^1_{\xi_0} z$ provides a first order approximation of the function $z$, with respect to the distance $d_{\xi_0}$.

LEMMA 4.1. *Assume that $a$ and $b$ satisfy*

$$a(\xi) = P^1_{\xi_0} a(\xi) + O\big(d^{\beta'}_{\xi_0}(\xi, \xi_0)\big), \quad b(\xi) = P^1_{\xi_0} b(\xi) + O\big(d^{\beta'}_{\xi_0}(\xi, \xi_0)\big), \tag{44}$$

*as $\xi \to \xi_0$, with $\beta' \le 2$. Let $z$ be a function such that*

$$Xz, Yz \in C^{\alpha}_{loc}(\Omega), \;\; \partial_t z \in L^q_{loc}(\Omega) \;\; with \;\; q > N/\beta'$$

*then for every ball $B_{\xi_0}(\xi_0, 4R)$ contained in $\Omega$, there exists a constant $\widetilde{C}$ depending on $||Xz||_{C^{\alpha}(B_{\xi_0}(\xi_0, 2R))}$, $||Yz||_{C^{\alpha}(B_{\xi_0}(\xi_0, 2R))}$, $||\partial_t z||_{L^q(B_{\xi_0}(\xi_0, 2R))}$ such that*

$$|z(\xi) - P^1_{\xi_0} z(\xi)| \le \widetilde{C} d^{1+\alpha'}_{\xi_0}(\xi, \xi_0),$$

*for every $\xi, \xi_0$ in $B_{\xi_0}(\xi_0, R)$, where $\alpha' = min(\alpha, \beta' - N/q)$.*

**Proof** We apply inequality (43) to the function

$$z_1(\xi) = z(\xi) - P^1_{\xi_0} z(\xi).$$

Setting $r = d_{\xi_0}(\xi, \xi_0)$, we get

$$|z_1(\xi)| \le \widetilde{C} \int_{B_{\xi_0}(\xi, 2r)} d^{-N+1}_{\xi_0}(\xi, \zeta) \Big( |Xz(\zeta) - Xz(\xi_0)| + |Yz(\zeta) - Yz(\xi_0)| \Big) + \tag{45}$$

$$+\widetilde{C} \int_{B_{\xi_0}(\xi, 2r)} d^{-N+1}_{\xi_0}(\xi, \zeta) \Big( |a(\zeta) - P^1_{\xi_0} a(\zeta)| + |b(\zeta) - P^1_{\xi_0} b(\zeta)| \Big) |\partial_t z(\zeta)|$$

$$+\widetilde{C} \int_{B_{\xi_0}(\xi, 2r)} d^{-N+3}_{\xi_0}(\xi, \zeta) |\partial_t z(\zeta)| \le$$

since $Xz, Yz \in C^{\alpha}$, and by the assumption (44) on $a$ and $b$,

$$\le \widetilde{C} \int_{B_{\xi_0}(\xi, 2r)} d^{-N+1}_{\xi_0}(\xi, \zeta) d^{\alpha}_{\xi_0}(\xi_0, \zeta) +$$

$$+\widetilde{C} \int_{B_{\xi_0}(\xi, 2r)} d^{-N+1}_{\xi_0}(\xi, \zeta) d^{\beta'}_{\xi_0}(\xi_0, \zeta) |\partial_t z(\zeta)| + \widetilde{C} \int_{B_{\xi_0}(\xi, 2r)} d^{-N+3}_{\xi_0}(\xi, \zeta) |\partial_t z(\zeta)| \le$$

(since $d_{\xi_0}(\xi_0,\zeta) \le d_{\xi_0}(\xi,\zeta) + d_{\xi_0}(\xi,\xi_0) \le d_{\xi_0}(\xi,\zeta) + r$, and $\beta \le 2$)

$$\le \widetilde{C} \int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+1}(\xi,\zeta)\big(d_{\xi_0}^{\alpha}(\xi,\zeta) + r^{\alpha}\big)+$$

$$+\widetilde{C} \int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+1}(\xi,\zeta)\big(d_{\xi_0}^{\beta'}(\xi,\zeta) + r^{\beta'}\big)|\partial_t z(\zeta)| \le$$

