Lipschitz continuous viscosity solutions for a class of fully nonlinear equations on Lie groups

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Abstract In this paper we prove existence and uniqueness of Lipschitz continuous viscosity solutions for Dirichlet problems involving a class a fully non-linear operators on Lie groups. In particular we consider the elementary symmetric functions of the eigenvalues of the Hessian built with left-invariant vector fields.

1 Introduction

Let $\mathbb{G} = (\mathbb{R}^n, *)$ be a Lie group with $\mathbb{R}^n$ as support and $*$ as group law. Let us denote by $\boldsymbol{l}$ the set of the left-invariant vector fields. If $E^l = \{X_1, \dots, X_n\}$ is any basis of $\boldsymbol{l}$ then we know that a Riemannian metric g on $\mathbb{G}$ is left-invariant if and only if the coefficients $g_{ij} := g(X_i, X_j)$ are constant functions: in particular each n-dimensional Lie group possesses a $n(n+1)/2$-dimensional family of distinct left-invariant metrics (see for instance [15] and the references therein). Let us fix any left-invariant metric g and let u be a smooth function, we will denote by $D_g u$ the gradient of u with respect to the metric g, that is: $g(D_g u, X) = Xu = du(X)$, for every vector field X. If ∇ is the Levi-Civita connection for g (we recall that the connection coefficients in term of any left-invariant basis are constant functions), then the metric Hessian of u is the tensor field of type $(0,2)$ defined by:

$$H_g u(X,Y) := XYu - (\nabla_X Y)u$$

for every pair of vector fields (X,Y); since ∇ is the Levi-Civita connection for g (that is $\nabla_X Y - \nabla_Y X = [X,Y]$), we note that $H_g u$ is always symmetric. We will denote by $D_g^2 u := g^{-1} H_g u$ the associated endomorphism. We explicitly note that the previous definition is intrinsic, namely the eigenvalues of $D_g^2 u$ do not change in a change of basis. Let us consider a coordinate frame $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$,

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referred to local coordinates (in our setting they are actually global), we have:

$$H_g u\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) = \frac{\partial^2 u}{\partial x_i x_j} - \Gamma^s_{ij} \frac{\partial u}{\partial x_s}$$

where Γ^s_{ij} are the Christoffel symbols for the metric g (they are not constant in general). Hence if Du, D^2u denote the usual Euclidian gradient and Hessian, then $D^2_g u$ reads in local coordinates

$$D^2_g u := G^{-1}(x)[D^2 u + B(x, Du)] \tag{1}$$

where we denoted by G^{-1} the matrix of coefficients of g^{-1} expressed in the coordinate frame (they are not constant functions in general) and the coefficients of the matrix B are given by $b_{ij} = \Gamma^s_{ij} \frac{\partial u}{\partial x_s}$.
Here we will consider also the case of strictly restrictions to some subspace of $\boldsymbol{l}$. Let then $m \leq n$ and let $E^l_m = \{X_1, \ldots, X_m\}$, we define the subspace of the left-invariant vector fields

$$H\mathbb{G} := span\{X_1, \ldots, X_m\}$$

Now we consider a left-invariant metric g_m on $H\mathbb{G}$, we can "complete" it to the full tangent space by defining the blocks metric:

$$g := \begin{pmatrix} g_m & 0 \\ 0 & Id_{n-m} \end{pmatrix}$$

where, for every integer n, Id_n denotes the identity matrix of order n. We define

$$H_{g,m} u(X, Y) := XYu - (\nabla_X Y)u, \qquad \forall\, X, Y \in H\mathbb{G}$$

and $D^2_{g,m} u := g_m^{-1} H_{g,m} u$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a positive function, here we are interested on the Dirichlet problem associated with equations of the following form

$$\big(\sigma_k(D^2_{g,m} u)\big)^{\frac{1}{k}} = f(x, u, D_g u), \qquad k = 1, \ldots, m \tag{2}$$

where, for every symmetric matrix M of order m, $\sigma_k(M)$ denotes the k-th elementary symmetric function in the eigenvalues of M.
Our motivation comes from the geometric theory of several complex variables, where fully nonlinear second order pde's appear, whose linearizations are nonvariational operators of Hömander type. (See [16] and references therein). These kinds of operators, also arising in many other theoretical and applied settings, have the form of (2)
A direct computation shows that equation (2) reads then in local coordinates:

$$\Big(\sigma_k\big(A_m(x)\, D^2 u\, A_m^T(x) + Q_m(x, Du)\big)\Big)^{\frac{1}{k}} = \widetilde{f}(x, u, Du) \tag{3}$$

for a suitable positive function $\widetilde{f}$ and where A_m is a $m \times n$ matrix and Q_m is a square matrix of order m, both with smooth coefficients: in particular Q_m is linear with respect to Du. When $m = n$ we will simply omit the index n: for instance we will write A in place of A_n. We will consider then the following Dirichlet problem:

$$\begin{cases} F(x,u,Du,D^2u) := -\big(\sigma_k(D^2_{g,m}u)\big)^{\frac{1}{k}} + f(x,u,D_g u) = 0, & \text{in}\,\Omega,\\ u = \phi, & \text{on}\,\partial\Omega, \end{cases} \tag{4}$$

with $\phi : \partial\Omega \to \mathbb{R}$.

We are looking for Lipschitz viscosity solutions of (4). We refer to [11], [8] for a full detailed exposition on the theory of viscosity solutions: we will give the basic definition of sub- and super-solution in the next section.

In analogy with [6], we define the open cone

$$\Gamma_k^m = \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m : \sigma_j(diag(\lambda)) > 0, \text{ for every}\, j = 1, \dots, k\},$$

where $diag(\lambda)$ is the $m \times m$ diagonal matrix, and we denote by $\overline{\Gamma_k^m}$ and $\partial\Gamma_k^m$ the closure and the boundary of Γ_k^m respectively. In order to state our main result we need the following

Definition 1.1. *Let $x_0 \in \mathbb{R}^n$ and let φ be a C^2 function in a neighborhood of x_0. We will say that φ is strictly F-admissible (respectively F-admissible) at x_0, if the vector $\lambda = (\lambda_1, \dots \lambda_m)$ of the eigenvalues of $D^2_{g,m}\varphi(x_0)$ belongs to the open cone Γ_k^m (respectively $\overline{\Gamma_k^m}$).[1] We will say that φ is strictly F-admissible (respectively F-admissible) in Ω if φ is strictly F-admissible (respectively F-admissible) at x_0 for every $x_0 \in \Omega$.*
Moreover if $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth defining function for Ω, that is

$$\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \qquad \partial\Omega = \{x \in \mathbb{R}^n : \rho(x) = 0\}$$

then we will say that the domain Ω is strictly F-admissible if ρ is strictly F-admissible.

We need to give also some structure conditions on the operator F, in particular on the growth of the function f. We will require that f is strictly increasing with respect to u, in particular let us set

$$\mu := \inf \frac{\partial f}{\partial u}. \tag{5}$$

We define the following function

$$f_\infty : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad f_\infty(x, r, \mathcal{P}) := \lim_{\lambda\to\infty} \frac{f(x, \lambda r, \lambda\mathcal{P})}{\lambda}$$

We can state our main result:

[1] Remark that the cone Γ_k^m is invariant with respect to permutation of λ_j.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and smooth open set, with defining function ρ strictly F-admissible, let $\phi \in C^{1,1}$ and let $f \in C^1$ be a positive function with $\mu > 0$. Moreover, let us suppose that f_∞ exists and it holds*

$$f_\infty(x, \rho, D_g\rho) < \left(\sigma_k(D^2_{g,m}\rho)\right)^{\frac{1}{k}}, \qquad \text{for every } x \in \partial\Omega \tag{6}$$

Then there exists a Lipschitz continuous viscosity solution u of the problem (4).

