Regularity properties of solutions of a class of elliptic-parabolic Levi type equations

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1 Introduction

In this note we study the regularity properties of solutions of a class of parabolic equations, which contains in particular the motion by Levi curvature, and the prescribed Levi curvature equation. We denote Ω an open set in R^3, (x_1, x_2, t) the elements of R^3 and for all $u : \Omega \to R$ we will introduce the vector fields

$$X_u = \partial_{x_1} + a\partial_t, \quad Y_u = \partial_{x_2} + b\partial_t, \tag{1}$$

where

$$a = \frac{u_{x_2} - u_{x_1}u_t}{1+u_t^2} \quad \text{and} \quad b = -\frac{u_{x_1} + u_{x_2}u_t}{1+u_t^2}. \tag{2}$$

Then the Levi operator is defined as

$$\mathcal{L}u = X_u^2 u + Y_u^2 u - (X_u a + Y_u b)\partial_t$$

and, the Levi curvature of the graph of u is

$$curv(u) = \frac{\mathcal{L}u}{(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2}}. \tag{3}$$

Here we study the equation:

$$\mathcal{L}u + h(a,b,\cdot)\partial_s u = k(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2}, \ in \ \Omega\times]-1,1[, \tag{4}$$

where $k : \Omega\times]-1,1[\to \mathbb{R}$, and $h : \mathbb{R}\times\mathbb{R}\times\Omega\times]-1,1[$ are smooth. When the solution is independent of s, then the equation becomes

$$\mathcal{L}u = k(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2} \ \ in \ \Omega, \tag{5}$$

which is, by (3) the equations of functions u whose graph has in any point prescribed curvature k. If $k = 0$ the equation becomes

$$\mathcal{L}u + h(a,b,\cdot)\partial_s u = 0, \ in \ \Omega\times]-1,1[, \tag{6}$$

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which describes the motion of the graph of u by Levi curvature. We refer the reader to [R], for a survey of the geometrical meaning and applications of these equations.

Existence results have been established for the solutions of (5) and (6), with different techniques. The Dirichlet problem for the stationary equation (5) has been studied by Bedford and Gaveau, who proved in [BG] the existence of classical solutions, less regular than the boundary datum, and by Slodkowsky and Tomassini, who proved in [ST1] the existence of a Lipschitz continuous solution under a requirement on the boundary datum much weaker than the preceding one. Analogous results hold for the existence of solutions of the Cauchy problem for the parabolic equation (6): Slodkowsky and Tomassini proved in [ST2] the existence of Lipschitz continuous solutions defined on $[0, \infty[$. This result was improved by Huisken and Klingenberg, who proved in [HK] the existence of a solution with the same regularity of the initial datum.

In all these works the regularity of the solutions is not due to the equation, but is inherited from the regularity of the boundary data. On the other hand, here we study the interior regularity properties of the solutions, using in an essential way the structure of the operator and of the vector fields. A linear operator with the same structure as $\mathcal{L} + h\partial_s$ would be

$$L_L u = X_L^2 u + Y_L^2 u - (X_L a + Y_L b)\partial_t + h\partial_s, \tag{7}$$

for linear vector fields $X_L = \partial_x + a\partial_t$ and $Y_L = \partial_y + b\partial_t$. When the vector fields are smooth, a sufficient condition for the hypoellipticity is the Hörmander condition (see [H]) and, in this case, the regularity of solutions has been studied by Folland [F], Folland and Stein [FS], Rothschild and Stein [RS]. When the vector fields are not smooth, the Hörmander condition is meaningless, since the Lie algebra generated by X_L and Y_L is not defined. On the other hand Franchi and Lanconelli in [FL] pointed out that the condition on the commutators can be replaced by the requirement that every couple of point ξ and $\bar{\xi}$ can be connected by an integral curve of the vector fields. Under this assumption they proved a regularity result for solutions of a class of equations associated to linear non smooth operators defined in terms of suitable vector fields, which however does not contain the one studied here.

A different and more general technique for studying linear operators represented as in (7) was introduced in [C] and [CM1]. They proved a first regularity result for the stationary operator:

If $X_L, Y_L, [X_L, Y_L]$, are linearly independent at every point and their coefficients belong to suitable $C_L^{k,\alpha}(\Omega)$ class of Hölder continuous functions, any solution $u \in C_L^{k-1,\alpha}(\Omega)$ of

$$X_L^2 u + Y_L^2 u - (X_L a + Y_L b)\partial_t u = f \quad \text{in } \Omega$$

is of class $C_L^{k+1,\alpha}(\Omega)$.

This result has been extended to the parabolic case by Montanari in [M], always under the assumption that $X_L, Y_L, [X_L, Y_L]$, are linearly independent at every point. We also refer to [CPP], where a simple presentation of the same method is provided, and it is applied to a parabolic equation, whose associated Lie algebra has step 3.

Due to the low regularity requirements in these kind of theorems, they can be naturally applied to nonlinear equations. It has been proved in [C] that in this case of the Levi equation assumptions on the commutators can be expressed in terms of the curvature of the graph of u:

$$[X_u, Y_u] = -\frac{\mathcal{L}u}{1+u_t^2}\partial_t = -\frac{curv(u)(1+a^2+b^2)^{3/2}}{(1+u_t^2)^{1/2}}\partial_t. \tag{8}$$

Hence, using the linear result, it has been proved in [C]

Theorem *If k never vanishes in Ω, any solution u of (5) of class $C^{2,\alpha}(\Omega)$, with $\alpha > \frac{1}{2}$, is of class $C^\infty(\Omega)$*(see [C]).

The analogous linear parabolic result, applied to the Levi equation holds (see [M]):

Theorem *If $k = 0$, and $curv(u)$ never vanishing in $\Omega\times]-1,1[$, any solution u of (6) of class $C_L^{2,\alpha}(\Omega\times]-1,1[)$ is of class $C^\infty(\Omega\times]-1,1[)$.* (su questo non sono sicura perche' nella copia che mi hai dato non lo trovo. L'affermazione pero' ci consentirebbe di risparmiare una definizione).

The problem of the regularity of classical solutions under general conditions on the commutators was still open. Here we extend our approach to linear parabolic equations in the case that the Lie algebra generated by X_L, Y_L and ∂_s has step 3. Applying these results to the nonlinear equation we prove

Theorem 1.1 *Assume that u is a solution of class $C^{3,\alpha}$, of equation (4) in an open subset $\Omega\times]-1,1[$ of $\mathbb{R}^4$, that $h \in C^\infty(\mathbb{R}\times\mathbb{R}\times\Omega\times]-1,1[)$ and $k \in C^\infty(\Omega\times]-1,1[)$. Assume that the following condition holds at every point $\xi \in \Omega\times]-1,1[$*

$$curv^2(u) + (X_u curv(u))^2 + (Y_u curv(u))^2 \neq 0. \tag{9}$$

Then $u \in C^\infty(\Omega\times]-1,1[)$.

Obviously, by the results in [C] and [M], we have only to study the regularity of the solution in a neighborhood of the points ξ_0, where $curv(u)(\xi_0) = 0$. At these points we can assume that $X_u(curv(u))(\xi_0) \neq 0$. Due to condition (8) we also have

$$[X_u, Y_u]_{|\xi_0} = 0,$$

and

$$[X_u, [X_u, Y_u]]_{|\xi_0} = X_u(curv(u))(\xi_0)\frac{(1+a^2+b^2)^{3/2}}{(1+u_t^2)^{1/2}}\partial_t,$$

so that X_u, Y_u and $[X_u, [X_u, Y_u]]$ are linearly independent at ξ_0, and, by continuity, in a neighborhood of this point.

Let us now state the linear result we need to prove Theorem 1.1. We fix an open set $\Omega \subset \mathbb{R}^3$, and functions

$$a, b, \in C^2(\Omega\times]-1,1[), c \in C^2(\Omega\times]-1,1[) \tag{10}$$

$$h \in C^2(\Omega \times]-1,1[) \;\; such \;\; that \;\; 1 \le h(\xi) \le 2 \; in \; \Omega \times]-1,1[.$$

We introduce, in analogy with (1) and (4) linear vector fields,

$$X_L = \partial_{x_1} + a\partial_t, \quad Y_L = \partial_{x_2} + b\partial_t, \quad S_L = h\partial_{x_3} - (X_L a + Y_L b - c)\partial_t, \quad (11)$$

and a linear operator

$$L_L = X_L^2 + Y_L^2 + S_L.$$

Then we study the regularity properties of solutions of

$$L_L u = g \quad in \;\; \Omega \times]-1,1[\qquad (12)$$

in a neighborhood of ξ_0. We assume that there exists $\xi_0 \in \Omega \times]-1,1[$ such that

$$[X_L, Y_L]_{|\xi_0} = 0, \quad [X_L, [X_L, Y_L]]_{|\xi_0} \ne 0. \qquad (13)$$

We can obviously always assume that $[X_L, [X_L, Y_L]]_{|\xi_0} \ne 0$ in a neighborhood of ξ_0, which we will be assumed to coincide with $\Omega \times]-1,1[$. Finally we assume that

$$X_L a + Y_L b, \;\; X_L b - Y_L a \in C^2(\Omega \times]-1,1[).$$

This last assumption, which seems to be very strong, is on the contrary very natural for our application, since, as we will see, it is satisfied by the nonlinear model.

In the following sections we will introduce suitable classes $C_L^{k,\alpha}$ of Hölder continuous functions and prove that

Theorem 1.2 *Let assume that the coefficients a, b, h in equation (12) are of class $C_L^{k-1,\alpha}(\Omega \times]-1,1[)$, assume that c and the second member g belong to $C_L^{k-2,\alpha}(\Omega \times]-1,1[)$, and that $\partial_t a, \partial_t b \in C_L^{\alpha}(\Omega \times]-1,1[)$. Let $u \in C_L^{k-1,\alpha}(\Omega \times]-1,1[)$, be a solution of (12), such that $\partial_t u \in C_L^{1,\alpha}(\Omega \times]-1,1[)$, $S_L u \in C_L^{1,\alpha}(\Omega \times]-1,1[)$. Then, $u \in C_L^{k,\beta}(\Omega \times]-1,1[)$, for every $\beta < \alpha$.*

The Lie algebra associated to the operator L is not free up to step 3 at every point. Hence we first apply the well known method of Rotschild and Stein, adding some more variables and lifting our vector fields to new vector fields X, Y S, such that the Lie algebra generated by X, Y S is free up to step 3. The associated operator will be

$$L = X^2 + Y^2 + S \;\; in \;\; U \subset \mathbb{R}^8.$$

In Section 2 we describe the properties of this new operator in R^8, and of the associated classes of Hölder continuous functions.

In Section 3 we will use a freezing method and we will associate to every point of the space a model operator $L_{\bar\xi}$, of Hörmander type whose associated Lie algebra is nilpotent and free, but, up to order 3, has the same structure as $\mathcal{G}$. Besides the choice of the new operator will be made in such a way that the distances associated to $L_{\bar\xi}$ and L are equivalent.

In Section 4 the fundamental solution of $L_{\bar{\xi}}$ will be used as a parametrix of the fundamental solution of L. Hence any solution of equation (4) admits an integral representation in terms of it, and suitable kernels $N_{\bar{\xi},i}$:

$$u(\xi) = \sum_{i=1}^{3} \int \Gamma_{\bar{\xi}}(\xi, \zeta) N_{\bar{\xi},i}(\zeta) d\zeta$$

(see Theorem 4.1). In [CPP] the authors provide necessary conditions on the kernels $N_{\bar{\xi},i}$, which ensures the differentiability of the functions u. Similar conditions are satisfied in our contest, and we can prove the regularity result stated in Theorem 1.2.

In Section 5 we conclude the proof of Theorem 1.1, applying these results to the nonlinear operator.

2 The lifting procedure

The behavior of the solutions of equations (12) are based on the structure of the Lie algebra generated by the vector fields X_L, Y_L and S_L. By condition (13) we need commutators of height at most 3 to generate the whole space, but the algebra is not free up to step 3. To overcome this problem we apply the lifting method, introduced by Rothschild and Stein in [RS], adding suitable new variables, and we lift our original vector fields to an extended space $\widetilde{\mathbb{R}^3 \times \mathbb{R}}$. In this way the Lie algebra generated by the new vector fields X, Y and S is free up to weight 3, i.e. X, Y, S and their commutators up to weight 3 satisfy the minimal number of relations at every point, and span the tangent space to $\widetilde{\mathbb{R}^3 \times \mathbb{R}}$.