(if we denote $q'$ the conjugate exponent of $q$)

$$\widetilde{C}\int_0^{2r}(\rho^{\alpha}+r^{\alpha})d\rho+\widetilde{C}\Big(\int_0^{2r}\rho^{(-N+1)q'+N-1}(\rho^{\beta'}+r^{\beta'})d\rho\Big)^{1/q'}||\partial_t z||_{L^q(B_{\xi_0}(\xi_0,R))} \le$$

$$\le \widetilde{C}(r^{1+\alpha} + r^{1+\beta'-N/q}||\partial_t z||_{L^q(B_{\xi_0}(\xi_0,R))}) \le \widetilde{C} d_{\xi_0}^{1+\alpha'}(\xi,\xi_0),$$

where $\alpha' = min(\alpha, \beta' - N/q)$, by the choice of $r$.

Lemma 4.2. *Assume that $a$ and $b$ satisfy (36) and (37), then*

$$a(\xi) = P^1_{\xi_0}a(\xi) + O\big(d_{\xi_0}^{1+\beta}(\xi,\xi_0)\big), \quad b(\xi) = P^1_{\xi_0}b(\xi) + O\big(d_{\xi_0}^{1+\beta}(\xi,\xi_0)\big),$$

*as $\xi \to \xi_0$.*

**Proof** The hypotheses on $a$ and $b$ ensure that condition (44) is satisfied. Indeed

$$|a(\xi) - P_{\xi_0}a(\xi)| \le$$

$$\le |a(\xi) - a(\xi_0)| + |x - x_0||Xa(\xi_0)| + |y - y_0||Ya(\xi_0)| \le$$

(if $d$ is the euclidean distance, by the hypothesis on $a$)

$$\le Cd^{\beta}(\xi,\xi_0) \le Cd_{\xi_0}^{\beta}(\xi,\xi_0).$$

Then, by the preceding lemma

$$a(\xi) = P^1_{\xi_0}a(\xi) + O\big(d_{\xi_0}^{1+\alpha'}(\xi,\xi_0)\big) \tag{46}$$

with $\alpha' = \beta - N/s$. Applying again the same lemma, we have that (46) holds with $\alpha' = min(\beta, \beta + 1 - N/s) = \beta$.

Finally in the following proposition we prove a Sobolev type embedding theorem, which will be used in the analytic estimate below.

PROPOSITION 4.2. *Assume that $a$ and $b$ satisfy (36) and (37). Let $z$ be a function such that*

$$Xz, Yz \in L^p_{loc}(\Omega), \ \ \partial_t z \in L^q_{loc}(\Omega) \ \ with \ \ 1 > N/p, 2+\beta > N/q$$

*then for every ball $B_{\xi_0}(\xi_0, R)$ contained in $\Omega$ there exists a constant $\widetilde{C}$ such that*

$$||z||_{L^\infty_{loc}(B_{\xi_0}(\xi_0,R))} \leq \widetilde{C}\big(||z||_{L^1(B_{\xi_0}(\xi_0,R))}+$$

$$+||Xz||_{L^p(B_{\xi_0}(\xi_0,R))}+||Yz||_{L^p(B_{\xi_0}(\xi_0,R))}+||\partial_t z||_{L^q(B_{\xi_0}(\xi_0,R))}\big).$$

**Proof** Applying Remark 4.1 we get

$$|z(\xi)| \leq \widetilde{C}\int_{B_{\xi_0}(\xi,2r)} |z(\zeta)| + \widetilde{C}\int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+1}(\xi,\zeta)\big(|Xz(\zeta)| + |Yz(\zeta)|\big)+$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+1}(\xi,\zeta)\Big(|a(\zeta) - P^1_{\xi_0}a(\xi)| + |b(\zeta) - P^1_{\xi_0}b(\xi)|\Big)|\partial_t z(\zeta)|$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+1}(\xi,\zeta)|\partial_t z(\zeta)| \leq$$

