Sobolev and Morrey estimates for non-smooth vector fields of step two

Annamaria Montanari and Daniele Morbidelli

1 Introduction

In this paper we are concerned with the following Sobolev and Morrey-type inequalities:

$$\Big(\int_\Omega |f(x)|^{\frac{pQ}{(Q-p)}}dx\Big)^{\frac{Q-p}{pQ}} \leq c_1\Big(\int_\Omega |Xf(x)|^p dx\Big)^{\frac{1}{p}},\ 1\leq p<Q, \tag{1}$$

$$|f(y)-f(z)| \leq c_2\Big(\int_\Omega |Xf(x)|^p dx\Big)^{\frac{1}{p}}\rho(y,z)^\alpha,\ y,z\in\Omega,\ p>Q. \tag{2}$$

Here $\Omega\subset\mathbb{R}^n$ is a bounded set, $f\in C_0^1(\Omega)$ and $Xf=(X_1f,\dots,X_mf)$ is the degenerate gradient associated with a family of vector fields of the form $X_j=\partial/\partial x_j+\sum_{k=m+1}^n a_j^k(x)\partial/\partial x_k$, $j=1,\dots,m$, $m\leq n$. In (1) and (2) $Q:=m+2(n-m)$ denotes the "homogeneous dimension" associated with the X_j's, $\alpha<1$ is a suitable positive number and $\rho(x,z)$ is a quasi–distance naturally associated with the vector fields.

Historically, the first embedding properties for degenerate Sobolev spaces are essentially contained in the papers [17], [6], [5] and [9], where the "subelliptic estimate"

$$\|f\|_{H^\varepsilon}\leq c\left(\|f\|_{L^2}+\|Xf\|_{L^2}\right),\quad f\in C_0^\infty(\Omega) \tag{3}$$

is proved. Here H^ε denotes the classical fractional Sobolev space of a suitable order $\varepsilon\in(0,1]$. Obviously, the classical embedding $H^\varepsilon\subset L^{2n/(n-2\varepsilon)}$ furnishes the estimate

$$\|f\|_{L^{2n/(n-2\varepsilon)}}\leq c\left(\|f\|_{L^2}+\|Xf\|_{L^2}\right),\quad f\in C_0^\infty(\Omega). \tag{4}$$

A different class of significant integral inequalities involving vector fields (see for example [9] and [18]) are the Poincaré-type inequalities

$$\int_B |f-f_B|\leq c_1 r\int_{c_2B}|Xf|,\quad f\in C^1(c_2B), \tag{5}$$

where B is a ball of radius r with respect to the Carnot–Carathéodory distance (see [5] for the definition), c_2B is the ball with same center of B and radius c_2r and $f_B=|B|^{-1}\int_B f$ denotes the average of f on the ball. In recent years, many papers have been devoted to the study of the so-called self-improving properties of Poincaré inequalities (see e.g. [10], [21], [12] and [16]). In these papers it is proved that the Poincaré inequality, together with the doubling property of the Lebesgue measure of the Carnot–Carathéodory balls, are sufficient to construct a satisfactory theory of first order Sobolev spaces.

While this last "axiomatic" aspect of the theory has been developed in great generality, it seems that the direct proof of inequalities (1), (2), (3) or (5) in concrete situations is a very complicate matter. There are essentially two classes of examples of vector fields in $\mathbb{R}^n$ for which the proof of the mentioned inequalities is available: the vector fields satisfying Hörmander condition (see [17] and [18]) and those of Grushin–Franchi–Lanconelli type (see [9] and [7]). In the first case smooth regularity is required, while in the latter the vector fields have to be of "diagonal form".

In the present paper we study embedding properties for Sobolev Spaces associated with families of step two which are in general non smooth and non diagonal. Before stating our results we briefly quote some recent papers in which this problem is addressed.

A first embedding result for non smooth and non diagonal vector fields of step two is contained in [19], where (3) is proved with $\varepsilon = 1/2$ by means of a Fourier analysis method. This procedure only requires that the coefficients of the vector fields have second derivatives in a suitable L^p–space. However, since the Sobolev–type inequality is obtained through (4), the embedding exponent is in general not sharp.

For a couple of C^1 vector fields of step two in $\mathbb{R}^3$, inequalities (1) and (2) are consequences of the Poincaré inequality (5), which has been proved in [20]. The technique of the paper [20] relies on precise estimates of the Carnot–Carathéodory distance. In the regularity assumptions of our paper it is not even clear if the mentioned distance is finite for each pair of points.

In the recent paper [3] the Sobolev inequality (1) is proved for a pair of Lipschitz continuous vector fields in $\mathbb{R}^3$, which arise in the study of regularity properties of real hypersurfaces in $\mathbb{C}^2$ with Levi curvature belonging to a suitable Sobolev space and different from zero at every point. This result shows that embedding theorems for Sobolev Spaces related to vector fields with non–smooth coefficients are an important tool in the study of regularity properties of some non linear elliptic degenerate equations. The paper [3] has motivated the present one.

Our main result is the following.

Theorem 1.1 *Consider the vector fields in* $\mathbb{R}^n$

$$X_j = \partial_j + \sum_{k=m+1}^{n} a_j^k \partial_k \tag{6}$$

with $a_j^k \in Lip(\mathbb{R}^n)$. *Let* $Q = m + 2(n-m)$. *Assume that, for almost every* $x \in \mathbb{R}^n$,

$$\partial_j = \sum_{1 \le i < k \le m} \lambda_j^{i,k}(x)[X_i, X_k](x), \qquad j = m+1, \dots, n, \tag{7}$$

where

$$\lambda_j^{i,k} \in L^\infty, \; X_h \lambda_j^{i,k} \in L^\infty \; j = m+1, ..., n, \quad i,k,h = 1, \dots, m. \tag{8}$$

(here X_h and $[X_i, X_k] = X_i X_k - X_k X_i$ denote distributional derivatives). Then, for any bounded open set $\Omega \subset \mathbb{R}^n$, the following statements hold:

(A) (Sobolev inequality). *If $1 \le p < Q$, then*

$$\|f\|_{L^{pQ/(Q-p)}} \le c_1 \|Xf\|_{L^p}, \quad \forall f \in C_0^1(\Omega);$$

(B) (Trudinger inequality). *If $p = Q$, then*

$$\int_\Omega \exp\Big(\Big(\frac{|f(x)|}{c_2\|Xf\|_Q}\Big)^{Q/(Q-1)}\Big) dx \le c_3, \qquad \forall f \in C_0^1(\Omega);$$

(C) (Morrey–type estimate). *There exists a quasi–distance ρ in Ω satisfying*

$$k_1|x-y| \le \rho(x,y) \le k_2|x-y|^{1/2}$$

for all $x, y \in \Omega$, (with $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$) and such that, if $Q < p < \infty$, then

$$|f(x) - f(z)| \le c_4 \|Xf\|_{L^p} \rho(x,z)^\alpha, \quad \forall x, z \in \Omega \quad \forall f \in C_0^1(\Omega)$$

where $\alpha = \min\{1/2, 1 - Q/p\}$;

(D) if $1 < p < Q$, then the space $W_0^{1,p}$ is compactly embedded in L^q for any $q < Qp/(Q-p)$. (Here $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|Xf\|_{L^p(\Omega)}$).

The constants k_1 and k_2 depend on Ω and on the Lipschitz constants of the a_j^k's in (6)*. The constants c_1, c_2, c_3 and c_4 depend on p, Ω, on the Lipschitz constants of the coefficients a_j^k's and on the $L^\infty-$norm of the functions in* (8)*.*

Hypotheses (6) and (7) in the smooth case ensure Hörmander condition of step 2. Let us remark that hypothesis (6) is not restrictive. Indeed, given a family $X_1, \dots, X_m$ of Lipschitz continuous vector fields linearly independent at a point $\bar{x}$, we can find a new family $Y_j = \sum_{j=1}^m M_{j,k} X_k$, $j = 1, \dots, m$ of the form (6), where $(M_{j,k})_{j,k=1,\dots,m}$ is a non-singular matrix with Lipschitz continuous entries in a neighborhood of $\bar{x}$.

Our method is inspired to some results of Rothschild and Stein [22] and to those of Citti, Lanconelli and Montanari [3], where a version of (1) for a pair of vector fields in $\mathbb{R}^3$ is proved. The main idea is to approximate the operators X_j at each fixed point with a family of vector fields which are left invariant on a nilpotent Lie group of step two. However, Rothschild and Stein in [22] first introduce local coordinates naturally associated with the vector fields and then, in this new system of canonical coordinates, they approximate the vector fields. Their procedure requires high regularity on the coefficients. Since our coefficients are only Lipschitz continuous, we will first approximate the vector fields with a family of smooth and left invariant first order operators on a nilpotent Lie group of step two and then we will introduce a change of variables. Our "freezing method" has been first introduced by Citti in [1] for $C^{1,\alpha}$ coefficients and then adapted to Lipschitz continuous vector fields in [2]. A consequence of our different approximation appears for $p > 2Q$, since the exponent $\alpha = \min(1/2, 1-Q/p)$ in Morrey estimate, for a function f with degenerate gradient in L^p, is worse than the usual exponent $1 - Q/p$, which arises in the smooth case.