The choice of new vector fields is not unique. We add the variables $(x_4, \cdots x_7)$ and we order the components of each point in such a way that t is the last variable. Then for any fixed point $\bar{\xi} = (\bar{x}_1, \cdots \bar{x}_7, t)$, we consider the following vectors:

$$\begin{cases} X = X_L - x_2 \partial_{x_4} - \frac{x_2^2}{2} \partial_{x_7} \\ Y = Y_L - x_3 \partial_{x_5} \\ S = S_L + x_1 \partial_{x_6}. \end{cases} \tag{14}$$

Correspondingly we can introduce a lifted operator

$$L = X^2 + Y^2 + S, \tag{15}$$

and consider the equation

$$Lu = g \text{ in } U \subset \Omega \times]-1, 1[\times R^4. \tag{16}$$

Remark 2.1 *Assume that u is a solution of (12) of class $C^3(\Omega \times]-1, 1[)$. We can set $\widetilde{\xi}_0 = (\xi_0, 0) \in U$, then the function u can be trivially extended to a function (always denoted u), solution of (16) in $\Omega \times]-1, 1[\times R^4$ and independent of the new variables $x_3, \cdots, x_7$. Hence in the sequel we will study the more general equation (16).*

In order to simplify notations we will denote

$$D_{(0)} = identity, \quad D_{(1)} = X, \quad D_{(2)} = Y, \quad D_{(3)} = S. \tag{17}$$

Besides for every multi-index $\sigma = (\sigma_1, \ldots, \sigma_m)$, with $\sigma_r \in \{0,1,2,3\}$, we set

$$D_\sigma = D_{\sigma_1} \cdots D_{\sigma_m},$$

and we call weight of σ and of D_σ the natural number

$$|(1)| = |(2)| = 1, \quad |(3)| = 2, \quad |\sigma| = \sum_{r=1}^{m} |\sigma_r|. \tag{18}$$

The Lie derivative of a function f with respect to a vector field D_σ is defined as follows:

Definition 2.1 *Let $U \subset \Omega \times]-1,1[\times R^4$, $f : U \to R$, $\bar{\xi} \in U$ and γ be an integral curve of D_σ, starting at $\bar{\xi}$. If there exists*

$$\frac{d}{dh}(f \circ \gamma)_{|h=0},$$

then we denote it by $D_\sigma f(\bar{\xi})$, and we will call it derivative of f in the direction D_σ at the point $\bar{\xi}$.

A direct computation shows that the Lie algebra generated by (D_σ) with $\sigma \in \{(1),(2),(3)\}$ is free up to weight 3, and a possible choice of a basis is the following one

$$D_{(4)} = [X,Y] = (Xb - Ya)\partial_t + x_2 \partial_{x_7} + \partial_{x_4},$$

$$D_{(5)} = [Y,S] = Yh\partial_{x_3} - Y(Xa + Yb - c)\partial_t - Sb\partial_t + \partial_{x_5},$$

$$D_{(6)} = [X,S] = Xh\partial_{x_3} - X(Xa + Yb - c)\partial_t - Sa\partial_t + \partial_{x_6},$$

$$D_{(7)} = [Y,[X,Y]] = \Big(Y(Xb - Ya) - (Xb - Ya)\partial_t b\Big)\partial_t + \partial_{x_7},$$

$$D_{(8)} = [X,[X,Y]] = \Big(X(Xb - Ya) - (Xb - Ya)\partial_t a\Big)\partial_t.$$

According to Definition (18) we will also set

$$|(4)| = 2, \quad |(i)| = 3, \ for\ i = 5, \cdots, 8.$$

Then we can define the canonical coordinates of any point, and a natural distance associated to these vector fields as follows (see for example [NSW]):

Definition 2.2 *If $\bar{\xi}$ belongs to $\Omega \times]-1,1[\times R^4$, then the map*

$$F_{\bar{\xi}} : e \to exp\Big(\sum_{i=1}^{8} e_i D_{(i)}\Big)(\bar{\xi}) \tag{19}$$

is a diffeomorphism of a neighborhood of the origin of $\mathbb{R}^8$ *to a neighborhood* $U_{\bar{\xi}}$ *of* $\bar{\xi}$ *in* $\mathbb{R}^8$. *Its inverse function* $\Phi_{D,\bar{\xi}}$ *defines the canonical change of variable associated to the vector fields* D_σ *and center* $\bar{\xi}$. *The numbers*

$$e = \Phi_{D,\bar{\xi}}(\xi)$$

are called canonical coordinates of ξ *around* $\bar{\xi}$ *with respect to the vector fields* $D_{(\sigma)}$.

By the local invertibility theorem, the open set U_ξ *on which* $\Phi_{D,\bar{\xi}}$ *is defined continuously depends on* ξ. *Hence in the sequel we will assume that* U *is a fixed open set such that* $U \subset\subset \Omega \times]-1,1[\times \mathbb{R}^4$ *and*

$$\exists \Phi_{D,\bar{\xi}}(\xi) \;\; \forall \xi, \bar{\xi} \in U. \tag{20}$$

If we choose $N = 18$, *we can consider the homogeneous norm on* $\mathbb{R}^8$

$$||e|| = \Big(|e_1|^N + |e_2|^N + |e_3|^{N/2} + |e_4|^{N/2} + \sum_{i=5}^{8} |e_i|^{N/3}\Big)^{1/N}.$$

The natural distance induced by $\Phi_{D,\bar{\xi}}$ *is defined as follows:*

$$\forall \xi, \bar{\xi} \in U \;\; d(\xi, \bar{\xi}) = ||\Phi_{D,\bar{\xi}}(\xi)||.$$

In term of this distance we can define natural Hölder continuous classes as follows:

Definition 2.3 *Let* U *be fixed as in (20). We say that* $f \in C^\alpha_L(U)$ *if there exists* $C > 0$ *such that*

$$|f(\xi) - f(\bar{\xi})| \le C d^\alpha(\xi, \bar{\xi}) \quad \forall \xi, \bar{\xi} \in U.$$

We say that $f \in C^\alpha_i(U)$ *if there exists a positive constant* C *such that*

$$|u(\exp(hD_{(i)})(\bar{\xi})) - u(\bar{\xi})| \le C|h|^\alpha, \tag{21}$$

for every suitably small h, *for every* $\bar{\xi} \in U$.

We say that $u \in C^{1,\alpha}_L(U)$ *if* $Xu, Yu \in C^\alpha_L(U)$ *and* $u \in C_i^{\frac{1+\alpha}{2}}(U)$, *for* $i = 3$.

We say that $u \in C^{2,\alpha}_L(U)$ *if* $Xu, Yu \in C^{1,\alpha}_L(U)$, $Su \in C^\alpha_L(U)$.

Let $k \ge 3$ *and let* $a, b, h \in C^{k-1,\alpha}_L(U)$, $c \in C^{k-2,\alpha}_L(U)$. *We say that* $u \in C^{k,\alpha}_L(U)$ *if* $Xu, Yu \in C^{k-1,\alpha}_L(U)$ *and* $Su \in C^{k-2,\alpha}_L(U)$.

For sake of simplicity, in the sequel, when we write $u \in C^{k,\alpha}_L(U)$, *for* $k \ge 3$, *we will assume implicitly that* $a, b, h \in C^{k-1,\alpha}_L(U)$, $c \in C^{k-2,\alpha}_L(U)$.

Let us recall the following properties of the $C^{k,\alpha}_L(U)$ classes:

Remark 2.2 *If* $u \in C^{1,\alpha}_L(U)$ *then* $u \in C_i^{\frac{1+\alpha}{3}}(U)$ *for every* $i = 5, \cdots, 8$.

If $u \in C^{2,\alpha}_L(U)$ *then* $u \in C_i^{\frac{2+\alpha}{3}}(U)$, *for every* $i = 5, \cdots, 8$.

Remark 2.3 *If $0 < \alpha < 1$ and $k \geq 1$, then*

$$C_L^{k,\alpha}(U) \subseteq C_L^{k-1,\beta}(U), \qquad \forall \beta \in]0,1[. \tag{22}$$

$$C_L^{3k,\alpha}(U) \subseteq C^{k,\frac{\alpha}{3}}(U) \subseteq C_L^{k,\frac{\alpha}{3}}(U), \tag{23}$$

where $C^{k,\alpha}(U)$ is the space of functions with derivatives up to order k that are α-Hölder continuous functions in the Euclidean sense.

A function f belonging to a $C_L^{k,\alpha}(U)$ class, has a natural Taylor development.

Proposition 2.1 *Let U fixed as in (20). Let $f \in C_L^{k,\alpha}(U)$, with $k = 1, 2$, then f has the following Taylor development of order 1 or 2 in the direction of the vector fields:*

$$f(\xi) = \widetilde{P}^k_{\bar{\xi}} f(\xi) + O(d^{k+\alpha}(\xi, \bar{\xi})), \quad k = 1, 2,$$

where

$$\widetilde{P}^1_{\bar{\xi}} f(\xi) = f(\bar{\xi}) + \sum_{i=1}^{2} e_i D_{(i)} f(\bar{\xi}),$$

$$\widetilde{P}^2_{\bar{\xi}} f(\xi) = f(\bar{\xi}) + \sum_{i=1,}^{4} e_i D_{(i)} f(\bar{\xi}) + \frac{1}{2} \sum_{i,j=1}^{2} e_i e_j D_{(i)} D_{(j)} f(\bar{\xi})$$

and $(e_1, ... e_8)$ are the canonical coordinates of ξ around $\bar{\xi}$.

Proof Let us first prove the assertion for $k = 1$. Let ξ, $\bar{\xi}$ be fixed, and let

$$e = \Phi_{D,\bar{\xi}}(\xi) \quad and \quad \xi_2 = exp\Big(\sum_{i=1}^{2} e_i D_{(i)}\Big)(\bar{\xi}).$$

Then

$$\xi = exp\Big(\sum_{i=1}^{8} e_i D_{(i)}\Big)(\bar{\xi}) = exp\Big(\sum_{i=1}^{8} e_i D_{(i)}\Big) exp\Big(-\sum_{i=1}^{2} e_i D_{(i)}\Big)(\xi_2) =$$

$$= exp\Big(\sum_{i=3}^{8} \big(e_i + O(||e||^2)\big) D_{(i)}\Big)(\xi_2). \tag{24}$$

Then, if $f \in C_L^{1,\alpha}(U)$,

$$f(\xi) = f(\xi) - f(\xi_2) + f(\xi_2) - f(\bar{\xi}) + f(\bar{\xi}) =$$

(by the properties of the classes $C_L^{1,\alpha}(U)$ recalled in Remark 2.2, condition (24) and the definition of ξ_2)

$$= O\Big(\sum_{i=3}^{8} \big(|e_i|^{(1+\alpha)/2} + ||e||^{(1+\alpha)}\big)\Big) + f\Big(exp\Big(\sum_{i=1}^{2} e_i D_{(i)}\Big)(\bar{\xi})\Big) - f(\bar{\xi}) + f(\bar{\xi}) =$$

by definition of the distance d and the mean value theorem

$$= f(\bar\xi) + \sum_{i=1}^{2} e_i D_{(i)} f(\widetilde\xi) + O(d^{1+\alpha}(\xi,\bar\xi)) =$$

(where $\widetilde\xi$ satisfies $d(\widetilde\xi,\bar\xi) \le d(\xi_2,\bar\xi) \le d(\xi_2,\xi) + d(\xi,\bar\xi) \le 2d(\xi,\bar\xi)$, and, since $D_{(i)}f$ is Hölder continuous)

$$= P^1_{\bar\xi} f(\xi) + O(d^{1+\alpha}(\xi,\bar\xi)).$$

The proof for $k = 2$ is similar to the preceding one. We choose

$$\xi_3 = exp\Big(\sum_{i=3}^{4} e_i D_{(i)}\Big)(\xi_2)$$

and, as before we get

$$\xi = exp\Big(\sum_{i=1}^{8} e_i D_{(i)}\Big)(\bar\xi) = \tag{25}$$

$$= exp\Big(\sum_{i=1}^{8} e_i D_{(i)}\Big) exp\Big(-\sum_{i=1}^{2} e_i D_{(i)}\Big) exp\Big(-\sum_{i=3}^{4} e_i D_{(i)}\Big)(\xi_3) =$$

$$= exp\Big(\sum_{i=5}^{8} \big(e_i + O(||e||^3)\big) D_{(i)}\Big)(\xi_3).$$

We immediately deduce

$$f(\xi) = f(\xi) - f(\xi_3) + f(\xi_3) - f(\xi_2) + f(\xi_2) - f(\bar\xi) + f(\bar\xi) =$$

(by Remark 2.2 and condition (25))

$$= O\Big(\sum_{i=5}^{8} |e_i|^{(2+\alpha)/3} + ||e||^{2+\alpha}\Big) + f\Big(exp\Big(\sum_{i=3}^{4} e_i D_{(i)}\Big)(\xi_2)\Big) - f(\xi_2)$$

$$+ f\Big(exp(\sum_{i=1}^{2} e_i D_{(i)})(\bar\xi)\Big) - f(\bar\xi) + f(\bar\xi) =$$

(by the mean value theorem and (25))

$$= f(\bar\xi) + \sum_{i=1}^{2} e_i D_{(i)} f(\bar\xi) + \frac{1}{2}\sum_{ij,=1}^{2} e_i e_j D_{(i)} D_{(j)} f(\widetilde\xi_2) +$$

$$+ \sum_{i=3}^{4} e_i D_{(i)} f(\widetilde\xi_3) + O(d^{2+\alpha}(\xi,\bar\xi)) =$$

(since the derivatives of f are Hölder continuous)

$$= \widetilde{P}^2_{\bar{\xi}} f(\xi) + O(d^{2+\alpha}(\xi, \bar{\xi})).$$

Note that $\widetilde{P}^2_{\bar{\xi}} f$ is a polynomial in the canonical coordinates. If we need a polynomial in the euclidean coordinates, we have to approximate the canonical coordinates e_i of a point ξ in a neighborhood of $\bar{\xi}$, with polynomials.