(if $p'$ is the conjugate exponent of $p$)

$$\leq \widetilde{C}||z||_{L^1(B_{\xi_0}(\xi_0,R))}+\widetilde{C}\big(||Xz||_{L^p(B_{\xi_0}(\xi_0,R))}+||Yz||_{L^p(B_{\xi_0}(\xi_0,R))}\big)\Big(\int_0^{2r}\rho^{(-N+1)p'+N-1}\Big)^{1/p'}+$$

$$+\widetilde{C}\int_{B_{\xi_0}(\xi,2r)}\big(d_{\xi_0}^{-N+1}(\xi,\zeta)d_{\xi_0}^{-N+1}(\xi_0,\zeta) + d_{\xi_0}^{-N+3}(\xi,\zeta)\big)|\partial_t z(\zeta)|d\zeta \leq$$

$$\leq \widetilde{C}\big(||z||_{L^1(B_{\xi_0}(\xi_0,R))}+||Xz||_{L^p(B_{\xi_0}(\xi_0,R))}+||Yz||_{L^p(B_{\xi_0}(\xi_0,R))}+||\partial_t z||_{L^q(B_{\xi_0}(\xi_0,R))}\big).$$

## 5. ANALYTIC ESTIMATE AND FOLIATION

In this section we will always denote by $u$ a strong viscosity solution to (1) and prove our main result.

Let us start with an analytic estimate of the intrinsic derivatives:

Proposition 5.1. *If $B(R) \subset\subset \Omega$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that for every $k \in N$ and for every multi-index of length $k$*

$$||D_0^I z||_{L^\infty(B(R))} \leq C_1 k! C_2^k.$$

**Proof** This result immediately follows from Propositions 3.5 and 4.2.

### 5.1. Uniqueness of the Cauchy problem

We are now ready to study the uniqueness of Cauchy problem (11) and to conclude the proofs of Theorem 1.1 and Corollary 1.1.

Let us remark that by (9) and (35) the coefficients $a$ and $b$ of the vector fields are $C^\alpha_{loc}(\Omega)$ for every $\alpha < 1$, then the Cauchy problem (11) could have more than one solution. Let us study its properties:

*Remark 5. 1.* If $\gamma$ and $g$ are two solutions of

$$\begin{cases} \gamma' = X(\gamma) \\ \gamma(0) = \xi_0 \end{cases}, \tag{47}$$

then for every $\beta > 1$ there exists a positive constant $c$ such that

$$d(\gamma(s), g(s)) < cs^\beta, \tag{48}$$

where $d$ the euclidean distance.

**Proof** By definition

$$\gamma(s) = (\gamma_1, \gamma_2, \gamma_3)(s) = (x_0 + s, y_0, t_0 + \int_0^s a(\gamma(\tau))d\tau), \tag{49}$$

$$g(s) = (g_1, g_2, g_3)(s) = (x_0 + s, y_0, t_0 + \int_0^s a(g(\tau))d\tau),$$

so that $\gamma_1(s) = g_1(s)$, $\gamma_2(s) = g_2(s)$, and for all $s > 0$

$$d(\gamma(s), g(s)) = |\gamma_3(s) - g_3(s)| = |\int_0^s (a(\gamma(\tau)) - a(g(\tau)))d\tau| \tag{50}$$

$$\leq \int_0^s |a(\gamma(\tau)) - a(g(\tau))| d\tau$$

since $a, b \in C^\alpha_{loc}(\Omega)$

$$\leq c \int_0^s d^\alpha(\gamma(\tau), g(\tau))d\tau \leq c \int_0^s |\gamma_3(\tau) - g_3(\tau)|^\alpha d\tau.$$

