Existence and Uniqueness of Lipschitz Continuous Graphs with Prescribed Levi Curvature

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Abstract In this paper we prove comparison principles between viscosity semicontinuous sub- and supersolutions of the generalized Dirichlet problem (in the sense of viscosity solutions) for the *Levi Monge-Ampère* equation. As a consequence of this result and of the Perron's method we get the existence of a continuous solution of the Dirichlet problem related to the prescribed Levi curvature equation under suitable assumptions on the boundary data and on the Levi curvature of the domain. We also show that such a solution is Lipschitz continuous by building Lipschitz continuous barriers and by applying a weak Bernstein method introduced by Barles in [3].

Résumé Dans cet article, nous prouvons des principes de comparaison entre sous et sursolutions du problème de Dirichlet généralisé (dans le sens des solutions de viscosité) pour l'équation de *Levi Monge-Ampère*. Comme conséquence de ces résultats, nous obtenons l'existence d'une solution continue du problème de Dirichlet associé à l'équation de la courbure de Levi sous des hypothèses convenables sur les conditions au bord et sur l'ouvert. Nous prouvons que la solution est lipschitzienne par la méthode de Bernstein faible introduite par Barles dans [3].

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Contents

1 Introduction **2**

2 Graphs with prescribed Levi curvature in a viscosity sense **7**

3 Boundary conditions **10**

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4 Comparison principles and existence results **15**

5 Lipschitz estimates and proofs of Theorems 1.1, 1.2, 1.3 **21**

6 Non existence results on balls **29**

1 Introduction

If M is a hypersurface in $\mathbb{R}^n$, and if Π is its second fundamental form, then the eigenvalues of Π are the principal curvatures of M. It is well known that the trace of Π is called the mean curvature of M, and the determinant of Π is the Gauss-Kronecker curvature. The Dirichlet problem for a convex graph with prescribed curvature is classical (see for example [14]) and has been considered by many authors in the past (see [25] for a list of references), starting from the pioneer works by A. D. Aleksandrov and I. Ya. Bakelman.

For a real hypersurface $M \subset \mathbb{C}^{n+1}$, we let H denote the n-dimensional complex subspace of the tangent space to M. The second fundamental form of M restricts to a Hermitian form Λ on H, which is called the Levi form. Precisely, if M is a real manifold of class C^2 and ρ is a local defining function for it, then the Levi form $\Lambda(\rho)$ is the restriction to the complex tangent space H of the Hermitian form related to the complex Hessian matrix $Hess_{\mathbb{C}}\rho = \left(\frac{\partial^2 \rho}{\partial z_\ell \partial \bar{z}_p}\right)_{\ell,p=1}^{n+1}$ of ρ.

The Levi form is of great importance in the study of envelopes of holomorphy in the theory of holomorphic functions in $\mathbb{C}^{n+1}$ (see [12], [15], [19], [22] for details on this matter). It is a standard fact that the Levi form is the biholomorphic invariant part of the real Hessian of the defining function; one way to derive it is to seek for a biholomorphic invariant analogue of Euclidean convexity (see for example [19]). Since Λ is obtained from part of the second fundamental form of M, it is not unreasonable that it will have some properties similar to curvatures. However, Λ itself depends on the defining function for the domain. To avoid this obstacle one can argue as follows. If M is given locally as $\{\rho = 0\}$, with $\partial\rho \neq 0$, then one can define the normalized Levi form as $L(\rho) = \frac{\Lambda(\rho)}{|\partial\rho|}$, and an easy calculations shows that L is independent of the defining function ρ and only depends on the domain (a proof of this assertion can be found in [20, Proposition A.1]). Bedford and Gaveau were the first to remark this, and in [6] they bounded in term of the normalized Levi form the domain over which M can be defined as a non parametric surface of class C^2. The signature of L is a biholomorphic invariant of M, although L itself is not invariant.

We recall that a domain $\{\rho < 0\}$ is strongly pseudoconvex if the Levi form ρ (or equivalently the normalized Levi form) is positive definite on the boundary.

The eigenvalues of L correspond to mean curvatures in certain complex directions and, more generally, symmetric functions in the eigenvalues of L are complex curvatures of M. The product of the eigenvalues of L, corresponding to the complex version of the Gauss-Kronecker curvature of M, is the scalar function $k_M(\cdot)$ defined by

$$k_M(z) = -|\partial\rho|^{-n-2} \det \begin{pmatrix} 0 & \partial_{\bar{1}}\rho & \cdots & \partial_{\overline{n+1}}\rho \\ \partial_1\rho & \partial_{1\bar{1}}\rho & \cdots & \partial_{1\overline{n+1}}\rho \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{n+1}\rho & \partial_{n+1\bar{1}}\rho & \cdots & \partial_{n+1\overline{n+1}}\rho \end{pmatrix}. \tag{1.1}$$

We will call $k_M(z)$ the total *Levi curvature* of M at a point $z \in M$. In (1.1) $\partial_j, \partial_{\bar{j}}, \partial_{\ell\bar{j}}$ denote the derivatives $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial^2}{\partial z_\ell \partial \bar{z}_j}$ respectively, $\partial\rho = (\partial_1\rho, \ldots, \partial_{n+1}\rho)$ and the derivatives are computed at

z. To convince the reader that the total Levi curvature is the analogous of the Gauss curvature for the complex structure, we propose the following example.

Example 1.1 (Total Levi curvature of a ball). *If M is a ball of radius R with center at zero, then we can choose its defining function as $\rho = |z_1|^2 + \cdots + |z_{n+1}|^2 - R^2$ and an easy calculation gets $k \equiv R^{-n}$.*

However, a cylinder in $\mathbb{C}^{n+1}$ may not have zero total Levi curvature, as the following example shows.

Example 1.2 (Total Levi curvature of a cylinder). *Let $B(0,r) \subset \mathbb{C}^n \times \mathbb{R}$ be a ball of radius r. We consider the following cylinder*

$$B(0,r) \times i\mathbb{R} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : |z|^2 + \left(\frac{w+\bar{w}}{2}\right)^2 - r^2 < 0\}.$$

It is easy to check that

$$\frac{1}{2r^n} \le k_{\partial B(0,r)\times i\mathbb{R}}(z,w) = \frac{r^2 + (\mathrm{Re} w)^2}{2r^{n+2}} \le \frac{1}{r^n}$$

for any $(z,w) \in \partial B(0,r) \times i\mathbb{R}$.

If M is the graph of a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{C}^n \times \mathbb{R}$, then we shall say that u is Levi convex in Ω if $epi(u) = \{(z,w) \in \mathbb{C}^{n+1} \colon \mathrm{Im}\, w > u(z, \mathrm{Re}\, w)\}$ is pseudoconvex at every point of M, and in this situation

$$k_M = 2^{n+2}(1+|Du|^2)^{-(n+2)/2} \det \begin{pmatrix} 0 & u_{\bar{z}} & u_{\bar{w}} - i/2 \\ u_z & u_{z\bar{z}} & u_{z\bar{w}} \\ u_w + i/2 & u_{w\bar{z}} & u_{w\bar{w}} \end{pmatrix}.$$

The determinant on the right-hand side is often called the *Levi Monge-Ampère* operator $LMA(u)$, to emphasize the comparison with the Euclidean Monge-Ampère operator.

Even if the Levi curvature has some geometric properties similar to the Euclidean Gauss curvature we must stress that the *Levi Monge-Ampère* operator is never strictly elliptic, not even on the class of strictly convex functions.

In this paper we deal with the Dirichlet problem of finding a non parametric hypersurface with prescribed total Levi curvature k on a domain $\Omega \subset \mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^n \times \mathbb{C}$ such that $\Omega \times i\mathbb{R}$ is strongly pseudoconvex, and it takes the following form: Given $\varphi \in C(\partial\Omega)$ and $k \ge 0$ continuous, find $u \in C(\overline{\Omega})$ Levi convex such that

$$u|_{\partial\Omega} = \varphi \quad \text{and} \quad LMA(u) = k(\cdot, u)(1+|Du|^2)^{(n+2)/2} \,\text{on}\, \Omega. \tag{1.2}$$

In the sequel we denote by Du and D^2u respectively the gradient and the Hessian matrix of u. The Dirichlet problem for LMA for $n = 1$ was considered first by A. Debiard and Gaveau [13], who gave an estimate for the modulus of continuity of the solution, and by Z. Slodkowski and G. Tomassini in [23].

In [21] it is shown that LMA is degenerate elliptic in the set of Levi convex functions, namely if u, v are Levi convex and $L(u) \le L(v)$ then $LMA(u) \le LMA(v)$. Therefore one cannot expect in general C^∞ regularity result for this equation. Recall that if $k \equiv 0$ then every real function of the last variable $u(\mathrm{Re}\, w)$ is a solution $LMA(u) = 0$. Hence, in this case the regularity of a solution comes from the regularity of the boundary data. However, if $k > 0$ the missing ellipticity direction can be recovered by taking into account the CR structure of the hypersurface. This fact has been used in [21] by F. Lascialfari and the second author to prove that the Levi Monge Ampère operator has some hypoelliptic properties analogous to the Monge Ampère operator. Precisely, by denoting $C^{m,\alpha}$ the ordinary Hölder space with respect to the Euclidean metric, we have the following regularity result

Theorem. *If $u \in C^{2,\alpha}(\Omega)$ is a strictly Levi convex solution to the Levi Monge-Ampère equation*

$$LMA(u) = q(\cdot, u, Du) \tag{1.3}$$

in an open set $\Omega \subset \mathbb{R}^{2n+1}$ and $q \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{2n+1})$ $q > 0$, then $u \in C^\infty(\Omega)$.

The existence of classical solution of equation (1.3) is an interesting open problem for $n > 1$, while for $n = 1$ it has been solved by Citti, Lanconelli and the second author in [7].

In the paper [24], Z. Slodkowski and G. Tomassini defined generalized viscosity solutions of the Dirichlet problem (1.2) with the power $3n/2$ in the term $(1 + |Du|^2)$. The technique developed in [24] is to reduce (1.2) to a Bellman problem for a family of quasilinear degenerate elliptic operator L_ν, and to provide a priori estimates of solutions of the uniformly elliptic equation $L_\nu(u) + \varepsilon\Delta u = k^{1/n}(1 + |Du|^2)^{1/2}$ independent of ε and of ν. Their main result is the existence of a Lipschitz continuous viscosity solution u to (1.2). However, their method requires very strong conditions on k and on the growth of its first and second derivatives (see [24, Theorem 2.4 and condition (2.5) p. 488]). Such a solution is shown to be unique only in the particular case $k \equiv 0$, which seems to be of particular interest for complex analysis because if u is the solution of (1.2) with $k = 0$, then for $\lambda \geq \max_\Omega u$, $\Gamma_+^\lambda(u) = \{(x, is) \in \overline{\Omega} \times i\mathbb{R} \colon u(x) \leq s \leq \lambda\}$ is both the holomorphic hull and the envelope of holomorphy of $C_{\varphi,\lambda} = \big(\overline{\Omega} \times \{i\lambda\}\big) \cup \{(x, is) \in \partial\Omega \times i\mathbb{R} \colon \varphi(x) \leq s \leq \lambda\}$. However, in view of the regularity results in [21], the case $k \geq 0$ and $k \not\equiv 0$ seems to be significative from a PDE's point of view.