Remark 2.4 *If $f \in C^{2,\alpha}_L(U)$ we have:*

$$f(\xi) = P^2_{\bar{\xi}} f(\xi) + O(d^{2+\alpha}(\xi, \bar{\xi})),$$

where $P^2_{\bar{\xi}} f(\xi)$ is the polynomial in the euclidean coordinates

$$P^2_{\bar{\xi}} f(\xi) = f(\bar{\xi}) + \sum_{i=1, i\neq 3}^{4} e_i D_{(i)} f(\bar{\xi}) + \frac{x_3 - \bar{x}_3}{h(\bar{\xi})} D_{(3)} f(\bar{\xi}) + \frac{1}{2} \sum_{i,j=1}^{2} e_i e_j D_{(i)} D_{(j)} f(\bar{\xi}).$$

Besides

$$|P^2_{\bar{\xi}} f(\xi) - P^2_{\xi_1} f(\xi)| \leq C\Big(d^{2+\alpha}(\bar{\xi}, \xi_1) + d^{\alpha}(\bar{\xi}, \xi_1) d^2(\bar{\xi}, \xi)\Big). \tag{26}$$

If $f \in C^{3,\alpha}_L(U)$, or $f \in C^{2,\alpha}_{euclidean}(U)$ and $\xi = exp(sS)(\bar{\xi})$ then

$$|P^2_{\bar{\xi}} f(\xi) - P^2_{\xi_1} f(\xi)| \leq C s^{(2+\alpha)/2} + C s^{(1+\alpha)/2} d(\bar{\xi}, \xi). \tag{27}$$

Proof We first note that $P^2_{\bar{\xi}} f$ is a polynomial in the euclidean coordinates. Indeed

$$e_1 = x_1 - \bar{x}_1, \quad e_2 = x_2 - \bar{x}_2,$$

$$e_4 = x_4 - \bar{x}_4 + (x_1 - \bar{x}_1)\bar{x}_2 + \frac{(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)}{2}.$$

Besides the first assertion is a consequence of Proposition 2.1, since

$$e_3 = \frac{x_3 - \bar{x}_3}{h(\bar{\xi})} + O(d(\xi, \bar{\xi})^3).$$

We also have for every $i = 1, ...4$:

$$D_{(i)} P^2_{\bar{\xi}} f = P^{2-|i|}_{\bar{\xi}} D_{(i)} f.$$

Hence if we set

$$F(\xi) = P^2_{\bar{\xi}} f(\xi) - P^2_{\xi_1} f(\xi),$$

we get

$$P^2_{\bar{\xi}} f(\xi) - P^2_{\xi_1} f(\xi) = F(\xi) =$$

$$= F(\bar{\xi}) + \sum_{i=1,}^{4} e_i(\xi, \bar{\xi}) D_{(i)} F(\bar{\xi}) + \frac{1}{2} \sum_{i,j=1}^{2} e_i(\xi, \bar{\xi}) e_j(\xi, \bar{\xi}) D_{(i)} D_{(j)} F(\bar{\xi}) =$$

$$= f(\bar\xi) - P^2_{\xi_1} f(\bar\xi) + \sum_{i=1,}^{2} e_i(\xi,\bar\xi)\Big(D_{(i)} f(\bar\xi) - P^1_{\xi_1} D_{(i)} f(\bar\xi)\Big)$$

$$+ \sum_{i=3,}^{4} e_i(\xi,\bar\xi)\Big(D_{(i)} f(\bar\xi) - D_{(i)} f(\xi_1)\Big) +$$

$$+\frac{1}{2}\sum_{i,j=1}^{2} e_i(\xi,\bar\xi) e_j(\xi,\bar\xi)\Big(D_{(i)}D_{(j)} f(\bar\xi) - D_{(i)}D_{(j)} f(\xi_1)\Big) \le$$

$$\le C\Big(d^{2+\alpha}(\bar\xi,\xi_1) + d^\alpha(\bar\xi,\xi_1) d^2(\bar\xi,\xi)\Big),$$

and this proves (26). If $f \in C^{3,\alpha}_L(U)$ or $f \in C^{2,\alpha}_{euclidean}(U)$ and $\xi_1 = exp(sS)(\bar\xi)$, then

$$|D_i f(\bar\xi) - D_i f(\xi_1)| + |D_{i,j} f(\bar\xi) - D_{i,j} f(\xi_1)| \le C s^{(1+\alpha)/2}.$$

Inserting this estimate in the preceding inequality we deduce assertion (27).

3 The freezing procedure

In this section we introduce new vector fields $X_{\bar\xi}$, $Y_{\bar\xi}$, and $S_{\bar\xi}$, whose generated Lie algebra generated is nilpotent, and has the same structure as the Lie algebra generated by X, Y, and S. Since we need to compute commutators of order 3 of X and Y, we define our frozen vector fields substituting to the coefficients a and b, their Taylor expansion of second order, while we only have to compute second order commutators of S, so that we will consider first order development of its coefficient. With this choice of polynomial we define the vector fields as follows:

$$\begin{cases} X_{\bar\xi} = \partial_{x_1} + P^2_{\bar\xi} a\partial_t - x_2\partial_{x_4} - \frac{x_2^2}{2}\partial_{x_7} \\ Y_{\bar\xi} = \partial_{x_2} + P^2_{\bar\xi} b\partial_t - x_3\partial_{x_5} \\ S_{\bar\xi} = P^1_{\bar\xi} h\partial_{x_3} - P^1_{\bar\xi}(Xa + Yb - c)\partial_t + x_1\partial_{x_6}. \end{cases} \tag{28}$$

The frozen operator of the operator L defined in (15) will be defined in terms of the new vector fields $X_{\bar\xi}$, $Y_{\bar\xi}$ and $S_{\bar\xi}$ as follows:

$$L_{\bar\xi} = X^2_{\bar\xi} + Y^2_{\bar\xi} + S_{\bar\xi}. \tag{29}$$

$L_{\bar\xi}$ is C^∞ operator, satisfying the Hörmander condition, hence there exists a control distance $d_{\bar\xi}$ and a fundamental solution $\Gamma_{\bar\xi}$ associated to $L_{\bar\xi}$. Moreover, the homogeneous dimension of the space is $N = 18$ and $\Gamma_{\bar\xi}$ is locally equivalent to $d_{\bar\xi}^{-N+2}$. Let us now study in detail the properties of the distance $d_{\bar\xi}$, and of the fundamental solution.

In order to simplify notations introduce notations similar to (17):

$$D_{\bar\xi,(0)} = identity, \quad D_{\bar\xi,(1)} = X_{\bar\xi}, \quad D_{\bar\xi,(2)} = Y_{\bar\xi}, \quad D_{\bar\xi,(3)} = S_{\bar\xi}.$$

Remark 3.1 *For reader convenience we compute the commutators of the frozen vector fields. From their explicit expression the structure of the generated Lie algebra will be clear.*

$$D_{\bar{\xi},(4)} = [X_{\bar{\xi}}, Y_{\bar{\xi}}] = \Big(P^1_{\bar{\xi}}(Xb) - P^1_{\bar{\xi}}(Ya)\Big)\partial_t + x_2\partial_{x_7} + \partial_{x_4}$$

$$D_{\bar{\xi},(5)} = [Y_{\bar{\xi}}, S_{\bar{\xi}}] = Yh(\bar{\xi})\partial_{x_3} - Y(Xa + Yb - c)(\bar{\xi})\partial_t - Sb(\bar{\xi})\partial_t + \partial_{x_5}$$

$$D_{\bar{\xi},(6)} = [X_{\bar{\xi}}, S_{\bar{\xi}}] = Xh(\bar{\xi})\partial_{x_3} - X(Xa + Yb - c)(\bar{\xi})\partial_t - Sa(\bar{\xi})\partial_t + \partial_{x_6}$$

$$D_{\bar{\xi},(7)} = [Y_{\bar{\xi}}, [X_{\bar{\xi}}, Y_{\bar{\xi}}]] = \Big(Y(Xb - Ya)(\bar{\xi}) - (Xb - Ya)\partial_t b(\bar{\xi})\Big)\partial_t + \partial_{x_7}$$

$$D_{\bar{\xi},(8)} = [X_{\bar{\xi}}, [X_{\bar{\xi}}, Y_{\bar{\xi}}]] = \Big(X(Xb - Ya)(\bar{\xi}) - (Xb - Ya)\partial_t a(\bar{\xi})\Big)\partial_t.$$

From this expression it follows that the Lie algebra generated by $X_{\bar{\xi}}$, $Y_{\bar{\xi}}$ and $S_{\bar{\xi}}$ is a free nilpotent Lie algebra with three generators and of step three. The vectors $(D_{\bar{\xi},(i)})$ are a basis of the vector space underlying the Lie algebra. Hence the exponential mapping

$$F_{\bar{\xi},\xi} : e \longrightarrow \exp(\sum_{i=1}^{8} e_i D_{\bar{\xi},(i)})(\xi) \tag{30}$$

is a global diffeomorphism. Its inverse function, the canonical change of variable associated to the vector fields $D_{\bar{\xi}}$, and center ξ will be denoted

$$\Phi_{\bar{\xi},\xi} \quad or, if \;\; \xi = \bar{\xi}, \;\; \Phi_{\bar{\xi}} = \Phi_{\bar{\xi},\bar{\xi}}. \tag{31}$$

The control distance $d_{\bar{\xi}}$ associated to $D_{\bar{\xi},(i)}$ is equivalent to the quasi distance (see [NSW])

$$d_{\bar{\xi}}(\xi, \bar{\xi}) = ||\Phi_{\bar{\xi}}(\xi)||.$$

Proposition 3.1 *There exist two constants C_1 and C_2 such that*

$$C_1 d(\bar{\xi}, \xi) \le d_{\bar{\xi}}(\bar{\xi}, \xi) \le C_2 d(\bar{\xi}, \xi)$$

for all ξ, $\bar{\xi} \in U$.