We conclude the introduction by remarking that our results hold in the class of compactly supported functions. This is a consequence of the fact that our method relies on several integrations by part. Moreover, the quasi–distance ρ in Morrey estimate is not the Carnot–Carathéodory distance. It would be interesting to study the Carnot–Carathéodory distance and to discuss the validity of a Poincaré inequality (for non–compactly supported functions) in the regularity assumptions of the present paper.

This paper is organized as follows. In Section 2 we introduce at any point $\bar{x} \in \mathbb{R}^n$ some "frozen" vector fields $X_{j,\bar{x}}$ and we show that there exists an explicit change of variable which transforms them into a nilpotent family independent on the point $\bar{x}$. We then define the quasi–distance ρ and get various estimates of derivatives of the fundamental solution $\Gamma_{\bar{x}}$ of the operator $\sum_{j=1}^m X_{j,\bar{x}}^2$ in term of ρ.

In Section 3 we prove statements (A) and (B) of Theorem 1.1 by means of a representation formula for $f \in C_0^1(\Omega)$ in term of $\Gamma_{\bar{x}}$.

In Section 4 we prove statements (C) and (D) of Theorem 1.1 by varying the point $\bar{x}$ in our representation formula.

In this paper we will use the following notations: c is a generic constant which may change even in a single string of estimates; we write $a \simeq b$ to state that there exist two positive constants c_1 and c_2 such that $c_1 b \le a \le c_2 b$; $\|f\|_p$ or $\|f\|_{L^p}$ will denote the L^p-norm of a function f on the set Ω; $|A|$ stands for the Lebesgue measure of the set $A \subset \mathbb{R}^n$; a vector field $X = \sum_{k=1}^n a_k(x)\partial_k$ will be identified with the vector function $(a_1(x), \ldots, a_n(x))$.

Acknowledgments. Authors would like to thank Ermanno Lanconelli for his encouragement and Giovanna Citti for some discussions about the freezing method.

2 Freezing method and the quasi-distance ρ

In this section, starting from a family of m vector fields of the form (9), we introduce at any point $\bar{x} \in \mathbb{R}^n$ some "frozen" vector fields $X_{j,\bar{x}}$ and we show that the latter (modulo an explicit change of variable) can be transformed in a family H_j, $j = 1, \ldots, m$ which does not depend on the point $\bar{x}$.

Consider in $\mathbb{R}^n$ the vector fields

$$X_j = \partial_j + \sum_{k=m+1}^{n} a_j^k \partial_k, \qquad j = 1, \ldots, m, \tag{9}$$

where the $a_j^k = a_j^k(x)$ are Lipschitz continuous real valued functions defined on $\mathbb{R}^n$. Assume that the fields are of step 2, in the sense of (7). This ensures that the total number $m(m-1)/2$ of commutators $[X_i, X_j]$, $1 \le i < j$ satisfies $m(m-1)/2 \ge n-m$. Fix a family $\mathcal{A} \subset \{(i,l) : 1 \le i < l \le m\}$ with $n-m$ objects and a bijective correspondence

$$\begin{aligned} &g : \{m+1, \dots, n\} \to \mathcal{A}, \qquad g(k) = (g_1(k), g_2(k)),\\ &h = g^{-1} : \mathcal{A} \to \{m+1, \dots, n\}. \end{aligned} \tag{10}$$

For any $k = m+1, \dots, n$ and $i, j = 1, \dots, m$ let

$$b_{i,j}^k = \begin{cases} 1 & \text{if } (i,j) = g(k),\\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Define for any fixed $\bar{x} \in \mathbb{R}^n$

$$\begin{aligned} a_{j,\bar{x}}^k(x) &= a_j^k(\bar{x}) + \sum_{i=1}^m b_{i,j}^k (x_i - \bar{x}_i), \quad x \in \mathbb{R}^n\\ X_{j,\bar{x}} &= \partial_j + \sum_{k=m+1}^n a_{j,\bar{x}}^k \partial_k. \end{aligned} \tag{12}$$

In the sequel we will use the following elementary properties:

$$\begin{aligned} &|a_{j,\bar{x}}^k(x) - a_j^k(x)| \le c|x - \bar{x}|\\ &X_i a_{j,\bar{x}}^k(x) - X_i a_{j,\bar{y}}^k(x) = 0, \qquad \forall x, \bar{x}, \bar{y} \in \mathbb{R}^n, \end{aligned} \tag{13}$$

where the constant c depends on the Lipschitz constants of the a_j^k's.

If $1 \le l < j \le m$, then

$$\begin{aligned} [X_{l,\bar{x}}, X_{j,\bar{x}}] &= \partial_l \Big(\sum_{\substack{k=m+1,\dots,n\\ i=1,\dots,m}} b_{i,j}^k (x_i - \bar{x}_i) \partial_k \Big)\\ &\quad - \partial_j \Big(\sum_{\substack{k=m+1,\dots,n\\ i=1,\dots,m}} b_{i,l}^k (x_i - \bar{x}_i) \partial_k \Big)\\ &= \sum_{k>m} \Big(b_{l,j}^k \partial_k - b_{j,l}^k \partial_k \Big) = \sum_{k>m} b_{l,j}^k \partial_k = \frac{\partial}{\partial x_{h(l,j)}}. \end{aligned} \tag{14}$$

Define, for a fixed $\bar{x} \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^n$,

$$\begin{aligned} \phi_{\bar{x}}(u) &= \exp\Big(\sum_{j=1}^m u_j X_{j,\bar{x}} + \sum_{(l,j) \in \mathcal{A}} u_{h(l,j)} [X_{l,\bar{x}}, X_{j,\bar{x}}] \Big)(\bar{x})\\ &= \exp\Big(\sum_{j=1}^m u_j X_{j,\bar{x}} + \sum_{(l,j) \in \mathcal{A}} u_{h(l,j)} \partial_{h(l,j)} \Big)(\bar{x})\\ &= \exp\Big(\sum_{j=1}^m u_j X_{j,\bar{x}} + \sum_{k=m+1}^n u_k \partial_k \Big)(\bar{x}) \end{aligned}$$

The exponential map $\phi_{\bar{x}}(u)$ can be explicitly written by solving the Cauchy problem

$$\begin{cases} \dot{\gamma}(s) = & \displaystyle\sum_{j=1}^{m} u_j \Big\{ e_j + \sum_{k=m+1}^{n} \Big(a_j^k(\bar{x}) + \sum_{i=1}^{j-1} b_{i,j}^k (\gamma_i(s) - \bar{x}_i) \Big) e_k \Big\} \\ & \quad + \displaystyle\sum_{k=m+1}^{n} u_k e_k, \\ \gamma(0) = & \bar{x}, \end{cases}$$

and letting $\phi_{\bar{x}}(u) = \gamma(1)$ (e_j, $j = 1, \dots, n$ denotes the Euclidean versor). If $i \leq m$ then $\gamma_i(s) = \bar{x}_i + u_i s$, while, if $k > m$, $\dot{\gamma}_k(s) = \sum_{j=1}^{m} u_j a_j^k(\bar{x}) + \sum_{1 \leq i < j \leq m} u_j b_{i,j}^k u_i s + u_k$. Thus, integrating over $[0, 1]$

$$\begin{aligned} \phi_{\bar{x}}(u) = \gamma(1) = & \sum_{j=1}^{m} (\bar{x}_j + u_j) e_j \\ & + \sum_{j=m+1}^{n} \Big(\bar{x}_j + u_j + A^j(\bar{x}) \cdot u + \frac{1}{2} B^j u \cdot u \Big) e_j. \end{aligned} \tag{15}$$

In (15) we have introduced, for $j = m+1, \dots, n$, the notation

$$\begin{aligned} A^j(\bar{x}) \cdot u &= \sum_{i=1}^{m} a_i^j(\bar{x}) u_i, \\ B^j u \cdot v &= \sum_{i,l=1}^{m} b_{i,l}^j u_i v_l = u_{g_1(j)} v_{g_2(j)} \end{aligned}$$

(the last equality follows from (11)). Denote by $\psi_{\bar{x}}$ the inverse map of $\phi_{\bar{x}}$. Then

$$\begin{aligned} u = \psi_{\bar{x}}(x) = & \sum_{j=1}^{m} (x_j - \bar{x}_j) e_j + \sum_{j=m+1}^{n} \Big\{ x_j - \bar{x}_j \\ & - (x - \bar{x}) \cdot A^j(\bar{x}) - \frac{1}{2} B^j (x - \bar{x}) \cdot (x - \bar{x}) \Big\} e_j. \end{aligned} \tag{16}$$