If we call

$$G(s) := |\gamma_3(\tau) - g_3(\tau)|, \quad H(s) := \int_0^s G^\alpha(\tau)d\tau,$$

by (50) we have

$$G(s) \le c\int_0^s G^\alpha(\tau)d\tau = cH(s).$$

Hence

$$H'(s) = G^\alpha(s) \le c^\alpha H^\alpha(s)$$

so that

$$\Big(\frac{H^{1-\alpha}}{1-\alpha} - Cs\Big)' \le 0, \quad \text{and} \quad H(0) = 0.$$

Then we get

$$H^{1-\alpha}(s) \le (1-\alpha)Cs,$$

and

$$G(s) \le cH(s) \le c_\alpha s^{1/1-\alpha}.$$

Since $\alpha$ is arbitrary in $]0,1[$, the thesis follows.

*Remark 5. 2.* Let $\gamma$ be a solution of (47). For every $\beta < 1$ there exists $C_\beta > 0$ such that

$$d_{\xi_0}(\gamma(s), \xi_0) \le C_\beta s^\beta \tag{51}$$

where the distance $d_{\xi_0}$ is defined by (40).

**Proof** Let $\beta < 1$ be fixed. By (40) we have

$$d_{\xi_0}(\gamma(s), \xi_0) = ||\phi_{\xi_0}^{-1}(\gamma(s))|| =$$

(by (49) and (41))

$$= ||\phi_{\xi_0}^{-1}(x_0 + s, y_0, \gamma_3(s))|| = ||\big(s, 0, \gamma_3(s) - t_0 - s\big(a(\xi_0) + \frac{s}{2}Xa(\xi_0)\big)|| =$$

(by (39))

$$= (s^6 + V^2(s))^{1/6},$$

where we have denoted

$$V(s) = \gamma_3(s) - t_0 - s\big(a(\xi_0) + \frac{s}{2} X a(\xi_0)\big).$$

Since the application $s \to d_{\xi_0}(\gamma(s), \xi_0)$ is a continuous function, there exists a value $\bar{s} \in [0, s]$ such that

$$d_{\xi_0}(\gamma(\bar{s}), \xi_0) = \max_{\tau \in [0,s]} d_{\xi_0}(\gamma(\tau), \xi_0). \tag{52}$$

We now estimate

$$V(\bar{s}) = \gamma_3(\bar{s}) - t_0 - \bar{s}\big(a(\xi_0) + X a(\xi_0)\frac{\bar{s}}{2}\big)$$

(since $\gamma_3' = a(\gamma)$)

$$= \int_0^{\bar{s}} \big(a(\gamma(\tau)) - a(\xi_0) - X a(\xi_0)\tau\big) d\tau = \int_0^{\bar{s}} \big(a(\gamma(\tau)) - P^1_{\xi_0} a(\gamma(\tau))\big) d\tau$$

(if $\alpha' = 2 - 1/\beta$, using Lemma 4.2)

$$\leq c \int_0^{\bar{s}} d^{1+\alpha'}_{\xi_0}(\gamma(\tau), \xi_0) d\tau \leq c\bar{s} \max_{[0,\bar{s}]} d^{1+\alpha'}_{\xi_0}(\gamma(\tau), \xi_0) = c\bar{s} d^{1+\alpha'}_{\xi_0}(\gamma(\bar{s}), \xi_0).$$

Hence, for all $\epsilon \in ]0, 1[$ there exists $c_\varepsilon$ such that

$$d^3_{\xi_0}(\gamma(\bar{s}), \xi_0) \leq c\Big(\bar{s}^3 + |V((\bar{s})|\Big) \leq c_\varepsilon\Big(\bar{s}^3 + \bar{s}^{\frac{3}{2-\alpha'}}\Big) + \epsilon d^3_{\xi_0}(\gamma(\bar{s}), \xi_0)$$

and, if we choose $\varepsilon$ sufficiently small,

$$d_{\xi_0}(\gamma(\bar{s}), \xi_0) \leq c\bar{s}^{\frac{1}{2-\alpha'}}.$$

In particular

$$d_{\xi_0}(\gamma(s), \xi_0)) \leq cs^{\frac{1}{2-\alpha'}} = cs^\beta.$$

We can now give a new pointwise definition of derivative in the direction of vector fields $X$, $Y$.