Proof Let us call $e = \Phi_{D,\bar{\xi}}(\xi)$ and $\bar{e} = \Phi_{\bar{\xi}}(\xi)$. In order to estimate e and $\bar{e}$, we consider the following two curves

$$\gamma(s) = \exp(s\sum_{i=1}^{8} e_i D_{(i)})(\bar{\xi}) \quad and \quad \bar{\gamma}(s) = \exp(s\sum_{i=1}^{8} \bar{e}_i D_{\bar{\xi},(i)})(\bar{\xi}).$$

Then the first components satisfy the following equation

$$\gamma_1'(s) - \bar{\gamma}_1'(s) = e_1 - \bar{e}_1, \quad \big(\gamma_1 - \bar{\gamma}_1\big)(0) = \big(\gamma_1 - \bar{\gamma}_1\big)(1) = 0.$$

Hence

$$e_1 = \bar{e}_1. \tag{32}$$

Analogously

$$e_2 = \bar{e}_2, \quad and \quad e_4 = \bar{e}_4. \tag{33}$$

Let us consider the third component:

$$\gamma_3'(s) = e_3 h(\gamma) + e_5 Y h(\gamma) + e_6 X h(\gamma)$$

$$\bar{\gamma}_3'(s) = \bar{e}_3 P^1_{\bar{\xi}} h(\bar{\gamma}) + \bar{e}_5 Y h(\bar{\xi}) + \bar{e}_6 X h(\bar{\xi}).$$

Integrating we get

$$|e_3 - \bar{e}_3| \Big| \int_0^1 h(\gamma(s)) ds \Big| \leq$$

$$\leq |\bar{e}_3| \int_0^1 \Big| h(\gamma(s)) - P^1_{\bar{\xi}} h(\bar{\gamma}(s)) \Big| ds + |\bar{e}_5| \int_0^1 \Big| Y h(\gamma(s)) - Y h(\bar{\xi}) \Big| ds +$$

$$+ |\bar{e}_5 - e_5| + |\bar{e}_6| \int_0^1 \Big| X h(\gamma(s)) - X h(\bar{\xi}) \Big| ds + |\bar{e}_6 - e_6|.$$

Since $P^1_{\bar{\xi}} h$ depends on the first two coordinates of its argument, which coincide in γ and $\bar{\gamma}$, $P^1_{\bar{\xi}} h(\gamma(s)) = P^1_{\bar{\xi}} h(\bar{\gamma}(s))$. The function h is of class C^2 in euclidean sense, hence it belongs to $C^{1,\alpha}_L$ and, by Proposition 2.1,

$$|h(\gamma(s)) - P^1_{\bar{\xi}} h(\bar{\gamma}(s))| = |h(\gamma(s)) - P^1_{\bar{\xi}} h(\gamma(s))| \leq C d^{1+\alpha}(\gamma(s), \bar{\xi}) \leq C d^{1+\alpha}(\xi, \bar{\xi}). \tag{34}$$

Analogously, since Xh and Yh are of class C^1, then

$$|X h(\gamma(s)) - X h(\bar{\xi})| + |Y h(\gamma(s)) - Y h(\bar{\xi})| \leq C d(\gamma(s), \bar{\xi}) \leq C d(\xi, \bar{\xi}),$$

for a suitable constant C. Inserting in the previous estimate, and using the fact that $h \geq 1$, we get

$$|e_3 - \bar{e}_3| \leq |e_3 - \bar{e}_3| \Big| \int_0^1 h(\gamma(s)) ds \Big| \leq$$

$$\leq |\bar{e}_3| d^{1+\alpha}(\xi, \bar{\xi}) + \big(|\bar{e}_5| + |\bar{e}_6|\big) d(\xi, \bar{\xi}) + |\bar{e}_5 - e_5| + |\bar{e}_6 - e_6| \leq$$

$$\leq d^2_{\bar{\xi}}(\bar{\xi}, \xi) d^{1+\alpha}(\bar{\xi}, \xi) + d^{2+\alpha}_{\bar{\xi}}(\bar{\xi}, \xi) d(\bar{\xi}, \xi) + |\bar{e}_5 - e_5| + |\bar{e}_6 - e_6|. \tag{35}$$

Let us consider the component number 5:

$$\gamma_5'(s) = -e_2 \gamma_3(s) + e_5 \quad \bar{\gamma}_5'(s) = -\bar{e}_2 \bar{\gamma}_3(s) + \bar{e}_5.$$

Integrating and using the fact that $e_2 = \bar{e}_2$, we get

$$|\bar{e}_5 - e_5| = |e_2| \int_0^1 \Big| \gamma_3(s) - \bar{\gamma}_3(s) \Big| ds \leq |e_2| \int_0^1 \int_0^s \Big| \gamma_3'(\tau) - \bar{\gamma}_3'(\tau) \Big| d\tau ds \leq$$

$$\leq |e_2|\Big(|e_3-\bar e_3|+|\bar e_3|\int_0^1 |h(\gamma(\tau))-P^1_{\bar\xi}h(\bar\gamma(\tau))|d\tau\Big)+|e_2|\Big(|e_5|+|e_6|\Big)+|e_2|\Big(|\bar e_5|+|\bar e_6|\Big) \leq$$

(by (34))

$$\leq |e_2||e_3-\bar e_3| + |e_2||\bar e_3|d^{1+\alpha}(\xi,\bar\xi) + |e_2|\Big(|e_5|+|e_6|+|\bar e_5|+|\bar e_6|\Big) \leq \tag{36}$$

$$\leq |e_2||e_3-\bar e_3| + d_{\bar\xi}(\xi,\bar\xi)d^{2+\alpha}(\xi,\bar\xi) + d(\xi,\bar\xi)d^{2+\alpha}_{\bar\xi}(\xi,\bar\xi).$$

Analogously

$$|\bar e_6 - e_6| \leq |e_1||e_3-\bar e_3| + d_{\bar\xi}(\xi,\bar\xi)d^{2+\alpha}(\xi,\bar\xi) + d(\xi,\bar\xi)d^{2+\alpha}_{\bar\xi}(\xi,\bar\xi). \tag{37}$$

Inserting this estimate in (35), we get, for e_1, e_2 sufficiently small,

$$|\bar e_3 - e_3| \leq d(\xi,\bar\xi)d^{2+\alpha}_{\bar\xi}(\xi,\bar\xi) + d^{2+\alpha}(\xi,\bar\xi)d_{\bar\xi}(\xi,\bar\xi). \tag{38}$$

We can now consider the last component:

$$|e_8-\bar e_8| \leq e_1\int_0^1 |a(\gamma(s)) - P^2_{\bar\xi}a(\bar\gamma(s))|ds$$

$$+e_2\int_0^1 |b(\gamma(s)) - P^2_{\bar\xi}b(\bar\gamma(s))|ds$$

$$+\bar e_3\int_0^1 |\widetilde c(\gamma(s)) - P^1_{\bar\xi}\widetilde c(\bar\gamma(s))|ds + |e_3-\bar e_3|$$

$$+e_4\int_0^1 |d(\gamma(s)) - P^1_{\bar\xi}d(\bar\gamma(s))|ds+$$

$$+O(|e_5|+|e_6|+|e_7|)+O(|\bar e_5|+|\bar e_6|+|\bar e_7|),$$

where $\widetilde c = -(Xa+Yb-c)$, and $d = Xb - Ya$. Using again the fact that $P^1_{\bar\xi}$ depends only on the first two coordinates, and the properties of the Taylor polynomial)

$$\leq d_{\bar\xi}(\xi,\bar\xi)\int_0^1 d^{2+\alpha}(\gamma(s),\bar\xi)ds + d^2_{\bar\xi}(\xi,\bar\xi)\int_0^1 d^{1+\alpha}(\gamma(s),\bar\xi)ds+$$

$$+O(|e_5|+|e_6|+|e_7|)+O(|\bar e_5|+|\bar e_6|+|\bar e_7|)+|e_3-\bar e_3|.$$

Then

$$|\bar e_8 - e_8| \leq d(\xi,\bar\xi)d^2_{\bar\xi}(\xi,\bar\xi) + d^2(\xi,\bar\xi)d_{\bar\xi}(\xi,\bar\xi). \tag{39}$$

$$+O(|e_5|+|e_6|+|e_7|)+O(|\bar e_5|+|\bar e_6|+|\bar e_7|)+|e_3-\bar e_3|.$$

Finally

$$\bar e_7 = e_7. \tag{40}$$

From the estimates (32), (33), (40), and the definition of the distance, we get

$$d(\xi,\bar{\xi}) \le d_{\bar{\xi}}(\xi,\bar{\xi}) + |e_3-\bar{e}_3|^{1/2} + |e_5-\bar{e}_5|^{1/3} + |e_6-\bar{e}_6|^{1/3} + |e_8-\bar{e}_8|^{1/3} \le$$

(by (38), (36), (37), (39))

$$\le d_{\bar{\xi}}(\xi,\bar{\xi}) + d_{\bar{\xi}}^{1/3}(\xi,\bar{\xi})d^{2/3}(\xi,\bar{\xi}) + d^{1/3}(\xi,\bar{\xi})d_{\bar{\xi}}^{2/3}(\xi,\bar{\xi}).$$

Now the first inequality of the thesis follows. The other inequality follows in the same way from the estimates established.

By construction, the Lie algebra generated by $X_{\bar{\xi}}$, $Y_{\bar{\xi}}$ and $S_{\bar{\xi}}$ has the same structure at every point. Hence the canonical change of variables, changes $X_{\bar{\xi}}$, $Y_{\bar{\xi}}$ and $S_{\bar{\xi}}$in linear vector fields of class C^∞, nilpotent and independent of $\bar{\xi}$. Precisely, we have

Remark 3.2 *We say that a vector field $\bar{D}_{(i)}$ is the image of $D_{\bar{\xi},(i)}$ with respect to the canonical change of variables, defined in (31) if for every function $f \in C^1_L(U)$*

$$\bar{D}_{(i)} f_{\bar{\xi}} = D_{\bar{\xi},(i)} f, \quad where \quad f_{\bar{\xi}} = f \circ \Phi_{\bar{\xi}}^{-1}.$$

If $\Phi_{\bar{\xi},j}$ denotes the j-th component of $\Phi_{\bar{\xi}}$, then the vector $\bar{D}_{(i)}$ can be represented in coordinates as follows:

$$\bar{D}_{(i)} = \sum_{j=1}^{8} D_{\bar{\xi},(i)}(\Phi_{\bar{\xi},j})\frac{\partial}{\partial \bar{e}_j}.$$

Moreover, the function $D_{\bar{\xi},(i)}(\Phi_{\bar{\xi}j})$ is a polynomial $P(\bar{e})$, in the variables $\bar{e} = \Phi_{\bar{\xi}j}$, with coefficients independent of $\bar{\xi}$.

In fact, $\bar{D}_{(i)}$ can be obviously written

$$\bar{D}_{(i)} = \sum_{j=1}^{8} a_{ij}\frac{\partial}{\partial \bar{e}_j} \tag{41}$$

for suitable coefficients a_{ij}. On the other hand, if

$$f = \Phi_{\bar{\xi}}, \quad then \quad f_{\bar{\xi}} = identity.$$

Hence

$$D_{\bar{\xi},(i)}(\Phi_{\bar{\xi},k}) = \bar{D}_{(i)}(\bar{e}_k) = \sum_{j=1}^{8} a_{ij}\frac{\partial \bar{e}_k}{\partial \bar{e}_j} = a_{ik}.$$

Substituting in (41), we get the first assertion. Let us compute $D_{\bar{\xi},(i)}(\Phi_{\bar{\xi}})$. At a point ζ we have

$$D_{\bar{\xi},(i)}\Phi_{\bar{\xi}}(\zeta) = \lim_{s\to 0}\frac{\Phi_{\bar{\xi}}(\exp(sD_{\bar{\xi},(i)})(\zeta)) - \Phi_{\bar{\xi}}(\zeta)}{s}. \tag{42}$$

If we call $\bar{e}$ the canonical coordinates of ζ with respect to $D_{\bar{\xi}}$, the first term in the right hand side of (42) becomes

$$\Phi_{\bar{\xi}}(\exp(sD_{\bar{\xi},(i)})(\zeta)) = \Phi_{\bar{\xi}}\Big(\exp(sD_{\bar{\xi},(i)})(\exp(\bar{e}D_{\bar{\xi}})(\bar{\xi}))\Big) =$$

by Campbell-Hausdorff formula

$$= \Phi_{\bar{\xi}}(\exp(\eta D_{\bar{\xi}})(\bar{\xi})) = \eta(s),$$

for a polynomial $\eta(s)$ in the variables s and e. The derivative in (42) is the coefficient of s in $\eta(s)$, hence it is a polynomial in the components of $\bar{e}$. For reader convenience we compute explicitly these derivatives:

$$D_{\bar{\xi},1}(\bar{e}) = \Big(1, 0, 0, -\frac{\bar{e}_2}{2}, 0, -\frac{\bar{e}_3}{2}, -\frac{\bar{e}_2^2}{12}, -\frac{\bar{e}_4}{2} - \frac{\bar{e}_1\bar{e}_2}{12}\Big),$$

$$D_{\bar{\xi},2}(\bar{e}) = \Big(0, 1, 0, \frac{\bar{e}_1}{2}, -\frac{\bar{e}_3}{2}, 0, -\frac{\bar{e}_4}{2} + \frac{\bar{e}_1\bar{e}_2}{12}, -\frac{\bar{e}_1^2}{12}\Big),$$

$$D_{\bar{\xi},3}(\bar{e}) = \Big(0, 0, 1, 0, \frac{\bar{e}_2}{2}, \frac{\bar{e}_1}{2}, 0, 0\Big), \quad D_{\bar{\xi},4}(\bar{e}) = \Big(0, 0, 0, 1, 0, 0, \frac{\bar{e}_2}{2}, \frac{\bar{e}_1}{2}, \Big),$$

$D_{\bar{\xi},i}(\bar{e}_j) = \delta_{i,j}$, where δ is the Kroneker function.