The main aim of this paper is to show the existence and the uniqueness of a Lipschitz continuous viscosity solution of (1.2) under far less restrictive hypotheses on the prescribed function k, without imposing any growth conditions on its first and second derivatives. To this purpose we use some classical techniques in the framework of viscosity solutions' theory. We recall that the theory of viscosity solutions, which was initiated in the early 80's by the papers of M.G. Crandall and P.L. Lions [9], M.G. Crandall, L.C. Evans and P.L. Lions [8], provides not only a convenient partial differential equations framework for dealing with the lack of the existence of classical solutions, but also leads to the correct formulation of the "generalized" boundary conditions of fully nonlinear elliptic and parabolic pde's. For a complete survey of the results obtained within the theory of viscosity solutions for the first-order case we refer to the books of Bardi and Capuzzo Dolcetta [1] and Barles [2], while for the second-order case we refer to the "Users'guide" of Crandall, Ishii and Lions [10].

In this framework the standard Dirichlet boundary conditions has to be relaxed and read the viscosity sense as

$$\min(-LMA(u) + k(\cdot, u)(1 + |Du|^2)^{(n+2)/2}, u - \varphi) \leq 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

and

$$\max(-LMA(u) + k(\cdot, u)(1 + |Du|^2)^{(n+2)/2}, u - \varphi) \geq 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

Roughly speaking, these relaxed conditions mean that the equations has to hold up to the boundary, when the boundary condition is not assumed in the classical sense. We refer again to the above mentioned references on viscosity solutions for a more complete presentation of boundary conditions in viscosity sense.

One of the main tools to prove the existence and the uniqueness of a continuous solution to (1.2) is to provide a comparison principle between semicontinuous sub and supersolutions to (1.2). Indeed the existence follows easily through the Perron's method by Ishii [16] with the version up to the boundary obtained by the first author in [11].

Hereafter we suppose that $\Omega \subset \mathbb{R}^{2n+1}$ is a bounded domain with boundary of class C^2. We list below some basic assumptions we use throughout the paper.

We assume that $k\colon \overline{\Omega} \times \mathbb{R} \to [0, +\infty)$ is a continuous bounded function satisfying

(H1) for all $R > 0$, there exists $\ell_R > 0$, such that, for every $x \in \overline{\Omega}$, and $-R \le v \le u \le R$

$$\ell_R(u - v) \le k^{1/n}(\cdot, u) - k^{1/n}(\cdot, v) \tag{1.6}$$

(H2) for all $R > 0$, there exists a modulus of continuity ω_R such that $\omega_R(s) \to 0$ as $s \to 0^+$ and

$$|k^{1/n}(x, u) - k^{1/n}(y, u)| \le \omega_R(|x - y|)$$

for all $(x, y) \in \overline{\Omega}$ and $|u| \le R$.

Conditions **(H1)** and **(H2)** will be used in Section 3 to prove a comparison principle between viscosity semicontinuous sub- and supersolution to the problem (1.2).

We add the following assumption on k and Ω, which will be used in Section 4 to solve the Dirichlet problem via the Perron's method.

(H3) $\Omega \times i\mathbb{R}$ is strongly pseudoconvex and for all $\xi_0 \in \partial\Omega$, $\sup_{\overline{\Omega}\times\mathbb{R}} k < k_{\partial\Omega\times i\mathbb{R}}(\xi_0)$.

(H4) $\sup_{\overline{\Omega}\times\mathbb{R}} k < \dfrac{1}{r^n}$, where r is the radius of the minimum sphere containing $\overline{\Omega}$.

Precisely, condition **(H3)** will guarantee that there is no loss of boundary condition and will also allow to build local barriers to the problem (1.2). Instead **(H4)** will be used to prove the existence of a particular subsolution to (1.2) and thus it will permit to get the existence of a continuous solution by the Perron's method.

We shall prove the following theorems.

Theorem 1.1. **[The strictly monotone case]** *Assume* ***(H1)****–* ***(H4)*** *hold. Then for any $\varphi \in C(\partial\Omega)$ there exists a unique continuous viscosity solution u of (1.2). Moreover, if $k \in Lip(\bar{\Omega} \times W)$ for every $W \subset\subset \mathbb{R}$, and $\varphi \in C^{1,1}(\partial\Omega)$, then $u \in Lip(\overline{\Omega})$.*

In place of conditions **(H1)** we shall also consider the following

(H5) for all $R > 0$, for every $x \in \overline{\Omega}$, and $-R \le v \le u \le R$

$$0 \le k(\cdot, u) - k(\cdot, v). \tag{1.7}$$

The condition **(H5)** aims at including the case when the prescribed function k is constant. Indeed, if we add the following condition

(H6) $k(x, u) = k(u)$ for all $(x, u) \in \Omega \times \mathbb{R}$,

we prove

Theorem 1.2. **[The x-independent case]** *Assume that* ***(H2)*** *–* ***(H6)*** *hold. Then, for every $\varphi \in C(\partial\Omega)$, there exists a unique continuous viscosity solution u of (1.2). Moreover, if $\varphi \in C^{1,1}(\partial\Omega)$, then $u \in Lip(\overline{\Omega})$.*

The proofs of Theorems 1.1 and 1.2 follow classical arguments within viscosity solutions's theory (see e.g [10]). Instead the Lipschitz continuity of the solution is obtained by building local barriers on the boundary and by adapting to our setting the weak Bernstein method, introduced by Barles in [3] to get gradient bound for viscosity solutions to fully nonlinear degenerate elliptic pde's.

If the prescribed function k satisfies the following conditions

(H7) $k \in Lip(\overline{\Omega} \times W)$ for every $W \subset\subset \mathbb{R}$,

(H8) there are $\alpha \geq 0, L > 0$ such that

$$\frac{D_x k \cdot p + D_u k |p|^2}{(1+|p|^2)^{1/2}} + gnk^{1+1/n} \geq \alpha \tag{1.8}$$

for almost every $(x, u) \in \Omega \times \mathbb{R}$ and for all $|p| \geq L$, for some $g \leq g_0$, where g_0 is the universal constant $\sqrt{2}(2-\sqrt{2})(\sqrt{2}+1)^{-1}$,

we prove the existence of a solution of (1.2) by an approximating argument. More precisely by using local barriers and the weak Bernstein method, we get a priori estimates of the Lipschitz constant and of the L_∞ norm of the solution of the approximating problem and we get the following

Theorem 1.3. **[The Lipschitz continuous case]** *Assume* ***(H3)*** *–* ***(H5)*** *and* ***(H7)*** *–* ***(H8)*** *hold. For every $\varphi \in C^{1,1}(\partial\Omega)$ there exists a Lipschitz continuous viscosity solution u of (1.2). Moreover, if $k > 0$, then the solution is unique.*

We should stress that for general fully nonlinear pde's the weak Bernstein method requires the inequality in (1.8) to hold for some $\alpha > 0$. Instead here because of the particular structure of the Levi Monge-Ampère operator the constant α can be zero.

The uniqueness statement in Theorem 1.3 is obtained via a comparison principle between continuous sub- and supersolutions. In the case $k > 0$ a strong comparison principle between C^2 sub- and supersolutions has been proved in [20], by taking into account that the non ellipticity direction can be recovered by commutations.

Our paper is organized as follows. In Section 2 we give a precise viscosity formulation of the Dirichlet problem (1.2) and we show the equivalence with the one given in [24]. In Section 3 we analyze the loss of boundary conditions for the Dirichlet problem (1.2). The question of loss of boundary conditions have been addressed by the first author in [11] for general fully nonlinear second order degenerate elliptic and parabolic equations. As it is well known this fact may depend on various aspects, such as the geometry of the domains, the structural properties of the operator appearing in the equation and the value of the boundary data. The main result of this section is that under the hypothesis **(H3)** there is no loss of boundary condition for the Dirichlet problem (1.2).

In Section 4 we prove comparison principles between viscosity semicontinuous sub- and supersolutions to the problem (1.2) under either conditions **(H1)** and **(H2)**, or conditions **(H2)** and **(H5)–(H6)**. In this section we also prove a comparison principle between continuous sub- and supersolution for Lipschitz continuous $k > 0$ satisfying **(H7)**, by using a geometric property of the Levi curvature. This result gives the uniqueness of a solution of problem (1.2) in Theorem 1.3. As a by-product of the comparison results and the Perron's method, under the hypothesis **(H4)** we get the existence of a continuous solution to (1.2) for all continuous boundary data. In Section 5, we show the existence of a Lipschitz continuous solution to (1.2) by applying the weak Bernstein method introduced by Barles in [3]. Moreover by using an approximating argument, together with some a priori estimate for the Lipschitz constant of a solution, we also prove the existence part of Theorem 1.3. In Section 6 we shall give an estimate of the maximum ball contained in Ω for which the problem (1.2) is solvable in the class of Lipschitz continuous viscosity solutions. Our argument is inspired to [6, Theorem 1] and to [20, Corollary 1.1]. Moreover, when the domain Ω is a ball, we shall prove a non existence result which shows that conditions **(H3)** and **(H4)** cannot be significantly relaxed.

2 Graphs with prescribed Levi curvature in a viscosity sense

In this Section we give the definition of pseudoconvex domains and Levi convex functions in a generalized viscosity sense. We also give a precise formulation of the Dirichlet problem (1.2) in a viscosity sense.

We start with the following

Definition 2.1. An open set $D \subset \mathbb{C}^{n+1}$ is pseudoconvex in a generalized viscosity sense if for every $z_0 \in \partial D$ and for every $\phi \in C^2(\mathbb{C}^{n+1})$ such that $\partial_z \phi(z_0) \neq 0$ and $\{\phi(z) < \phi(z_0)\} \subseteq D$ near z_0, we have $L(\phi)(z_0) \geq 0$.

One can see that Definition 2.1 is equivalent to the definition of Hartogs pseudoconvexity given in the literature (see e.g. [19]). More precisely we have the following equivalences.