Remark 1.1. *If the function f does not depend on its first argument, then in Corollary 3.1 we relax the assumptions of Theorem 1.1 by requiring $\frac{\partial f}{\partial u} \geq 0$ and $f \geq 0$.*

We will prove our result by a regularization procedure and a gradient estimate in Section 3. Due to the fact that the equation is fully nonlinear, the regularization procedure will be a very delicate issue, in order to preserve some nice structural properties of F: roughly speaking, it won't be enough to consider $F_\varepsilon(x, u, Du, D^2u) = F(x, u, Du, D^2u) + \varepsilon\Delta u$ as for quasilinear pde's, because the solution of $F_\varepsilon = 0$ might be non admissible for F. Our strategy is new and we can briefly describe it as follows: first of all we will look at the k-th elementary symmetric function of the eigenvalues of a $m \times m$ Hermitian matrix H as the limit as $\varepsilon \to 0$ of σ_k of a $n \times n$ Hermitian matrix H^ε (see Example 3.1 for an easy explanation in a simple case). However, $\sigma_k(H^\varepsilon)$ is only elliptic when the eigenvalues of H^ε are in the cone Γ^n_k. Then we will follow an old idea of Trudinger [19] and we will look at $(\sigma_k)^{1/k}(H^\varepsilon)$ as the limit as $\delta \to 0$ of $(\sigma_k)^{1/k}(H^{\varepsilon,\delta})$, which is uniformly elliptic when the eigenvalues of $H^{\varepsilon,\delta} = H^\varepsilon + \delta\, trace\, H^\varepsilon$ are in the cone Γ^n_k (see Lemma 3.2). Moreover, in order to prove interior gradient estimates, we will differentiate the regularized equation with respect to *right* invariant vector fields of the group: our choice is motivated by the fact that on a Lie group any right invariant vector field commutes with any left invariant vector field (see Section 2, Lemma 2.1) and this key property greatly simplifies calculations when differentiating.

In Section 4 we first prove a comparison principle for viscosity solutions by simple testing that our operator F satisfies the structure conditions of [8, Theorem 3.3] when the prescribed function f is strictly monotone increasing with respect u. Anyway, since one can be interested even at the case of constant function f, we would like to have a comparison principle for F also when f is not strictly increasing with respect to u. It is a standard fact by now that in order to adapt the proof for the strictly monotone case in this situation, one needs to find (for instance) a strictly sub-solution for F. Thus, we will give a sufficient condition on the sectional curvature of the group in order to ensure the existence of a strictly sub-solution for F and in Theorem 4.2 we prove the comparison principle for f monotone increasing with respect u. We explicitly note that we are working in full generality: if one considers Lie groups with some additional structure, then one could remove the hypothesis on the sectional curvature and one can have the comparison principle for every domain in $\mathbb{R}^n$. For instance in the case of Hessian defined on the first layer (let us say of dimension m) of Carnot groups, then one can use as strictly sub-solution a quadratic function

depending only on the "first m variables" (see [2], [3], [4]). As a consequence of the comparison principle, via the Perron method and the construction in Lemma 3.5 of a particular sub-solution and a particular super-solution, we also get existence and uniqueness of a viscosity solution of the Dirichlet problem (4) for every continuous boundary data ϕ.
We conclude the Introduction by making some remarks on our operator. First we note that there is another recurrent definition of Hessian on Lie groups, let us call it the symmetrized Hessian $H^s u$, that is, for every smooth function u and for every pair of left-invariant vector fields X, Y:

$$H^s u(X,Y) = \frac{XYu + YXu}{2} \tag{7}$$

In particular there is a very large literature on questions involving this symmetrized Hessian on Carnot groups (see for instance [9], [12], [20], [22], [7], [1] and the references therein). An easy computation shows that our metric Hessian $H_g u$ coincides with $H^s u$ if and only if it holds:

$$\nabla_X Y = \frac{1}{2}[X,Y], \qquad X, Y \in \boldsymbol{l} \tag{8}$$

We note that if the metric g is bi-invariant then (8) holds, and it is a known fact that if the Lie group is compact then it admits a bi-invariant metric. This is not the case since our base manifold is $\mathbb{R}^n$: anyway if we allows more structure on our Lie groups, for instance if we consider the case of Carnot groups, then the two Hessian definitions coincide, as we will show in Example 2.1. Finally, in Example 2.2 we will give an example of a Lie group that does not satisfy (8).

2 Preliminaries and examples

Here we recall some basic facts about Lie Groups and we will prove a key lemma. Let $\mathbb{G} = (\mathbb{R}^n, *)$ be a Lie group with $\mathbb{R}^n$ as support and $*$ as group law. For fixed $p \in \mathbb{G}$, the left translation with respect to p, is the map:

$$L_p : \mathbb{G} \to \mathbb{G}, \quad L_p(\xi) = p * \xi$$

The right translation with respect to p, is the map

$$R_p : \mathbb{G} \to \mathbb{G}, \quad R_p(\xi) = \xi * p$$

By using one the several equivalent definition, we will say that a vector field X is left-invariant if

$$X_{L_p(\xi)} = dL_p(\xi) X_\xi, \qquad \forall\, \xi \in \mathbb{G} \tag{9}$$

Analogously, we will say that a vector field Y is right-invariant if

$$Y_{R_p(\xi)} = dR_p(\xi) Y_\xi, \qquad \forall\, \xi \in \mathbb{G} \tag{10}$$

We will denote by $\boldsymbol{l}$ the set of the left-invariant vector fields and by $\boldsymbol{r}$ the set of the right-invariant vector fields. We have that any left invariant vector field commutes with any right invariant vector field. We have not been able to find any references with some explicit proof of this nice property: hence we prove it in the following Lemma.

Lemma 2.1. *Let us consider any couple of vector fields X, Y with $X \in \boldsymbol{l}$ and $Y \in \boldsymbol{r}$. Then*

$$[X, Y] = 0$$

Proof. If φ_t^X and φ_t^Y denote respectively the flow generated by $X \in \boldsymbol{l}$ and the one generated by $Y \in \boldsymbol{r}$, then by using equations (9) and (10) one can show that these flows read as

$$\varphi_t^X(p) = L_p\big(\varphi_t^X(0)\big), \qquad \varphi_t^Y(p) = R_p\big(\varphi_t^Y(0)\big)$$

In fact, by definition φ_t^X solves the following Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}\varphi_t^X(p) = X_{\varphi_t^X(p)} \\[2ex] \varphi_0^X(p) = p \end{cases}$$

Now, let us define $\psi_t(p) := L_p\big(\varphi_t^X(0)\big)$. We have

$$\begin{cases} \dfrac{d}{dt}\psi_t(p) = dL_p\big(\varphi_t^X(0)\big)\dfrac{d}{dt}\varphi_t^X(0) = dL_p\big(\varphi_t^X(0)\big)X_{\varphi_t^X(0)} = X_{\psi_t(p)} \\[2ex] \psi_0(p) = L_p\big(\varphi_0^X(0)\big) = L_p\big(0\big) = p \end{cases}$$