Because of the previous remark the operator

$$\bar{L} = \bar{X}^2 + \bar{Y}^2 + \bar{S}$$

is a sum of squares of linear and nilpotent vector fields of class C^∞ independent of $\bar{\xi}$. If we denote $\bar{\Gamma}$ its fundamental solution, and $\bar{d}$ its natural distance, then $\bar{\Gamma}$ is locally equivalent to $\bar{d}^{2-N}$. Moreover,

$$\Gamma_{\bar{\xi}}(\xi, \zeta) = \frac{1}{h(\bar{\xi})(Xb - Ya)(\bar{\xi})}\bar{\Gamma}(\Phi_{\bar{\xi}}(\xi), \Phi_{\bar{\xi}}(\zeta))$$

and

$$d_{\bar{\xi}}(\xi, \zeta) = \bar{d}(\Phi_{\bar{\xi}}(\xi), \Phi_{\bar{\xi}}(\zeta)). \tag{43}$$

Let us now study the dependence of the canonical coordinates $\Phi_{\bar{\xi}}(\zeta)$ on the variable $\bar{\xi}$, and as a consequence we will deduce the dependence of the fundamental solution $\Gamma_{\bar{\xi}}(\bar{\xi}, \zeta)$ on the same variable.

Lemma 3.1 *Let U be defined as in (20), and let $\xi, \bar{\xi}$ be fixed in U. Then there exists a constant C_3 independent of ξ and $\bar{\xi}$ such that for every $\zeta \in U$*

$$\bar{d}(\Phi_{\bar{\xi}}(\zeta), \Phi_{\xi_1}(\zeta)) \le C_3\Big(d_{\bar{\xi}}(\bar{\xi}, \xi) + d_{\bar{\xi}}^{\alpha/3}(\bar{\xi}, \xi) d_{\bar{\xi}}^{1-\alpha/3}(\bar{\xi}, \zeta)\Big).$$

Precisely, if w denotes the canonical coordinate of $\Phi_{\bar{\xi}}(\zeta)$ around $\Phi_{\xi_1}(\zeta)$ with respect to the vector fields $\bar{D}$,

$$|w_i| \le C_3\Big(d_{\bar{\xi}}^{|(i)|}(\bar{\xi}, \xi) + d_{\bar{\xi}}^{\alpha}(\bar{\xi}, \xi) d_{\bar{\xi}}^{|(i)|-\alpha}(\bar{\xi}, \zeta)\Big),$$

for every $i = 1, \cdots, 8$. *Moreover if* $\xi = exp(sS)(\bar{\xi})$, *then for every* i *such that* $|(i)| \leq 1$, $w_i = 0$. *For* $|(i)| > 1$

$$|w_i| \leq C_3 \Big(s d^{|(i)|-2}(\zeta, \xi) + s^{(1+\alpha)/2} d^2(\zeta, \xi) \Big).$$

Proof By definition of canonical coordinates w can be expressed as follows

$$w = \Phi_{\bar{\xi}, \zeta} \Big(\Phi_{\bar{\xi}}^{-1} \Phi_{\xi}(\zeta) \Big).$$

The distances are expressed in terms of canonical coordinates, if we denote

$$\eta = \Phi_{\xi}(\zeta), \quad \theta = \Phi_{\bar{\xi}}(\xi),$$

we have to estimate w in terms of η and θ. Since $\Phi_{\bar{\xi}, \zeta}$ is the inverse mapping of the exponential, we have

$$\exp(w D_{\bar{\xi}})(\zeta) = \Phi_{\bar{\xi}}^{-1}(\Phi_{\xi}(\zeta))$$

(by definition of η)

$$= \Phi_{\bar{\xi}}^{-1}(\eta) = \exp(\eta D_{\bar{\xi}})(\bar{\xi}) =$$

(since $\bar{\xi} = \exp(-\theta D_{\bar{\xi}})(\xi)$)

$$= \exp(\eta D_{\bar{\xi}}) \exp(-\theta D_{\bar{\xi}})(\xi) =$$

$$= \exp(\eta D_{\bar{\xi}})(\exp(-\theta D_{\bar{\xi}}) \exp(-\eta D_{\xi})(\zeta) =$$

$$= \exp(\eta D_{\bar{\xi}})(\exp(-\theta D_{\bar{\xi}}) \exp(-\eta D_{\bar{\xi}} - \eta(D_{\xi} - D_{\bar{\xi}}))(\zeta).$$

If we denote f_i and g_i the coefficients of ∂_t and ∂_{x_3} in the vector fields $D_{\bar{\xi},i}$, we get

$$\eta(D_{\xi} - D_{\bar{\xi}}) = \sum_{i=1}^{8} \eta_i \Big(P_{\xi}^{3-|(i)|} f_i(\zeta) - P_{\bar{\xi}}^{3-|(i)|} f_i(\zeta) \Big) \partial_t +$$

$$+ \sum_{i=3}^{8} \frac{\eta_i}{h} \Big(P_{\xi}^{3-|(i)|} g_i(\zeta) - P_{\bar{\xi}}^{3-|(i)|} g_i(\zeta) \Big) \Big(D_{\bar{\xi},(3)} - \frac{f_3}{f_8} D_{\bar{\xi},(8)} \Big) =$$

(by Remark 2.4)

$$= \sum_{i=1}^{8} \eta_i O\big(||\theta||^{\alpha} \; (||\eta||^{3-|(i)|} + ||\theta||^{3-|(i)|}) \big) (D_{\bar{\xi},(8)} + D_{\bar{\xi},(3)}) =$$

$$= O\big(||\eta|| ||\theta||^{\alpha+2} \; + ||\theta||^{\alpha} ||\eta||^3 \big) (D_{\bar{\xi},(8)} + D_{\bar{\xi},(3)}).$$

Hence, if $*$ denotes the operation induced by the Campbell-Hausdorff formula, we get

$$\exp(w D_{\bar{\xi}})(\zeta) = \exp \Big(\eta D_{\bar{\xi}} * (-\theta D_{\bar{\xi}}) * (-\eta D_{\bar{\xi}} + O\big(||\theta||^{\alpha+3} + ||\theta||^{\alpha} ||\eta||^3 \big) (D_{\bar{\xi},(8)} + D_{\bar{\xi},(3)}) \Big)(\zeta) =$$

$$= \exp\Big(-\theta D_{\bar\xi} + (\eta_1\theta_2 - \theta_1\eta_2) D_{\bar\xi,(4)} + \sum_{i=3}^{8} O\big(||\theta||^2||\eta|| + ||\theta||^\alpha ||\eta||^{3-\alpha}\big) D_{\bar\xi,(i)}\Big).$$

Since the canonical coordinates are unique, this implies that

$$w_i = \theta_i \quad for \quad |i| = 1,$$

$$|w_i| = |\theta_i| + O(||\theta||^{|i|-1}||\eta|| + ||\theta||^\alpha ||\eta||^{|i|-\alpha}) \quad for \quad |i| > 1.$$

Writing this same assertion in terms of the distances, we get the first part of the thesis. If $\xi = exp(sS)(\bar\xi)$ then arguing as before, we get

$$\eta(D_{\bar\xi} - D_\xi) = O\big((||\eta|| s^{\frac{2+\alpha}{2}} + ||\eta||^2 s^{\frac{1+\alpha}{2}})(D_{\bar\xi,8} + D_{\bar\xi,3})\big)$$

and consequently

$$exp(wD_{\bar\xi})(\zeta) = exp(-sS + \eta_2 s D_{\bar\xi,(5)} + \eta_1 s D_{\bar\xi,(6)}) =$$

$$= O\big((||\eta|| s^{\frac{2+\alpha}{2}} + ||\eta||^2 s^{\frac{1+\alpha}{2}})(D_{\bar\xi,8} + D_{\bar\xi,3})\big).$$

Then

$$|w_i| = -s\delta_{i3} + O(||\eta|| s + ||\eta||^2 s^{\frac{1+\alpha}{2}}),$$

where δ is the Kroneker function. The second assertion of the thesis is proved.

An Hölder continuous dependence on the variable $\bar\xi$ can now be proved also for $\Gamma_{\bar\xi}$. Let C_1, C_2, C_3 be the constants introduced in Proposition 3.1 and Lemma 3.1, and let C_4 be a fixed constant such that

$$C_4 > 1 + \frac{2C_2C_3}{C_1}, \quad C_4 > 1 + \frac{(2C_3)^{3/(3-\alpha)} C_2}{C_1}.$$

Then

Proposition 3.2 *Let U be fixed as in (20). There is a constant C_5 independent of ξ and $\bar\xi$ such that for every couple of points $\bar\xi$ and ξ in U, $\Gamma_{\bar\xi}$ and Γ_ξ satisfy the following estimate:*

$$|\Gamma_{\bar\xi}(\bar\xi,\zeta) - \Gamma_\xi(\xi,\zeta)| \le C_5 \left(\frac{d(\bar\xi,\xi)}{d^{N-1}(\bar\xi,\zeta)} + \frac{d^\alpha(\bar\xi,\xi)}{d^{N-2}(\bar\xi,\zeta)} \right), \tag{44}$$

for every ζ in the set $U_{\bar\xi,\xi} = \{\zeta \in U : d(\xi,\zeta) \ge C_4 d(\xi,\bar\xi)\}$. If $\xi = exp(sS)(\bar\xi)$, then

$$|\Gamma_{\bar\xi}(\bar\xi,\zeta) - \Gamma_\xi(\xi,\zeta)| \le C_5 \left(\frac{s}{d^{N}(\bar\xi,\zeta)} + \frac{s^{(1+\alpha)/2}}{d^{N-2}(\bar\xi,\zeta)} \right).$$

Proof By (43) we have

$$|\Gamma_{\bar\xi}(\bar\xi,\zeta) - \Gamma_\xi(\xi,\zeta)| = \Big| \frac{\bar\Gamma(0,\Phi_{\bar\xi}(\zeta))}{h(\bar\xi)(Xb - Ya)(\bar\xi)} - \frac{\bar\Gamma(0,\Phi_\xi(\zeta))}{h(\xi)(Xb - Ya)(\xi)} \Big|, \tag{45}$$

by the mean value theorem

$$= |w\bar{D}\bar{\Gamma}(0,\theta)| + \frac{d^{\alpha}(\xi,\bar{\xi})}{d^{N-2}(\zeta,\bar{\xi})},$$

where w are the canonical coordinate of $\Phi_{\bar{\xi}}(\zeta)$ around $\Phi_{\xi}(\zeta)$ with respect to the vector fields $\bar{D}$, and θ is a suitable point belonging to the integral curve $exp(\tau w D)(\Phi_{\xi}(\zeta))$. Consequently

$$\bar{d}(\theta,0) \geq \bar{d}(0,\Phi_{\bar{\xi}}(\zeta)) - \bar{d}(\theta,\Phi_{\bar{\xi}}(\zeta)) \geq$$

$$\geq \bar{d}(0,\Phi_{\bar{\xi}}(\zeta)) - \bar{d}(\Phi_{\xi}(\zeta),\Phi_{\bar{\xi}}(\zeta)) \geq$$

(by (43) and Lemma 3.1)

$$\geq d_{\bar{\xi}}(\bar{\xi},\zeta) - C_3 d_{\bar{\xi}}(\bar{\xi},\xi) - C_3 d_{\bar{\xi}}^{\alpha/3}(\bar{\xi},\zeta) d_{\bar{\xi}}^{1-\alpha/3}(\bar{\xi},\xi). \tag{46}$$

Note that

$$d_{\bar{\xi}}(\bar{\xi},\zeta) \geq \quad \text{(by Proposition 3.1)} \quad C_1 d(\bar{\xi},\zeta) \geq$$

$$C_1\Big(d(\xi,\zeta) - d(\xi,\bar{\xi})\Big) \geq \quad (\text{if } \ \zeta \in D) \quad C_1(C_4-1)d(\xi,\bar{\xi})$$

$$\text{(by Proposition 3.1)} \quad \geq \frac{C_1(C_4-1)}{C_2} d_{\bar{\xi}}(\xi,\bar{\xi}).$$

Then by (46)

$$\bar{d}(\theta,0) \geq d_{\bar{\xi}}(\bar{\xi},\zeta) - C_3 d_{\bar{\xi}}(\bar{\xi},\xi) - C_3 d_{\bar{\xi}}^{\alpha/3}(\bar{\xi},\zeta) d_{\bar{\xi}}^{1-\alpha/3}(\bar{\xi},\xi)$$

$$\geq \frac{1}{2}\big(d_{\bar{\xi}}(\bar{\xi},\zeta) - 2C_3 d_{\bar{\xi}}(\bar{\xi},\xi)\big) + \frac{1}{2}\big(d_{\bar{\xi}}^{1-\alpha/3}(\bar{\xi},\zeta) - 2C_3 d_{\bar{\xi}}^{1-\alpha/3}(\bar{\xi},\xi)\big) d_{\bar{\xi}}^{\alpha/3}$$

$$\geq \frac{1}{2} d_{\bar{\xi}}(\bar{\xi},\zeta)\Big(1 - \frac{2C_3C_2}{C_1(C_4-1)}\Big) + \frac{1}{2} d_{\bar{\xi}}(\bar{\xi},\zeta)\Big(1 - 2C_3\big(\frac{C_2}{C_1(C_4-1)}\big)^{1-\alpha/3}\Big)$$

$$(\text{for the choice of } \ C_4\ ) \geq \frac{1}{2} d_{\bar{\xi}}(\bar{\xi},\zeta).$$

By the standard estimates of $\bar{\Gamma}$ in the directions of the vector fields we get

$$|\bar{D}_{\sigma}\bar{\Gamma}(0,\theta)| \leq C d_{\bar{\xi}}^{-N+2-|\sigma|}(\bar{\xi},\zeta) \tag{47}$$

By Lemma 3.1 we conclude that

$$|\omega\bar{D}\bar{\Gamma}(0,\theta)| \leq \frac{d(\bar{\xi},\xi)}{d^{q-1}(\bar{\xi},\zeta)} + \frac{d^{\alpha}(\bar{\xi},\xi)}{d^{q-2}(\bar{\xi},\zeta)}.$$

If $\xi = exp(sS)(\bar{\xi})$, then, by (45), (47) and Lemma 3.1 we conclude the second part of the thesis.