Proposition 2.1. *Let $D \subset \mathbb{C}^{n+1}$ be an open set. The following conditions are equivalent:*

1. *D is pseudoconvex in a generalized viscosity sense;*
2. *For every $z_0 \in \partial D$ and for every quadratic polynomial q with $q(z_0) = 0$, $\partial_z q(z_0) \neq 0$, such that $\{z : q(z) < 0\}$ is contained in D near z_0, then $L(q)(z_0) \geq 0$;*
3. *D is Hartogs pseudoconvex.*

Proof. The proof of Proposition 2.1 is implicitly contained in [18, Theorem 4.1.27]) and we leave details to the reader. □

If M is non parametric hypersurface, then locally M is the graph of a C^2 function $u : \Omega \to \mathbb{R}$, with Ω an open bounded subset in $\mathbb{R}^N$ with $N = 2n + 1$. Then we can choose the defining function of M as $\rho = u(x, y, t) - s$, $M = \{s = u(x_1, y_1, \ldots, x_n, y_n, t)\}$. The coefficients $A_{\ell\overline{p}}(Du, D^2u)$ of the Levi form $L(u)$ are quasilinear partial differential operators. Precisely, the real part and the imaginary part of $A_{\ell\overline{p}}(Du, D^2u)$ are:

$$\begin{aligned} Re(A_{\ell\bar{p}}(Du, D^2u)) =& (\partial_{x_\ell x_p} u + \partial_{y_\ell y_p} u + a_\ell \partial_{x_p t} u + a_p \partial_{x_\ell t} u \\ & + b_\ell \partial_{y_p t} u + b_p \partial_{y_\ell t} u + (a_\ell a_p + b_\ell b_p)\partial_t^2 u) \\ Im(A_{\ell\bar{p}}(Du, D^2u)) =& (\partial_{x_\ell y_p} u - \partial_{x_p y_\ell} u - a_p \partial_{y_\ell t} u + a_\ell \partial_{y_p t} u \\ & + b_p \partial_{x_\ell t} u - b_\ell \partial_{x_p t} u + (b_p a_\ell - b_\ell a_p)\partial_t^2 u) \end{aligned} \tag{2.9}$$

where

$$a_\ell = a_\ell(Du) = \frac{\partial_{y_\ell} u - \partial_{x_\ell} u \, \partial_t u}{1 + (\partial_t u)^2}, \quad b_\ell = b_\ell(Du) = \frac{-\partial_{x_\ell} u - \partial_{y_\ell} u \, \partial_t u}{1 + (\partial_t u)^2}. \tag{2.10}$$

In particular for every $l = 1, \dots, n$, the diagonal coefficient $A_{\ell\bar{l}}(Du, D^2u)$ is a degenerate elliptic second order operator, whose characteristic form

$$\xi = (\xi_1, \dots, \xi_N) \longrightarrow (\xi_{2l-1} + a_\ell \xi_N)^2 + (\xi_{2l} + b_\ell \xi_N)^2,$$

is non negative definite for every $\xi \in \mathbb{R}^N$, but has $2n-1$ eigenvalues identically zero. In accordance with the notations of the Introduction, we define the *Levi Monge-Ampère* operator as

$$LMA(u) = (1 + u_t^2) \det(A_{\ell,\bar{p}}(Du, D^2u)). \tag{2.11}$$

Definition 2.2. *We say that a function $u \in C^2(\overline{\Omega})$ is Levi convex (strictly Levi convex) at $\xi_0 \in \Omega$ if $L(u)(\xi_0) \geq 0$ (> 0) and Levi convex (strictly Levi convex) in Ω if $L(u)(\xi) \geq 0$ (> 0) for every $\xi \in \Omega$.*

Remark 2.1. *The following conditions are equivalent (see [21]):*

1. *u is Levi convex in Ω,*
2. *the matrix $A(Du, D^2u) = (A_{\ell,\bar{p}}(Du, D^2u))_{\ell,p=1\dots,n}$ is non negative definite in Ω,*
3. *the epigraph of u is pseudoconvex.*

In [21] it has been proved that if $u \in C^2(\overline{\Omega})$ is convex in the classical sense, then u is Levi convex. In particular, if $D^2u \geq 0$ as a quadratic form, then $A(Du, D^2u) \geq 0$. The converse obviously is not true. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set and let $u\colon \Omega \to \mathbb{R}$ be a C^2 function. We say that u is a solution of the prescribed curvature equation if, given a continuous function $k : \Omega \times \mathbb{R} \to \mathbb{R}$

$$\det A(Du, D^2u) = k(\xi, u) f(Du), \tag{2.12}$$

where

$$f(Du) = 2^n \frac{(1 + |Du|^2)^{\frac{n+2}{2}}}{(1 + (\partial_t u)^2)}.$$

For any $O \subseteq \mathbb{R}^m$, we denote by $BUSC(O)$ the set of bounded and upper semicontinuous functions in O and by $BLSC(O)$ the set of bounded lower semicontinuous functions in O.

Definition 2.2 can be generalized to upper semicontinuous functions as follows (see also [24])

Definition 2.3. We say that a function $u \in USC(\overline{\Omega})$ is Levi convex (strictly Levi convex) in a viscosity sense at $\xi_0 \in \Omega$ if for all $\phi \in C^2(\overline{\Omega})$ and for all local maximum ξ_0 of $u - \phi$ we have $L(\phi)(\xi_0) \geq 0$ (> 0) and Levi convex (strictly Levi convex) in Ω if $L(\phi)(\xi) \geq 0$ (> 0)

Now we give a definition of viscosity subsolution and supersolution to the equation (2.12). Our definition extends the one in Ishii and Lions in [17] in the case of the classical Monge-Ampère equation, and it is analogous to that given by Slodkowski and Tomassini in [24] for the Dirichlet problem (1.2) .

Definition 2.4. We say that $u \in USC(\overline{\Omega})$ (resp. $v \in LSC(\overline{\Omega})$) is a viscosity subsolution (resp. supersolution) of (2.12) if for all $\phi \in C^2(\overline{\Omega})$ the following holds: at each local maximum ξ_0 (resp. local minimum) point of $u - \phi$ ($v - \phi$) then

$$\begin{aligned} \det A(D\phi, D^2\phi)(\xi_0) &\geq k(\xi_0, u(\xi_0)) f(D\phi(\xi_0)) \\ &\text{and} \\ L(\phi)(\xi_0) &\geq 0 \end{aligned}$$

(resp. either $L(\phi)(\xi_0)$ is not semidefinite positive or

$$\begin{array}{rcl}\det A(D\phi, D^2\phi)(\xi_0) & \leq & k(\xi_0, u(\xi_0))f(D\phi(\xi_0)) \\ & \text{and} & \\ L(\phi)(\xi_0) & \geq & 0\,).\end{array}$$

We recall that for all $A \in \mathcal{S}^n$ (the set of $n \times n$ symmetric matrices) we have the following matrix identity:

$$(\det A)^{1/n} = \begin{cases} \inf\{\mathrm{Tr}(AB) : B \in \mathcal{S}^n, B \geq 0, \det B = n^{-n}\}, & \text{if } A \geq 0 \\ -\infty, & \text{otherwise.} \end{cases} \tag{2.13}$$

In view of the identity (2.13), we give another viscosity formulation of the Dirichlet problem (1.2). To this purpose we consider the operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$ with $N = 2n+1$, defined by

$$F(\xi, u, p, X) := \begin{cases} k^{1/n}(\xi, u)f^{1/n}(p) - (\det A(p, X))^{1/n}\,, & \text{if } A(p, X) \geq 0 \\ +\infty, & \text{otherwise} \end{cases} \tag{2.14}$$

and the Dirichlet problem

$$\begin{cases} F(\xi, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u(\xi) = \varphi(\xi), & \text{in } \partial\Omega, \end{cases} \tag{DP}$$

where $\varphi \in C(\partial\Omega)$ the solution u is a scalar function and Du and D^2u denote respectively its gradient and Hessian matrix.

We set F^* and F_* the usc and lsc envelope of F respectively.

Definition 2.5. A function $u \in BUSC(\overline{\Omega})$ (resp. $v \in BLSC(\overline{\Omega})$) is said to be a viscosity subsolution (resp. supersolution) of (DP) in a generalized sense iff the following property holds :

for all $\phi \in C^2(\overline{\Omega})$, at each maximum point $\xi_0 \in \overline{\Omega}$ of $u - \phi$ we have

$$F_*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) \leq 0, \text{ if } \xi_0 \in \Omega$$

$$\min(F_*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)), u(\xi_0) - \varphi(\xi_0)) \leq 0 \quad \text{if } \xi_0 \in \partial\Omega,$$

(resp. for all $\phi \in C^2(\overline{\Omega})$, at each minimum point $\xi_0 \in \overline{\Omega}$ of $u - \phi$ we have

$$F^*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) \geq 0, \text{ if } \xi_0 \in \Omega$$

$$\max(F^*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)), u(\xi_0) - \varphi(\xi_0)) \geq 0 \quad \text{if } \xi_0 \in \partial\Omega)$$

Proposition 2.2. *Every solution u of (2.12) at $\xi_0 \in \Omega$ in the sense of Definition 2.4 is a solution of (DP) in a generalized sense and viceversa.*

Proof. We show that every generalized solution of (DP) at $\xi_0 \in \Omega$ is a solution of (2.12) in the sense of Definition 2.4 , the other implication being evident. Let u be a generalized solution of (DP) and let $\phi \in C^2(\Omega)$ be such that $u - \phi$ has a maximum at $\xi_0 \in \Omega$, then the inequality $F_*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) \leq 0$ implies that $L(\phi)(\xi_0) \geq 0$. Thus we have

$$\begin{array}{rcl}\det A(D\phi, D^2\phi)(\xi_0) & \geq & k(\xi_0, u(\xi_0))f(D\phi(\xi_0)) \\ & \text{and} & \\ L(\phi)(\xi_0) & \geq & 0\end{array}$$

Now suppose that $u - \phi$ has a minimum at $\xi_0 \in \Omega$ and that $F^*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) = +\infty$, (the case $F^*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) < +\infty$ being trivial). We distinguish the following two cases:

1) $L(\phi)(\xi_0) \geq 0$ and $L(\phi)(\xi_0)$ has at least one null eigenvalue. In this case we have

$$0 = \det A(D\phi, D^2\phi)(\xi_0) \leq k(\xi_0, u(\xi_0))f(D\phi(\xi_0)).$$

2) $L(\phi)(\xi_0) > 0$, then there is a ball $B(\xi_0, r)$, $r > 0$, such that $L(\phi)(y) > 0$ for all $y \in B(\xi_0, r)$. It follows that

$$F^*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) = F(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0))$$

and the inequality $F^*(\xi_0, u(\xi_0), D\phi(\xi_0), D^2\phi(\xi_0)) \geq 0$ implies

$$\begin{array}{rcl} \det A(D\phi, D^2\phi)(\xi_0) & \leq & k(\xi_0, u(\xi_0))f(D\phi(\xi_0)) \\ & \text{and} & \\ L(\phi)(\xi_0) & \geq & 0. \end{array}$$

Hence we can conclude. □

In the sequel when we talk about sub- and supersolutions of (DP), we will always mean in a viscosity sense.

We explicitly remark subsolutions of (DP) are Levi convex in a viscosity sense. Moreover, standard calculations show that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is Levi convex, then u is a classical solution of (DP) iff u is a viscosity solution of (DP) (see [24]).

3 Boundary conditions

Let Ω be a bounded open set in $\mathbb{R}^{2n+1}$ with C^2 boundary. Denote by d a smooth function agreeing in a neighborhood $\mathcal{W}$ of $\partial\Omega$ with the signed distance function to $\partial\Omega$ which is positive in Ω and negative in $I\!R^N \setminus \overline{\Omega}$ and we denote by $n(\xi) := -Dd(\xi)$ for all $\xi \in \mathcal{W}$. If $\xi \in \partial\Omega$, $n(\xi)$ is just the unit outward normal to $\partial\Omega$ at ξ.

