Equazioni a derivate parziali. - *Smooth regularity for solutions of the Levi Monge-Ampère equation.* Nota di FRANCESCA LASCIALFARI e ANNAMARIA MONTANARI

ABSTRACT. - We present a smooth regularity result for strictly Levi convex solutions to the Levi Monge-Ampère equation. It is a fully nonlinear PDE which is degenerate elliptic. Hence elliptic techniques fail in this situation and we build a new theory in order to treat this new topic. Our technique is inspired to those introduced in [3] and [8] for the study of degenerate elliptic quasilinear PDE's related to the Levi mean curvature equation. When the right hand side has the meaning of total curvature of a real hypersurface in $\mathbb{C}^{n+1}$, the Levi Monge-Ampère equation arises in the study of envelopes of holomorphy and has important applications in the theory of holomorphic functions of several complex variables.

KEY WORDS: Levi Monge-Ampère equation; Fully nonlinear degenerate elliptic PDE; Non-linear vector fields; Schauder-type estimate; Smooth regularity of strictly Levi convex solutions.

RIASSUNTO. - *Regolarità C^∞ delle soluzioni dell'equazione di Levi Monge-Ampère.* In questa nota presentiamo un risultato di regolarità C^∞ delle soluzioni strettamente Levi convesse dell'equazione di Levi Monge-Ampère. Si tratta di un'equazione alle derivate parziali totalmente non lineare e ellittico-degenere. Per studiare questa equazione non si possono utilizzare le tecniche classiche delle equazioni ellittiche, pertanto costruiamo una teoria della regolarità *ad hoc* in spazi di funzioni hölderiane modellati sulla geometria del problema. La nostra tecnica si ispira a quella introdotta in [3] e [8] per operatori quasilineari ellittico-degeneri che intervengono nello studio dell'equazione con assegnata curvatura media di Levi. Quando il secondo membro ha il significato di curvatura totale di un'ipersuperficie reale in $\mathbb{C}^{n+1}$, l'equazione di Levi Monge-Ampère contiene informazioni sui domini di olomofia ed ha importanti applicazioni nella teoria delle funzioni olomorfe di più variabili complesse.

1 Introduction

The Levi Monge-Ampère equation naturally arises in the study of envelopes of holomorphy in the theory of holomorphic functions in $\mathbb{C}^{n+1}$ (see [12]).

Let $M = \{\rho = 0\}$ be a real hypersurface in $\mathbb{C}^{n+1}$ and assume that M is locally the graph of a C^2 function $u : \Omega \to \mathbb{R}$, with Ω an open bounded subset in $\mathbb{R}^{2n+1}$. Let us denote by $z = (z_1, \dots, z_{n+1})$ a point of $\mathbb{C}^{n+1}$, with $z_l = x_l + iy_l$ for every $l = 1, \dots, n+1$, and write $M = \{y_{n+1} = u(x_1, y_1, \dots, x_n, y_n, x_{n+1})\}$. If $z_0 \in M$, we denote by $T_0^{\mathbb{C}}$ the complex tangent hyperplane to M at z_0, by $\mathcal{U} = \{h_l, l = 1, \dots, n\}$ a complex basis of it and by $L(u)$ the Levi form of $\rho = u - y_{n+1}$ restricted to $T_0^{\mathbb{C}}$ (see [14]). $L(u)$ is a hermitian form in n variables whose coefficients

$$A_{l\overline{p}}(u) = \langle (Hess_{\mathbb{C}}\rho)h_l, h_p\rangle, \quad \forall\, h_l, h_p \in \mathcal{U}$$

are quasilinear partial differential operators and $A_{l\bar{l}}(u)$ are degenerate elliptic. Here we have denoted by

$$Hess_{\mathbb{C}}\rho = \left(\frac{\partial^2 \rho}{\partial z_l \partial \overline{z}_p}\right)_{l,p=1}^{n+1}.$$