Finally, we recall a property of the Taylor polynomial, in the frozen directions:

Remark 3.3 *Let* $f \in C_L^{k,\alpha}$ *and* $k \leq 3$, *let* $P_{\bar\xi}^k f$ *be the polynomial*

$$P_{\bar\xi}^k f(\xi) = \sum_{i=1}^{k} \frac{1}{i!} \sum_{lengh(\sigma)=i, |(\sigma)|=i} \bar e_\sigma D_\sigma^k f(\bar\xi)$$

where $\bar e$ *are the frozen canonical coordinates. Then the following relation holds:*

$$f(\xi) = P_{\bar\xi}^k f(\xi) + O(d^{k+\alpha}(\xi, \bar\xi)),$$

and

$$|P_\xi^k f(\zeta) - P_{\bar\xi}^k f(\zeta)| \leq d^\alpha(\xi, \bar\xi)\Big(d^k(\xi, \bar\xi) + d^k(\bar\xi, \zeta)\Big).$$

Besides, if $f \in C_L^{k+1,\alpha}$ *and* $\xi = exp(sS)(\bar\xi)$, *then*

$$|P_\xi^k f(\zeta) - P_{\bar\xi}^k f(\zeta)| \leq s^{(1+\alpha)/2}\Big(d^k(\xi, \bar\xi) + d^k(\bar\xi, \zeta)\Big).$$

Proof Let us fix two points ξ and $\bar\xi$, and denote $\bar e = \Phi_{\bar\xi}(\xi)$. Then, using (32), (33), (38), (36), (37), (39)), (40), is easy to recognize that for every $i = 3, \cdots 8$

$$\bar e_3 = e_i + O(d^{3+\alpha}(\xi, \bar\xi),$$

Hence, by Proposition 2.1, the assertion is true for $k = 1, 2$. We omit the proof for $k = 3$, since it is similar to the proof of Proposition 2.1.

4 Regularity results for the linear operator

Our regularity result will be achieved by a technique introduced in [CM2]. In order to obtain the regularity of solutions of the linear equation with low regular coefficients we follow the formulation of this method provided in [CPP]. It is based on the following two results:

Definition 4.1 *Let* (F_ε) *be a family of continuous functions on* $U \times U$ *and let* $f : U \to \mathbb{R}$. *We say that*

$$F_\varepsilon(\xi; \bar\xi) \to f(\bar\xi) \quad \varepsilon \to 0,$$

locally uniformly of order $k + \alpha$ *if for every compact set* $K \subset\subset U$ *there exist* $C > 0$, $\nu > 0$ *such that* $\forall \xi, \bar\xi \in K$ $d_{\bar\xi}(\xi, \bar\xi) \leq \nu\varepsilon$

$$|F_\varepsilon(\xi, \bar\xi) - f(\bar\xi)| \leq C\varepsilon^{k+\alpha}.$$

Besides we have:

Lemma 4.1 *Let* $|\sigma| \geq 1$ *and let* $u \in C_L^{|\sigma|-1,\alpha}(U)$, *a function which can represented as*

$$u(\xi) = I_1(\xi, \bar\xi) + I_2(\xi, \bar\xi)$$

where I_1 is of class $C_L^{|\sigma|,\alpha}(U)$ as a function of ξ in U. Let us denote $D_\sigma(\xi)I_1(\cdot,\bar\xi)$ the derivative with respect to the first variable, and assume that the function $\xi \to D_\sigma(\xi)I_1(\cdot,\bar\xi)$ is continuous in ξ uniformly in $\bar\xi$. Assume that there exists a family $I_{2,\varepsilon}$ of smooth functions and a continuous function I_2^σ such that

$$I_{2,\varepsilon}(\xi,\bar\xi) \to I_2(\bar\xi,\bar\xi) \qquad of \ \ order |\sigma| + \alpha$$

$$D_\sigma I_{2,\varepsilon}(\xi,\bar\xi) \to I_2^\sigma(\bar\xi) \qquad of \ \ order \alpha.$$

Then $u \in C_L^{|\sigma|}(U)$ and for every $\bar\xi$ in U

$$D_\sigma u(\bar\xi) = D_\sigma(\bar\xi)I_1(\cdot,\bar\xi) + I_2^\sigma(\bar\xi).$$

Using this lemma we will be able to obtain the regularity of the solutions.

4.1 Representation formulas

In order to apply Lemma 4.1, we need a representation formula for solutions of (12), of class $u \in C^{3,\alpha}_{L,loc}(\Omega\times]-1,1[$. This will be obtained using the fundamental solution $\Gamma_{\bar\xi}$ as a parametrix of the fundamental solution of L.

We can extend u, and all the coefficients of the operator on $\Omega\times]-1,1[\times R^4$, as constant functions of the variables $(x_4,\cdots,x_7)$. We can obviously study the regularity of u on the open set U defined in (20). We then fix $U \subset\subset U_1 \subset\subset \Omega\times R^5$, and we choose a function $\phi \in C_0^\infty(\Omega\times R^5)$ such that $\phi\equiv 1$ on U_1, so that $v_{|U} = (v\phi)_{|U}$.

Theorem 4.1 *Let $3 \le k \le 7$. Let assume that the coefficients a, b, h in equation (12) are of class $C^{2,\alpha}(U)\cap C_L^{k-1,\alpha}(U)$, $c, g \in C_L^{k-2,\alpha}(U)$ that $\partial_t a \in C_L^\alpha(U)$, $\partial_t b \in C_L^\alpha(U)$. Let $u \in C_L^{k-1,\alpha}$, be a classical solution of (12) such that $\partial_t u \in C_L^{1,\alpha}(U)$, $Su \in C_L^{1,\alpha}(U)$. For every $\bar\xi \in U_2$, $u = u\phi$ can be represented as*

$$u(\xi) = u\phi(\xi) = \int \Gamma_{\bar\xi}(\xi,\zeta)N_1(\bar\xi,\zeta)d\zeta \tag{48}$$

$$+\sum_{|\rho|,|\theta|\le 1}\int e_\rho D_{\bar\xi,\theta}\Gamma_{\bar\xi}(\xi,\zeta)N_{2,k,\theta,\rho}(\bar\xi,\zeta)d\zeta + \sum_{|\rho|\le 1}\int D_{\bar\xi,\rho}\Gamma_{\bar\xi}(\xi,\zeta)N_{3,k,\rho}(\bar\xi,\zeta)d\zeta,$$

where $N_{i,k,\rho}(\cdot,\bar\xi)$ are supported in the support of ϕ, and
(i) $\operatorname{supp}(N_1(\cdot,\bar\xi)) \subset \operatorname{supp}(\nabla\phi)$

$$\left|N_1(\xi,\zeta) - N_1(\bar\xi,\zeta)\right| \le C d_{\bar\xi}(\bar\xi,\xi)^\alpha, \tag{49}$$

(ii) $N_{2,k,\theta,\rho}(\bar\xi,\cdot)$ is of class $C_L^{k-|\rho|+|\theta|-1,\alpha}$ and for every multi-index σ such that $|\sigma| \le k-|\rho|+|\theta|-1$

$$\left|D_\sigma(\zeta)N_{2,k,\theta,\rho}(\bar\xi,\cdot) - D_\sigma(\zeta)N_{2,k,\theta,\rho}(\xi,\cdot)\right| \le C d_{\bar\xi}(\bar\xi,\xi)^\alpha, \qquad \forall \xi,\bar\xi \in U_2;$$

(iii) there exists a constant C, only dependent on the choice of ϕ and U_1, such that

$$\left|N_{3,k,\rho}(\bar\xi,\zeta)\right| \le C d_{\bar\xi}^{k-2+\alpha+|\rho|}(\bar\xi,\zeta), \tag{50}$$

and

$$\left|N_{3,k,\rho}(\xi,\zeta)-N_{3,k,\rho}(\bar\xi,\zeta)\right|\le Cd_{\bar\xi}(\bar\xi,\xi)^{\alpha}\Big(d_{\bar\xi}(\bar\xi,\zeta)^{k-2+|\rho|}+d_{\xi}(\xi,\zeta)^{k-2+|\rho|}\Big), \tag{51}$$

for every $\xi,\bar\xi\in U_2$.

Proof If $k\le 4$ we set $i=3, j=4$, and if $k\ge 5$ we set $i=j=k-1$. In this way we have in any case

$$Su\in C_L^{i-2,\alpha}(U),\partial_t u\in C_L^{j-3,\alpha}(U),$$

By definition of fundamental solution we have

$$u\phi(\xi)=\int\Gamma_{\bar\xi}(\xi,\zeta)L_{\bar\xi}(u\phi)(\zeta)d\zeta.$$

Since u is a solution of equation (12), we get

$$L_{\bar\xi}(u\phi)=u(\zeta)L_{\bar\xi}\phi(\zeta)+2X_{\bar\xi}u(\zeta)X_{\bar\xi}\phi(\zeta)+2Y_{\bar\xi}u(\zeta)Y_{\bar\xi}\phi(\zeta)+ \tag{52}$$

$$+(L_{\bar\xi}u-Lu+g)\phi(\zeta)$$

(with a direct computation we obtain - see for example [C] where the computation is carried out in details)

$$=\Big(u(\zeta)L_{\bar\xi}\phi(\zeta)+2X_{\bar\xi}u(\zeta)X_{\bar\xi}\phi(\zeta)+2Y_{\bar\xi}u(\zeta)Y_{\bar\xi}\phi(\zeta)\Big)+$$

$$-(a-P^2_{\bar\xi}a)X\partial_t u\phi(\zeta)-(a-P^2_{\bar\xi}a)X_{\bar\xi}\partial_t u\phi(\zeta)+$$

$$-(b-P^2_{\bar\xi}b)Y\partial_t u\phi(\zeta)-(b-P^2_{\bar\xi}b)Y_{\bar\xi}\partial_t u\phi(\zeta)-\frac{(h-P^1_{\bar\xi}h)}{h}Su\phi(\zeta)-$$

$$-\frac{(h-P^1_{\bar\xi}h)}{h}(Xa+Yb-c)\partial_t u\phi(\zeta)+g\phi(\zeta).$$

Integrating each term separately, and calling $A_0,\cdots,A_7$ the resulting term we get:

$$u\phi(\xi)=\sum_{i=0}^{7}A_i(\xi,\bar\xi). \tag{53}$$

Now we can study each of them separately.

$$A_0(\xi,\bar\xi)=\int\Gamma_{\bar\xi}(\xi,\zeta)\Big(u(\zeta)L_{\bar\xi}\phi(\zeta)+2X_{\bar\xi}u(\zeta)X_{\bar\xi}\phi(\zeta)+2Y_{\bar\xi}u(\zeta)Y_{\bar\xi}\phi(\zeta)\Big)$$

is of type N_1.

$$A_1(\xi,\bar\xi)=-\int\Gamma_{\bar\xi}(\xi,\zeta)(a-P^2_{\bar\xi}a)(\zeta)X\partial_t u(\zeta)\phi(\zeta)d\zeta=$$

$$= -\int \Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)(\zeta)\Big(X\partial_t u(\zeta) - P^{j-4}_{\bar{\xi}}(X\partial_t u)(\zeta)\Big)\phi(\zeta)d\zeta$$
$$-\int \Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)(\zeta)P^{j-4}_{\bar{\xi}}(X\partial_t u)(\zeta)\phi(\zeta)d\zeta.$$

By the properties of the fundamental solution, and of the Taylor polynomials proved in Remark 3.3, we have

$$|a - P^2_{\bar{\xi}}a||X\partial_t u - P^{j-4}_{\bar{\xi}}(X\partial_t u)| \le d^{k-2+\alpha}_{\bar{\xi}}(\bar{\xi},\zeta)$$

Besides

$$\Big|(a - P^2_{\bar{\xi}}a)\Big(X\partial_t u - P^{j-4}_{\bar{\xi}}(X\partial_t u)\Big) - (a - P^2_{\xi}a)\Big(X\partial_t u - P^{j-4}_{\xi}(X\partial_t u)\Big)\Big| \le$$

$$|P^2_{\xi}a - P^2_{\bar{\xi}}a|\Big|X\partial_t u - P^{j-4}_{\bar{\xi}}(X\partial_t u)\Big| + |a - P^2_{\xi}a|\Big|P^{j-4}_{\bar{\xi}}(X\partial_t u) - P^{j-4}_{\xi}(X\partial_t u)\Big| \le$$
$$\le C d_{\bar{\xi}}(\bar{\xi},\xi)^{\alpha}\Big(d_{\bar{\xi}}(\bar{\xi},\zeta)^{k-2} + d_{\xi}(\xi,\zeta)^{k-2}\Big).$$

Hence the kernel in first integral is of type $N_{3,k,0}$, while the kernel in the second integral, is of type $N_{2,k,0,0}$, always by Remark 3.3.