Therefore φ_t^X and ψ_t solve the same Cauchy problem and since X has smooth coefficients, then they have to coincide. The same holds for φ_t^Y.
Since the group law $*$ is associative, at every point p we have

$$\begin{aligned} (\varphi_t^Y \circ \varphi_t^X)(p) &= \varphi_t^Y(\varphi_t^X(p)) = R_{\varphi_t^X(p)}\varphi_t^Y(0) = \varphi_t^Y(0) * \varphi_t^X(p) \\ &= \varphi_t^Y(0) * \big(p * \varphi_t^X(0)\big) = \big(\varphi_t^Y(0) * p\big) * \varphi_t^X(0) \\ &= \varphi_t^Y(p) * \varphi_t^X(0) = L_{\varphi_t^Y(p)}\varphi_t^X(0) = \varphi_t^X(\varphi_t^Y(p)) \\ &= (\varphi_t^X \circ \varphi_t^Y)(p) \end{aligned}$$

and we get:

$$\varphi_t^Y \circ \varphi_t^X = \varphi_t^X \circ \varphi_t^Y.$$

Hence

$$\varphi_{-t}^Y \circ \varphi_{-t}^X \circ \varphi_t^Y \circ \varphi_t^X = \varphi_{-t}^Y \circ \varphi_{-t}^X \circ \varphi_t^X \circ \varphi_t^Y = Id.$$

Therefore

$$[X, Y] = \frac{1}{2}\Big(\frac{d^2}{dt^2}\big(\varphi_{-t}^Y \circ \varphi_{-t}^X \circ \varphi_t^Y \circ \varphi_t^X\big)\Big)\Big|_{t=0} \equiv 0$$

□

Next we consider a couple of significative examples in which we show structural properties related to the elegant equality (8).

Example 2.1. *Let $\mathbb{G}$ be a Carnot group on $\mathbb{R}^n$ (see for instance [5]). Let us consider the Jacobian basis E^l for* $\mathfrak{l}$ *and let g be the metric that makes orthonormal the vector fields of this basis. If ∇ denotes the Levi-Civita connection for g, by the very definition of the connection coefficients we have, for $i, j, k = 1, \dots, n$:*

$$g(\nabla_{X_i} X_j, X_k) = \frac{1}{2}\big(g([X_i, X_j], X_k) - g([X_j, X_k], X_i) + g([X_k, X_i], X_j)\big) \quad (11)$$

Now let us suppose that the first layer of the stratification V_1^m has dimension $m < n$ and it is spanned by the first m vector fields of the basis, namely $E^l_m = \{X_1, \dots, X_m\}$, and let us consider the Hessian defined on this layer, that is we consider $H_{g,m}u(X_i, X_j)$, with $i, j = 1, \dots, m$: we know that $[X_i, X_k]$ never belongs to V_1^m, for any $k = 1, \dots, n$, hence by the formula (11) we get

$$g(\nabla_{X_i} X_j, X_k) = \frac{1}{2} g([X_i, X_j], X_k), \quad i, j = 1, \dots, m, \quad k = 1, \dots, n$$

therefore (8) holds.

The following is an example in which (8) is not fulfilled.

Example 2.2. *We consider the Lie group in $\mathbb{R}^2$ given by the following group law $*$ (see also [5], pg.21): for any $(x, y), (t, s) \in \mathbb{R}^2$*

$$(x, y) * (t, s) = (x + t, y + se^x)$$

A basis for $\mathfrak{l}$ *is given by*

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}$$

We consider the metric g that makes orthonormal the vector fields X, Y, and we denote by ∇ the Levi-Civita connection for g. Therefore a direct computation shows that

$$0 = 2\nabla_X Y \neq [X, Y] = -Y$$

hence (8) is not satisfied.

3 Regularization and gradient estimates

We first recall the definition of sub- and super-solution in the viscosity sense.

Definition 3.1. *Let us consider the equation*

$$F(x, u, Du, D^2u) = 0, \qquad \text{in } \Omega, \quad (12)$$

We say that a function $u \in USC(\Omega)$ is a viscosity sub-solution for (12) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $x_0 \in \Omega$ is a local maximum for the function $u - \varphi$, then φ is F-admissible at x_0 and

$$F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \le 0 \tag{13}$$

We say that a function $u \in LSC(\Omega)$ is a viscosity super-solution for (12) if for every $\varphi \in C^2(\Omega)$, it holds the following: if $x_0 \in \Omega$ is a local minimum for the function $u - \varphi$, then either φ is F-admissible at x_0 and

$$F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \ge 0 \tag{14}$$

or φ is not F-admissible at x_0.
A continuous function u is a viscosity solution for (12) if it is either a viscosity sub-solution and a viscosity super-solution for (12).
We say that a function $u \in USC(\overline{\Omega})$ is a viscosity sub-solution for (4) if u is a viscosity sub-solution for (12) and in addition $u \le \phi$ on $\partial\Omega$.
We say that a function $u \in LSC(\overline{\Omega})$ is a viscosity super-solution for (4) if u is a viscosity super-solution for (12) and in addition $u \ge \phi$ on $\partial\Omega$.
A viscosity solution for (4) is either a viscosity sub-solution and a viscosity super-solution for (4).

Now we start our regularization procedure. For $\varepsilon > 0$, let us define the following blocks matrix

$$I_m^\varepsilon := \begin{pmatrix} Id_m & 0 \\ 0 & \varepsilon Id_{n-m} \end{pmatrix}$$

We will consider the approximated endomorphism given by the following $n \times n$ matrix

$$D^2_{g,m,\varepsilon} u := I_m^\varepsilon \, D^2_g u \, I_m^\varepsilon$$

Therefore we will deal with the approximated operators

$$F^\varepsilon(x, u, Du, D^2u) := -\big(\sigma_k(D^2_{g,m,\varepsilon} u)\big)^{\frac{1}{k}} + f(x, u, D_g u)$$

The following example shows the idea of the regularization procedure in a simple case.

Example 3.1. *Let us consider the Heisenberg group $\mathbb{H}^1$, with vector fields given by:*

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}$$

Now, let $det(Hu)$ be the determinant of the horizontal Hessian defined by the vector fields X, Y (since $\mathbb{H}^1$ is a Carnot group, by Lemma 2.1 we can consider the symmetrized Hessian as well):

$$Hu := \begin{pmatrix} X^2u & \frac{XYu+YXu}{2} \\ \frac{XYu+YXu}{2} & Y^2u \end{pmatrix}$$

By using the previous regularization procedure, we can think of $det(Hu)$ *as the limit of* $\sigma_2(H^\varepsilon u)$*, as* ε *approaches zero, where:*

$$H^\varepsilon u := \begin{pmatrix} X^2u & \frac{XYu+YXu}{2} & \varepsilon\frac{XTu+TXu}{2} \\ \frac{XYu+YXu}{2} & Y^2u & \varepsilon\frac{YTu+TYu}{2} \\ \varepsilon\frac{XTu+TXu}{2} & \varepsilon\frac{YTu+TYu}{2} & \varepsilon^2T^2u \end{pmatrix}$$

We will prove that there exists a sequence of Lipschitz continuous viscosity solutions of the problem with $F^\varepsilon = 0$: then by taking the uniform limit as ε approaches zero, we will find a Lipschitz continuous viscosity solutions u of the original problem with $F = 0$. In fact we have