Precisely the real part and the imaginary part of $A_{l\bar{p}}$ are:

$$\begin{aligned} Re(A_{l\bar{p}}(u)) =&(1+(\partial_{x_{n+1}}u)^2)(\partial_{x_l x_p}u + \partial_{y_l y_p}u + a_l\partial_{x_p x_{n+1}}u + a_p\partial_{x_l x_{n+1}}u \\ &+ b_l\partial_{y_p x_{n+1}}u + b_p\partial_{y_l x_{n+1}}u + (a_l a_p + b_l b_p)\partial^2_{x_{n+1}}u) \\ Im(A_{l\bar{p}}(u)) =&(1+(\partial_{x_{n+1}}u)^2)(\partial_{x_l y_p}u - \partial_{x_p y_l}u - a_p\partial_{y_l x_{n+1}}u + a_l\partial_{y_p x_{n+1}}u \\ &+ b_p\partial_{x_l x_{n+1}}u - b_l\partial_{x_p x_{n+1}}u + (b_p a_l - b_l a_p)\partial^2_{x_{n+1}}u) \end{aligned} \tag{1}$$

where

$$a_l = \frac{\partial_{y_l}u - \partial_{x_l}u\,\partial_{x_{n+1}}u}{1+(\partial_{x_{n+1}}u)^2}, \quad b_l = \frac{-\partial_{x_l}u - \partial_{y_l}u\,\partial_{x_{n+1}}u}{1+(\partial_{x_{n+1}}u)^2}. \tag{2}$$

In the sequel we also use the following notation:

$$a = (a_1, \dots, a_n), \qquad b = (b_1, \dots, b_n).$$

We are now ready to define the Levi Monge-Ampère operator as

$$LMA(u) = det(A_{l\overline{p}}(u)). \tag{3}$$

Moreover, we say that a function $u \in C^2(\Omega)$ is Levi convex (strictly Levi convex) at ξ_0 if $L(u)(\xi_0) \geq 0$ (> 0) and Levi convex (strictly Levi convex) if $L(u)(\xi) \geq 0$ (> 0) for every $\xi \in \Omega$.

In [12] Slodkowski and Tomassini generalized these definitions to continuous functions and proved the existence of a viscosity solution $u \in Lip(\bar{\Omega})$ to the Dirichlet problem

$$\begin{cases} LMA(u) = k(\cdot, u)(1+|Du|^2)^{\frac{n+2}{2}} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ u \quad \text{is Levi convex} & \end{cases} \tag{4}$$

where Du is the euclidean gradient of u in $\mathbb{R}^{2n+1}$, $g \in C(\partial\Omega)$, $k \in C(\Omega \times \mathbb{R})$ and $k \geq 0$. Here k represents a sort of “total Levi curvature” of M and it is the analogous of the Gauss curvature for the classical Monge-Ampère equation (see [10]). Further regularity of their Lipschitz continuous viscosity solution is an interesting open problem.

In this note we give a first positive response to this question. Precisely we announce the following result:

Theorem 1.1 *If $u \in C^{2,\alpha}(\Omega)$ is a strictly Levi convex solution to the Levi Monge-Ampère equation*

$$LMA(u) = q(\cdot, u, Du) \tag{5}$$

in an open set $\Omega \subset \mathbb{R}^{2n+1}$ and $q \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{2n+1})$ is positive, then $u \in C^\infty(\Omega)$.

Here we have denoted by $C^{m,\alpha}$ the ordinary Hölder space with respect to the euclidean metric.
Let us remark that, even if u is strictly Levi convex, the Levi Monge-Ampère operator $LMA(u)$ is degenerate elliptic. Indeed, if we call D^2u the euclidean Hessian matrix of u, then by (1) there exists a function F, which is smooth with respect to its arguments, such that

$$LMA(u) = F(Du, D^2u).$$

Moreover, if $r_{ij} = D_{ij}u$, by computing the real matrix $\left(\frac{\partial F}{\partial r_{ij}}\right)_{i,j=1}^{2n+1}$, we recognize that

$$\frac{\partial F}{\partial r_{ij}}(Du, D^2u) \geq 0$$

with minimum eigenvalue identically zero. Hence we are forced to develop techniques that are very different from the ones used to study the classical Monge-Ampère equation and the complex Monge-Ampère equation, which are elliptic PDE's if evaluated on strictly convex functions and on strictly plurisubharmonic functions respectively (see [10], [1], [2]).