In order to write now the vector field $X_{j,\bar{x}}$ in the coordinates u recall first that, if $j = m+1, \dots, n$, then $\partial/\partial x_j = \partial/\partial u_j$. If $j \leq m$ the derivative ∂_j transforms as follows:

$$
\begin{aligned}
\frac{\partial}{\partial x_j} &= \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} = \sum_{k=1}^{m} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} + \sum_{k=m+1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} \\
&= \frac{\partial}{\partial u_j} + \sum_{k=m+1}^{n} \frac{\partial}{\partial x_j}\Big\{(x-\bar{x})_k - A^k(\bar{x})\cdot(x-\bar{x}) \\
&\qquad - \frac{1}{2}B^k(x-\bar{x})\cdot(x-\bar{x})\Big\}\frac{\partial}{\partial u_k} \\
&= \frac{\partial}{\partial u_j} + \sum_{k=m+1}^{n} \Big\{-a_j^k(\bar{x}) - \frac{\partial}{\partial x_j}\Big(\frac{1}{2}B^k(x-\bar{x})\cdot(x-\bar{x})\Big)\Big\}\frac{\partial}{\partial u_k} \\
&= \frac{\partial}{\partial u_j} - \sum_{k=m+1}^{n} a_j^k(\bar{x})\frac{\partial}{\partial x_k} \\
&\qquad - \frac{1}{2}\sum_{k=m+1}^{n} \frac{\partial}{\partial x_j}\Big((x-\bar{x})_{g_1(k)}\cdot(x-\bar{x})_{g_2(k)}\Big)\frac{\partial}{\partial x_k},
\end{aligned}
$$

where we used the property $B^k\xi\cdot\xi = \xi_{g_1(k)}\xi_{g_2(k)}$. Then the definition of $X_{j,\bar{x}}$ gives

$$
\begin{aligned}
X_{j,\bar{x}} &= \frac{\partial}{\partial x_j} + \sum_{k=m+1}^{n} a_j^k(\bar{x})\frac{\partial}{\partial x_k} + \sum_{\substack{k=m+1,\dots,n\\ i=1,\dots,m}} b_{i,j}^k(x_i-\bar{x}_i)\frac{\partial}{\partial x_k} \\
&= \frac{\partial}{\partial u_j} - \frac{1}{2}\sum_{k=m+1}^{n} \frac{\partial}{\partial x_j}(x-\bar{x})_{g_1(k)}(x-\bar{x})_{g_2(k)}\frac{\partial}{\partial x_k} \\
&\quad + \sum_{\substack{k=m+1,\dots,n\\ i=1,\dots,m}} b_{i,j}^k(x-\bar{x})_i\frac{\partial}{\partial x_k} \\
&= \frac{\partial}{\partial u_j} - \frac{1}{2}\sum_{\{k:g_1(k)=j\}} (x-\bar{x})_{g_2(k)}\frac{\partial}{\partial x_k} \\
&\quad - \frac{1}{2}\sum_{\{k:g_2(k)=j\}} (x-\bar{x})_{g_1(k)}\frac{\partial}{\partial x_k} + \sum_{k=m+1}^{n}\sum_{\{i:(i,j)=g(k)\}} (x-\bar{x})_i\frac{\partial}{\partial x_k} \\
&= \frac{\partial}{\partial u_j} - \frac{1}{2}\sum_{\{k:g_1(k)=j\}} u_{g_2(k)}\frac{\partial}{\partial u_k} + \frac{1}{2}\sum_{\{k:g_2(k)=j\}} u_{g_1(k)}\frac{\partial}{\partial u_k} \\
&= \frac{\partial}{\partial u_j} - \frac{1}{2}\sum_{\{l:(j,l)\in\mathcal{A}\}} u_l\frac{\partial}{\partial u_{h(j,l)}} + \frac{1}{2}\sum_{\{l:(l,j)\in\mathcal{A}\}} u_l\frac{\partial}{\partial u_{h(l,j)}} = H_j.
\end{aligned}
$$

The vector fields H_j, $j = 1,\dots,m$ are homogeneous of degree 1 with respect to the family of dilations $(D(\lambda))_{\lambda>0}$ defined by

$$
D(\lambda)(\xi_1,\dots,\xi_n) = (\lambda\xi_1,\dots,\lambda\xi_m,\lambda^2\xi_{m+1},\dots,\lambda^2\xi_n), \quad \xi\in\mathbb{R}^n.
$$

Moreover, the vector fields H_j are left invariant on the Lie Group $(\mathbb{R}^n,\circ)$, where the law "$\circ$" is defined for $u, v \in \mathbb{R}^n$ by

$$u \circ v = \sum_{j=1}^{m} (u_j + v_j) e_j + \sum_{j=m+1}^{n} \Big(u_j + v_j + \frac{1}{2}(B^j u \cdot v - B^j v \cdot u) \Big) e_j.$$

Obviously $u^{-1} = -u$ and

$$u^{-1} \circ v = \sum_{j=1}^{m} (v_j - u_j) e_j + \sum_{j=m+1}^{n} \Big(v_j - u_j - \frac{1}{2}(B^j u \cdot v - B^j v \cdot u) \Big) e_j. \tag{17}$$

We now use the change of variable $\phi_{\bar{x}}$ in order to get precise estimates of the fundamental solution (and of its derivatives) of the operator $L_{\bar{x}} = \sum_{j=1}^{m} X_{j,\bar{x}}^2$ in term of a suitable quasi–distance.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Let $a_j^k \in Lip(\Omega)$, $j = 1, \dots, m$, $k = m+1, \dots, n$. Consider the function*

$$\rho(x,y) = \sum_{j=1}^{m} |y_j - x_j| + \sum_{j=m+1}^{n} \Big| (y-x)_j - A^j(x) \cdot (y-x) \Big|^{1/2}, \tag{18}$$

$x, y \in \Omega$. Then there exist two positive constants c_1 and c_2 depending on Ω and on the Lipschitz norm of a_j^k, such that

$$\begin{aligned} &\rho(x,y) \le c_1 \rho(y,x), \quad x, y \in \Omega \\ &\rho(x,y) \le c_2 (\rho(x,z) + \rho(z,y)), \quad x, y, z \in \Omega. \end{aligned}$$

Moreover there are positive constants c_3 and c_4 such that

$$c_3^{-1} |x-y| \le \rho(x,y) \le c_3 |x-y|^{1/2}, \qquad x, y \in \Omega \tag{19}$$

$$|\{ y : \rho(x,y) < r \}| = c_4 r^Q, \quad r > 0, \quad Q = m + 2(n-m). \tag{20}$$

Proof. We start by showing (19).

$$\begin{aligned} |x-y| \le& \sum_{j=1}^{m} |x_j - y_j| \\ &+ \sum_{j=m+1}^{n} \Big\{ |x_j - y_j - A^j(x) \cdot (y-x)| + |A^j(x) \cdot (y-x)| \Big\} \\ \le& \sum_{j=1}^{m} |x_j - y_j| \\ &+ c \sum_{j=m+1}^{n} |x_j - y_j - A^j(x) \cdot (y-x)|^{1/2} + c \sum_{j=1}^{m} |y_j - x_j| \\ \le& \, c \rho(x,y). \end{aligned}$$

Here we have used the boundedness of Ω and of $|A^j(x)|$, $x \in \Omega$. Again by the boundedness of Ω and of $|A^j(x)|$, $x \in \Omega$, together with the Euclidean triangle inequality, we get the estimate $\rho(x,y) \leq c|x-y|^{1/2}$.

In order to check that ρ is a quasi–distance we first prove that $\rho(x,y) \leq c\rho(y,x)$.

$$\begin{aligned}
\rho(x,y) &= \sum_{j=1}^{m} |x_j - y_j| + \sum_{j=m+1}^{n} |y_j - x_j - A^j(x)\cdot(y-x)|^{1/2} \\
&\leq \sum_{j=1}^{m} |x_j - y_j| + \sum_{j=m+1}^{n} |y_j - x_j - A^j(y)\cdot(y-x)|^{1/2} \\
&\qquad + \sum_{j=m+1}^{n} |(A^j(x) - A^j(y))\cdot(y-x)|^{1/2} \\
&\leq \rho(y,x) + c|x-y| \leq c\rho(y,x),
\end{aligned}$$

by (19).