Let us now integrate the second term at the right hand side of (52).

$$A_2(\xi,\bar{\xi}) = -\int \Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)X_{\bar{\xi}}\partial_t u\phi(\zeta)d\zeta =$$

(integrating by parts)

$$= \int X_{\bar{\xi}}\Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)\phi(\zeta)d\zeta +$$

$$+\int \Gamma_{\bar{\xi}}(\xi,\zeta)(X_{\bar{\xi}}a - X_{\bar{\xi}}P^2_{\bar{\xi}}a)\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)\phi(\zeta)d\zeta +$$

$$+\int \Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)X_{\bar{\xi}}\phi(\zeta)d\zeta$$

(since $X_{\bar{\xi}} = X - (a - P^2_{\bar{\xi}}a)\partial_t$, we can change the second term as follows)

$$= \int X_{\bar{\xi}}\Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)\phi(\zeta)d\zeta$$

$$+\int \Gamma_{\bar{\xi}}(\xi,\zeta)(Xa - P^1_{\bar{\xi}}(Xa))\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)\phi(\zeta)d\zeta$$

$$-\int \Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)\partial_t a(\zeta)\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)\phi(\zeta)d\zeta +$$

$$+\int \Gamma_{\bar{\xi}}(\xi,\zeta)(a - P^2_{\bar{\xi}}a)\Big(\partial_t u - \partial_t u(\bar{\xi})\Big)X_{\bar{\xi}}\phi(\zeta)d\zeta =$$

(now we add and subtract some term in each of the previous integral, in order to use their Taylor development. For simplicity we denote $P^0_{\bar\xi}a = 0$.)

$$= \int X_{\bar\xi}\Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi}a)\Big(\partial_t u - P^{j-3}_{\bar\xi}(\partial_t u)\Big)\phi(\zeta)d\zeta$$

$$+ \int X_{\bar\xi}\Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi}a)\Big(P^{j-3}_{\bar\xi}(\partial_t u) - \partial_t u(\bar\xi)\Big)\phi(\zeta)d\zeta +$$

$$+ \int \Gamma_{\bar\xi}(\xi,\zeta)(Xa - P^1_{\bar\xi}(Xa))\Big(\partial_t u - P^{j-3}_{\bar\xi}(\partial_t u)\Big)\phi(\zeta)d\zeta$$

$$+ \int \Gamma_{\bar\xi}(\xi,\zeta)(Xa - P^1_{\bar\xi}(Xa))\Big(P^{j-3}_{\bar\xi}(\partial_t u) - \partial_t u(\bar\xi)\Big)\phi(\zeta)d\zeta$$

$$- \int \Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi}a)\partial_t a(\zeta)\Big(\partial_t u - P^{j-3}_{\bar\xi}(\partial_t u)\Big)\phi(\zeta)d\zeta -$$

$$- \int \Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi}a)\Big(\partial_t a(\zeta) - P^{k-4}_{\bar\xi}(\partial_t a)\Big)\Big(P^{j-3}_{\bar\xi}\partial_t u - \partial_t u(\bar\xi)\Big)\phi(\zeta)d\zeta =$$

$$- \int \Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi}a)P^{k-4}_{\bar\xi}\partial_t a(\zeta)\Big(P^{j-3}_{\bar\xi}\partial_t u - \partial_t u(\bar\xi)\Big)\phi(\zeta)d\zeta =$$

$$+ \int \Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi}a)\Big(\partial_t u - \partial_t u(\bar\xi)\Big)X_{\bar\xi}\phi(\zeta)d\zeta.$$

The first kernel is of type $N_{3,k,1}$, the second of type $N_{2,k,1,1}$ kernels 3, 5, 6, of type $N_{3,k,0}$, the kernel 4 of type $N_{2,k,0,1}$, the kernel 7 of type $N_{2,k,0,0}$ and the kernel 8 of type N_1.

The third term in (52) can be handled exactly as $A_1(\xi,\bar\xi)$, while the fourth has the same behavior as $A_2(\xi,\bar\xi)$. Let us consider A_5, and denote $m = min(3, i-2)$:

$$A_5(\xi,\bar\xi) = -\int \Gamma_{\bar\xi}(\xi,\zeta)\Big(1 - \frac{P^1_{\bar\xi}h}{h}\Big)\Big(Su - P^m_{\bar\xi}(Su)\Big)\phi(\zeta)d\zeta$$

$$- \int \Gamma_{\bar\xi}(\xi,\zeta)\Big(1 - \frac{P^1_{\bar\xi}h}{h}\Big)P^m_{\bar\xi}(Su)\phi(\zeta)d\zeta.$$

Here the first term have the behavior of $N_{3,k,0}$, while the second is of type to $N_{2,k,0,0}$.

$$A_6(\xi,\bar\xi) = -\int \Gamma_{\bar\xi}(\xi,\zeta)\Big(1 - \frac{P^1_{\bar\xi}h}{h}\Big)(Xa + Yb - c)\partial_t u\phi(\zeta)d\zeta =$$

$$- \int \Gamma_{\bar\xi}(\xi,\zeta)\Big(1 - \frac{P^1_{\bar\xi}h}{h}\Big)(Xa + Yb - c)\big(\partial_t u - P^{j-3}_{\bar\xi}\partial_t u\big)\phi(\zeta)d\zeta$$

$$- \int \Gamma_{\bar\xi}(\xi,\zeta)\Big(1 - \frac{P^1_{\bar\xi}h}{h}\Big)\big((Xa + Yb - c) - P^{k-2}_{\bar\xi}(Xa + Yb - c)\big)P^{j-3}_{\bar\xi}\partial_t u\phi(\zeta)d\zeta$$

$$-\int \Gamma_{\bar\xi}(\xi,\zeta)\Big(1-\frac{P^1_{\bar\xi}h}{h}\Big)P^{k-2}_{\bar\xi}(Xa+Yb)P^{j-3}_{\bar\xi}\partial_t u\phi(\zeta)d\zeta$$

The kernels in the first two integrals are of type $N_{3,k,0}$, while the last of type $N_{2,k,0,0}$.

$$A_7(\xi,\bar\xi)=\int \Gamma_{\bar\xi}(\xi,\zeta)g\phi(\zeta)d\zeta,$$

which of type $N_{2,k,0,0}$.

We will need also an other representation of u:

Theorem 4.2 *Let $3\le k\le 7$. Let assume that the coefficients a,b,h in equation (12) are of class $C^{k-1,\alpha}_L(U)$, $c,g\in C^{k-2,\alpha}_L(U)$ that $\partial_t a\in C^\alpha_L(U)$, $\partial_t b\in C^\alpha_L(U)$. Let $u\in C^{k-1,\alpha}_L$, be a classical solution of (12) such that $\partial_t u\in C^{1,\alpha}_L(U)$, $Su\in C^{1,\alpha}_L(U)$. For every $\bar\xi\in U_2$, $u=u\phi$ can be represented as*

$$u(\xi)=u\phi(\xi)=\int \Gamma_{\bar\xi}(\xi,\zeta)N_1(\bar\xi,\zeta)d\zeta \tag{54}$$

$$+\sum_{|\rho|,|\theta|\le 1}\int e_\rho D_{\bar\xi,\theta}\Gamma_{\bar\xi}(\xi,\zeta)\widetilde N_{2,k,\theta,\rho}(\bar\xi,\zeta)d\zeta+\sum_{|\rho|\le 1}\int D_{\bar\xi,\rho}\Gamma_{\bar\xi}(\xi,\zeta)\widetilde N_{3,k,\rho}(\bar\xi,\zeta)d\zeta,$$

where $\widetilde N_{i,k,\rho}(\cdot,\bar\xi)$ are supported in the support of ϕ, and
(i) $\mathrm{supp}(N_1(\cdot,\bar\xi))\subset\mathrm{supp}(\nabla\phi)$ *if $\xi=\exp(sS)(\bar\xi)$, and, if $\xi=\exp(sS)(\bar\xi)$ then*

$$\big|N_1(\xi,\zeta)-N_1(\bar\xi,\zeta)\big|\le Cs^{(1+\alpha)/2};$$

(ii) $\widetilde N_{2,k,\theta,\rho}(\cdot,\bar\xi)$ is of class $C^{k-|\rho|+|\theta|-2,\alpha}_L$ and for every multi-index σ such that $|\sigma|\le k-|\rho|+|\theta|-2$ for every $\xi=\exp(sS)(\bar\xi)$

$$\Big|D_\sigma(\xi_1)\widetilde N_{2,k-1,\theta,\rho}(\bar\xi,\cdot)-D_\sigma(\xi_1)\widetilde N_{2,k-1,\theta,\rho}(\xi,\cdot)\Big|\le Cs^{(1+\alpha)/2}$$

(iii) there exists a constant C, only dependent on the choice of ϕ and K_1, such that

$$\Big|\widetilde N_{3,k,\rho}(\bar\xi,\zeta)\Big|\le Cd^{k-3+\alpha}_{\bar\xi}(\bar\xi,\zeta), \tag{55}$$

and

$$\Big|\widetilde N_{3,k-1,\rho}(\bar\xi,\zeta)-\widetilde N_{3,k-1,\rho}(\xi,\zeta)\Big|\le Cs^{(\alpha+1)/2}\Big(d_{\bar\xi}(\bar\xi,\zeta)^{k-3+\alpha}+d_\xi(\xi,\zeta)^{k-3+\alpha}\Big),$$

if $\xi=\exp(sS)(\bar\xi)$.

Proof Let us represent the function u as in (53), with the same notations as in the preceding theorem. The function A_1 can be also expressed as follows:

$$A_1(\xi,\bar\xi)=-\int \Gamma_{\bar\xi}(\xi,\zeta)(a-P^2_{\bar\xi}a)(\zeta)\Big(X\partial_t u(\zeta)-P^{j-5}_{\bar\xi}(X\partial_t u)(\zeta)\Big)\phi(\zeta)d\zeta$$

$$-\int \Gamma_{\bar\xi}(\xi,\zeta)(a - P^2_{\bar\xi} a)(\zeta) P^{j-5}_{\bar\xi}(X\partial_t u)(\zeta)\phi(\zeta)d\zeta.$$

Since $X\partial_t u$ is of class $C^{j-4,\alpha}_L$, and we consider the Taylor polynomial of order $j-5$, then these two terms are of type $\widetilde{N}_{3,k,0}$, $\widetilde{N}_{2,k,0,0}$ respectively.

Analogously, if we represent A_2 as before, but using Taylor developments of $\partial_t u$ of order $j-4$, it turns out to be the sum of 8 terms, of the following type. The first kernel is of type $\widetilde{N}_{3,k,1}$, the second of type $\widetilde{N}_{2,k,1,1}$ kernels 3, 5, 6, of type $\widetilde{N}_{3,k,0}$, the kernel 4 of type $\widetilde{N}_{2,k,0,1}$, the kernel 7 of type $\widetilde{N}_{2,k,0,0}$ and the kernel 8 of type N_1.

Using Taylor expansion of Su of order $i-3$, we represent A_5 as a sum of terms of type $\widetilde{N}_{3,k,0}$ and $\widetilde{N}_{2,k,0,0}$ respectively.

4.2 Low order regularity

Theorem 4.3 *Let $3 \le k \le 7$. Let assume that the coefficients a, b, h in equation (12) are of class $C^{2,\alpha}(U) \cap C^{k-1,\alpha}_L(U)$, $c, g \in C^{k-2,\alpha}_L(U)$ that $\partial_t a \in C^{\alpha}_L(U)$, $\partial_t b \in C^{\alpha}_L(U)$. Let $u \in C^{k-1,\alpha}_L$, be a classical solution of (12) such that $\partial_t u \in C^{1,\alpha}_L(U)$, $Su \in C^{1,\alpha}_L(U)$. Then $u \in C^{k,\beta}_L(U)$ for every $\beta < \alpha$.*

The proof is obtained in two steps: first we compute the derivatives of u by means of Lemma 4.1. Then we study the regularity of this term, which is explicitly known.