In this section we analyze the loss of boundary conditions for the Dirichlet problem (DP) where F is given by (2.14). The question of loss of boundary conditions have been addressed by the first author in [11] for general fully nonlinear second order degenerate elliptic and parabolic equations. As it is well known this fact may depend on various aspects, such as the geometry of the domains, the structural properties of the operator appearing in the equation and the value of the boundary data (see e.g. the example in [5]). Here we are going to test on the operator (2.14) the conditions which have been found in [11] implying that the Dirichlet boundary conditions are assumed continuously by the solutions of (DP). To this end we introduce the following subsets of the boundary $\partial\Omega$: we denote by Σ_- the set of the points $\xi \in \partial\Omega$ such that, for all $R > 0$ either

$$\liminf_{\substack{w \to \xi \\ \alpha \downarrow 0}} \left\{ F(w, -R, \frac{-n(w) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2} n(w) \otimes n(w) + \frac{o_\alpha(1)}{\alpha^2}) \right\} > 0 \tag{3.15}$$

or

$$\liminf_{\substack{w \to \xi \\ \alpha \downarrow 0}} \left\{ F(w, -R, \frac{-n(w) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha} D^2 d(w) + \frac{o_\alpha(1)}{\alpha}) \right\} > 0 \tag{3.16}$$

and we denote by Σ_+ the set of the points $\xi \in \partial\Omega$ such that, for all $R > 0$

$$\limsup_{\substack{w\to\xi \\ \alpha\downarrow 0}} \left\{ F(w, R, \frac{n(w) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha^2} n(w) \otimes n(w) + \frac{o_\alpha(1)}{\alpha^2}) \right\} < 0 \tag{3.17}$$

or

$$\limsup_{\substack{w\to\xi \\ \alpha\downarrow 0}} \left\{ F(w, R, \frac{n(w) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha} D^2 d(w) + \frac{o_\alpha(1)}{\alpha}) \right\} < 0, \tag{3.18}$$

where $o_\alpha(1) \to 0$ as $\alpha \downarrow 0$ and $p \otimes p$ is the matrix $(p_i p_j)_{i,j=1}^{2n+1}$, for all $p = (p_1, \dots, p_{2n+1})$. Finally we set

$$\Sigma := \partial\Omega \setminus (\Sigma_- \cup \Sigma_+).$$

We premise some comments on the sets $\Sigma_\pm$. In the Section 4 of [11], it is proved that there cannot be loss of boundary conditions respectively for the sub- and supersolutions of (DP), namely for any $\xi \in \Sigma_-$ (resp. Σ_+) and any subsolutions u (resp. supersolutions v) we have $u(\xi) \le \varphi(\xi)$ $(v(x) \ge \varphi(\xi))$.

In the sequel we shall show that condition **(H3)** on $\partial\Omega$ is enough to guarantee that $\Sigma = \emptyset$.

Now we are going to test the conditions (3.15), (3.16), (3.17) and (3.18) in the case of the operator (2.14). We first note that if u is a defining function for Ω then $\rho(x, y, t, s) = u(x, y, t) - s$ is a defining function for a real manifold $M \subseteq R^{2n+1}$ and for every $(\xi, s) \in M$ $(\xi = (x, y, t))$ we have that

$$k_M^\infty(\xi) := \lim_{\eta\to\infty} k_M(\xi, \eta s) \tag{3.19}$$

is exactly the Levi curvature of the cylinder $\partial\Omega \times i\mathbb{R} = \{(\xi, s) : u(\xi) = 0\}$. The Levi curvature (1.1) of M can be represented as follows

$$k_M(\xi, u) = -h(Du) \det B(Du, D^2u) \tag{3.20}$$

with $h(Du) := 2^{n+2}(1 + |Du|^2)^{-(\frac{n+2}{2})}$ and

$$B(Du, D^2u) = \begin{pmatrix} 0 & \partial_{\bar 1} u & \cdots & \frac{\partial_t u - i}{2} \\ \partial_1 u & \partial_{1\bar 1} u & \cdots & \frac{\partial_{1t} u}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial_t u + i}{2} & \frac{\partial_{t\bar 1} u}{2} & \cdots & \frac{\partial_{tt} u}{4} \end{pmatrix}. \tag{3.21}$$

We first observe that

$$h(\eta Du) = 2^{n+2} \eta^{-(n+2)} (\eta^{-2} + |Du|^2)^{-(\frac{n+2}{2})}$$

$$\det B(\eta Du, \eta D^2u) = \eta^{n+2} \det \begin{pmatrix} 0 & \partial_{\bar 1} u & \cdots & \frac{\partial_t u - i/\eta}{2} \\ \partial_1 u & \partial_{1\bar 1} u & \cdots & \frac{\partial_{1t} u}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial_t u + i/\eta}{2} & \frac{\partial_{t\bar 1} u}{2} & \cdots & \frac{\partial_{tt} u}{4} \end{pmatrix}.$$

Thus we have

$$k_M^\infty(\xi) = -2^n |Du|^{-n+2} \det B_\infty(Du, D^2u) \tag{3.22}$$

with

$$B_\infty(Du, D^2u) := \begin{pmatrix} 0 & \partial_{\bar{1}}u & \cdots & \partial_t u \\ \partial_1 u & \partial_{1\bar{1}}u & \cdots & \partial_{1t}u \\ \vdots & \vdots & \ddots & \vdots \\ \partial_t u & \partial_{t\bar{1}}u & \cdots & \partial_{tt}u \end{pmatrix}.$$

By algebraic computations one can rewrite k_M also in the following way (see e.g [21])

$$k_M(\xi, u) = \frac{1}{2^n}\frac{(1+u_t^2)}{(1+|Du|^2)^{n+2/2}}\det A(Du, D^2u) \tag{3.23}$$

where $A(p, X)$ is the $n \times n$ Hermitian matrix defined in (2.9).

We list below some facts on the matrix A that will be useful also in the next Sections. First we have

$$A(\eta p, \eta X) = \frac{\eta}{(\eta^{-2}+u_t^2)^2}A'(p, X, \eta) \tag{3.24}$$

where the coefficients of $A'(p, X, \eta)$ are given by

$$\begin{aligned} Re(A'_{\ell\bar{p}}(Du, D^2u, \eta)) =& (\eta^{-2}+u_t^2)^2[\partial_{x_\ell x_p}u + \partial_{y_\ell y_p}u + a'_\ell\partial_{x_p t}u + a'_p\partial_{x_\ell t}u \\ &+ b_\ell\partial_{y_p t}u + b_p\partial_{y_\ell t}u] + (a'_\ell a'_p + b'_\ell b'_p)\partial_t^2 u \\ Im(A'_{\ell\bar{p}}(Du, D^2u, \eta)) =& (\eta^{-2}+u_t^2)^2[\partial_{x_\ell y_p}u - \partial_{x_p y_\ell}u - a'_p\partial_{y_\ell t}u + a'_\ell\partial_{y_p t}u \\ &+ b'_p\partial_{x_\ell t}u - b_\ell\partial_{x_p t}u] + (b'_p a'_\ell - b'_\ell a'_p)\partial_t^2 u \end{aligned} \tag{3.25}$$

and

$$a'_\ell = a'_\ell(Du, \eta) = \eta^{-1}\partial_{y_\ell}u - \partial_{x_\ell}u\,\partial_t u, \quad b'_\ell = b'_\ell(Du, \eta) = -\eta^{-1}\partial_{x_\ell}u - \partial_{y_\ell}u\,\partial_t u. \tag{3.26}$$

Moreover $A'(p, X, \eta)$ converges to $A_\infty(p, X)$ as $\eta \to \infty$ locally uniformly in (p, X), where the real part and the imaginary part of $(A_\infty)_{\ell\bar{p}}(Du, D^2u)$ are given by

$$\begin{aligned} Re((A_\infty)_{\ell\bar{p}}(Du, D^2u)) =& (u_t)^4[\partial_{x_\ell x_p}u + \partial_{y_\ell y_p}u + a_\ell^\infty\partial_{x_p t}u + a_p^\infty\partial_{x_\ell t}u \\ &+ b'_\ell\partial_{y_p t}u + b_p^\infty\partial_{y_\ell t}u] + (a_\ell^\infty a_p^\infty + b_\ell^\infty b_p^\infty)\partial_t^2 u \\ Im((A_\infty)_{\ell\bar{p}}(Du, D^2u)) =& (u_t)^4[\partial_{x_\ell y_p}u - \partial_{x_p y_\ell}u - a_p^\infty\partial_{y_\ell t}u + a_\ell^\infty\partial_{y_p t}u \\ &+ b_p\partial_{x_\ell t}u - b_\ell\partial_{x_p t}u] + (b_p^\infty a_\ell^\infty - b_\ell^\infty a_p^\infty)\partial_t^2 u \end{aligned} \tag{3.27}$$

with

$$a_\ell^\infty = a_\ell^\infty(Du) = -\partial_{x_\ell}u\,\partial_t u, \quad b_\ell^\infty = b_\ell^\infty(Du) = -\partial_{y_\ell}u\,\partial_t u. \tag{3.28}$$

Next we start analyzing the two conditions (3.15), (3.17). Standard computations show that $\det B_\infty(n, n\otimes n) = \det B_\infty(-n, -n\otimes n) = 0$.

Now we take $\xi_0 \in \partial\Omega$ and we distinguish two cases.

CASE 1: for all $\alpha > 0$ small and for all w close to ξ_0 the matrix $A(\frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})$ is not semidefinite positive. In this case we trivially have

$$\liminf_{\substack{w\to\xi \\ \alpha\downarrow 0}}\left\{F(w, R, \frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})\right\} \geq 0$$

CASE 2: there are subsequences $\alpha_n \to 0$ and $w_n \to \xi_0$ (that we continue to denote by α and w) such that the matrix $A(\frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})$ is semidefinite positive. In this case the following estimate holds.

$$\begin{aligned}
&F(w, -R, \frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})\\
&= f^{1/n}\left(\frac{-n(w)+o_\alpha(1)}{\alpha}\right)\Big\{k^{1/n}(y,-R)\\
&- \left[\det(A(\frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})f^{-1}(\frac{-n(w)+o_\alpha(1)}{\alpha})\right]^{1/n}\Big\}.
\end{aligned}$$

By using the identity (3.23) and $\det B_\infty(-n, -n\otimes n) = 0$, we get

$$\begin{aligned}
&\liminf_{\substack{w\to\xi\\ \alpha\downarrow 0}} -\left[\det(A(\frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})f^{-1}(\frac{-n(w)+o_\alpha(1)}{\alpha})\right]^{1/n}\\
&\quad + k^{1/n}(w,-R)\\
&\geq \liminf_{\substack{w\to\xi\\ \alpha\downarrow 0}}[-2(-\det(B_\infty(-n(w), -n(w)\otimes n(w)))^{1/n} + k^{1/n}(w,-R)]\\
&= k^{1/n}(\xi,-R) \geq 0.
\end{aligned}$$

Thus since $f^{1/n} \geq 0$ we get

$$\begin{aligned}
&\liminf_{\substack{w\to\xi_0\\ \alpha\downarrow 0}}\left\{F(w, -R, \frac{-n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})\right\}\\
&\geq \liminf_{\substack{w\to\xi_0\\ \alpha\downarrow 0}} f^{1/n}(\frac{-n(w)+o_\alpha(1)}{\alpha})[-2(-\det(B_\infty(-n(\xi_0), -n(\xi_0)\otimes n(\xi_0)))^{1/n} + k^{1/n}(\xi_0,-R)] \geq 0.
\end{aligned}$$

In a similar way one sees that

$$\limsup_{\substack{w\to\xi_0\\ \alpha\downarrow 0}}\left\{F(w, R, \frac{n(w)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha^2}n(w)\otimes n(w) + \frac{o_\alpha(1)}{\alpha^2})\right\} \geq 0$$

Thus the conditions (3.15) and (3.17) are not satisfied.

This implies that we have to impose some suitable conditions on the *Levi curvature* of the domain in order that both conditions (3.16) and (3.18) hold.