The "triangle inequality" can be proved as follows:

$$\begin{aligned}
\rho(x,y) &= \sum_{j=1}^{m} |x_j - y_j| + \sum_{j=m+1}^{n} |y_j - x_j - A^j(x)\cdot(y-x)|^{1/2} \\
&\leq \sum_{j=1}^{m} (|y_j - z_j| + |z_j - x_j|) \\
&\quad + \sum_{j=m+1}^{n} \Big\{ |y_j - z_j - A^j(z)\cdot(y-z)|^{1/2} \\
&\quad + |z_j - x_j + A^j(z)\cdot(y-z) - A^j(x)\cdot(y-x)|^{1/2} \Big\} \\
&= \rho(z,y) + \sum_{j=1}^{m} |z_j - x_j| + \sum_{j=m+1}^{n} |z_j - x_j - A^j(x)\cdot(z-x)|^{1/2} \\
&\quad + \sum_{j=m+1}^{n} |(A^j(x) - A^j(z)\cdot(z-y)|^{1/2} \\
&\leq \rho(z,y) + \rho(x,z) + c|x-z|^{1/2}\,|z-y|^{1/2} \\
&\leq c(\rho(z,y) + \rho(x,z)).
\end{aligned}$$

The estimate (20) of the measure can be obtained by the change of variable $y \mapsto \xi$, $\xi_j = y_j - x_j$, if $1 \leq j \leq m$ and $\xi_j = y_j - x_j - A^j(x)\cdot(y-x)$, if $m+1 \leq j \leq n$. The change of variable has Jacobian 1 and thus

$$|\{y : \rho(x,y) < r\}| = \int_{\rho(x,y)<r} dy = \int_{\sum_{j\leq m}|\xi_j| + \sum_{j>m}|\xi_j|^{1/2} < r} d\xi = cr^Q.$$

□

By Proposition 2.1, ρ is equivalent to the symmetric function $\tilde{\rho}(x,y) = \rho(x,y) + \rho(y,x)$ which turns out to be a quasi–distance. The set Ω, equipped with the Lebesgue measure and the quasi–distance $\tilde{\rho}$ is a Homogeneous Space, in the sense of Coifman and Weiss. Then the following result holds.

Corollary 2.2 *The "fractional integration" operator defined by*

$$T_\mu f(x) = \int_\Omega \frac{f(y)dy}{\rho(x,y)^{Q-\mu}}, \qquad x \in \Omega, \qquad 0 < \mu < Q$$

satisfies

$$\|T_\mu f\|_{pQ/(Q-\mu p)} \leq c_5 \|f\|_p, \quad 1 < p < Q/\mu, \quad f \in L^p(\Omega), \tag{21}$$

$$\sup_{\lambda > 0} \lambda |\{x : |T_\mu f(x)| > \lambda\}|^{(Q-\mu)/Q} \leq c_6 \|f\|_{L^1}, \qquad f \in L^1(\Omega), \tag{22}$$

$$\int_\Omega \exp\Big(\frac{|T_\mu f(x)|}{c_7 \|f\|_{Q/\mu}}\Big)^{Q/(Q-\mu)} \leq c_8, \quad f \in L^{Q/\mu}(\Omega). \tag{23}$$

Proof. The proof of (21) and (22) are contained in [14], while (23) is proved in [15]. $\square$

Denote by $\Gamma_{\bar{x}}(x, \cdot)$ the fundamental solution of the hypoelliptic operator $\sum_{j=1}^m X_{j,\bar{x}}^2$ with pole at x. In the following proposition we give some pointwise estimates concerning $\Gamma_x(x,y)$.

Proposition 2.3 *Assume that $a_j^k \in Lip(\mathbb{R}^n)$ and define $a_{j,x}^k$ as in* (12). *Given an integer $k > 0$, denote by X_x^k any derivative of the form $X_{j_1,x} X_{j_2,x} \cdots X_{j_k,x}$, $j_i \in \{1, ..., m\}$. Denote also by $(a_x(y) - a(y))^k$ any product of the form $\prod_{l=1}^k (a_{j_l,x}^{k_l}(y) - a_{j_l}^{k_l}(y))$, $j_i \in \{1, ..., m\}$, $k_i \in \{m+1, ..., n\}$. Then there exists $c > 0$ such that*

$$|X_x^k \Gamma_x(x,y)| \leq c \frac{1}{\rho(x,y)^{Q-2+k}}, \quad x, y \in \Omega \tag{24}$$

Moreover, there exists a constant $\varepsilon \in (0,1)$ such that, if $x, y, z \in \Omega$ and $\rho(z,x) \leq \varepsilon \rho(x,y)$, then

$$|X_x^k \Gamma_x(x,y) - X_z^k \Gamma_z(z,y)| \leq c \rho(z,x)^{1/2} \frac{1}{\rho(x,y)^{Q+k-3/2}} \tag{25}$$

Remark 2.4 *In the smooth case estimate the right hand side in* (25) *has the better form $c\rho(z,x)/\rho(x,y)^{Q+k-1}$.*

Proof of Proposition 2.3. It is easy to check that the fundamental solution of $\sum_{j=1}^m X_{j,\bar{x}}^2$ can be written as $\Gamma_{\bar{x}}(x,y) = \Gamma_H(\psi_{\bar{x}}(x), \psi_{\bar{x}}(y))$, where Γ_H denotes the fundamental solution of the sublaplacian $\sum H_j^2$. In particular, if $x = \bar{x}$, we have $\Gamma_x(x,y) = \Gamma_H(\psi_x(x), \psi_{\bar{x}}(y)) = \Gamma_H(0, \psi_{\bar{x}}(y)) = \Gamma_H(\psi_{\bar{x}}(y))$, by a slight abuse of notation. Thus

$$|X_x^k \Gamma_x(x,y)| = |H^k \Gamma_H(\psi_x(y))| \leq c \frac{1}{\|\psi_x(y)\|^{Q-2+k}}.$$

The last inequality is a standard estimate concerning the fundamental solution of a sub-Laplacian on a Carnot Group (see [6]). Here we let $\|u\| = \sum_{j=1}^{n} |u_j| + \sum_{j=m+1}^{n} |u_j|^{1/2}$. The function $u \mapsto \|u\|$ is homogeneous of degree 1 with respect to the family of dilations $(D(\lambda))_{\lambda>0}$. Hence in the sequel we will call it homogeneous norm.

Now, by the explicit form (16) of $\psi_x(y)$, the proof of (24) can be concluded as follows:

$$\begin{aligned}
\|\psi_x(y)\| &= \Big\| \sum_{j=1}^{m} (y-x)_j e_j + \sum_{j=m+1}^{n} \Big\{ (y-x)_j - (y-x)\cdot A^j(x) \\
&\qquad + \frac{1}{2} B^j(y-x)\cdot(y-x) \Big\} e_j \Big\| \\
&= \sum_{j=1}^{m} |y_j - x_j| + \sum_{j=m+1}^{n} \Big| (y-x)_j - (y-x)\cdot A^j(x) \\
&\qquad + \frac{1}{2} B^j(y-x)\cdot(y-x) \Big|^{1/2} \\
&\simeq \sum_{j=1}^{m} |y_j - x_j| + \sum_{j=m+1}^{n} \Big| (y-x)_j - (y-x)\cdot A^j(x) \Big|^{1/2} \\
&= \rho(x,y).
\end{aligned}$$

The last equivalence is a consequence of the estimate $|B^j(y-x)\cdot(y-x)|^{1/2} \leq c\sum_{k=1}^{m} |y_k - x_k|$.

In order to prove (25) we first prove that

$$\rho(x,z) \leq \varepsilon\rho(x,y) \Rightarrow \|\psi_x(y)^{-1} \circ \psi_z(y)\| \leq c\sqrt{\varepsilon}\|\psi_x(y)\|, \tag{26}$$

where c does not depend on $x, y \in \Omega$. The group law (17) gives

$$\begin{aligned}
&\psi_x(y)^{-1} \circ \psi_z(y) \\
&= \sum_{j=1}^{m} \left[(y_j - z_j) - (y_j - x_j) \right] e_j \\
&\quad + \sum_{j=m+1}^{n} \Big\{ \Big[y_j - z_j - (y-z)\cdot A^j(z) - \frac{1}{2} B^j(y-z)\cdot(y-z) \Big] e_j \\
&\quad - \Big[(y-x)_j - (y-x)\cdot A^j(x) - \frac{1}{2} B^j(y-x)\cdot(y-x) \Big] e_j \\
&\quad - \frac{1}{2} \Big[B^j(y-x)\cdot(y-z) - B^j(y-z)\cdot(y-x) \Big] \Big\} e_j \\
&= \sum_{j=1}^{m} (x_j - z_j) e_j + \sum_{j=m+1}^{n} \Big\{ x_j - z_j - A^j(z)\cdot(y-z) + A^j(x)\cdot(y-x) \\
&\quad - \frac{1}{2} \Big[B^j(y-x)\cdot(x-z) + B^j(y-z)\cdot(x-z) \Big] \Big\}.
\end{aligned}$$