Proposition 3.1. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded and smooth open set, strictly* F*-admissible. Let us suppose that there exists a sequence of Lipschitz continuous viscosity solutions* $\{u^\varepsilon\}$ *of the problem*

$$\begin{cases} F^\varepsilon(x, u, Du, D^2u) = 0, & in\,\Omega, \\ u = \phi, & on\,\partial\Omega, \end{cases} \tag{15}$$

with Lipschitz constant independent of ε*: then if* u *denotes the uniform limit of* u^ε*, as* ε *approaches zero, then* u *is a Lipschitz continuous viscosity solution of the problem (4).*

Remark 3.1. *Since the defining function of the domain* Ω *is smooth, then one can choose* ε *small enough such that* Ω *is actually strictly* F^ε*-admissible.*

We will give the proof for the following slightly more general operator. Let $k \le m \le n$ and define

$$F(x, r, p, M) = -\Big(\sigma_k\big(H_m(x, r, p, M)\big)\Big)^{\frac{1}{k}} + f(x, r, p, M)$$

where f is a continuous and positive function, and where H_m is the first minor $m\times m$ (that is $H_m = \{h_{ij}\}_{i,j=1,\dots,m}$) of a $n\times n$ symmetric matrix H, with smooth coefficients eventually depending on x, r, p, M. Then we define the perturbed operator

$$F^\varepsilon(x, r, p, M) = -\Big(\sigma_k\big(H^\varepsilon_m(x, r, p, M)\big)\Big)^{\frac{1}{k}} + f(x, r, p, M)$$

with $H^\varepsilon_m := I^\varepsilon_m\ H\ I^\varepsilon_m$. Let $x_0 \in \mathbb{R}^n$ and let φ be a C^2 function in a neighborhood of x_0, we will say that φ is F^ε-admissible at x_0 if the eigenvalues of $H^\varepsilon_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))\big)$ belong to the cone $\overline{\Gamma^n_k}$. The set of F^ε-admissible functions continuously deform into the set of F-admissible functions, but we note that for $\varepsilon_1 \neq \varepsilon_2$, there are no inclusions in general between the sets of F^{ε_1}-admissible functions and of F^{ε_2}-admissible functions.

We want to prove the following

Lemma 3.1. *Let $\{u_\varepsilon\}$ be a sequence of viscosity solutions for the problem $F^\varepsilon = 0$, then if we call u the uniform limit of u_ε, as ε goes to zero, then u is a viscosity solution of the problem $F = 0$.*

Proof. We need to prove that u is either a viscosity sub-solution and super-solution. We use definition (3.1). First we prove that u is a viscosity sub-solution. Let then $x_0 \in \mathbb{R}^n$ and let φ be a C^2 function in a neighborhood of x_0 such that $u - \varphi$ has a local maximum at x_0. We can choose a sequence $x_\varepsilon \to x_0$, as ε approaches zero, such that $u_\varepsilon - \varphi$ has a local maximum at x_ε. Since u_ε is a viscosity sub-solution for $F^\varepsilon = 0$, then it holds

$$\begin{cases} \varphi \,\text{is}\, F^\varepsilon\text{−admissible at}\, x_\varepsilon \\ F^\varepsilon\big(x_\varepsilon, u_\varepsilon(x_\varepsilon), D\varphi(x_\varepsilon), D^2\varphi(x_\varepsilon)\big) \le 0 \end{cases} \tag{16}$$

Passing to the limit in (16), as ε goes to zero, we get

$$\begin{cases} \varphi \,\text{is}\, F\text{−admissible at}\, x_0 \\ F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \le 0 \end{cases} \tag{17}$$

Therefore u is a viscosity sub-solution for $F = 0$. Now, let $x_0 \in \mathbb{R}^n$ and let φ be a C^2 function in a neighborhood of x_0 such that $u - \varphi$ has a local minimum at x_0. As before we can choose a sequence $x_\varepsilon \to x_0$, as ε approaches zero, such that $u_\varepsilon - \varphi$ has a local minimum at x_ε. In this situation, since u_ε is a viscosity super-solution for $F^\varepsilon = 0$, we have to distinguish two cases. In the first one, namely if φ is F^ε-admissible in x_ε, then we have

$$\begin{cases} \varphi \,\text{is}\, F^\varepsilon\text{−admissible at}\, x_\varepsilon \\ F^\varepsilon\big(x_\varepsilon, u_\varepsilon(x_\varepsilon), D\varphi(x_\varepsilon), D^2\varphi(x_\varepsilon)\big) \ge 0 \end{cases} \tag{18}$$

Again passing to the limit in (18), as ε goes to zero, we get

$$\begin{cases} \varphi \,\text{is}\, F\text{−admissible at}\, x_0 \\ F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \ge 0 \end{cases} \tag{19}$$

Therefore u is a viscosity super-solution for $F = 0$. In the second case we have φ is not F^ε-admissible in x_ε. Three situations can occur passing to the limit:

(i) φ is strictly F-admissible in x_0,

(ii) the m vector of the eigenvalues of $H_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))$ does not belong to $\overline{\Gamma_k^m}$

(iii) the m vector of the eigenvalues of $H_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))$ belongs to $\partial\Gamma_k^m$.

In the first case, since the eigenvalues of $H_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))$ belong to the open cone Γ_k^m and φ is C^2, we can choose $\varepsilon > 0$ small enough such that the eigenvalues of $H_m^\varepsilon(x_\varepsilon, \varphi(x_\varepsilon), D\varphi(x_\varepsilon), D^2\varphi(x_\varepsilon))$ belong to the open cone Γ_k^n and we have again (18). In the second case, since the m vector of the eigenvalues of $H_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))$ does not belong to $\overline{\Gamma_k^m}$, then u is a viscosity super-solution for $F = 0$ at x_0 by definition. In the third case, since the m vector of the eigenvalues of $H_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))$ belongs to $\partial\Gamma_k^m$, there exists at least one index j in $\{1, \dots, k\}$ such that

$$\sigma_j\big(H_m(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))\big) = 0$$

Now, since $j \leq k$

$$0 \leq \Big(\frac{\sigma_k\big(H_m\big)}{\binom{n}{k}}\Big)^{\frac{1}{k}} \leq \Big(\frac{\sigma_j\big(H_m\big)}{\binom{n}{j}}\Big)^{\frac{1}{j}}$$

we obtain

$$\begin{cases} \varphi \,\text{is}\, F\text{−admissible at}\, x_0 \\ F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) = 0 + f\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \geq 0 \end{cases} \tag{20}$$

Therefore u is a viscosity super-solution for $F = 0$. □

Proof of Proposition 3.1. First of all since the Lipschitz constant for the sequence $\{u_\varepsilon\}$ is independent of ε, then the uniform limit u is still a Lipschitz continuous function. Then, by the previous Lemma 3.1 we have that u is a viscosity solution of $F = 0$. Finally we note that since u_ε is either a viscosity sub-solution and super-solution for the Dirichlet problem with $F^\varepsilon = 0$, the inequalities on the boundary conditions simply pass to the limit. □

In order to find the sequence $\{u^\varepsilon\}$ we will need a further (uniformly elliptic) regularization. Let us fix $\varepsilon > 0$, and let us denote by $\Delta_{g,m,\varepsilon}u$ the perturbed g-Laplacian of the group $\mathbb{G}$, namely:

$$\Delta_{g,m,\varepsilon}u := trace(D^2_{g,m,\varepsilon}u)$$

For $\delta > 0$, we define

$$D^2_{g,m,\varepsilon,\delta}u := D^2_{g,m,\varepsilon}u + \delta(\Delta_{g,m,\varepsilon}u)Id_n$$

where Id_n denotes the identity operator of order n. Hence we define the operator

$$F^{\varepsilon,\delta}(x, u, Du, D^2u) := -\big(\sigma_k(D^2_{g,m,\varepsilon,\delta}u)\big)^{\frac{1}{k}} + f(x, u, D_gu)$$