To this end we assume that Ω satisfies **(H3)**. Then since $-d$ is a defining function of Ω the following two conditions holds for every $s \in \mathbb{R}$ and $\xi_0 \in \partial\Omega$:

$$A_\infty(n(\xi_0), -D^2 d(\xi_0)) > 0 \tag{3.29}$$

and

$$2\left(-\det(B_\infty(n(\xi_0), -D^2 d(\xi_0))\right)^{1/n} > k^{1/n}(\xi_0, s) \tag{3.30}$$

Proposition 3.1. *Assume* **(H3)** *then both the conditions (3.16) and (3.18) are satisfied.*

Proof. We first notice that $A_\infty(-n(\xi_0), D^2d(\xi_0)) = -A_\infty(n(\xi_0), -D^2d(\xi_0))$, thus $A_\infty(-n(\xi_0), D^2d(\xi_0))$ is not semidefinite positive.

We claim that the matrix $A(\frac{-n(w)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha})$ is not semidefinite positive as well for $\alpha \to 0$ and $w \to \xi_0$.

Indeed if $A_\infty(-n(\xi_0), D^2d(\xi_0))$ is not semidefinite positive, then there is at least one eigenvalue which is strictly negative. Now from (3.24) it follows that

$$\begin{aligned} & A(\frac{-n(w)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha}) \\ & = \frac{\alpha^{-1}}{(\alpha^2 + (d_t + o_\alpha(1)^2)^2} A'(Dd + o_\alpha(1), D^2d + o_\alpha(1), \alpha^{-1}) \end{aligned} \tag{3.31}$$

Moreover

$$A'(Dd(w) + o_\alpha(1), D^2d(w) + o_\alpha(1), \alpha^{-1}) \to A_\infty(Dd(\xi_0), D^2d(\xi_0))$$

as $\alpha \to 0$ and $w \to \xi_0$. Furthermore one can see that there are $r > 0$ and α_0 such that for all $0 < \alpha \leq \alpha_0$ and for all $w \in B(\xi_0, r)$ the matrix $A'(Dd(w) + o_\alpha(1), D^2d(w) + o_\alpha(1), \alpha^{-1})$ is not semidefinite positive. From (3.31) it follows that the matrix $A(\frac{-n(w)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha})$ is not semidefinite positive and we prove the claim.

Hence we have

$$\liminf_{\substack{w\to\xi_0 \\ \alpha\downarrow 0}} \left\{ F(w, -R, \frac{-n(w)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha}) \right\} = +\infty \tag{3.32}$$

and (3.16) holds.

On the other hand if $A_\infty(n(\xi_0), D^2d(\xi_0))$ is definite positive then the matrix

$$A(\frac{n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha}) > 0$$

for $\alpha \to 0$ and $w \to \xi_0$. To show this fact one argues exactly as above. Thus we have

$$\begin{aligned} & \limsup_{\substack{w\to\xi_0 \\ \alpha\downarrow 0}} \left\{ F(w, R, \frac{n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha}) \right\} \\ & = \limsup_{\substack{w\to\xi_0 \\ \alpha\downarrow 0}} f^{1/n}(\frac{n(w)+o_\alpha(1)}{\alpha}) \cdot \\ & \limsup_{\substack{w\to\xi_0 \\ \alpha\downarrow 0}} \{ - \left(\det(A(\frac{n(w)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2d(w) + \frac{o_\alpha(1)}{\alpha}) f^{-1}(\frac{n(w)+o_\alpha(1)}{\alpha}) \right)^{1/n} \\ & + k^{1/n}(y, R) \} \\ & \leq \limsup_{\substack{w\to\xi_0 \\ \alpha\downarrow 0}} f^{1/n}(\frac{-n(w)+o_\alpha(1)}{\alpha}) \cdot \\ & [-2\left(-\det(B_\infty(n(\xi_0), -D^2d(\xi_0)))\right)^{1/n} + k^{1/n}(\xi_0, R)] < 0\,. \end{aligned}$$

where the last inequality follows by combining (3.30) and the fact that $f^{1/n} \geq 2$. Thus (3.18) is satisfied and we conclude. ∎

4 Comparison principles and existence results

In this section we provide two comparison principles between viscosity semicontinuous subsolutions and supersolutions of (DP) under the hypothesis **(H3)**, which guarantees that the boundary data is assumed continuously.

As a by-product of the these comparison results and the Perron 's method we get the existence of a unique continuous viscosity solution of (DP).

The first comparison result of this section is the following theorem, which holds under the assumptions that the function k is strictly increasing with respect to u. The proof of this result is standard and we provide it for the reader's convenience.

Theorem 4.1. *Assume* **(H1)**–**(H3)**. *Let* $u \in BUSC(\overline{\Omega})$ *and* $v \in BLSC(\overline{\Omega})$ *be respectively a viscosity subsolution and supersolution of* (DP). *Then* $u \leq v$ *in* $\overline{\Omega}$.

Proof. We suppose by contradiction that $\max_{\overline{\Omega}}(u-v) = M > 0$. By **(H3)** such maximum is achieved at an interior point ξ_0. For all $\varepsilon > 0$ we consider the auxiliary function

$$\Phi_\varepsilon(\xi,\zeta) = u(\xi) - v(\zeta) - \frac{|\xi-\zeta|^2}{\varepsilon^2}.$$

and let $(\xi_\varepsilon,\zeta_\varepsilon)$ be a maximum of Φ_ε in $\overline{\Omega}\times\overline{\Omega}$. By standard arguments we get, up to subsequences, $\xi_\varepsilon, \zeta_\varepsilon \to \tilde{\xi} \in \overline{\Omega}$, and

$$\begin{aligned}
&\frac{|\xi_\varepsilon-\zeta_\varepsilon|^2}{\varepsilon^2} = O(\varepsilon) \quad \text{as } \varepsilon \to 0,\\
&u(\xi_\varepsilon) - v(\zeta_\varepsilon) \to u(\tilde{\xi}) - v(\tilde{\xi}) = M\\
&u(\xi_\varepsilon) \to u(\tilde{\xi}),\ v(\zeta_\varepsilon) \to v(\tilde{\xi}).
\end{aligned}$$

We observe that since $u \leq v$ on $\partial\Omega$ we have $\tilde{\xi} \in \Omega$, thus for ε small enough $\xi_\varepsilon,\ \zeta_\varepsilon \in \Omega$ as well. Hence the equation holds for both u and v respectively at ξ_ε and ζ_ε.

Set $\phi(\xi,\zeta) = \dfrac{|\xi-\zeta|^2}{\varepsilon^2}$. For all $\alpha > 0$ there exist $X, Y \in \mathcal{S}(N)$ such that, if $p_\varepsilon := 2\dfrac{(\xi_\varepsilon-\zeta_\varepsilon)}{\varepsilon^2}$, we have

$$(p_\varepsilon, X) \in \mathcal{J}^{2,+}u(\xi_\varepsilon), \quad (p_\varepsilon, Y) \in \mathcal{J}^{2,-}v(\zeta_\varepsilon),$$

$$-(\frac{1}{\alpha} + ||A||)Id \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \alpha A^2 \tag{4.33}$$

where $A = D^2\phi(\xi,\zeta)$, and

$$F_*(\xi_\varepsilon, u(\xi_\varepsilon), p_\varepsilon, X) \leq 0 \quad \text{and} \quad F^*(\zeta_\varepsilon, v(\zeta_\varepsilon), p_\varepsilon, Y) \geq 0 \tag{4.34}$$

We note that (4.33) implies $X \leq Y$, thus $A(p_\varepsilon, X) \leq A(p_\varepsilon, Y)$ for all ε as well. By subtracting the two inequalities in (4.34) and by using **(H1)** we get

$$\ell_R(u(\tilde{\xi}) - v(\tilde{\xi})) \leq o(1) \quad \text{as } \varepsilon \to 0$$

and we obtain a contradiction by letting $\varepsilon \to 0$. $\square$

Next we are going to prove a comparison result by assuming the weaker condition **(H5)**. When there is not a strict monotonicity with respect to u, one of the classical approaches within the theory of viscosity solutions, is to try to find a strict subsolution or supersolution either of the original equation or of a suitable approximation of it. Here we extend the techniques used in Ishii and Lions [17] for quasilinear equations.

To this purpose we need the following two Lemmas.

Lemma 4.1. *There is a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\inf_{p\in\mathbb{R}^{2n+1}} \left(\det A(p, D^2\psi)\right)^{1/n} = \nu > 0.$$

Proof. Let us take $\psi(x, y, t) = g\left(\frac{\|x\|^2+\|y\|^2}{2}\right)$, with $g \in C^2(\mathbb{R})$ and $g', g'' > 0$. We note that for all $p \in \mathbb{R}^N$ and $X \in \mathcal{S}^N$ we have $A(p, Y) = \sigma(p)Y\overline{\sigma}^T(p)$ where σ is the $n \times N$ matrix given by

$$\sigma(p) = \begin{pmatrix} I_n, & -iI_n, & a(p) - ib(p) \end{pmatrix} \tag{4.35}$$

$a(\cdot), b(\cdot)$ being defined in (2.9). By using the above identity one can readily see that we have

$$A(p, D^2\psi) = 2g'\left(I_n + \frac{g''}{2g'}(x - iy) \otimes (x + iy)\right)$$

and

$$(\det A(p, D^2\psi))^{1/n} \geq 2g'\left(1 + \frac{g''}{2g'}(\|x\|^2 + \|y\|^2)\right)^{1/n} = \nu.$$

□

Lemma 4.2. *If $u \in BUSC(\overline{\Omega})$ is a viscosity subsolution of $F = 0$, then $u_m = u + \frac{1}{m}\psi$, with ψ as in the previous lemma, is a strictly viscosity subsolution of*

$$F(\xi, u_m, Du_m - D\psi/m, D^2u_m) + f^{1/n}(Du_m - D\psi/m)\left(k^{1/n}(\xi, u_m - \psi/m) - k^{1/n}(\xi, u_m)\right) = -\frac{\nu}{m}.$$

Proof. We notice that for all $\phi \in C^2(\overline{\Omega})$, ξ_0 is a maximum point of $u_m - \phi$, iff ξ_0 is a maximum point of $u - (\phi - \psi/m)$. Thus, since u is a viscosity subsolution of $F = 0$, at each maximum point of $u_m - \phi$, we have

$$F\left(\xi_0, u(\xi_0), (D\phi - \frac{D\psi}{m})(\xi_0), (D^2\phi - D^2\psi/m)(\xi_0)\right) \leq 0.$$

Therefore, by the convexity of the function $A \longmapsto -(\det A)^{1/n}$ and by Lemma 4.1, we get

$$\begin{aligned}
&F(\xi_0, u_m, D\phi - D\psi/m, D^2\phi) + f^{1/n}(D\phi - D\psi/m)\left(k^{1/n}(\xi_0, u_m - \psi/m) - k^{1/n}(\xi_0, u_m)\right) \\
&\quad = -\left(\det A(D\phi - D\psi/m, D^2\phi)\right)^{1/n} + f^{1/n}(D\phi - D\psi/m)k^{1/n}(\xi_0, u) \\
&\quad \leq -\left(\det A(D\phi - D\psi/m, D^2\phi - D^2\psi/m)\right)^{1/n} - \left(\det A(D\phi - D\psi/m, D^2\psi/m)\right)^{1/n} \\
&\qquad + f^{1/n}(D\phi - D\psi/m)k^{1/n}(\xi_0, u) \\
&\quad = F\left(\xi_0, u, D\phi - \frac{D\psi}{m}, D^2\phi - D^2\psi/m\right) - \left(\det A(D\phi - D\psi/m, D^2\psi/m)\right)^{1/n} \\
&\quad \leq -\left(\det A(D\phi - D\psi/m, D^2\psi/m)\right)^{1/n} \leq -\frac{\nu}{m}.
\end{aligned}$$

□

Now we shall prove a comparison principle, by assuming that **(H6)** holds, i.e. $k\colon \mathbb{R} \to [0, +\infty)$ is a continuous function which does not depend on ξ.