Let us remark that in the case $n = 1$ the operator $LMA(u)$ defined in (3) coincides with the Levi form $L(u)$ and equation (5) is a quasilinear PDE (see, for instance, [13]). Regularity properties of its solutions have been studied in [3], [4], [5], [6], [7], [9]. Moreover, the analogous of the equation with prescribed mean curvature for a real hypersurface in $\mathbb{C}^{n+1}$ has been studied in [8], where smooth regularity of classical solutions is proved.

In order to sketch the proof of our result we introduce some notations. Let $u \in C^{2,\alpha}(\Omega)$ be a strictly Levi convex solution to (5) and define for every $l = 1, \ldots, n$ the first order vector fields

$$X_l = \partial_{x_l} + a_l \partial_{x_{n+1}}, \quad Y_l = \partial_{y_l} + b_l \partial_{x_{n+1}}, \tag{6}$$

whose coefficients a_l and b_l are the smooth functions of the gradient of u given by (2).
Since the fixed solution u belongs to $C^{2,\alpha}(\Omega)$, then the coefficients a_l, b_l are $C^{1,\alpha}(\Omega)$ functions. So we write the coefficients of the Levi form $A_{l\bar{p}}$ in terms of the vector fields in (6):

$$A_{l\bar{l}}(u) = (1 + (\partial_{x_{n+1}} u)^2)^2 (X_l^2 u + Y_l^2 u), \tag{7}$$

$$Re(A_{l\overline{p}}(u)) = \frac{(1 + (\partial_{x_{n+1}} u)^2)^2}{2} (X_l X_p u + X_p X_l u + Y_l Y_p u + Y_p Y_l u),$$

$$Im(A_{l\overline{p}}(u)) = \frac{(1 + (\partial_{x_{n+1}} u)^2)^2}{2} (X_l Y_p u + Y_p X_l u - Y_l X_p u - X_p Y_l u).$$

Since u is strictly Levi convex in Ω, then by (7)

$$A_{l\bar{l}} > 0, \quad l = 1, \ldots, n. \tag{8}$$

For every $l = 1, \ldots, n$ we put

$$Z_{2l} = Y_l, \quad Z_{2l-1} = X_l, \qquad Z = (Z_1, Z_2, \ldots, Z_{2n}), \qquad Z^2 u = (Z_l Z_p u)_{l,p=1}^{2n}. \tag{9}$$

Then we prove that

$$\frac{LMA(u)}{(1 + (\partial_{x_{n+1}} u)^2)^{2n}} = \mathcal{H}(Z^2 u),$$

with $\mathcal{H}$ a smooth function of its arguments. We also prove that there exists a smooth positive function K such that

$$\frac{q(\cdot, u, Du)}{(1 + (\partial_{x_{n+1}} u)^2)^{2n}} = K(\cdot, u, Zu, \partial_{x_{n+1}} u). \tag{10}$$

A smooth function K such that (10) holds always exists because, by (2) and (6)

$$\partial_{x_i} u = X_i u - (\partial_{x_{n+1}} u)\, Y_i u, \qquad \partial_{y_i} u = Y_i u + (\partial_{x_{n+1}} u)\, X_i u.$$

For example, if $q = k(\cdot, u)(1 + |Du|^2)^{\frac{n+2}{2}}$ as in (4), then

$$K = k(\cdot, u)(1 + |Zu|^2)^{\frac{n+2}{2}} (1 + (\partial_{x_{n+1}} u)^2)^{\frac{2-3n}{2}}. \tag{11}$$

Hence we write the fully nonlinear equation in (5) as

$$\mathcal{H}(Z^2 u) = K(\cdot, u, Zu, \partial_{x_{n+1}} u). \tag{12}$$

Since u is strictly Levi convex in Ω, then there exists a positive constant M such that

$$\sum_{i,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{ij}}(Z^2u)\eta_i\eta_j \geq M \sum_{j=1}^{2n} \eta_j^2, \quad \forall \eta = (\eta_1, \dots, \eta_{2n}) \in \mathbb{R}^{2n},$$