The homogeneous norm is

$$
\begin{aligned}
&\|\psi_x(y)^{-1} \circ \psi_z(y)\| \\
&= \sum_{j=1}^{m} |x_j - z_j| + \sum_{j=m+1}^{n} \Big| x_j - z_j - A^j(z) \cdot (y-z) + A^j(x) \cdot (y-x) \\
&\qquad - \frac{1}{2} \Big[B^j(y-x) \cdot (x-z) + B^j(y-z) \cdot (x-z) \Big] \Big|^{1/2} \\
&\le \sum_{j=1}^{m} |x_j - z_j| + \sum_{j=m+1}^{n} \Big| x_j - z_j - A^j(z) \cdot (x-z) \Big|^{1/2} \\
&\quad + \sum_{j=m+1}^{n} \Big| (A^j(z) - A^j(x)) \cdot (x-y) \\
&\quad - \frac{1}{2} [B^j(y-x) \cdot (x-z) + B^j(y-z) \cdot (x-z)] \Big|^{1/2} \\
&\le \rho(z,x) + c(|z-x|\,|x-y|)^{1/2} \le c\rho(z,x) + \rho(x,y)^{1/2}\rho(z,x)^{1/2} \\
&\le c\sqrt{\varepsilon}\rho(x,y),
\end{aligned}
\tag{27}
$$

as soon as $\rho(x,z) \le \varepsilon\rho(x,y)$. Thus (26) is proved. Then, if $\rho(x,z) \le \varepsilon\rho(x,y)$ and $\varepsilon > 0$ is small enough, by (26), (27) and the standard Calderon–Zygmund type estimate for the derivatives of the fundamental solution of sub-laplacians on Carnot Groups, we get

$$
\begin{aligned}
|X_x^k \Gamma_x(x,y) - X_z^k \Gamma_z(z,y)| &= |H^k \Gamma_H(\psi_x(y)) - H^k \Gamma_H(\psi_z(y))| \\
&\le c \frac{\|\psi_x(y)^{-1} \circ \psi_z(y)\|}{\|\psi_x(y)\|^{Q+k-1}}.
\end{aligned}
$$

Deleting the last line in (27) we also get the estimate $\|\psi_x(y)^{-1} \circ \psi_z(y)\| \le c\rho(x,y)^{1/2}\rho(z,x)^{1/2}$, which ends the proof of (25). □

3 Sobolev and Trudinger inequalities

In this section we prove part (A) and (B) of Theorem 1.1. The proof will be a consequence of the following proposition.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Denote by ρ the function introduced in Proposition 2.1. Then there exists a constant $c > 0$ such that, for any $f \in C_0^\infty(\Omega)$ the following estimate holds:*

$$
|f(x)| \le c \int_\Omega \frac{|Xf(y)|}{\rho(x,y)^{Q-1}} dy.
$$

Proof. Representing the function $f \in C_0^\infty(\Omega)$ by means of the fundamental solution $\Gamma_x(x,y)$ of the operator $\sum_{j=1}^m X_{j,x}^2$ with pole at x we have

$$\begin{aligned} f(x) &= \sum_{j=1}^m \int_\Omega X_{j,x}\Gamma_x(x,y)X_{j,x}f(y)dy \\ &= \sum_{j=1}^m \int_\Omega X_{j,x}\Gamma_x(x,y)(X_{j,x}-X_j+X_j)f(y)dy \\ &= \sum_{j=1}^m \int_\Omega X_{j,x}\Gamma_x(x,y) \sum_{k=m+1}^n (a_{j,x}^k(y)-a_j^k(y))\frac{\partial}{\partial y_k}f(y)dy \\ &\quad + \sum_{j=1}^m \int_\Omega X_{j,x}\Gamma_x(x,y)X_jf(y)dy \\ &:= I_1(x)+I_1'(x). \end{aligned} \tag{28}$$

Proposition 2.3 gives the desired estimate of I_1':

$$|I_1'(x)| \le c\int \frac{|Xf(y)|}{\rho(x,y)^{Q-1}}.$$

Rewrite now (7) in the form

$$\partial_{y_k} = \sum_{i,l=1}^m \tilde\lambda_k^{i,l}(y)X_iX_l, \tag{29}$$

where we let $\tilde\lambda_k^{i,l}=\lambda_k^{i,l}$ if $i<l$, $\tilde\lambda_k^{i,l}=-\lambda_k^{l,i}$ if $i>l$ and $\tilde\lambda_k^{i,i}=0$. The functions $\lambda_k^{i,l}$ also satisfy hypothesis (8). Thus the evaluation of I_1 will be ensured by the estimate of the following integral

$$J(x) := \int X_{j,x}\Gamma_x(x,y)(a_{j,x}^k(y)-a_j^k(y))\tilde\lambda_k^{i,l}(y)X_iX_lf(y)dy, \tag{30}$$

$i,j,l=1,\dots,m$, $k=m+1,\dots,n$.

We now integrate by part in (30). In order to deal correctly with the singular terms we first truncate the domain of integration. Write for brevity

$$g(x,y) = X_{j,x}\Gamma_x(x,y)(a_{j,x}^k(y)-a_j^k(y))\tilde\lambda_k^{i,l}(y). \tag{31}$$

Since by (24) and (20) the function $y \mapsto X_{j,x}\Gamma_x(x,y)$ is integrable in Ω and all the remaining functions appearing in integral (30) are bounded we have $J(x)=\lim_{\varepsilon\to 0}J_\varepsilon(x)$, where

$$J_\varepsilon(x) = \int_{\Omega_\varepsilon} g(x,y)X_iX_lf(y)dy$$

and $\Omega_\varepsilon = \Omega\setminus\{y : \Gamma_x(x,y) > \varepsilon^{2-Q}\}$. Recall that the functions $\tilde\lambda_k^{i,l}$ are bounded together with their distributional derivatives $X_i\tilde\lambda_k^{i,l}$. Then, by the results of [11] and [13], there exists a family $(\tilde\lambda_k^{i,l})_\sigma$, $\sigma>0$ of smooth functions on Ω such that $(\tilde\lambda_k^{i,l})_\sigma \to \tilde\lambda_k^{i,l}$ and $X_i(\lambda_k^{i,l})_\sigma \to X_i\tilde\lambda_k^{i,l}$ in $L^2(K)$, as $\sigma\to 0$, where K is the support of f. Approximating the functions $\tilde\lambda_k^{i,l}$ in the described way, we can apply on the set Ω_ε the following integration by part formula

$$\begin{aligned}\int_{\Omega_\varepsilon} g(y)Xv(y)dy = &\int_{\partial\Omega_\varepsilon} g(y)v(y)X(y)\cdot\nu(y)d\sigma(y)\\ &-\int_{\Omega_\varepsilon}(Xg(y)+\mathrm{div}X(y)g(y))v(y)dy,\end{aligned}$$

where v and g are Lipschitz continuous functions, $X=\sum_{j=1}^n a^j\partial_j$ is a Lipschitz continuous vector field, $\mathrm{div}X(y)=\sum_{j=1}^n \partial_j a^j(y)$, $\nu(y)$ is the outer normal to Ω_ε at the point y, $d\sigma$ denotes the surface-measure and $X(y)\cdot\nu(y)=\sum_{j=1}^n a_j(y)\nu_j(y)$.

Thus

$$\begin{aligned}J_\varepsilon(x) &= \int_{\partial\Omega_\varepsilon} g(x,y)X_l f(y)X_i(y)\cdot\nu(y)d\sigma(y)\\ &\quad -\int_{\Omega_\varepsilon}(X_i+\mathrm{div}X_i)g(x,y)X_l f(y)dy\\ &= J'_\varepsilon(x)+J''_\varepsilon(x).\end{aligned}$$

We first show that $J'_\varepsilon(x)\to 0$, as $\varepsilon\to 0$. Denote by ∇ the Euclidean gradient, then $X_i(y)\cdot\nu(y)=X_i(y)\cdot\frac{\nabla\Gamma_x(x,y)}{|\nabla\Gamma_x(x,y)|}=\frac{X_i\Gamma_x(x,y)}{|\nabla\Gamma_x(x,y)|}$. Moreover, by also using (14) and (10)

$$\begin{aligned}X_i\Gamma_x(x,y) &= (X_i-X_{i,x})\Gamma_x(x,y)+X_{i,x}\Gamma_x(x,y)\\ &= \sum_{k=m+1}^n (a_i^k-a_{i,x}^k)(y)\partial_{y_k}\Gamma_x(x,y)+X_{i,x}\Gamma_x(x,y)\\ &= \sum_{k=m+1}^n (a_i^k-a_{i,x}^k)(y)[X_{g_1(k),x},X_{g_2(k),x}]\Gamma_x(x,y)\\ &\quad + X_{i,x}\Gamma_x(x,y),\end{aligned}$$

hence

$$\begin{aligned}|g(x,y)X_i\Gamma_x(x,y)| &\le c\rho(x,y)\,|X_{j,x}\Gamma_x(x,y)|\Big(|X_{i,x}\Gamma_x(x,y)|\\ &\quad +\rho(x,y)\sum_{k=m+1}^n |[X_{g_1(k),x},X_{g_2(k),x}]\Gamma_x(x,y)|\Big)\\ &\le c(\rho(x,y))^{3-2Q}.\end{aligned}$$

The function $X_l f(y)$ is bounded on Ω and vanishes on $\partial\Omega$. Thus, for any fixed x we have

$$\begin{aligned}|J'_\varepsilon(x)| &= \left|\int_{\{\Gamma_x(x,y)=\varepsilon^{2-Q}\}} g(x,y)X_l f(y)X_i(y)\cdot\nu(y)d\sigma(y)\right|\\ &\le \frac{c}{\varepsilon^{2Q-3}}\int_{\{\Gamma_x(x,y)=\varepsilon^{2-Q}\}}\frac{1}{|\nabla\Gamma_x(x,y)|}d\sigma(y).\end{aligned}$$

Define

$$F(r) = \frac{1}{r^{Q-1}} \int_{\{\Gamma_x(x,y)=r^{2-Q}\}} \frac{1}{|\nabla \Gamma_x(x,y)|} d\sigma(y),$$

then, by the coarea formula [4, p. 118] with $f(y) = (\Gamma_x(x,y))^{1/(2-Q)}$ and by the change of variables (16), we get

$$\begin{aligned}
\int_0^\varepsilon F(r)dr &= (Q-2)\int_0^\varepsilon \Big(\int_{\{r=f(y)\}} \frac{1}{|\nabla f(y)|} d\sigma(y)\Big) dr \\
&= (Q-2)\int_{\{\Gamma_x(x,y)>\varepsilon^{2-Q}\}} dy \\
&= (Q-2)\int_{\{\Gamma_H(\xi)>\varepsilon^{2-Q}\}} d\xi = (Q-2)\varepsilon^Q \int_{\{\Gamma_H(\xi)>1\}} d\xi
\end{aligned}$$

Hence, by differentiating the previous equality with respect to ε, we get $F(\varepsilon) = c\varepsilon^{Q-1}$ and

$$|J'_\varepsilon(x)| \le \frac{c}{\varepsilon^{Q-2}} F(\varepsilon) = c\varepsilon$$

The last term tends to zero, as $\varepsilon \to 0$.