Theorem 4.2. *Assume* **(H2)**–**(H3)** *and* **(H5)**–**(H6)**. *Let* $u \in BUSC(\overline{\Omega})$ *and* $v \in BLSC(\overline{\Omega})$ *be respectively viscosity sub- and supersolution of* (DP). *Then* $u \leq v$ *in* $\overline{\Omega}$.

Proof. We consider $u_m = u + \frac{1}{m}\psi$ with ψ as in Lemma 4.1. We may suppose without restriction that $(|x|^2 + |y|^2) \neq 0$ in $\overline{\Omega}$, otherwise in the definition of ψ we replace $(|x|^2 + |y|^2)$ with $(|x - x_0|^2 + |y - y_0|^2)$with a suitable (x_0, y_0). Moreover we choose g in such a way that $\|\psi\|_\infty < +\infty$. Our aim is to show that $\sup_\Omega(u_m - v) \leq \frac{1}{m}\|\psi\|_\infty$. Suppose by contradiction that for all m large enough we have $M_m = \max_{\overline{\Omega}}(u_m - v) > \frac{1}{m}\|\psi\|_\infty$. Since by **(H3)** we have $u(x) \leq \varphi(x) \leq v(x)$ for all $x \in \partial\Omega$, such a maximum is achieved at an interior point ξ_0 (depending on m). For all $\varepsilon > 0$ let us consider the auxiliary function

$$\Phi_\varepsilon(\xi, \zeta) = u_m(\xi) - v(\zeta) - \frac{|\xi - \zeta|^2}{\varepsilon^2}.$$

Let $(\xi_\varepsilon, \zeta_\varepsilon)$ be a maximum of Φ_ε in $\overline{\Omega} \times \overline{\Omega}$. By standard arguments we get, up to subsequences, $\xi_\varepsilon, \zeta_\varepsilon \to \tilde{\xi} \in \overline{\Omega}$, and

$$\begin{aligned}
&\frac{|\xi_\varepsilon - \zeta_\varepsilon|^2}{\varepsilon^2} = o_\varepsilon(1) \quad \text{as } \varepsilon \to 0,\\
&u_m(\xi_\varepsilon) - v(\zeta_\varepsilon) \to u_m(\tilde{\xi}) - v(\tilde{\xi}) = M_\varepsilon\\
&u_m(\xi_\varepsilon) \to u_m(\tilde{\xi}), \ v(\zeta_\varepsilon) \to v(\tilde{\xi}).
\end{aligned}$$

Since $\tilde{\xi}$ is necessarily in Ω, for ε small enough we have $\xi_\varepsilon, \ \zeta_\varepsilon \in \Omega$ as well. Hence the equation holds for both u_m and v respectively in ξ_ε and ζ_ε.

There exist $X, Y \in \mathcal{S}^N$ such that, if $p_\varepsilon := 2\dfrac{(\xi_\varepsilon - \zeta_\varepsilon)}{\varepsilon^2}$, we have

$$(p_\varepsilon, X) \in \mathcal{J}^{2,+}u_m(\xi_\varepsilon), \quad (p_\varepsilon, Y) \in \mathcal{J}^{2,-}v(\zeta_\varepsilon),$$

$$-\frac{8}{\varepsilon^2} Id \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{4.36}$$

and by Lemma 4.2

$$\begin{aligned}
F(\xi_\varepsilon, u_m, p_\varepsilon - D\psi/m, X) + f^{1/n}(p_\varepsilon - D\psi/m)\Big(k^{1/n}(u_m - \psi/m) - k^{1/n}(u_m)\Big) < -\frac{\nu}{m}\\
F(\zeta_\varepsilon, v, p_\varepsilon, Y) \geq 0.
\end{aligned} \tag{4.37}$$

Moreover, $(p_\varepsilon - \frac{D\psi}{m}, X - D^2\psi/m) \in \mathcal{J}^{2,+}u(\xi_\varepsilon)$, and

$$F_*(\xi_\varepsilon, u(\xi_\varepsilon), p_\varepsilon - D\psi/m, X - D^2\psi/m) \leq 0. \tag{4.38}$$

As we observe in Lemma 4.1 we have $A(p_\varepsilon, Y) = \sigma(p_\varepsilon) Y \overline{\sigma}^T(p_\varepsilon)$ where σ is the $n \times N$ matrix defined in (4.35). Set $\Sigma_1 = \sigma(p_\varepsilon - \frac{1}{m}D\psi)$ and $\Sigma_2 = \sigma(p_\varepsilon)$. Multiply both sides of the inequality (4.36) by the matrix $\begin{pmatrix}\Sigma_1 & \Sigma_2\end{pmatrix}$ on the left, and by the transpose of its conjugate on the right, to get

$$\Sigma_1 X \overline{\Sigma}_1^T - \Sigma_2 Y \overline{\Sigma}_2^T \leq \frac{3}{\varepsilon^2}(\Sigma_1 - \Sigma_2)(\overline{\Sigma_1 - \Sigma_2})^T = \frac{1}{\varepsilon^2}\eta \otimes \bar{\eta} \tag{4.39}$$

with

$$\begin{aligned}
\eta &= [a(p_\varepsilon - \frac{1}{m}D\psi) - a(p_\varepsilon)] - i[b(p_\varepsilon - \frac{1}{m}D\psi) - b(p_\varepsilon)] \\
&= -\frac{g'}{m}\left(\frac{y - x(p_\varepsilon)_{2n+1} - i(x + y(p_\varepsilon)_{2n+1})}{1 + (p_\varepsilon)^2_{2n+1}}\right) \\
&= \frac{g'}{m}\left(\frac{(p_\varepsilon)_{2n+1} - i}{1 + (p_\varepsilon)^2_{2n+1}}\right)(x - iy).
\end{aligned}$$

Thus,

$$\Sigma_1 X \overline{\Sigma}_1^T - \Sigma_2 Y \overline{\Sigma}_2^T \leq \frac{(g')^2}{\varepsilon^2 m^2 (1 + (p_\varepsilon)^2_{2n+1})}(\|x\|^2 + \|y\|^2) Id.$$

¿From (4.38) it follows that

$$\Sigma_1 (X - D^2\psi/m)\overline{\Sigma}_1^T \geq 0$$

and

$$\Sigma_1 X \overline{\Sigma}_1^T \geq \frac{\Sigma_1 D^2\psi \overline{\Sigma}_1^T}{m} = \frac{1}{m}(g''(x - iy) \otimes (x + iy) + 2g' Id) > 0.$$

We will choose the function g in a such way that, for m large enough and for all ε, we have

$$\left(\frac{g''}{m} - \frac{(g')^2}{m^2\varepsilon^2(1 + (p_\varepsilon)^2_{2n+1})}\right) \geq 0. \tag{4.40}$$

Thus the following estimate holds

$$\begin{aligned}
\Sigma_2 Y \overline{\Sigma}_2^T &\geq \Sigma_1 X \overline{\Sigma}_1^T - \frac{1}{\varepsilon^2}\eta \otimes \bar{\eta} \\
&\geq \frac{1}{m}(g''(x - iy) \otimes (x + iy) + 2g' Id) - \frac{1}{\varepsilon^2}\eta \otimes \bar{\eta} \\
&= \left(\frac{g''}{m} - \frac{(g')^2}{m^2\varepsilon^2(1 + (p_\varepsilon)^2_{2n+1})}\right)(x - iy) \otimes (x + iy) + \frac{2g'}{m} Id \\
&\geq \frac{2g'}{m} Id = \gamma Id > 0,
\end{aligned}$$

Now, set $\lambda = \dfrac{(g')^2}{\varepsilon^2 m^2 (1 + (p_\varepsilon)^2_{2n+1})}(\|x\|^2 + \|y\|^2)$, we have

$$\begin{aligned}
(\det(\Sigma_1 X \Sigma_1^T))^{1/n} - (\det(\Sigma_2 Y \Sigma_2^T))^{1/n} &\leq (\det(\Sigma_2 Y \Sigma_2^T + \lambda Id))^{1/n} - (\det(\Sigma_2 Y \Sigma_2^T))^{1/n} \\
&= (\det(\Sigma_2 Y \Sigma_2^T))^{1/n}\left(\det(Id + \lambda(\Sigma_2 Y \Sigma_2^T)^{-1})^{1/n} - 1\right) \\
&\leq (\det(\Sigma_2 Y \Sigma_2^T))^{1/n}\left(\frac{\mathrm{trace}(Id + \lambda(\Sigma_2 Y \Sigma_2^T)^{-1})}{n} - 1\right) \\
&\leq \frac{8}{\varepsilon^2}(\det(\Sigma_2 \Sigma_2^T))^{1/n}\left(\frac{\mathrm{trace}(Id + \lambda(\gamma)^{-1})}{n} - 1\right) \\
&\leq \frac{8}{\varepsilon^2}(\det(\Sigma_2 \Sigma_2^T))^{1/n}\left(\frac{\lambda m}{2g'}\right) \\
&\leq \frac{16}{\varepsilon^2}(1 + \frac{1}{2}(|a(p_\varepsilon)|^2 + |b(p_\varepsilon)|^2))^{1/n}\left(\frac{g'(\|x\|^2 + \|y\|^2)}{2m\varepsilon^2(1 + (p_\varepsilon)^2_{2n+1})}\right) \\
&\leq \frac{C}{\varepsilon^2 \varepsilon^{2/n}}\left(\frac{g'(\|x\|^2 + \|y\|^2)}{2m\varepsilon^2}\right),
\end{aligned} \tag{4.41}$$

where, in the last inequality, we use the estimate $|p_{\varepsilon}| \leq \dfrac{C}{\varepsilon}$, with C independent of ε, m.

Set $\xi = (x, y, 0)$, we also have

$$\begin{aligned} f^{1/n}(p_\varepsilon) - f^{1/n}(p_\varepsilon - \frac{D\psi}{m}) &= f^{1/n}(p_\varepsilon) - f^{1/n}(p_\varepsilon - g'\xi/m) \\ &\leq C\frac{1}{\varepsilon^{2/n}}\frac{g'}{m}(\|x\|^2 + \|y\|^2)^{1/2}(1 + o_\varepsilon(1)) \end{aligned} \tag{4.42}$$

By subtracting the two inequalities in (4.37) and by using **(H2)**, **(H5)** and (4.41), (4.42) we finally obtain

$$\begin{aligned} \frac{\nu}{m} = \frac{2g'\left(1 + \frac{g''}{2g'}(\|x\|^2 + \|y\|^2)\right)^{1/n}}{m} &\leq \frac{C}{\varepsilon^{2/n}}\frac{g'}{m}(\|x\|^2 + \|y\|^2)^{1/2}(1 + o_\varepsilon(1)) \\ &+ \frac{C}{\varepsilon^2\varepsilon^{2/n}}\left(\frac{g'(\|x\|^2 + \|y\|^2)}{2m\varepsilon^2}\right). \end{aligned} \tag{4.43}$$

Now we take $g(s) = \exp(\beta s - \alpha)$ with β and α to be determined as follows. We have $g' = \beta g$, and $g'' = \beta^2 g$. Thus since $(x, y) \neq (0, 0)$, if we choose $m = \beta^{\frac{2n+1}{2n}}$ and $\varepsilon = \beta^{-\frac{1}{6n}}$, then for β large enough we get a contradiction in (4.43). We finally choose α (depending on β and the diameter of Ω) in such a way that $g \leq 1$. We point out that by this choice of g, ε and m the inequality (4.40) is satisfied. Thus we can conclude. □

Remark 4.1. One can prove a variant of Theorems 4.1 and 4.2 in which the condition $u \leq v$ on $\partial\Omega$ is dropped and the conclusion is changed to $u - v \leq \sup_{\partial\Omega}(u - v)^+$, (see e.g. User's guide [10]).