It turns out that $-F^{\varepsilon,\delta}$ is uniformly elliptic in the set of $F^{\varepsilon,\delta}$-admissible functions with ellipticity constant depending on ε and δ, in particular we have:

Lemma 3.2. *There exist constants $0 < \lambda_{\varepsilon,\delta} < \Lambda_{\varepsilon,\delta}$, depending on ε and δ, such that*

$$\lambda_{\varepsilon,\delta}\ trace(N) \leq -F^{\varepsilon,\delta}(x, r, p, M + N) + F^{\varepsilon,\delta}(x, r, p, M) \leq \Lambda_{\varepsilon,\delta}\ trace(N)$$

for every symmetric and positive definite $n \times n$ matrix N, and for every $x \in \Omega, r \in \mathbb{R}, p \in \mathbb{R}^n$ and for every symmetric $n \times n$ matrix M such that the eigenvalues of $H_m^{\varepsilon,\delta}(x,p,M) := H_m^{\varepsilon}(x,p,M) + \delta\, trace H_m^{\varepsilon}(x,p,M)$ are in the cone Γ_k^n. Here we have denoted by

$$\begin{aligned} &H_m^{\varepsilon}(x,p,M) := A^{\varepsilon}(x) M A^{\varepsilon T}(x) + Q^{\varepsilon}(x,p), \\ &A^{\varepsilon}(x) := I^{\varepsilon} A_n(x),\ Q^{\varepsilon}(x,p) := I^{\varepsilon} Q_n(x,p) I^{\varepsilon} \end{aligned}$$

(see (3)*).*

Proof. It's a straightforward computation, by taking into account that the functions $\big(\sigma_k\big)^{\frac{1}{k}}$ are homogeneous of degree one, monotone increasing and concave, and the fact that the matrix $A^{\varepsilon}(x)$ is strictly positive definite and smooth on the compact domain $\overline{\Omega}$. Precisely, let $0 < \lambda^{\varepsilon} < \Lambda^{\varepsilon}$ be positive constants such that

$$\lambda^{\varepsilon} Id_n \le A^{\varepsilon}(x) A^{\varepsilon T}(x) \le \Lambda^{\varepsilon} Id_n$$

as quadratic forms and for all $x \in \bar{\Omega}$. By the concavity and the monotonicity of $\big(\sigma_k\big)^{\frac{1}{k}}$ we have

$$\begin{aligned} &- F^{\varepsilon,\delta}(x,r,p,M+N) + F^{\varepsilon,\delta}(x,r,p,M) \\ &= \Big(\sigma_k\left(H_m^{\varepsilon,\delta}(x,p,M+N)\right)\Big)^{\frac{1}{k}} - \Big(\sigma_k\left(H_m^{\varepsilon,\delta}(x,p,M)\right)\Big)^{\frac{1}{k}} \\ &= \Big(\sigma_k\left(H_m^{\varepsilon}(x,p,M+N) + \delta\, trace(H_m^{\varepsilon}(x,p,M+N)) Id_n\right)\Big)^{\frac{1}{k}} \\ &- \Big(\sigma_k\left(H_m^{\varepsilon}(x,p,M) + \delta\, trace(H_m^{\varepsilon}(x,p,M)) Id_n\right)\Big)^{\frac{1}{k}} \\ &\ge \Big(\sigma_k\left(A^{\varepsilon} N A^{\varepsilon T} + \delta\, trace(A^{\varepsilon} N A^{\varepsilon T}) Id_n\right)\Big)^{\frac{1}{k}} \\ &\ge \Big(\sigma_k\left(\delta\, trace(A^{\varepsilon} N A^{\varepsilon T}) Id_n\right)\Big)^{\frac{1}{k}} \\ &\ge \delta \lambda^{\varepsilon}\, trace(N) \Big(\sigma_k\left(Id_n\right)\Big)^{\frac{1}{k}} = \lambda_{\varepsilon,\delta}\, trace(N) \end{aligned}$$

Moreover, by the monotonicity, the homogeneity and the concavity of $\big(\sigma_k\big)^{\frac{1}{k}}$

and by Lagrange theorem there is $\theta \in]0,1[$ such that

$$
\begin{aligned}
& - F^{\varepsilon,\delta}(x,r,p,M+N) + F^{\varepsilon,\delta}(x,r,p,M) \\
& = \Big(\sigma_k\left(H_m^{\varepsilon,\delta}(x,p,M+N)\right)\Big)^{\frac{1}{k}} - \Big(\sigma_k\left(H_m^{\varepsilon,\delta}(x,p,M)\right)\Big)^{\frac{1}{k}} \\
& = \Big(\sigma_k\left(H_m^{\varepsilon}(x,p,M+N) + \delta\, trace(H_m^{\varepsilon}(x,p,M+N)) Id_n\right)\Big)^{\frac{1}{k}} \\
& - \Big(\sigma_k\left(H_m^{\varepsilon}(x,p,M) + \delta\, trace(H_m^{\varepsilon}(x,p,M)) Id_n\right)\Big)^{\frac{1}{k}} \\
& \leq \Big(\sigma_k\left(H_m^{\varepsilon}(x,p,M) + \ \delta trace\left(H_m^{\varepsilon}(x,p,M)\right) Id_n + (1+\delta) trace\left(A^{\varepsilon} N A^{\varepsilon T}\right) Id_n\right)\Big)^{\frac{1}{k}} \\
& - \Big(\sigma_k\left(H_m^{\varepsilon}(x,p,M) + \delta\, trace(H_m^{\varepsilon}(x,p,M)) Id_n\right)\Big)^{\frac{1}{k}} \\
& = \Big(\partial_{r_{jj}}\left(\sigma_k\right)^{\frac{1}{k}}\left(H_m^{\varepsilon,\delta}(x,p,M) + (1+\delta)\theta\, trace\left(A^{\varepsilon} N A^{\varepsilon T}\right) Id_n\right)\Big) \\
& \qquad \cdot (1+\delta)\, trace\left(A^{\varepsilon} N A^{\varepsilon T}\right) \\
& \leq \Big(\partial_{r_{jj}}\left(\sigma_k\right)^{\frac{1}{k}}\left((1+\delta)\theta\, trace\left(A^{\varepsilon} N A^{\varepsilon T}\right) Id_n\right)\Big) \cdot (1+\delta)\, trace\left(A^{\varepsilon} N A^{\varepsilon T}\right) \\
& = \Big(\partial_{r_{jj}}\left(\sigma_k\right)^{\frac{1}{k}}\left(Id_n\right)\Big) \cdot (1+\delta)\, trace\left(A^{\varepsilon} N A^{\varepsilon T}\right) \\
& \leq (1+\delta)\Lambda^{\varepsilon}\, trace\,(N) = \Lambda_{\varepsilon,\delta}\, trace\,(N)
\end{aligned}
$$

where $\partial_{r_{jj}}\Big(\left(\sigma_k(r)\right)^{\frac{1}{k}}\Big)$ denotes the sum in j of partial derivatives of $\left(\sigma_k\right)^{\frac{1}{k}}$ with respect to r_{jj} and we have used its positiveness and its decreasing monotonicity in the set of k-admissible matrices [2], and its homogeneity of degree zero. □