Definition 5.1. Let $\xi_0 = (x_0, y_0, t_0) \in \Omega$ and $\gamma$ be a solution to problem (47).

We say that a function $f \in C^\alpha_{loc}(\Omega)$, with $\alpha \in ]0, 1[$, has Lie-derivative in the direction of the vector field $X$ in $\xi_0$ if there exists

$$\frac{d}{dh}(f \circ \gamma)_{|h=0},$$

and we will denote its value by $Xf(\xi_0)$. The Lie-derivative with respect to $Y$ is defined in the same matter.

This definition is well-posed. Indeed, if $f$ is $\alpha-$hölder continuous and $\gamma$ and $g$ are two different solutions of (47), we can choose $\beta > 1/\alpha$ such that (48) is satisfied. Then

$$\frac{f(\gamma(s)) - f(\gamma(0))}{s} = \frac{f(g(s)) - f(g(0))}{s} + \frac{f(\gamma(s)) - f(g(s))}{s}$$

where

$$\Big|\frac{f(\gamma(s)) - f(g(s))}{s}\Big| \le c\frac{d^\alpha(\gamma(s), g(s))}{s} \le cs^{\alpha\beta-1} \to 0,$$

as $s \to 0$. It follows that, if the Lie-derivative $Xf(\xi_0)$ exists, its value does not depend on the choice of the integral curve $\gamma$.

If the weak derivative of a function $f$ is sufficiently regular, then the two notions of derivatives coincide.

PROPOSITION 5.2. *If $f \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in ]0,1[$ and its weak derivatives $Xf, Yf \in C^\alpha_{loc}(\Omega), \partial_t f \in L^p_{loc}(\Omega)$ with $p > 1/\alpha$, then for all $\xi \in \Omega$ the Lie-derivatives $Xf(\xi), Yf(\xi)$ exist and coincide with the weak ones.*

**Proof** Let us prove the statement for the Lie derivative with respect to $X$. We can choose $\beta > \frac{1}{1+\alpha}$ such that (51) is satisfied. By Lemma 4.2

$$\frac{f(\gamma(s)) - f(\xi_0)}{s} = Xf(\xi_0) + \frac{O(d^{1+\alpha}_{\xi_0}(\gamma(s), \xi_0))}{s} = \tag{53}$$

$$= Xf(\xi_0) + O(s^{\beta(1+\alpha)-1}).$$

Letting $s \to 0$ we get the thesis.

THEOREM 5.1. *For every $x$, $y$ the Cauchy problem (11) has a unique solution.*

**Proof** Let us first consider the Cauchy problem (47), and let $\gamma$ be a solution. By Proposition 3.4 for every $k$ the function $X^k a$ satisfies the hypotheses of Proposition 5.2. Then

$$(a \circ \gamma)^{(k)}(s) = X^k a(\gamma(s)),$$

and by virtue of the analytic estimate in Proposition 5.1,

$$|(a \circ \gamma)^k(s)| \le k!c^k.$$

It follows that $a \circ \gamma$ is analytic and

$$(a \circ \gamma)(s) = \sum_{k=0}^{\infty} X^{(k)} a(\xi_0) \frac{s^k}{k!}$$

and since the right hand side depends only on $a$, also the solution of

$$\gamma' = (1, 0, a \circ \gamma), \quad \gamma(0) = \xi_0$$

is unique. The same proof also apply to the solutions of the more general problem (11). Indeed for $x$ and $y$ fixed, the vector field $Z = (x-x_0)X+(y-y_0)Y$ has the same summability properties and satisfies the same analytic estimate as $X$.