$\frac{\partial}{\partial z_{ij}}$ being the derivative with respect to Z_iZ_ju.
Moreover, we recognize that (see also [3])

$$[Z_{2l-1}, Z_{2l}] = [X_l, Y_l] = -(X_l^2u + Y_l^2u)\partial_{x_{n+1}}, \qquad l = 1, \dots, n \tag{13}$$

so that by (6), (7), (8) and (13) the vector fields

$$Z_1, \dots, Z_{2n}, [Z_1, Z_2] \tag{14}$$

are linearly independent at every point and span $\mathbb{R}^{2n+1}$.
Let us explicitly remark that, even if the coefficients of Z were smooth, $\mathcal{H}$ would not satisfy Hörmander's condition of hypoellipticity since it is not a sum of squares of first order vector fields. Moreover, to the best of our knowledge, no regularity result has been published about fully nonlinear equation of the type (12) even in the case of smooth vector fields Z.

The sketch of the proof of Theorem 1.1 is organized as follows. In Section 2, by arguing as in [8], [9], we build a regularity theory, in some spaces of Hölder continuous functions, for the linear operator

$$H = \sum_{i,j=1}^{2n} h_{ij} Z_i Z_j - \lambda \partial_{x_{n+1}} \tag{15}$$

with low regular coefficients and with

$$\sum_{i,j=1}^{2n} h_{ij}\eta_i\eta_j \geq M \sum_{j=1}^{2n} \eta_j^2, \quad \forall \eta = (\eta_1, \dots, \eta_{2n}) \in \mathbb{R}^{2n} \tag{16}$$

for a positive constant M. The main result of this section is an interior Schauder-type estimate for classical solutions of $Hv = f$, with $h_{ij}, \lambda, f \in C^\alpha$, and the coefficients a, b of Z of class $C^{1,\alpha}$. Schauder estimates for sum of squares of smooth vector fields satisfying Hörmander condition have been proved in [15], but that technique does not seem to work in this situation because the coefficients a, b of Z are only $C^{1,\alpha}$. Moreover, under our assumption on the coefficients h_{ij}, there exists no change of variables which transforms the linear operator H in (15) into the linear operator defined in [8].

In Section 3 we prove Theorem 1.1 by using a non standard bootstrap method.

2 Linear Theory and Schauder-type estimate

In this section we first introduce some class $C_Z^{m,\alpha}$ of Hölder continuous functions naturally arising from the geometry of the problem. We then build a regularity theory for the linear operator H defined in (15) in these spaces.

For every $l = 1, \dots, n$ let us define the first order vector fields Z_l as in (9) with coefficients $a, b \in C^{1,\alpha}(\Omega)$. Moreover, let us assume that the vector fields $Z_1, \dots, Z_{2n}, [Z_1, Z_2]$ are linearly independent at every point and span $\mathbb{R}^{2n+1}$.
If the coefficients of the vector fields were smooth, then the linear operator H would satisfy Hörmander's condition of hypoellipticity. In our context the coefficients are only $C^{1,\alpha}(\Omega)$. However, for every

$\xi, \xi_0 \in \Omega$ there exists $\gamma : [0,1] \to \mathbb{R}^{2n+1}$ integral curve of the vector fields introduced in (14) which connects ξ_0 and ξ. Then there exists a control distance $d_Z(\xi, \xi_0)$ naturally associated to the geometry of the problem (see for example the distance ϱ_4 defined in [11], page 113).
We now define the class of Hölder continuous functions in terms of d_Z: for $0 < \alpha < 1$

$$\begin{aligned} C_Z^\alpha(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \text{ s.t. there exists a constant } c > 0 : \\ |v(\xi) - v(\xi_0)| \le c\, d_Z^\alpha(\xi, \xi_0) \text{ for all } \xi, \xi_0 \in \Omega \Big\} \end{aligned} \tag{17}$$

and

$$C_Z^{1,\alpha}(\Omega) = \{ v : \Omega \to \mathbb{R} \,:\, \exists\, Z_j v \in C_Z^\alpha(\Omega) \quad \forall\, j = 1, \dots, 2n \}.$$