Bounds for the second derivatives and for their Hölder seminorms now follow from the uniformly elliptic theory in [18], [13], since $F^{\varepsilon,\delta}$ is uniformly elliptic in the sense of [18]. Estimates for higher derivatives follow from the linear uniformly elliptic theory [10, Lemma 17.16]. These estimates, together with the fact that $F^{\varepsilon,\delta} = f > 0$ when the eigenvalues of $H^{\varepsilon,\delta} \in \partial\Gamma_k^n$, allow us to apply the method of continuity [10, Theorem 17.8] Therefore for every $\phi_\delta \in C^{2,\alpha}$ there exists a classical solution $u^{\varepsilon,\delta} \in C^{2,\alpha}$ of the Dirichlet problem related to $F^{\varepsilon,\delta} = 0$ (from further regularity results, $u^{\varepsilon,\delta}$ is actually C^{∞}); moreover $u^{\varepsilon,\delta}$ is strictly F-admissible.
We will prove a gradient bound for $u^{\varepsilon,\delta}$, uniform in ε, δ. Thus, with fixed $\varepsilon > 0$, by taking the uniform limit as δ goes to zero we will find a Lipschitz continuous viscosity solution u^{ε} of the Dirichlet problem related to $F^{\varepsilon} = 0$; in this last limit process we can use the stability property of the viscosity solutions with respect to the uniform convergence, since the sets of $F^{\varepsilon,\delta}$-admissible functions satisfy a crucial property of inclusion as δ decreases, for fixed ε (see for instance [21]).
The idea is to consider the strictly elliptic regularization $F^{\varepsilon,\delta}$ and provide uniform gradient estimates on the smooth solution via a Bernstein method, by using derivatives along right-invariant vector fields. First we note the following

[2] We recall that A is k-admissible if $\sigma_j(A) \geq 0$ for every $j = 1, \ldots, k$

fact: for any two Riemannian metric $g, \widetilde{g}$, we have by the very definition of the gradient, that for every smooth function u, it holds $g(D_g u, \cdot) = \widetilde{g}(D_{\widetilde{g}} u, \cdot)$. Now, if we denote by $|D_g u|_g^2 = g(D_g u, D_g u)$ (and the same for $|D_{\widetilde{g}} u|_{\widetilde{g}}$), therefore since $\overline{\Omega}$ is a compact subset of $\mathbb{R}^n$, we have that there exist two positive constants $C_1 \le C_2$ such that on $\overline{\Omega}$ it holds:

$$C_1 |D_g u|_g \le |D_{\widetilde{g}} u|_{\widetilde{g}} \le C_2 |D_g u|_g$$

Therefore fixed any metric $\widetilde{g}$, having an estimate on $|D_{\widetilde{g}} u|_{\widetilde{g}}$ will imply an estimate on any other metric gradient, in particular on the Euclidean one Du. Let us denote by $\mathbf{r}$ the set of the right-invariant vector fields and by $E^r = \{Y_1, \dots, Y_n\}$ any basis of $\mathbf{r}$. We will denote by $\widetilde{g}$ the right-invariant metric that makes orthonormal the vector fields of the basis E^r. Hence, in the basis E^r we have

$$D_{\widetilde{g}} u = \sum_{k=1}^{n} (Y_k u) Y_k, \qquad |D_{\widetilde{g}} u|_{\widetilde{g}}^2 = \sum_{k=1}^{n} (Y_k u)^2$$

Next we have:

Proposition 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and smooth open set, strictly F^{ε}-admissible, let $\phi \in C^{1,1}(\Omega)$ and $f \in C^1$ with $\mu > 0$. Moreover suppose that (6) holds. Then there exists a Lipschitz continuous viscosity solution u^{ε} of the problem (15).*

We will prove the Proposition (3.2) with the help of several lemmas. For $\delta > 0$, let us consider the operator $F^{\varepsilon,\delta}$. First of all we explicitly note that if a function is strictly F^{ε}-admissible with respect to $D^2_{g,m,\varepsilon}$, then it is also strictly $F^{\varepsilon,\delta}$-admissible. Since $-F^{\varepsilon,\delta}$ is strictly elliptic there exists a smooth solution $u^{\varepsilon,\delta}$ of the problem

$$\begin{cases} F^{\varepsilon,\delta}(x, u, Du, D^2 u) = 0, & \text{in } \Omega, \\ u = \phi_\delta, & \text{on } \partial\Omega, \end{cases} \tag{21}$$

where ϕ_δ is the convolution of ϕ with the usual Euclidean mollifier. Let us denote by f_{Y_k} the derivative of f with respect to its first argument along the vector field Y_k, and define $M := \sup \left(\frac{1}{2} \sum_{k=1}^{n} |f_{Y_k}|^2 \right)$. We have

Lemma 3.3. *We assume the same hypotheses as in Proposition (3.2). Let $w = \frac{1}{2} |D_{\widetilde{g}} u^{\varepsilon,\delta}|_{\widetilde{g}}^2$ Then it holds:*

$$\sup_{\Omega} w \le \max \left\{ \sup_{\partial\Omega} w, M/\mu^2 \right\} \tag{22}$$

Proof. If $w \le M/\mu^2$ there is nothing to prove; thus let us assume the converse. We apply the Bernstein technique, that is: we differentiate the equation $F^{\varepsilon,\delta}(x, u, Du^{\varepsilon,\delta}, D^2 u^{\varepsilon,\delta}) = 0$ along Y_k, then we multiply by $Y_k(u^{\varepsilon,\delta})$ and we take the sum over k from 1 to n. Since the coefficients of the left-invariant metric g are constant functions, the only terms of second order we need to differentiate are of the form:

$$X_i X_j u^{\varepsilon,\delta} - (\nabla_{X_i} X_j) u^{\varepsilon,\delta}, \qquad X_i,\ X_j \in \mathbf{l}$$

Since ∇ is the Levi-Civita connection for g, we have that also the vector field $\nabla_{X_i} X_j$ is left-invariant. Then, by Lemma (2.1) we have:

$$Y_k(X_iX_ju^{\varepsilon,\delta} - (\nabla_{X_i}X_j)u^{\varepsilon,\delta}) = X_iX_jY_ku^{\varepsilon,\delta} - (\nabla_{X_i}X_j)Y_ku^{\varepsilon,\delta}$$

Thus, by taking into account that $u^{\varepsilon,\delta}$ is strictly $F^{\varepsilon,\delta}$-admissible, we have

$$a_{ij} := \left\{ \frac{\partial}{\partial h_{sj}} \left(\left(\sigma_k(D^2_{g,m,\varepsilon,\delta}u^{\varepsilon,\delta}) \right)^{\frac{1}{k}} \right) \right\} > 0 \tag{23}$$

where h_{ij} are the entries of $D^2_{g,m,\varepsilon,\delta}u^{\varepsilon,\delta}$. Now we define the linear operator

$$Lv := -\sum_{i,j=1}^{n} a_{ij}\{D^2_{g,m,\varepsilon,\delta}v\}_{ij} + \sum_{i=1}^{n} \frac{\partial f}{\partial \mathcal{P}_i}\{D_g v\}_i$$

A straightforward computation shows that

$$Lw \le -2\partial_u f w - \sum_{k=1}^{n} Y_k u^{\varepsilon,\delta} f_{Y_k} \le \frac{M}{\mu} - w\mu \le 0, \tag{24}$$

therefore by a standard application of the classical maximum principle we obtain (22). ☐

Next, if ν denotes the inward normal direction to $\partial\Omega$ with respect to $\widetilde{g}$, we need to find an estimate for the normal component of $D_{\widetilde{g}}u^{\varepsilon,\delta}$, that is $\nu(u^{\varepsilon,\delta})$. We have

Lemma 3.4. *Again we assume the same hypotheses as in Proposition (3.2). Therefore it holds:*

$$\sup_{\partial\Omega} \left| \nu(u^{\varepsilon,\delta}) \right| \le C \tag{25}$$

with C depending on $\|D\phi\|_\infty, \|D^2\phi\|_\infty, \|u^{\varepsilon,\delta}\|_\infty$.