In the general case when k depends on x and it is not strict monotone with respect to u we are able to prove a comparison result between continuous sub and supersolution of (DP) by following a dilation argument (see e.g [25, Theorem 2.2],[17, 4]).

Theorem 4.3. *Assume* **(H5)** *and* **(H7)** *and suppose that* $k > 0$. *Let* $u, v \in C(\bar{\Omega})$ *be respectively viscosity sub- and supersolution of* (DP). *Then*

$$\sup_{\Omega}(u - v) \leq \sup_{\partial\Omega}(u - v)^+.$$

Proof. Let $R = \max(||u||_\infty, ||v||_\infty)$. It is not restrictive to assume u is non negative in Ω, otherwise we replace u, v with $u + R, v + R$, respectively. In this case F will be replaced by $G(\xi, s, p, X) = F(\xi, s - R, p, X)$. Moreover, it is not restrictive to assume $|\xi| < \mathrm{diam}(\Omega)$ for every $\xi \in \Omega$, otherwise we replace F with $G(\xi, s, p, X) = F(\xi - \xi_0, s, p, X)$ for a suitable ξ_0.

For all $r > 1$ we set $\Omega_r = r^{-1}\Omega$ and we introduce the function

$$u_r(x) = r^{-1}u(rx) \quad x \in \overline{\Omega}_r \tag{4.44}$$

We claim that there is $\delta < 0$ such that for all $r > 1$ close to 1 u_r is a viscosity solution of

$$F(x, u_r(x), Du_r, D^2u_r) \leq -\delta(r - 1) \quad \text{in } \Omega_r.$$

Indeed let $\phi \in C^2(\overline{\Omega}_r)$ and $\xi \in \Omega_r$ such that $u_r - \phi$ has a local maximum at ξ. Then $u(y) - r\phi(r^{-1}y)$ has a local maximum at $r\xi$. Since u is a subsolution of (DP) we have

$$F_*(r\xi, u(r\xi), D\phi, r^{-1}D^2\phi) \leq 0\,.$$

The above inequality implies that $L\phi(\xi) \geq 0$ and

$$-\det A(D\phi, D^2\phi) + r^n k(r\xi, u(r\xi)) f(D\phi) \leq 0.$$

Set $L_\Omega = \sup_{\Omega\times\Omega\times[0,R]} \frac{k(\xi,u)-k(\xi_0,u)}{|\xi-\xi_0|}$ and suppose for the moment that

$$L_\Omega < \frac{n \inf_{\Omega\times\mathbb{R}} k}{\mathrm{diam}(\Omega)} \tag{4.45}$$

The following estimate holds

$$\begin{aligned}
-\det A(D\phi, D^2\phi) &+ k(\xi, u_r(\xi)) f(D\phi) \leq k(\xi, u_r(\xi)) f(D\phi) - r^n k(r\xi, u(r\xi)) f(D\phi) \\
&= (1-r^n) k(r\xi, u(r\xi)) f(D\phi) \\
&+ [k(\xi, u_r(\xi)) - k(\xi, r u_r(\xi))] f(D\phi) \\
&+ [k(\xi, r u_r(\xi)) - k(r\xi, u(r\xi))] f(D\phi) \\
&\leq f(D\phi)[(1-r^n)(\inf_{\Omega\times\mathbb{R}} k) + (r-1) L_\Omega |\xi|] \\
&\leq (r-1) f(D\phi)[-n(\inf_{\Omega\times\mathbb{R}} k) + L_\Omega \mathrm{diam}(\Omega)].
\end{aligned}$$

From (4.45) and the fact that $f(D\phi) \geq 1$, it follows that there is $\delta > 0$ such that we have

$$-\det A(D\phi, D^2\phi) + k(\xi, u_r(\xi)) f(D\phi) \leq -\delta(r-1).$$

Now by arguing as in the proof of Theorem 4.1 one gets

$$\sup_{\Omega\cap\Omega_r} (u_r - v) \leq \sup_{\partial(\Omega\cap\Omega_r)} (u_r - v)^+ \tag{4.46}$$

and the conclusion follows by letting $r \to 1^+$.

If (4.45) is not satisfied one proceeds by covering Ω with small balls of radius $r < \frac{n \inf_{\Omega\times\mathbb{R}} k}{L_\Omega}$. $\square$

Remark 4.2. *We remark that Theorem 4.3 is a consequence of the fact that we are considering a curvature equation. Precisely set $\rho_r(z) = \rho(rz)$, with $\rho(z) = u(\xi) - s$ and let K the Levi curvature of $\{\rho = 0\}$. Then we have*

$$\partial_z \rho_r(z) = r(\partial_z \rho)(rz), \quad \partial_{z\bar{z}} \rho_r(z) = r^2 (\partial_{z\bar{z}} \rho)(rz)$$

and

$$\begin{aligned}
-\left\{ |\partial\rho_r|^{-n-2} \det \begin{pmatrix} 0 & \partial_{\bar{z}}\rho_r \\ \partial_z \rho_r & \partial_{z\bar{z}}\rho_r \end{pmatrix} \right\}(z) &= -r^n \left\{ |\partial\rho|^{-n-2} \det \begin{pmatrix} 0 & \partial_{\bar{z}}\rho \\ \partial_z \rho & \partial_{z\bar{z}}\rho \end{pmatrix} \right\}(rz) \\
&= r^n K(rz).
\end{aligned}$$

By the comparison results and the Perron's method we get the existence of a unique continuous solution of (DP).

Corollary 4.1. *Assume either the hypotheses of Theorem 4.1 or of Theorem 4.2 and suppose* **(H4)** *holds. Then for any $\varphi \in C(\partial\Omega)$ there exists a unique continuous viscosity solution of (DP).*

Proof. We use the Perron's method introduced for viscosity solutions by H. Ishii [16] with the version up to the boundary of the first author [11]. We observe that if M is large enough then the function $\overline{v}(\xi) = M$ is a supersolution of (DP). Let us denote by ξ_0 the center of the sphere of minimum radius r containing $\overline{\Omega}$. Under the assumption **(H4)** the function $\underline{v}(\xi) = (r^2 - |\xi - \xi_0|^2)^{1/2} - M$ is subsolution of (DP). Indeed if we set $\rho(\xi, s) = (r^2 - |\xi - \xi_0|^2)^{1/2} - M - s$ then the zero level set of ρ is a subset of a sphere of radius r and one knows that in this case $k_{\{\rho=0\}}(z) = \dfrac{1}{r^n}$. Thus $F(\xi, \underline{v}(\xi), D\underline{v}(\xi), D^2\underline{v}(\xi)) = -\dfrac{1}{r} + k^{1/n}(\xi, \underline{v}(\xi)) < 0$.

The Perron's method provides us with a (possibly discontinuous) solution u of (DP) such that $\underline{v} \le u \le \overline{v}$ in $\overline{\Omega}$. The condition that $\Sigma = \emptyset$ implies that there is no loss of boundary condition on $\partial\Omega$ and therefore, every subsolution ω and every supersolution w of (DP) satisfies

$$\omega \le \varphi \le w \quad \text{on } \partial\Omega\,.$$

The first consequence of this inequality is that $u_* = u^* = \varphi$ on $\partial\Omega$ and therefore u is continuous at points of $\partial\Omega$. The second one is the uniqueness of the continuous solution u of (DP) which follows from either Theorem 4.1 or Theorem 4.2. □

5 Lipschitz estimates and proofs of Theorems 1.1, 1.2, 1.3

In this section we shall always denote by x a point in $\mathbb{R}^N$, with $N = 2n + 1$. We shall prove the existence of a Lipschitz continuous viscosity solution of (DP) under suitable assumptions on k and geometric conditions on the domain. To this purpose we follow two different approaches. More precisely in the particular case that k does not depend on the variable x we adapt the method of translation (see e.g. [17]), whereas in the case when k depends on x and u, since it is not possible in general to obtain the existence through the Perron's method we use a Bernstein type method and a proper approximation argument that we explain later.

Throughout this section we assume that Ω satisfies **(H3)**. We recall that under **(H3)** the boundary data are assumed in a classical sense by the viscosity subsolutions and supersolution of (DP), and the conditions (3.29) and (3.30) are satisfied.

We introduce the following notation : for $\gamma > 0$ we set

$$\Omega_\gamma := \{x \in \overline{\Omega}\ : d(x) < \gamma\}.$$

We observe that since $\partial\Omega$ is of class C^2 then for $\gamma > 0$ small the distance function $d \in C^2(\Omega_\gamma)$.

We start with the following lemma.

Lemma 5.1. *Assume* **(H3)**, $\varphi \in C^{1,1}(\partial\Omega)$. *Then there are* $\lambda' > 0$, *and* $0 < \gamma' \le \gamma$ *such that for all* $\lambda \ge \lambda'$ *the functions* $\underline{u}(x) = \varphi(x) - \lambda d(x)$, *and* $\overline{u}(x) = \varphi(x) + \lambda d(x)$ *are respectively classical subsolution and supersolution of* (DP) *in* $\Omega_{\gamma'}$ *and* $\underline{u}(x) = \overline{u}(x) = \varphi(x)$ *in* $\partial\Omega$. *Moreover* $\underline{u}$ *and* $\overline{u}$ *are Lipschitz continuous in* $\overline{\Omega}_\gamma$.

Proof. Let us continue to denote by φ the smooth extension of φ to $\overline{\Omega}$.

Subsolution case: We show that there are $\gamma' \le \gamma$ and $\lambda' > 0$ such that for all $\lambda \ge \lambda'$, $\underline{u}(x)$ is a classical subsolution of (DP) in $\Omega_{\gamma'}$. We have $D\underline{u}(x) = D\varphi(x) - \lambda Dd(x)$, $D^2\underline{u}(x) = D^2\varphi(x) - \lambda D^2 d(x)$. From the condition (3.29) and the continuity of A_∞ there is $r_0 > 0$ such that for all $x \in B(x_0, r_0)$ we have $A_\infty(-Dd(x), -D^2d(x)) > 0$. We notice that

$$A(D\underline{u}(x), D^2\underline{u}(x)) = \frac{\lambda}{[(\lambda^{-2}) + (\lambda^{-1}D\underline{u}(x)))^2_N]^2} A'(\lambda^{-1}D\underline{u}(x), \lambda^{-1}D^2\underline{u}(x), \lambda) \qquad (5.47)$$

and $A'(\lambda^{-1}D\underline{u}(x), \lambda^{-1}D^2\underline{u}(x), \lambda)$ converges to $A_\infty(Dd(x), D^2d(x))$ uniformly in $B(x_0, r_0)$ as $\lambda \to \infty$. Thus there exists $\lambda_0 := \lambda(x_0, r)$ such that for all $\lambda \geq \lambda_0$ and for all $x \in B(x_0, r_0)$ the matrix $A'(\lambda^{-1}D\varphi(x) - Dd(x), \lambda^{-1}D^2\varphi(x) - D^2d(x))$ is definite positive as well. Since $\partial\Omega$ is compact, we can find $\gamma' < \gamma$ and λ' such that for all $\lambda \geq \lambda'$ and for all $x \in \Omega_{\gamma'}$ the matrix $A'(\lambda^{-1}\underline{u}(x), \lambda^{-1}D^2\underline{u}(x))$ is definite positive and the same holds for the matrix $A(D\underline{u}(x), D^2\underline{u}(x))$ because of (5.47).