In order to rewrite $J''_\varepsilon = J''_\varepsilon(x)$ we use the explicit form (31) of g:

$$\begin{aligned}
J''_\varepsilon &= -\int_{\Omega_\varepsilon} X_i \Big[X_{j,x}\Gamma_x(x,y)(a^k_{j,x}(y) - a^k_j(y))\tilde{\lambda}(y)\Big] X_l f(y) dy \\
&\quad - \int_{\Omega_\varepsilon} \mathrm{div} X_i(y) \Big[X_{j,x}\Gamma_x(x,y)(a^k_{j,x}(y) - a^k_j(y))\tilde{\lambda}^{i,l}_j(y)\Big] X_l f(y) dy \\
&= -\int_{\Omega_\varepsilon} X_i X_{j,x}\Gamma_x(x,y)(a^k_{j,x}(y) - a^k_j(y))\tilde{\lambda}^{i,l}_j(y) X_l f(y) dy \\
&\quad - \int_{\Omega_\varepsilon} X_{j,x}\Gamma_x(x,y)(X_i a^k_j(y) - X_i a^k_{j,x}(y))\tilde{\lambda}^{i,l}_j(y) X_l f(y) dy \\
&\quad - \int_{\Omega_\varepsilon} X_{j,x}\Gamma_x(x,y)(a^k_{j,x}(y) - a^k_j(y)) X_i \tilde{\lambda}^{i,l}_j(y) X_l f(y) dy \\
&\quad - \int_{\Omega_\varepsilon} \mathrm{div} X_i(y) \Big[X_{j,x}\Gamma_x(x,y)(a^k_{j,x}(y) - a^k_j(y))\tilde{\lambda}^{i,l}_j(y)\Big] X_h f(y) dy \\
&:= B_\varepsilon(x) + B_{1,\varepsilon}(x) + B_{2,\varepsilon}(x) + B_{3,\varepsilon}(x).
\end{aligned}$$

Rewrite the term B_ε using $X_i = X_i - X_{i,x} + X_{i,x}$:

$$\begin{aligned}
B_\varepsilon(x) &= \sum_{h=m+1}^{n} \int_{\Omega_\varepsilon} (a^h_i(y) - a^h_{i,x}(y))\partial_{y_h} X_{j,x}\Gamma(x,y)(a^k_{j,x}(x) - a^k_{j,x}(y)) \\
&\qquad \cdot \tilde{\lambda}^{i,l}_k(y) X_l f(y) dy \\
&\quad + \int_{\Omega_\varepsilon} X_{i,x} X_{j,x}\Gamma(x,y)(a^k_{j,x}(x) - a^k_{j,x}(y))\tilde{\lambda}^{i,h}_k(y) X_l f(y) dy \\
&= B_{4,\varepsilon}(x) + B_{5,\varepsilon}(x)
\end{aligned}$$

We now show that each of the integrals $B_{k,0}(x)$, $k = 1, \dots, 5$ is well defined. This guarantees that $B_{k,0}(x) = \lim B_{k,\varepsilon}(x)$, as $\varepsilon \to 0$. More precisely we will show that $|B_{k,0}(x)| \le T_1(|Xf|)(x)$. Since $\tilde{\lambda}_k^{i,h} \in L^\infty$ and $|X_i a_{j,x}^k| \le c$, we see that

$$|B_{1,0}(x)| \le c \int_\Omega |X_{j,x}\Gamma(x,y)||X_l f(y)|dy.$$

Then the term $B_{1,0}$ can be estimated using the same arguments of I_1' (see (28)).

The estimate of the term $B_{2,0}$ can be done using the Lipschitz continuity of the fields (which, by also (19), gives $|a_{j,x}^k(y) - a_j^k(y)| \le c|x-y| \le c\rho(x,y)$) and the boundedness of $X_i\tilde{\lambda}_k^{i,h}$. Then

$$\begin{aligned}|B_{2,0}(x)| &\le c \int_\Omega |X_{j,x}\Gamma_x(x,y)|\rho(x,y)|X_i\tilde{\lambda}_k^{i,h}(y)||X_l f(y)|dy \\ &\le cT_2(|Xf|)(x) \le cT_1(|Xf|)(x).\end{aligned}$$

The term $B_{3,0}$ can be evaluated with an argument similar to the one used for $B_{2,0}$.

By (14) and (10) the term $B_{4,0}$ is controlled by an expression of the form:

$$|B_{4,0}(x)| \le c \int_\Omega |X_x^3\Gamma_x(x,y)(a_x - a)^2(y)|Xf(y)|dy \le cT_1(|Xf|)(x).$$

We used here the notations introduced in the statement of Proposition 2.3.

Finally, the estimate of $B_{5,0}$ can be obtained with a similar argument. $\square$

Proof of Sobolev and Trudinger inequality. The proof of Sobolev inequality for $1 < p < Q$ and Trudinger inequality follows from the "representation formula" proved in the previous proposition and from the continuity estimates of Corollary 2.2.

In the case $p = 1$ Corollary 2.2 provides the weak inequality

$$\sup_{\lambda > 0} \lambda|\{|f| > \lambda\}|^{(Q-1)/Q} \le c\|Xf\|_{L^1},$$

which, by the truncation technique used in [8, p. 564], can be improved to the desired estimate

$$\Big(\int_\Omega |f|^{Q/(Q-1)}\Big)^{(Q-1)/Q} \le c\int_\Omega |Xf|.$$

4 Morrey estimate and compact embedding

In the following proposition we give an estimate of the difference between the values of a function f at two different points in term of integrals involving the degenerate gradient of f and suitable kernels. This will enable us to prove the Morrey estimate (C) in Theorem 1.1 and a fractional type estimate which will provide the compact embedding (D) in Theorem 1.1.

Proposition 4.1 *Let $f \in C_0^\infty(\Omega)$. Then*

$$|f(x)-f(z)| \le c\int_\Omega K(x,z;y)|Xf(y)|dy, \qquad x,z\in\Omega,$$

where the kernel K is a sum of a finite number of terms of the form

$$\begin{aligned} R_k(x,z;y) = \Big| & X_x^{k+1}\Gamma_x(x,y)(a_x(y)-a(y))^k \\ & - X_z^{k+1}\Gamma_z(z,y)(a_z(y)-a(y))^k\Big|, \end{aligned} \tag{32}$$

k=0,1,2, and

$$R(x,z;y) = \Big|X_x\Gamma_x(x,y)(a_x(y)-a(y)) - X_z\Gamma_z(z,y)(a_z(y)-a(y))\Big|. \tag{33}$$