Proof. Let $\rho \in C^{2,\alpha}$ be a defining function for Ω, strictly F-admissible. Then ρ is $F^{\varepsilon,\delta}$-admissible for ε and δ small. Let us consider for a small $\gamma > 0$

$$\Omega_\gamma = \{x \in \mathbb{R}^n : -\gamma < \rho(x) < 0\}$$

and let us define the following functions, for some $\lambda > 0$

$$\overline{u} = \phi_\delta - \lambda\rho, \qquad \underline{u} = \phi_\delta + \lambda\rho$$

where we have also denoted by ϕ_δ a smooth extension on the whole domain Ω. We have $\underline{u} = \overline{u} = \phi_\delta = u^{\varepsilon,\delta}$ on $\partial\Omega$ and $\underline{u} \le u^{\varepsilon,\delta} \le \overline{u}$ on $\{\rho = -\delta\}$ for

$$\lambda > \frac{1}{\gamma} \max\{(\max_{\overline{\Omega}} \phi + \max_{\overline{\Omega}} |u^{\varepsilon,\delta}|), (\min_{\overline{\Omega}} \phi - \max_{\overline{\Omega}} |u^{\varepsilon,\delta}|)\}$$

Then $\underline{u} \le u^{\varepsilon,\delta} \le \overline{u}$ on $\partial\Omega_\delta$. Now, since $D^2_{g,m,\varepsilon,\delta}\underline{u} = D^2_{g,m,\varepsilon,\delta}\phi''''_\delta + \lambda D^2_{g,m,\varepsilon,\delta}\rho$, by a direct computation we have $\sigma_k(D^2_{g,m,\varepsilon,\delta}\underline{u}) = \lambda^k \sigma_k(D^2_{g,m,\varepsilon,\delta}\rho) + o(\lambda^k)$, as

$\lambda \to +\infty$. Therefore, for $\lambda > 0$ large enough and by using condition (6), $\underline{u}$ is a sub-solution of

$$\begin{cases} F^{\varepsilon,\delta}(x,u,Du,D^2u) = 0, & \text{in}\, \Omega_\gamma, \\ u = u^{\varepsilon,\delta}, & \text{on}\, \partial\Omega_\gamma, \end{cases} \tag{26}$$

Now, let u_l be the solution of the semilinear strictly elliptic problem

$$\begin{cases} -\dfrac{1}{n}\Big[\dbinom{n}{m}\Big]^{\frac{1}{k}} trace(D^2_{g,m,\varepsilon,\delta}u) + f(x,u,D_g u) = 0, & \text{in}\, \Omega_\gamma, \\ u = u^{\varepsilon,\delta}, & \text{on}\, \partial\Omega_\gamma \end{cases} \tag{27}$$

Then, from the inequality

$$\big(\sigma_k(D^2_{g,m,\varepsilon,\delta}u)\big)^{\frac{1}{k}} \le \frac{1}{n}\Big[\binom{n}{m}\Big]^{\frac{1}{k}} trace(D^2_{g,m,\varepsilon,\delta}u)$$

we have that u_l is a super-solution of

$$\begin{cases} F^{\varepsilon,\delta}(x,u,Du,D^2u) = 0, & \text{in}\, \Omega_\gamma, \\ u = u^{\varepsilon,\delta}, & \text{on}\, \partial\Omega_\gamma, \end{cases} \tag{28}$$

Finally, since $trace(D^2_{g,m,\varepsilon,\delta}\overline{u}) = -\lambda trace(D^2_{g,m,\varepsilon,\delta}\rho) + o(\lambda)$, as $\lambda \to +\infty$, we also have that $\overline{u}$ is a super-solution of

$$\begin{cases} -\dfrac{1}{n}\Big[\dbinom{n}{m}\Big]^{\frac{1}{k}} trace(D^2_{g,m,\varepsilon,\delta}u) + f(x,u,D_g u) = 0, & \text{in}\, \Omega_\gamma, \\ u = u^{\varepsilon,\delta}, & \text{on}\, \partial\Omega_\gamma \end{cases} \tag{29}$$

for $\lambda > 0$ large enough. By the classical comparison principle, we get:

$$\underline{u} \le u^{\varepsilon,\delta} \le u_l \le \overline{u}, \qquad \text{in}\Omega_\delta$$

therefore

$$\nu(\underline{u}) \le \nu(u^{\varepsilon,\delta}) \le \nu(\overline{u})$$

on $\partial\Omega$. This ends the proof. □

Last, we estimate $u^{\varepsilon,\delta}$.

Lemma 3.5. *Under the hypotheses of Proposition (3.2), it holds:*

$$\sup_\Omega |u^{\varepsilon,\delta}| \le C \tag{30}$$

where C is a positive constant not depending on ε, δ.

Proof. We need to find a bounded sub- and super-solution to (21). By direct computation, we have that

i) the function

$$\underline{u} = \inf_{\partial\Omega} \phi + c\rho$$

is a sub-solution to (21), with

$$c \geq \frac{(\sup_\Omega f)^k}{c_\rho}, \qquad (\inf \sigma_k(D^2_{g,m,\varepsilon,\delta}\rho) =: c_\rho > 0)$$

ii) the function

$$\overline{u} = \sup_{\partial\Omega} \phi$$

is a super-solution to (21)

Since $\underline{u}$ and $\overline{u}$ are bounded on Ω, we get (30). □

Proof of Proposition 3.2 . By putting together Lemma 3.3, 3.4, 3.5, we have the thesis. □

We are now ready to give the proof of our main theorem.

Proof of Theorem 1.1 . By Proposition 3.2 there exists a Lipschitz viscosity solution u^ε of the problem (15). By Proposition 3.1 u^ε uniformly converges to a Lipschitz viscosity solution u of the problem (4) □

The following Corollary captures the case f constant.

Corollary 3.1. *If f is independent of x and $f \geq 0, \partial_u f \geq 0$ then Theorem 1.1 still holds.*

Proof. First of all assume $f > 0, \partial_u f \geq 0$ and remark that $M = 0$ in (24). Then assume $f \geq 0, \partial_u f \geq 0$. For $0 < s < 1$, let $f^s = f + s$ and u^s be the viscosity solution given by Theorem 1.1 of the problem

$$\begin{cases} -\big(\sigma_k(D^2_{g,m}u)\big)^{\frac{1}{k}} + f^s(x, u, D_g u) = 0, & \text{in } \Omega, \\ u = \phi, & \text{on } \partial\Omega. \end{cases} \tag{31}$$

By Lemma 3.3, 3.4, 3.5 u^s uniformly converges to a Lipschitz viscosity solution u of the problem (4). □

4 Comparison Principles

Now we give a sufficient condition to ensure the uniqueness of the viscosity solution of the Dirichlet problem (4). By following the analysis on comparison principle in [8] and [11], we see that if the prescribed function f is continuous, positive and strictly increasing with respect to u, then F is proper in the set of F-admissible functions, according the definition in [8]. However, in order to prove the comparison principle for F in the set of F-admissible functions we also need the following non trivial lemma. Let us denote by

$$H_m(x,p,X) := A_m(x)XA_m^T(x) + Q_m(x,p)$$

with A_m and Q_m as in (3).