### 5.2. Proof of the Main result

We are now ready to prove our main result.

**Proof of Theorem 1.1** For all $\xi_0 = (x_0, y_0, t_0) \in \Omega$ there exists a neighborhood $B$ of $(x_0, y_0)$ in $\mathbb{R}^2$ such that for every $(x, y) \in B$ the exponential $\exp((x-x_0)X + (y-y_0)Y)(\xi_0)$ is well defined. We compute the derivatives with respect to the variables $x, y$ of the exponential map as follows

$$\partial_x \exp((x - x_0)X + (y - y_0)Y)(\xi_0) =$$

(since $X$ and $Y$ commute)

$$= \partial_x \exp((x - x_0)X)(\exp((y - y_0)Y)(\xi_0)) =$$

by definition of the exponential mapping

$$= X \exp((x-x_0)X)(\exp(y-y_0)Y)(\xi_0)) = X(\exp((x-x_0)X+(y-y_0)Y)(\xi_0))$$

$$= (1, 0, a(\exp((x - x_0)X + (y - y_0)Y)(\xi_0))).$$

Analogously

$$\partial_y \exp((x - x_0)X + (y - y_0)Y)(\xi_0) = Y(\exp((x - x_0)X + (y - y_0)Y)(\xi_0))$$

$$= (0, 1, b(\exp((x - x_0)X + (y - y_0)Y)(\xi_0))).$$

Because of the particular structure of the vector fields $X$ and $Y$ we have

$$exp((x - x_0)X + (y - y_0)Y)(\xi_0) = (x, y, t_{\xi_0}(x, y)),$$

for a particular function $t_{\xi_0}$, so that

$$\partial_x t_{\xi_0}(x,y)) = a(x,y,t_{\xi_0}(x,y)), \quad \partial_y t_{\xi_0}(x,y)) = b(x,y,t_{\xi_0}(x,y)),$$

and, if we denote $\nabla_{xy} = (\partial_x, \partial_y)$,

$$\nabla^I_{xy}\partial_x t_{\xi_0}(x,y) = D^I_0 a(x,y,t_{\xi_0}(x,y)), \quad \nabla_{xy}\partial_y t_{\xi_0}(x,y)) = D^I_0 b(x,y,t_{\xi_0}(x,y)).$$

Hence, by Proposition 5.1, and (17) the function $t_{\xi_0}(x,y)$ is an analytic function of $(x,y)$ and

$$M_{\xi_0} := \{(x,y,t_{\xi_0}(x,y)) : (x,y) \in B\}$$

is a real analytic two dimensional manifold. Also note that the restriction $u|_{M_{\xi_0}}$ is harmonic with respect to the variables $(x,y)$. Indeed

$$\nabla^I_{xy}(u(x,y,t_{\xi_0}(x,y))) = (D^I_0 u)(x,y,t_{\xi_0}(x,y)),$$

hence by Proposition 3.4

$$\Delta_{(x,y)} u(x,y,t_{\xi_0}(x,y)) = (X^2 u + Y^2 u)(x,y,t_{\xi_0}(x,y)) = 0.$$

Proposition 5.3. *For every multi-index $I$ the function $D^I_0 u(\xi)$ is Lipschitz continuous, with respect to the euclidean distance of $\mathbb{R}^3$.*