In (32) and (33) we used the notations of Proposition 2.3. Moreover, if $\rho(x,z)\le \varepsilon\rho(x,y)$, then

$$K(x,y;z)\le c\frac{\rho(z,x)^{1/2}}{\rho(x,y)^{Q-1/2}}. \tag{34}$$

Proof. For x, $z\in\Omega$ and $f\in C_0^\infty$ write $f(x)$ and $f(z)$ as in (28). Then

$$f(x)-f(z) = (I_1(x)-I_1(z)) + (I_1'(x)-I_1'(z)).$$

The term $(I_1'(x)-I_1'(z))$ has the appropriate form, since

$$|I_1'(x)-I_1'(z)| \le \sum_{j=1}^m\int_\Omega \Big|X_{j,x}\Gamma_x(x,y)-X_{j,z}\Gamma_z(z,y)\Big||X_jf(y)|dy,$$

and the kernel $\Big|X_{j,x}\Gamma_x(x,y)-X_{j,z}\Gamma_z(z,y)\Big|$ is of type (32) with $k=0$. Now we will integrate by part the term $(I_1(x)-I_1(z))$ by using (29):

$$\begin{aligned} I_1(x)-I_1(z) &= \sum_{j=1}^m\sum_{k=m+1}^n\int\Big\{X_{j,x}\Gamma_x(x,y)(a_{j,x}^k-a_j^k)(y) \\ &\qquad - X_{j,z}\Gamma_z(x,y)(a_{j,z}^k-a_j^k)(y)\Big\}\partial_{y_k}f(y)dy \\ &= \sum_{j=1}^m\sum_{k=m+1}^n\sum_{i,l=1}^m\int\Big\{X_{j,x}\Gamma_x(x,y)(a_{j,x}^k-a_j^k)(y) \\ &\qquad - X_{j,z}\Gamma_z(x,y)(a_{j,z}^k-a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y)X_iX_lf(y)dy \\ &= \int S(x,z;y)X_lf(y)dy, \end{aligned}$$

where the kernel S is defined by the last equality through an integration by part. Denote $X_i^* = -X_i - \mathrm{div}(X_i)$ and write for brevity Γ_x instead of $\Gamma_x(x,y)$. By omitting the summation symbol on the indices j, i, k and l, the kernel $S = S(x,z;y)$ has the following form (take also (13) into account)

$$
\begin{aligned}
S &= X_i^*\Big\{\Big(X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) - X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big)\tilde\lambda_k^{i,l}(y)\Big\} \\
&= -\Big\{X_{j,x}\Gamma_x(X_i a_{j,x}^k - X_i a_j^k)(y) - X_{j,z}\Gamma_z(X_i a_{j,z}^k - X_i a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big\{X_i X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) - X_i X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big\{X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) - X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}X_i\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big(X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) - X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big)\mathrm{div}X_i(y)\tilde\lambda_k^{i,l}(y) \\
&= -\Big\{\Big(X_{j,x}\Gamma_x - X_{j,z}\Gamma_z\Big)(X_i a_{j,x}^k - X_i a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big\{(X_i - X_{i,x})X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) \\
&\qquad - (X_i - X_{i,z})X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big\{X_{i,x}X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) \\
&\qquad\qquad - X_{i,z}X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big\{X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) - X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}X_i\tilde\lambda_k^{i,l}(y) \\
&\quad - \Big\{X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) - X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}\mathrm{div}X_i(y) \\
&= -A_1 - A_2 - A_3 - A_4 - A_5.
\end{aligned}
$$

Recall that, by (14) and (10), for any $\xi \in \mathbb{R}^n$, $s = m+1, \dots, n$, $\frac{\partial}{\partial y_s} = [X_{g_1(s),\xi}, X_{g_2(s),\xi}]$. Write this last equality with $\xi = x$ and $\xi = z$. Thus

$$
\begin{aligned}
A_2 = \sum_{s=m+1}^{n} &\Big\{(a_i^s - a_{i,x}^s)(y)[X_{g_1(s),x}, X_{g_2(s),x}]X_{j,x}\Gamma_x(a_{j,x}^k - a_j^k)(y) \\
&- (a_i^s - a_{i,z}^s)(y)[X_{g_1(s),z}, X_{g_2(s),z}]X_{j,z}\Gamma_z(a_{j,z}^k - a_j^k)(y)\Big\}\tilde\lambda_k^{i,l}(y).
\end{aligned}
$$

Then each of the terms $|A_1|$, $|A_2|$, $|A_3|$ can be estimated (using $\tilde\lambda_j^{i,l} \in L^\infty$) by a kernel of the form $R_k(x,z;y)$ in (32) with $k = 0,1,2$.

The terms $|A_4|$ and $|A_5|$ can be estimated (using $X_j\tilde\lambda_k^{i,l}$, $\mathrm{div}X_j \in L^\infty$) by the less singular kernel R defined in (33). The estimate of the kernel R_k can be obtained by means of (25):

$$
\begin{aligned}
R_k(x,z;y) &\le \big|X_x^{k+1}\Gamma_x(x,y) - X_z^{k+1}\Gamma_z(z,y)\big|\,|(a_x(y) - a(y))^k| \\
&\quad + \big|X_z^{k+1}\Gamma_z(z,y)\big|\,\big|(a_x(y) - a(y))^k - (a_z(y) - a(y))^k\big| \\
&\le c\frac{\rho(z,x)^{1/2}}{\rho(x,y)^{Q+k-1/2}}\rho(x,y)^k \\
&\quad + c\frac{1}{\rho(z,y)^{Q+k-1}}|(a_x(y) - a(y))^k - (a_z(y) - a(y))^k|.
\end{aligned}
\tag{35}
$$

The estimate of the last line is non trivial only if $k=2$. By (12) and (13)

$$\begin{aligned}
&|(a_x(y)-a(y))^2-(a_z(y)-a(y))^2| \\
&=|(a^{k_1}_{j_1,x}-a^{k_1}_{j_1})(y)(a^{k_2}_{j_2,x}-a^{k_2}_{j_2})(y)-(a^{k_1}_{j_1,z}-a^{k_1}_{j_1})(y)(a^{k_2}_{j_2,z}-a^{k_2}_{j_2})(y)| \\
&\le|\{(a^{k_1}_{j_1,x}-a^{k_1}_{j_1})(y)-(a^{k_1}_{j_1,z}-a^{k_1}_{j_1})(y)\}(a^{k_2}_{j_2,z}-a^{k_2}_{j_2})(y)| \\
&\quad+|(a^{k_1}_{j_1,z}-a^{k_1}_{j_1})(y)\{(a^{k_2}_{j_2,z}-a^{k_2}_{j_2})(y)-(a^{k_2}_{j_2,x}-a^{k_2}_{j_2})(y)\}| \\
&\le c|z-y||z-x|.
\end{aligned} \tag{36}$$

□

Proof of the Morrey estimate. Let $f\in C_0^\infty(\Omega)$. Write

$$I_k(x,z)=\int_\Omega R_k(x,z;y)|Xf(y)|dy.$$

Starting from Proposition 4.1 estimate (C) of Theorem 1.1 will be proved as soon as we will be able to show that $|I_k(x,z)|\le c\rho^\alpha(x,z)$. Fix $\varepsilon>0$ and write

$$\begin{aligned}
I_k(x,z)=&\int_{\rho(x,z)\ge\varepsilon\rho(x,y)} R_k(x,z;y)|Xf(y)|dy \\
&+\int_{\rho(x,z)\le\varepsilon\rho(x,y)} R_k(x,z;y)|Xf(y)|dy \\
=&I'_k(x,z)+I''_k(x,z).
\end{aligned}$$

The estimate of I'_k is easy: indeed, by (24) and by the Lipschitz continuity of the a^k_j's we can estimate I'_k by the Hölder inequality:

$$\begin{aligned}
I'_k\le c\|Xf\|_{L^p}\Big(&\int_{\rho(x,z)\ge\varepsilon\rho(x,y)}\frac{dy}{\rho(x,y)^{(Q-1)p/(p-1)}} \\
&+\int_{\rho(x,z)\ge\varepsilon\rho(x,y)}\frac{dy}{\rho(z,y)^{(Q-1)p/(p-1)}}\Big)^{(p-1)/p}
\end{aligned} \tag{37}$$

Both the integrals in (37) are well defined (the function $1/\rho^\beta$ is integrable near 0 iff $\beta<Q$ and $p>Q\Leftrightarrow\frac{(Q-1)p}{p-1}<Q$). By the triangle inequality

$$\rho(z,y)\le c(\rho(z,x)+\rho(x,y))\le c\rho(z,x),$$

on the integration set. Then $\{\rho(x,z)>\varepsilon\rho(x,y)\}$ is included in $\{\rho(x,z)\ge c\rho(z,y)\}$. Thus the second integral in (37) can be estimated by the first. Using polar coordinates we have

$$\begin{aligned}
I'_k(x,z)&\le\|Xf\|_{L^p}\Big(\int_{\rho(x,y)\le c\rho(x,z)}\frac{dy}{\rho(x,y)^{(Q-1)p/(p-1)}}\Big)^{(p-1)/p} \\
&\le c\|Xf\|_{L^p}\Big(\int_0^{c\rho(x,z)}\frac{t^{Q-1}dt}{t^{(Q-1)p/(p-1)}}\Big)^{(p-1)/p} \\
&=c\|Xf\|_{L^p}\rho(x,z)^{1-Q/p}.
\end{aligned}$$

By estimate (34) and Hölder inequality we get

$$I_k'' \le c\|Xf\|_{L^p}\rho(x,z)^{1/2}\Big(\int\limits_{\rho(x,z)\le\varepsilon\rho(x,y)} \frac{dy}{\rho(x,y)^{(Q-1/2)p/(p-1)}}\Big)^{(p-1)/p}$$

Using polar coordinates we have

$$\begin{aligned} I_k''(x,z) &\le c\rho(x,z)^{1/2}\|Xf\|_{L^p}\Big(\int\limits_{c\rho(x,z)}^{M} \frac{t^{Q-1}dt}{t^{(Q-1/2)p/(p-1)}}\Big)^{(p-1)/p} \\ &\le c\|Xf\|_{L^p}\rho(x,z)^{\alpha} \end{aligned}$$

with $\alpha = \min\{1/2, 1-Q/p\}$. □

Proof of the compact embedding. We prove that for every $1 < p < Q$, $q < Qp/(Q-p)$ there exists $s > 0$ such that the following inequality holds:

$$\Big(\int_{\Omega\times\Omega} \frac{|f(x)-f(z)|^q}{|x-z|^{n+qs}}dxdz\Big)^{1/q} \le c\Big(\int_{\Omega}|Xf(x)|^p dx\Big)^{1/p}. \tag{38}$$

This, together with the classical compact embedding $W_0^{s,q}(\Omega) \subset L^q(\Omega)$ will prove the claim.