STEP 1. In order to differentiate v, we use the representation formula proved in Theorem 4.1. The function

$$\xi \to I_1(\xi,\bar\xi) = \int \Gamma_{\bar\xi}(\xi,\zeta) N_1(\bar\xi,\zeta)d\zeta + \sum_{|\rho|\le 1}\int D_{\bar\xi,\rho}\Gamma_{\bar\xi}(\xi,\zeta)N_{2,k,\rho}(\bar\xi,\zeta)d\zeta$$

is of class C^∞. Let us study

$$I_2(\xi,\bar\xi) = \sum_{|\rho|\le 1}\int D_{\bar\xi,\rho}\Gamma_{\bar\xi}(\xi,\zeta)N_{3,k,\rho}(\bar\xi,\zeta)d\zeta.$$

We fix a function θ in $C^\infty(R)$ satisfying $0 \le \theta \le 1$, $\theta(\tau) = 0\ \forall \tau \le 1$ and $\theta(\tau) = 1\ \forall\tau \ge 1$, and define for every $\epsilon > 0$

$$I_{2,\varepsilon}(\xi,\bar\xi) = \sum_{|\rho|\le 1}\int \theta\Big(\frac{d_{\bar\xi}(\xi,\zeta)}{\epsilon}\Big) D_{\bar\xi,\rho}\Gamma_{\bar\xi}(\xi,\zeta)N_{3,k,\rho}(\bar\xi,\zeta)d\zeta$$

Then it is not difficult to show that

$$\sup_{\delta_{\bar\xi}(\xi,\bar\xi)<\varepsilon} |I_{2,\epsilon}(\xi) - I_2(\xi,\bar\xi)| \le C\epsilon^{p+\gamma}. \tag{56}$$

Finally, if we set

$$I_2^{\sigma}(\bar{\xi}) = \sum_{|\rho|\leq 1} \int D_\sigma D_{\bar{\xi},\rho} \Gamma_{\bar{\xi}}(\bar{\xi},\zeta) N_{3,k,\rho}(\bar{\xi},\zeta) d\zeta$$

then we obtain:

$$\sup_{\delta_{\bar{\xi}}(\xi,\bar{\xi})<\varepsilon} |D_\sigma I_{2,\epsilon}(\xi,\bar{\xi}) - I_2^{\sigma}(\bar{\xi})| \leq C\epsilon^{\gamma} \tag{57}$$

All the Hypotheses of Lemma 4.1 are satisfied. Hence the derivative exists.

STEP 2. The Hölder continuity of derivatives is in turn organized in two parts. First, using the Hölder estimates provided in Theorem 4.1, and arguing as in [C], we obtain that the derivatives $D_\sigma u$ with $|\sigma| = k$ are of class C_L^{β} for $\beta < \alpha$. Then, using the estimates of Theorem 4.2, and arguing as in [M], we obtain that for every σ such that $|\sigma| = k-1$, the derivatives $D_\sigma u$ belong to $C_S^{(1+\beta)/2}(U)$ for every $\beta < \alpha$. These two estimates together imply that $u \in C_L^{k,\beta}(U)$ for every $\beta < \alpha$.

In view of the iteration we need to perform in the next section we recall that if u is a solution of (12), then its derivatives satisfy the same equation, with a different second member (see [M] for a detailed proof of the following lemma).

Lemma 4.2 *If u is a solution of (12) then the function $v_8 = \partial_t u$ is a solution of $Lv_8 = \partial_t g + F_{8,c}$, where*

$$F_{8,c} = -2\partial_t a X \partial_t u - 2\partial_t b Y \partial_t u + \partial_t h \partial_{x_3} u + \partial_t c \partial_t u,$$

the function $\widetilde{v}_3 = \partial_{x_3} u$ where is a solution of $L\widetilde{v}_3 = \partial_{x_3} g + F_{3,c}$

$$F_{3,c} = -2\partial_{x_3} a X \partial_t u - 2\partial_{x_3} b Y \partial_t u + \partial_{x_3} h \partial_{x_3} u + \partial_{x_3} c \partial_t u.$$

The function $v_1 = Xu$ is a solution of $Lv_1 = Xg + F_{1,c}$, where

$$F_{1,c} = Y(q\partial_t u) + q\partial_t Y u + \frac{h}{1+u_t^2}\Big(Y\partial_{x_3} u - X\partial_{x_3} u \partial_t u\Big)\partial_t u$$

$$-\frac{q\partial_t u - c}{1+u_t^2}\Big(Y\partial_t u - X\partial_t u \partial_t u\Big)\partial_t u - Xh\partial_{x_3} u + X(q\partial_t u)\partial_t u - c\partial_t u,$$

where $q = h\partial_{x_3} u - (Xa + Yb)\partial_t u - g$.

Finally the function $v_2 = Yu$, is a solution of $Lv_2 = Yg + F_{2,c}$, where

$$F_{2,c} = X(q\partial_t u) + q\partial_t X u - \frac{h}{1+u_t^2}\Big(X\partial_{x_3} u + Y\partial_{x_3} u \partial_t u\Big)\partial_t u$$

$$+\frac{q\partial_t u - c}{1+u_t^2}\Big(X\partial_t u + Y\partial_t u \partial_t u\Big)\partial_t u - Yh\partial_{x_3} u + Y(q\partial_t u)\partial_t u - c\partial_t u.$$

5 The nonlinear operator

In this section we prove Theorem 1.1. We will recognize that the derivatives of the function u satisfy a linear equation, and we apply to it the results of the preceding section.

Let us fix a solution u of class $C^{3,\alpha}(\Omega)$ of equation (4), let us define a, b in terms of the solution u as in (2), and let us choose

$$c = ku_t \frac{(1+a^2+b^2)^{3/2}}{1+u_t^2)^{1/2}},$$

and let $h = h(a, b, \cdot)$ be the function defined in the introduction. Hence a, b, c and h are of class $C^{2,\alpha}(\Omega)$. Let X, Y, S be the vector fields defined in (11) in terms of a, b, c and h and let

$$L = X^2 + Y^2 + S.$$

We recall that (see for example [CM]) that $a = Yu$, and $b = -Xu$, so that

$$Xa + Yb = XYu - YXu = -\frac{\mathcal{L}u}{1+u_t^2}\partial_t u =$$

$$= \frac{k(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2} + h(a,b,\cdot)\partial_{x_3}u}{1+u_t^2} \in C^{2,\alpha}(\Omega),$$

$$Xb - Ya = -(X^2u + Y^2u) = -\frac{\mathcal{L}u}{1+u_t^2} \in C^{2,\alpha}(\Omega).$$

Hence the operator L satisfies all the assumptions of Theorem 1.2.

We will denote $C^{k,\alpha}_{\mathcal{L}}$ the classes of Hölder continuous functions introduced in Definition 2.3. We will also call $\widetilde{D}_1 = X$, $\widetilde{D}_2 = Y$, $\widetilde{D}_3 = \partial_{x_3}$ and $\widetilde{D}_8 = \partial_t$. For every multiindex $\sigma = (\sigma_1, \ldots, \sigma_m)$, with $\sigma_r \in \{1,2,3,4\}$, let

$$\widetilde{D}_\sigma = \widetilde{D}_{\sigma_1} \cdots \widetilde{D}_{\sigma_m},$$

and we call length of σ the number m. Moreover we will denote $\widetilde{D}^k$ the vector whose components are all the derivatives of length at most k.

By (3) and (5) u is a solution of

$$X^2u + Y^2u - (Xa+Yb)\partial_t u - h(a,b,\cdot)\partial_{x_3}u = k(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2},$$

and in the following proposition we show that its derivatives are solution of an equation associated to the operator L.

Proposition 5.1 *Let k be a natural number, and let σ be a multiindex with $\sigma_i \in \{1,2,3,8\}$ and $l(\sigma) = k$. Assume that the function*

$$v_\sigma = \widetilde{D}_\sigma u \tag{58}$$

has all the derivatives of length 2 in $C^{\beta}_{\mathcal{L}}(\Omega)$ with $\beta \in]0,1[$. Then it is a solution of

$$L(v_\sigma) = f_\sigma, \quad where \quad f_\sigma = f_\sigma(\widetilde{D}^k u, X\widetilde{D}^k u, Y\widetilde{D}^k u),$$

and f_σ is a smooth function of its arguments.

Proof Let us prove it by induction on k. By Lemma 4.2 it follows that the functions v_i with $i \in \{1,2,3,8\}$ are solutions of

$$(X^2 + Y^2 - (Xa + Yb)\partial_t + h\partial_{x_3})v_i =$$

$$= \widetilde{D}_i\Big(k(1+a^2+b^2)^{3/2}\Big)(1+u_t^2)^{1/2} +$$

$$+ku_t\frac{(1+a^2+b^2)^{3/2}}{(1+u_t^2)^{1/2}}[\widetilde{D}_i, \partial_t]u + ku_t\frac{(1+a^2+b^2)^{3/2}}{(1+u_t^2)^{1/2}}\partial_t(\widetilde{D}_i u) + F_{i,0},$$

where $F_{i,0}$ is defined in Lemma 4.2. Bringing at the left hand side the second term in the right and using the definition of c we get

$$L(v_i) = \widetilde{D}_i\Big(k(1+a^2+b^2)^{3/2}\Big)(1+u_t^2)^{1/2} + ku_t\frac{(1+a^2+b^2)^{3/2}}{(1+u_t^2)^{1/2}}[\widetilde{D}_i, \partial_t]u + F_{i,0}.$$

The right hand side depends on

$$\widetilde{D}u, X\widetilde{D}u, X\widetilde{D}u, \partial_t a, \partial_t b, \partial_{x_3} a, \partial_{x_3} b,$$

and for every $i \in \{3,8\}$, it has been proved in [M] that

$$\widetilde{D}_i a = \widetilde{D}_i Y u = \frac{Y\widetilde{D}_i u - \partial_t u X\widetilde{D}_i u}{1+u_t^2}, \quad \widetilde{D}_i b = -\widetilde{D}_i X u = -\frac{X\widetilde{D}_i u + \partial_t u Y\widetilde{D}_i u}{1+u_t^2}.$$

It is then clear that $\widetilde{D}_i u$ is a solution of the required equation, and the thesis is proved for $k=1$.

Let us assume that the thesis is true for k and prove it for $k+1$. Let $l(\sigma) = k$ and v_σ be a solution of $L(v_\sigma) = f_\sigma$. By Lemma 4.2 for every $i \in \{1,2,3,8\}$ the function $\widetilde{D}_i v_\sigma$ is a solution of

$$L(v_\sigma) = \widetilde{D}_i(f_\sigma) + F_{i,c},$$

and the thesis follows as before, by commuting the derivative $\widetilde{D}_i$ with the vector fields X and Y.

Proof of Theorem 1.1 To prove Theorem 1.1 we will use a bootstrap method. Since $u \in C^{3,\alpha}(\Omega)$, then for every $i \in \{1,2,3,8\}$, $\widetilde{D}_i u \in C^{2,\alpha}(\Omega)$ and by Proposition 5.1 $L(\widetilde{D}_i u) \in C^{1,\alpha}(\Omega)$. Hence we can apply Theorem 4.3 with $k=3$ to get $\widetilde{D}_i u \in C^{3,\beta}_L(\Omega)$. In particular $u \in C^{4,\beta}_L(\Omega)$ and $a,b,h,c \in C^{3,\beta}_L(\Omega)$. Hence we can apply Theorem 4.3 with $k=4$ to the function $\widetilde{D}_i u \in C^{3,\beta}_L(\Omega)$, to get $\widetilde{D}_i u \in C^{3,\beta}_L(\Omega)$, for every $i \in \{1,2,3,8\}$. Iterating this procedure we get

$$\widetilde{D}_i u \in C^{3,\beta}_L(\Omega),$$

for every $k \leq 7$ and for every $\beta < \alpha$. This prove the thesis for $k = 1$. Now we assume that the thesis is true for k and prove it for $k+1$. Let $v_\sigma = \widetilde{D}_\sigma u \in C_L^{4,\beta}(\Omega)$, with $l(\sigma) = k+1$. By Proposition 5.1 $Lv_\sigma = f_\sigma \in C_L^{3,\beta}(\Omega)$. Hence, by Theorem 4.3 v_σ is $C_L^{6,\beta}(\Omega)$ and f_σ is of class $C_L^{5,\beta}(\Omega)$. Finally, always by Theorem 4.3, $v_\sigma \in C_L^{7,\beta}(\Omega)$.

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