Lemma 4.1. *There is a function $\omega : [0,\infty[\to [0,\infty[$ that satisfies $\omega(0+) = 0$ such that*

$$F(y,r,\alpha(x-y),Y) - F(x,r,\alpha(x-y),X) \le \omega(\alpha|x-y|^2 + |x-y|) \qquad (32)$$

whenever $x, y \in \Omega$, $r \in \mathbb{R}$, X, Y are Hermitian matrices such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \qquad (33)$$

and the eigenvalues of $H_m(x,\alpha(x-y),X), H_m(y,\alpha(x-y),Y)$ are in $\overline{\Gamma_k^m}$.

Proof. Since $Q_m(x,p)$ is smooth and linear with respect to p, there is $L > 0$ such that

$$-L\alpha|x-y|^2 I_m \le Q_m(x,\alpha(x-y)) - Q_m(y,\alpha(x-y)) \le L\alpha|x-y|^2 I_m \quad (34)$$

and since A_m is smooth, by (33) with a standard argument there is $\tilde{L} > 0$ such that

$$A_m(x)XA_m^T(x) - A_m(y)YA_m^T(y) \le \tilde{L}\alpha|x-y|^2 I_m, \qquad (35)$$

(see [8, equation (3.17)]) . By putting together (34) and (35) we get

$$H_m(x,\alpha(x-y),X) \le H_m(y,\alpha(x-y),Y) + (L+\tilde{L})\alpha|x-y|^2 I_m. \qquad (36)$$

Now, recall that in the set of Hermitian k-admissible matrices the function $G(A) = -(\sigma_k(A))^{1/k}$ is monotone decreasing, convex and homogeneous of degree one, i.e. $G(\lambda A) = \lambda G(A)$ for all $\lambda \in \mathbb{R}^+$. Define $g(t) = G(A + t\lambda I_m)$. We have that g is continuous in $[0,1]$ and differentiable in $]0,1[$. Thus, by Lagrange Theorem, there is $\theta \in]0,1[$ such that $g(1) = g(0) + g'(\theta)$ and by the convexity and the homogeneity of G we have

$$\begin{aligned} G(A+\lambda I_m) =& G(A) + \frac{\partial G}{\partial a_{ij}}(A+\theta\lambda I_m)\lambda\delta_{ij} \\ \ge& G(A) + \lambda\frac{\partial G}{\partial a_{jj}}(\theta\lambda I_m) \\ =& G(A) + \lambda\frac{\partial G}{\partial a_{jj}}(I_m) = G(A) - \lambda C_{m,k} \end{aligned} \qquad (37)$$

where $C_{m,k}$ is a positive constant only depending on m and k.

By applying (37) to the right hand side of (36) and by the monotonicity of G we get

$$G(H_m(x,\alpha(x-y),X)) \ge G(H_m(y,\alpha(x-y),Y)) - C_{m,k}(L+\tilde{L})\alpha|x-y|^2. \quad (38)$$

Finally, by (38) and the continuity of f we have (32) □

We can conclude that the following comparison principle holds

Proposition 4.1 (First comparison principle)**.** *If the prescribed function f is continuous, positive and strictly increasing with respect to u, then if $\underline{u}$ and $\overline{u}$ are respectively viscosity sub- and super-solution of $F = 0$ in Ω, such that $\underline{u}(y) \leq \overline{u}(y)$ for all $y \in \partial\Omega$, then $\underline{u}(x) \leq \overline{u}(x)$ for every $x \in \overline{\Omega}$.*

Proof. F is proper in the set of F-admissible functions, and by Lemma 4.1 F satisfies [8, (3.14)] and therefore the hypotheses of [8, Theorem 3.3] are full satisfied. □

A comparison principle in the class of uniformly horizontal convex sub- and super-solution of the Monge-Ampère equation in Carnot groups has been proved in [4]. We remark that Theorem 4.1 refines such result because we do not require a uniform F admissibility condition. Since we are interested even at the case of constant function f, we would like to have a comparison principle for F also when f is not strictly increasing with respect to u. It is a standard fact by now that in order to adapt the proof for the strictly monotone case in this situation, one needs to find (for instance) a strictly sub-solution for F. Let us denote by K the sectional curvature related to g: the following Lemma is standard in Riemannian geometry and we refer to [17] for the proof.

Lemma 4.2. *Let us suppose that there exists a constant $M \geq 0$ such that $K \leq M$. Then there exists a constant $r_M > 0$ (the injectivity radius) only depending on M such that for every $x_0 \in \mathbb{G}$, the squared distance function $\varphi := d_g^2(x_0, \cdot)$ is well defined and smooth on the geodesic ball $B(x_0, r_M)$. Moreover φ is strictly convex on $B(x_0, r_M)$, namely there exists a constant c_M only depending on M, such that*

$$H_g\varphi \geq c_M g, \qquad \text{on } B(x_0, r_M)$$

In particular if $M = 0$ then $r_M = \infty$.

Therefore we have

Proposition 4.2 (Second comparison principle)**.** *Let us suppose that there exists a constant $M \geq 0$ such that $K \leq M$ and let Ω be a bounded open set contained in the geodesic ball $B(x_0, r_M)$, for some $x_0 \in \mathbb{G}$. Let f be a continuous function, positive and non-decreasing with respect to u. Then the comparison principle for F holds.*

Proof. We only need the prove it when f is not strictly decreasing with respect to u. By the previous Lemma 4.2 we see that $D_g^2\varphi \geq c_M Id_n$ on Ω. One can build then a strictly sub-solution and the proof follows in a standard way. □

We explicitly note that we are working in full generality: if one considers Lie groups with some additional structure, then one could remove the hypothesis on the sectional curvature and one can have the comparison principle for every domain in $\mathbb{R}^n$. For instance in the case of Hessian defined on the first layer (let us say of dimension m) of Carnot groups, then one can use as strictly sub-solution a quadratic function depending only on the "first m variables" (see [2],

[3], [4]). However the next example shows that also in general Lie group one can have the comparison principle for every domain in $\mathbb{R}^n$.

Example 4.1. *We consider again the Example 2.2 and we explicitly note that it is not a Carnot group. Since we have only two (orthonormal) vector fields, X, Y, we need to compute the sectional curvature only on this pair. We obtain by a straightforward computation $K(X,Y) = -1$. Since the sectional curvature is a negative constant, therefore for every $x_0 \in \mathbb{R}^2$, the squared distance function $d_g^2(x_0, \cdot)$ is well defined, smooth and strictly convex on the whole $\mathbb{R}^2$*

Finally we remark that by our comparison principles via the Perron method and the existence of $\overline{u}, \underline{u}$ in Lemma 3.5 we also get the following existence result for the Dirichlet Problem with continuous boundary data.

Proposition 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and C^2 open set, with defining function ρ strictly F-admissible. Moreover, assume that comparison holds Then, for every $\phi \in C(\partial\Omega)$ and $f \in C(\Omega \times \mathbb{R})$ a positive function, there exists a unique continuous viscosity solution u of the problem (4).*

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