Let $\bar{s} = \bar{s}(p,q) \in (0,1)$ be the solution of

$$\frac{pQ}{Q-(1-\bar{s})p} = q, \quad \text{i.e.} \quad \bar{s} = 1 - \frac{q-p}{pq}Q. \tag{39}$$

This choice of $\bar{s}$ ensures that $\|T_{1-\bar{s}}f\|_q \le c\|f\|_p, \quad f \in L^p$ (by (21) with $\mu = 1-\bar{s}$). Fix now $s < \min\{\bar{s}/3, 1/8\}$, and estimate the fractional norm of a function f as follows: by Proposition 4.1, $|f(x)-f(z)|$ is less than a sum of a finite number of terms of the form $\int_\Omega R_k(x,z;y)|Xf(y)|dy$, with R_k as in (32), plus a finite number of less singular terms which we will not take into account in the sequel. Thus it suffices to evaluate the integral

$$J := \int_{\Omega\times\Omega} \frac{1}{|x-z|^{n+qs}}\Big|\int R_k(x,z;y)|Xf(y)|dy\Big|^q dxdz. \tag{40}$$

For any integral of the form (40) we can assume by symmetry that the integration takes place over the set where

$$|X_x^{k+1}\Gamma_x(x,y)(a_x-a)^k(y)| \ge |X_z^{k+1}\Gamma_z(z,y)(a_z-a)^k(y)|. \tag{41}$$

The change of variable $z = x+h$ and an application of the Minkowski inequality give

$$\begin{aligned} J &\le c\int_\Omega\int_{\{h:x+h\in\Omega\}} \frac{1}{|h|^{n+qs}}\Big|\int_\Omega R_k(x,x+h;y)|Xf(y)|dy\Big|^q dhdx \\ &\le \int_\Omega\Big[\int_\Omega |Xf(y)|\Big\{\int_{A_{x,y}} \frac{R_k(x,x+h;y)^q}{|h|^{n+qs}}dh\Big\}^{1/q}dy\Big]^q dx, \end{aligned}$$

where we let $A_{x,y} = \{h : x + h \in \Omega$ and (41) holds$\}$. Now write (for a suitably small $\varepsilon > 0$)

$$\begin{aligned}\int_{A_{x,y}} \frac{R_k(x, x+h; y)^q}{|h|^{n+qs}} dh &= \int_{\{\rho(x,x+h)\geq\varepsilon\rho(x,y)\}\cap A_{x,y}} \frac{R_k(x, x+h; y)^q}{|h|^{n+qs}} dh \\ &\quad + \int_{\{\rho(x,x+h)\leq\varepsilon\rho(x,y)\}\cap A_{x,y}} \frac{R_k(x, x+h; y)^q}{|h|^{n+qs}} dh \\ &= I_1 + I_2.\end{aligned}$$

By (41) $R_k(x, x+h; y) \leq c/\rho(x, y)^{Q-1}$ and the inequality $\rho(x, x+h) \leq c|h|^{1/2}$ furnishes

$$\begin{aligned}I_1 &\leq c \int_{|h|^{1/2}\geq\varepsilon\rho(x,y)} \frac{dh}{|h|^{n+qs}\rho(x, y)^{(Q-1)q}} \\ &= \frac{c}{\rho(x, y)^{(Q-1+2s)q}} \leq \frac{c}{\rho(x, y)^{(Q-(1-\bar{s}))q}},\end{aligned}$$

because $2s < \bar{s}$. The estimate of I_2 follows from the estimates of R_k given in Proposition 4.1. More precisely (recall also that, since $s < \frac{1}{8}$, then $\frac{q}{2} - 4qs > 0$),

$$\begin{aligned}I_2 &\leq \int_{\rho(x,x+h)\leq\varepsilon\rho(x,y)} \frac{1}{|h|^{n+qs}} \frac{\rho(x, x+h)^{q/2}}{\rho(x, y)^{q(Q-1/2)}} dh \\ &\leq \int_{\rho(x,x+h)\leq\varepsilon\rho(x,y)} \frac{c}{|h|^{n+qs}} \frac{\rho(x, x+h)^{4qs}\rho(x, x+h)^{q/2-4qs}}{\rho(x, y)^{q(Q-1/2)}} dh \\ &\leq \int_{|h|\leq c\varepsilon\rho(x,y)} \frac{|h|^{2qs}}{|h|^{n+qs}} \frac{\rho(x, y)^{q/2-4qs}}{\rho(x, y)^{q(Q-1/2)}} dh \\ &= c\int_{|h|<c\rho(x,y)} \frac{dh}{|h|^{n-qs}} \frac{c}{\rho(x, y)^{(Q-1+4s)q}} \\ &= \frac{c}{\rho(x, y)^{q(Q-(1-3s))}} \leq \frac{c}{\rho(x, y)^{q(Q-(1-\bar{s}))}},\end{aligned}$$

because $3s < \bar{s}$.

Thus, the estimates of I_1 and I_2, together with (39), give

$$\begin{aligned}\left(\int_{\Omega\times\Omega} \frac{dxdz}{|x - z|^{n+qs}} |f(x) - f(z)|^q\right)^{1/q} &\leq c\|T_{1-\bar{s}}(|Xf|)\|_q \\ &\leq c\|Xf\|_p.\end{aligned}$$

□

References

[1] G. Citti, C^∞ *regularity of solutions of the Levi equation,* Ann. Inst. H. Poincaré, Anal. non Linéaire, 15, 4, 1998, 517-534.

[2] G. Citti, E. Lanconelli and A. Montanari, *On the interior C^∞-solvability of the Dirichlet problem for the prescribed Levi-curvature equation,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 10, (1999), 61-68

[3] G. Citti, E. Lanconelli and A. Montanari, *Smoothness of Lipschitz continuous graphs with non vanishing Levi curvature*, preprint 2000.

[4] L. C. Evans and R. F. Gariepy *Measure Theory and Fine Properties of Functions,* CRC Press, New York, 1992.

[5] C. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems,* Conference on Harmonic Analysis in honor of Antoni Zygmund, Wadsworth Mathematical Series, Wadsworth, Belmont, Calif. 1983, 590–606.

[6] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie gropus,* Ark. Mat. 13 (1975), 161-207.

[7] B. Franchi, *Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations,* Trans. Am. Math. Soc. **327** (1991), 125–158.

[8] B. Franchi, S. Gallot and R. L. Wheeden, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. **300** (1994), 557–571.

[9] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 523–541.

[10] B. Franchi, G. Lu and R. L. Wheeden, *A relationship between Poincaré type inequalities and representation formulas in spaces of homogeneous type*, Internat. Math. Res. Notices, 1996, no. 1, 1–14.

[11] B. Franchi, R. Serapioni and F. Serra Cassano, *Approximation and embedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields*, Boll. Un. Mat. Ital. B (7) 11 (1997), 83–117.

[12] N. Garofalo and D. M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[13] N. Garofalo and D. M. Nhieu, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory Spaces*, J. Anal. Math. **74** (1998), 67–97.

[14] A. E. Gatto and S. Vági, *Fractionals integrals on spaces of homogeneous type*, in: Analysis and Partial Differential Equations, C. Sadosky (ed.), Dekker, New York 1990, 171–216.

[15] A. E. Gatto and S. Vági, *On the exponential integrability of fractional integrals on spaces of homogeneous type*, Colloq. Math. **64** (1993), 121–127.

[16] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **688** (2000).

[17] L. Hörmander, *Hypoelliptic second order differential equations,* Acta Math. 119 (1967), 147-171.

[18] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition,* Duke Math J. **53** (1986), 503–523.

[19] E. Lanconelli and A. Montanari, *An embedding theorem for non smooth vector fields of step two,* Le Matematiche **LIV** (1999) – Supplemento, 111–124.

[20] E. Lanconelli and D. Morbidelli, *On the Poincaré inequality for vector fields,* to appear on Ark. Mat.

[21] P. Maheux and L. Saloff-Coste, *Analyse sur le boules d'un opérateur sous-elliptique*, Math. Ann. 303 (1995), 713–746.

[22] L. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotents groups,* Acta Math. **137** (1976), 247–320.

Annamaria Montanari
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato, 5, 40127 Bologna, Italy
E-Mail: montanar@dm.unibo.it

Daniele Morbidelli
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato, 5, 40127 Bologna, Italy
E-Mail: morbidel@dm.unibo.it