**Proof** For the continuous dependence of the Cauchy problem, for every $\xi_0 \in \Omega$ there exist an euclidean ball $B(r)$ in $\mathbb{R}^2$ and a real number $\delta > 0$ such that for every $\tau \in ]t_0 - \delta, t_0 + \delta[$ the function

$$(x,y) \to t_{(x_0,y_0,\tau)}(x,y)$$

is defined on $B(r)$. Moreover

$$V_{\xi_0} = \{(x,y,t_{(x_0,y_0,\tau))}) : (x,y) \in B(r), \tau \in ]t_0 - \delta, \tau_0 + \delta[\}$$

is an open neighborhood of $\xi_0$. For every fixed $\tau \in ]t_0 - \delta, t_0 + \delta[$ the function

$$v_\tau : B(r) \to \mathbb{R} \quad v_\tau(x,y) = v(x,y,t_{(x_0,y_0,\tau)}(x,y))$$

satisfies

$$\Delta_{xy} v_\tau = 0.$$

Hence it is harmonic, and there exists a constant independent of $\tau$ such that

$$sup|\nabla^I_{xy} v_\tau| \le C_I sup|v_\tau| \le \frac{\pi}{2} C_I.$$

It follows that for every $\xi \in V_{\xi_0}$

$$|D^I_0 v(\xi)| \le C,$$

and, by definition of $\omega$,

$$|D^I_0 \omega(\xi)| \le C.$$

By (31), this ensures that $D^I_0 u$ are Lipschitz continuous.

Let us now operate the same identification as in section 1.2: $\mathbb{C}^2 = \mathbb{C} \times \mathbb{R} \times i\mathbb{R}$, and recall the definition of foliation:

DEFINITION 5.2. Let $B$ be the complex unit ball. A real hypersurface $\Gamma$ of $\mathbb{C}^2$ is foliated by holomorphic curves if for any point $\zeta \in \Gamma$ there exists a neighborhood $U$ of $\zeta$ in $\Gamma$ and a function

$$\Phi : B \times ]-1, 1[ \to U,$$

which is a homeomorphism of $B \times ]-1, 1[$ onto the domain $U$, such that $\Phi(., \tau)$ is a complex analytic function on $B$ for every fixed $\tau \in ]-1, 1[$.

**Proof of Corollary 1.1** We identify $B$ with a complex disc and choose $\delta$ small, then the function

$$\Psi : B \times ]t_0 - \delta, t_0 + \delta[ \to \mathbb{R}^3$$

$$\Psi(x, y, \tau) := \exp((x - x_0)X + (y - y_0)Y)(x_0, y_0, \tau)$$

is an homeomorphism of $B \times ]t_0 - \delta, t_0 + \delta[$ in a neighborhood of $\xi_0$ in $\mathbb{R}^3$. $\Psi$ is continuous, and it inverse is

$$\Psi^{-1}(x, y, t) = \big(x, y, t_{(x,y,t)}(2x_0 - x, 2y_0 - y)\big).$$

Besides, there exists a neighborhood $U$ of $(\xi_0, u(\xi_0))$ in $\mathbb{C}^2$ such that the application

$$\Phi : B \times ]t_0 - \delta, t_0 + \delta[ \to \Gamma(u) \cap U \quad \Phi = (\Psi, u \circ \Psi)$$

is a homeomorphism. Moreover, for any fixed $\tau$, the function $\Phi_\tau : B \to \mathbb{C}^2, \Phi_\tau(x, y) := \Phi(x, y, \tau)$, is an holomorphic function. The first component $\Phi_{\tau,1}(x, y) = (x, y)$ is obviously holomorfic. The second component

$$\Phi_{\tau,2}(x, y) = (t_{\xi_0}(x, y), u(x, y, t_{\xi_0}(x, y))$$

satisfies the Cauchy-Riemann equations

$$\partial_x t_{\xi_0}(x,y) = a(x,y,t_{\xi_0}(x,y))$$

(by (9))

$$= (Yu)(x,y,t_{\xi_0}(x,y)) = \partial_y u(x,y,t_{\xi_0}(x,y)),$$

$$\partial_y t_{\xi_0}(x,y) = b(x,y,t_{\xi_0}(x,y))$$

(by (9))

$$= -(Xu)(x,y,t_{\xi_0}(x,y)) = -\partial_x u(x,y,t_{\xi_0}(x,y)).$$

Hence, according with Definition 5.2, the graph of $u$ is foliated by analytic complex curves.

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