If the coefficients $a, b \in C_Z^{m-1,\alpha}(\Omega)$,$m \ge 2$, we define

$$C_Z^{m,\alpha}(\Omega) = \{ v \in C_Z^{m-1,\alpha}(\Omega) \,:\, Z_j v \in C_Z^{m-1,\alpha}(\Omega) \quad \forall\, j = 1, \dots, 2n \}.$$

Obviously (see [8])

$$C^{m,\alpha}(\Omega) \subset C_Z^{m,\alpha}(\Omega) \subset C^{m/2,\alpha/2}(\Omega).$$

For every $m \ge 0$ we also define spaces of locally Hölder continuous functions :

$$C_{Z,loc}^{m,\alpha}(\Omega) = \{ v : \Omega \to \mathbb{R} \,:\, v \in C_Z^{m,\alpha}(\Omega') \quad \forall\, \Omega' \subset\subset \Omega \}.$$

If $v \in C_Z^\alpha(\Omega)$ we define

$$[v]_{\alpha;\Omega}^Z = \sup_{\xi,\zeta \in \Omega} \frac{|v(\xi) - v(\zeta)|}{d_Z^\alpha(\xi, \zeta)}.$$

Let $I = (i_1, \dots, i_m)$ be a multi-index of length $|I| = m$ and denote by

$$Z^I = Z_{i_1} Z_{i_2} \cdots Z_{i_m}.$$

If $v \in C_Z^{m,\alpha}(\Omega)$, with $m = 0, 1, 2, \dots$, and $0 < \alpha < 1$ we define the seminorm

$$[v]_{m,\alpha;\Omega}^Z = \sup_{|I| = m} [Z^I v]_{\alpha;\Omega}^Z,$$

and the norms

$$|v|_{m;\Omega}^Z = \sum_{j=0}^m \left(\sup_{|I|=j} \sup_\Omega |Z^I v| \right),$$

$$|v|_{m,\alpha;\Omega}^Z = |v|_{m;\Omega}^Z + [v]_{m,\alpha;\Omega}^Z.$$

We are now ready to state our interior Schauder-type estimate for solutions of $Hv = f$ with H as in (15).

Proposition 2.1 *Let $h_{ij}, \lambda \in C_Z^\alpha(\Omega)$, $a, b \in C_Z^{1,\alpha}(\Omega)$ and $v \in C_Z^{2,\alpha}(\Omega)$ be a solution of equation $Hv = f \in C_Z^\alpha(\Omega)$. Then if $\Omega' \subset\subset \Omega$ with $d_Z(\Omega', \partial\Omega) \ge \delta > 0$, there is a positive constant c such that for every $\beta \in (0, \alpha)$*

$$\delta |Zv|_{0;\Omega'}^Z + \delta^2 |Z^2 v|_{0;\Omega'}^Z + \delta^{2+\beta} [Z^2 v]_{\beta;\Omega'}^Z \le c(\sup_\Omega |v| + |f|_{0,\alpha;\Omega}^Z) \tag{18}$$

where c depends only on the constant M in (16), on $|h_{ij}|_{0,\alpha;\Omega}^Z$, $|\lambda|_{0,\alpha;\Omega}^Z$, $|a|_{1,\alpha;\Omega}^Z$, $|b|_{1,\alpha;\Omega}^Z$ as well as on $n, \alpha, \delta, \Omega$.

We next define the difference quotient of v at ξ in the direction Z_l as

$$\Delta_l^h v(\xi) = \frac{v(\exp(hZ_l)(\xi)) - v(\xi)}{h},$$

where $\exp(hZ_l)(\xi) = \gamma(h)$ denotes the solution of the following Cauchy problem

$$\begin{cases} \gamma' & = Z_l\, \gamma \\ \gamma(0) & = \xi. \end{cases}$$

We then apply Proposition 2.1 to $\Delta_l^h v$ for every $l = 1, \ldots, 2n$ to obtain, by an iteration process:

Proposition 2.2 *Let $h_{ij}, \lambda \in C^{m-1,\alpha}_{Z,loc}(\Omega)$ $a, b \in C^{m,\alpha}_{Z,loc}(\Omega)$, $m \geq 2$ and $v \in C^{2,\alpha}_{Z,loc}(\Omega)$ be a solution of equation $Hv = f$ with $f \in C^{m-1,\alpha}_{Z,loc}(\Omega)$. Then the solution v belongs to $C^{m+1,\beta}_{Z,loc}(\Omega)$ for every $\beta \in (0, \alpha)$.*

3 The Bootstrap Method

In this section, by mean of a bootstrap argument, we sketch the proof of Theorem 1.1. Let us fix a strictly Levi convex $C^{2,\alpha}(\Omega)$ solution u to equation (5), which we write as in (12), and define the vector fields Z as in (9) with $a = a(Du), b = b(Du)$ as in (2). Since u is strictly Levi convex in Ω then by (13) the vector fields in (14) are linearly independent. Let d_Z be the associated control distance and define the spaces $C^{m,\alpha}_Z$ in term of it as in Section 2. For all B and B' in Ω such that $B' \subset B \subset\subset \Omega$ we define $h_0 = d_Z(B', \partial B) > 0$, and for every $h \in \mathbb{R}$ such that $0 < |h|^{1/2} < h_0$ we define

$$w_h(\xi) = \frac{u(\xi + he_j) - u(\xi)}{h}$$

with e_j the unit coordinate vector in $\mathbb{R}^{2n+1}$ in the j direction, $j = 1, \ldots, 2n+1$. Hence we recognize that w_h is a solution of

$$H_u w_h = F_h$$

with

$$H_u = \sum_{i,j=1}^{2n} a_{ij} Z_i Z_j$$

for suitable Hölder-continuous coefficients a_{ij}.
Since u is strictly Levi convex in Ω then we recognize that there exists a positive constant M such that

$$\sum_{i,j=1}^{2n} a_{ij}\eta_i\eta_j \geq M \sum_{i=1}^{2n} \eta_i^2, \quad \forall \eta = (\eta_1, \ldots, \eta_{2n}) \in \mathbb{R}^{2n}.$$

Moreover $|F_h|_{0,\alpha;B'}$, $|a_{ij}|_{0,\alpha;B}$ are bounded by a positive constant independent of h. Hence, by applying Proposition 2.1 to w_h, we may assert that for all $B'' \subset\subset B'$ there exists a subsequence of $Z_i Z_l w_h$ which uniformly converges in $C^\beta_Z(B'')$ to $Z_i Z_l D_j u$ for every $\beta < \alpha$, for all $i, l = 1, \ldots, 2n$, and $j = 1, \ldots, 2n+1$. In particular we get:

Proposition 3.1 *If $u \in C^{2,\alpha}(\Omega)$ is a strictly Levi convex solution to (5), then $Du \in C^{2,\beta}_{Z,loc}(\Omega)$ for every $\beta \in (0, \alpha)$.*

Now let us define H in terms of u as in (15), with

$$h_{ij} = \frac{\partial \mathcal{H}}{\partial z_{ij}}(Z^2 u),$$

$$\lambda = nK\partial_{x_{n+1}}u + \frac{\partial K}{\partial u_{x_{n+1}}}(1 + (\partial_{x_{n+1}}u)^2), \tag{19}$$

and K defined as in (10). As an immediate consequence of Proposition 3.1 the coefficients $a, b, \lambda \in C^{2,\beta}_{Z,loc}(\Omega)$, while $\frac{\partial \mathcal{H}}{\partial z_{ij}}(Z^2u) \in C^{1,\beta}_{Z,loc}(\Omega)$. Moreover, the function

$$v = (v_1, \dots, v_{2n}, v_{2n+1}) = (Z_1 u, \dots, Z_{2n}u, \arctan u_{x_{n+1}}) \tag{20}$$

is a $C^{2,\beta}_{Z,loc}(\Omega)$ solution to

$$Hv = f(\cdot, u, v, Zv), \tag{21}$$

with $f = (f_1, \dots, f_{2n}, f_{2n+1})$ a smooth function of its arguments. Since the right hand side in (21) is of class $C^{1,\beta}_{Z,loc}(\Omega)$, we apply Proposition 2.2 with $m = 2$ to get $v \in C^{3,\gamma}_{Z,loc}(\Omega)$ for every $\gamma \in (0, \beta)$.
Then we conclude the proof by induction. Let us assume that the function v defined in (20) belongs to $C^{m,\alpha}_{Z,loc}(\Omega)$ and prove that $v \in C^{m+1,\beta}_{Z,loc}(\Omega)$ for every $\beta \in (0, \alpha)$. Indeed $a, b, \lambda \in C^{m,\alpha}_{Z,loc}(\Omega)$, $\frac{\partial \mathcal{H}}{\partial z_{ij}} \in C^{m-1,\alpha}_{Z,loc}(\Omega)$ and v is a solution to (21) with right hand side of class $C^{m-1,\alpha}_{Z,loc}(\Omega)$. Hence, by Proposition 2.2, $v \in C^{m+1,\beta}_{Z,loc}(\Omega)$ for every $\beta \in (0, \alpha)$ and Theorem 1.1 is proved.

References

[1] Z. Blocki, *On the regularity of the Complex Monge-Ampère operator* Kim, Kang-Tae (ed.) et al., Complex geometric analysis in Pohang. POSTECH-BSRI SNU-GARC international conference on several complex variables, Pohang, Korea, June 23–27, 1997. Providence, RI: American Mathematical Society. Contemp. Math. 222, 181-189 (1999).

[2] L. Caffarelli, J.J. Kohn, L. Niremberg, J. Spruck, *The Dirichlet problem for non-linear second order elliptic equations II: Complex Monge-Ampère and uniformlyelliptic equations,* Comm. Pure Appl. Math. **38** (1985), 209-252.

[3] G. Citti, *C^∞ regularity of solutions of a quasilinear equation related to the Levi operator* Ann. Scuola Norm. Sup. di Pisa Cl. Sci., Serie **4** Vol. XXIII (1996), 483-529.

[4] G. Citti, *C^∞ regularity of solutions of the Levi equation*, Ann. Inst. H. Poincare, Anal. non Linéaire, 15, 4 (1998) 517-534.

[5] G. Citti, E. Lanconelli, A. Montanari, *On the smoothness of viscosity solutions of the prescribed Levi-curvature equation*, Rend. Mat. Acc. Lincei s.9, v.10 (1999) 61-68.

[6] G. Citti, A. Montanari, *Strong solutions for the Levi curvature equation*, Adv. in Diff. Eq. Vol **5** (1-3) (2000), 323-342.

[7] G. Citti, A. Montanari, *Regularity properties of Levi flat graphs* C.R. Acad. Sci. Paris, t. 329, Série 1 (1999) 1049-1054.

[8] G. Citti, A. Montanari, C^∞ *regularity of solutions of an equation of Levi's type in* $\mathbb{R}^{2n+1}$, to appear on Ann. Mat. Pura Appl.

[9] G. Citti, A. Montanari, *Regularity properties of solutions of a class of elliptic-parabolic nonlinear Levi type equations*, Preprint.

[10] D. Gilgarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehrer der Math. Wiss. vol. 224, Springer-Verlag, New York (1977).

[11] A. Nagel, E.M. Stein, S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985) 103-147.

[12] Z. Slodkowski, G. Tomassini, *The Levi equation in higher dimension and relationships to the envelope of holomorphy*, American Journal of Mathematics **116** (1994), 479-499.

[13] Z. Slodkowski, G. Tomassini, *Weak solutions for the Levi equation and Envelope of Holomorphy*, J. Funct. Anal. **101**, no. **4** (1991), 392-407.

[14] G. Tomassini, *Geometric Properties of Solutions of the Levi equation*, Ann. Mat. Pura Appl. (4), **152** (1988), 331-344.

[15] C.J. Xu, *Regularity for Quasilinear Second-Order Subelliptic Equations*, Comm. Pure and Appl. Math., **45** (1992), 77-96.

Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta S. Donato, 5 - 40127 Bologna
lascia@dm.unibo.it
montanar@dm.unibo.it