On the other hand one sees that

$$h(D\underline{u}(x)) \det(B(D\underline{u}(x), D^2\underline{u}(x))) \to 2^n \det(B_\infty(Dd(x), D^2d(x)))$$

as $\lambda \to \infty$ uniformly in $x \in B(x_0, r_0)$. Since $\partial\Omega$ is a compact set, the condition (3.30) implies that for λ large enough and for all $x \in \Omega_{\gamma'}$ we have

$$(-h(D\underline{u}(x)) \det(B(D\underline{u}(x), D^2\underline{u}(x)))^{1/n} > k^{1/n}(x, \underline{u}(x)).$$

In particular this yields

$$\left(\det(A(D\underline{u}(x), D^2\underline{u}(x))f^{-1}(D\underline{u}(x))\right)^{1/n} > k^{1/n}(x, \underline{u}(x))$$

for all $x \in \Omega_{\gamma'}$. Thus for $\lambda > 0$ large enough and for all $x \in \Omega_{\gamma'}$ we have

$$\begin{aligned} &F_*(x, \underline{u}(x), D\underline{u}(x), D^2\underline{u}(x))) = \\ &\quad f^{1/n}(D\underline{u}(x)) \cdot \left\{ - \left(\det(A(D\underline{u}(x), D^2\underline{u}(x))f^{-1}(D\underline{u}(x))\right)^{1/n} + k^{1/n}(x, \underline{u}) \right\} < 0. \end{aligned}$$

This proves that $\overline{u}$ is a classical subsolution of (DP) in $\Omega_{\gamma'}$.

Supersolution case: Let us consider the function $\overline{u}(x) = \varphi(x) + \lambda d(x)$. We first notice that for any $x_0 \in \partial\Omega$, $A_\infty(Dd, D^2d) = -A_\infty(-Dd, -D^2d)$ is not semidefinite positive. This means that there exists at least one eigenvalue which is strictly negative. By analogous arguments as above one can show that for $\lambda > 0$ large enough and for γ' small the matrix $A(D\varphi(x) + \lambda Dd(x), D^2\varphi(x) + \lambda D^2d(x))$ is not semidefinite positive for all $x \in \Omega_{\gamma'}$. This implies that $F^*(x, \overline{u}, D\overline{u}, D^2\overline{u}) = +\infty$. Finally the Lipschitz continuity of $\underline{u}$ and $\overline{u}$ follows from the fact that $\varphi \in C^{1,1}(\overline{\Omega}_\gamma)$ and $d \in C^2(\overline{\Omega}_\gamma)$. Thus we can conclude. $\square$

Next we prove the Lipschitz continuity of the solution to (DP) under the assumption that k does not depend on x.

Theorem 5.1. [The x-independent case] *Assume* **(H2)**–**(H6)**, $\varphi \in C^{1,1}(\partial\Omega)$. *Then there exists a unique Lipschitz continuous viscosity solution u of (DP).*

Proof. Let us continue to denote by φ the smooth extension of φ to $\overline{\Omega}$. The existence of a continuous solution u to (DP) follows from Corollary 4.1. Moreover by comparing u with the barriers defined in Corollary 4.1 one gets that $||u||_\infty < R$ for some $R > 0$. Now we consider the functions $\underline{u}$ and $\overline{u}$ defined in Lemma 5.1. We have $\underline{u} = u = \overline{u}$ on $\partial\Omega$, and $\underline{u} \leq u \leq \overline{u}$ on $d(x) = \gamma'$ provided $\lambda > \dfrac{||u||_\infty + ||\varphi||_\infty}{\gamma'}$. Theorem 4.2 yields that $\underline{u} \leq u \leq \overline{u}$ in $\Omega_{\gamma'}$. To show the Lipschitz continuity of u we adapt the method of translations (see [17]): let $h \in \mathbb{R}^N$, since the equation does not depend on x the function $u(\cdot + h)$ is a viscosity solution of the same equation as that for u but set in $\Omega - h$. Theorem 4.2 and Remark 4.1 yield

$$\begin{aligned} \sup_{\Omega_{\gamma'} \cap (\Omega_{\gamma'} - h)} |u - u(\cdot + h)| &\leq \sup_{\partial(\Omega_{\gamma'} \cap (\Omega_{\gamma'} - h))} |u - u(\cdot + h)| \\ &\leq \sup_{\partial(\Omega_{\gamma'} \cap (\Omega_{\gamma'} - h))} \max\{|\overline{u} - \underline{u}(\cdot + h)|, |\underline{u} - \overline{u}(\cdot + h)|\} \\ &\leq C|h|. \end{aligned}$$

Thus u is Lipschitz continuous in $\Omega_{\gamma'}$. Next we show that this implies that u is Lipschitz continuous in Ω. Indeed by Theorem 4.2 and Remark 4.1 we have

$$\sup_{\Omega\cap(\Omega-h)} |u - u(\cdot + h)| \leq \sup_{\partial(\Omega\cap(\Omega-h))} |u - u(\cdot + h)| \,. \tag{5.48}$$

We estimate the l.h.s of (5.48). If $h \leq \gamma'$, then $\sup_{\partial(\Omega\cap(\Omega-h))} |u - u(\cdot + h)| \leq C|h|$ by the above estimates. Otherwise $|u(x) - u(x+h)| \leq 2||u||_\infty \leq 2||u||_\infty \dfrac{h}{\gamma'}$. In any case we have

$$\sup_{\Omega\cap(\Omega-h)} |u - u(\cdot + h)| \leq C|h|,$$

and we can conclude. □

Proof of Theorem 1.2. In view of (2.11) and of (2.12), the Dirichlet problem (1.2) is equivalent to (DP). Hence Theorem 1.2 follows from Theorem 5.1. □

Next we prove the existence of a Lipschitz continuous solution to (DP) under more general conditions on k for which we cannot applying directly the Perron's method not having a comparison result. To this end we use a "weak Bernstein method" introduced by Barles in [3] to obtain gradient bound for viscosity solutions to fully nonlinear pde's. Roughly speaking in [3] it is shown that if a continuous degenerate elliptic operator $G\colon \overline{\Omega} \times \mathbb{R} \times I\!R^N \times \mathcal{S}^N \to \mathbb{R}$ satisfies in a neighborhood of the set of $\{(x, u, p, M) \ : |u| \leq R, |p| \geq L, G = 0\}$ the condition

$$D_x G \cdot p + D_u G |p|^2 - g D_M G \cdot M^2 > \alpha > 0 \tag{5.49}$$

for some constants $\alpha, g > 0$ and L large and for all $R > 0$ then any viscosity solution of $G = 0$ satisfies

$$\sup_{\overline{\Omega}} |Du(x)| \leq \max(L, \sup_{\partial\Omega} |Du(x)|). \tag{5.50}$$

Next we will use the hypothesis **(H8)**.

Remark 5.1. We observe that

(i) if $k \in C^1(\Omega \times \mathbb{R}, [0, +\infty))$, then **(H8)** is satisfied if for instance one the following conditions hold:

1. $D_u > 0$;
2. $|D_x k| \leq L D_u k + \tilde{g} n k^{1+1/n}$ for some $L > 0$ and $\tilde{g} < g_0$.

(ii) if k satisfies **(H5)**, **(H6)** and **(H7)**, then it satisfies **(H8)** too. Hence, the following approach provides an alternative proof of Theorem 5.1 for Lipschitz continuous k.

We first show that if $k \in C^1(\Omega \times \mathbb{R}, [0, +\infty))$ satisfies **(H8)** for some $\alpha > 0$, then a Lipschitz continuous solution of (DP) satisfies (5.50).

To this end we denote

$$\tilde{F}(x, u, p, X) = -\det(A(p, X)) + k(x, u) f(p).$$

We start with the following Lemma in which we prove that if $k \in C^1(\Omega \times \mathbb{R}, [0, +\infty))$ satisfies **(H8)** for some $\alpha > 0$, then the operator $\tilde{F}$ satisfies the condition (5.49) in a neighborhood of the set $V(R, L) = \{(x, u, p, M) \ : |u| \leq R, |p| \geq L, F(x, u, p, M) = 0\}$.

Lemma 5.2. *If $k \in C^1(\Omega \times \mathbb{R}, [0, +\infty))$ satisfies* **(H8)** *for some $\alpha > 0$, then the condition (5.49) is satisfied by $\tilde{F}$ in a neighborhood $W(R, L)$ of the set $V(R, L)$.*

Proof. We start by showing that (5.49) holds in the set $V(R, L)$.

We first observe that if $(x, u, p, M) \in V(R, L)$ then we have $A(p, X) \geq 0$ and $\tilde{F}(x, u, p, X) = -\det A(p, X) + k(x, u)f(p) = 0$. The following equalities hold.

$$D_x\tilde{F} = f(p)D_xk(x, u), \quad D_u\tilde{F} = f(p)D_uk(x, u) \tag{5.51}$$

Now set $\Sigma(p) = \overline{\sigma}^T(p)\sigma(p)$, where $\sigma(p)$ is the matrix defined in (4.35). We notice that $\Sigma(p) \geq 0$ with minimum eigenvalue identically zero and

$$\Sigma(p) \leq \mathrm{Tr}\Sigma(p) \cdot I_N, \tag{5.52}$$

where $\mathrm{Tr}\Sigma(p) = 2n + |a(p)|^2 + |b(p)|^2 = 2n + \dfrac{\sum_1^{2n} p_j^2}{(1 + p_N^2)}$.

By denoting $A^*(p, X)$ the cofactor matrix of $A(p, X)$, recalling that $\det A^*(p, X) = (\det A(p, X))^{n-1}$ and using the inequality (5.52), we have

$$\begin{aligned}
-D_M\tilde{F} \cdot M^2 &= \mathrm{Tr}[(A^*(p, M))\sigma(p)M^2\overline{\sigma}^T(p)] \\
&\geq n(\det[(A^*(p, M))(p)\sigma(p)M^2\overline{\sigma}^T])^{1/n} \\
&\geq n[\mathrm{Tr}\Sigma(p)]^{-1/n}\left(\det[(A^*(p, M))\sigma(p)M\Sigma(p)M\overline{\sigma}^T(p)]\right)^{1/n} \\
&= n\det(A(p, M))^{1+1/n}[\mathrm{Tr}\Sigma(p)]^{-1/n} \\
&= n(k(x, u)f(p))^{1+1/n}[\mathrm{Tr}\Sigma(p)]^{-1/n}.
\end{aligned} \tag{5.53}$$

In (5.53) we use the fact that for all Hermitian matrices $A, B \geq 0$ we have

$$\det(AB) \leq \left(\frac{\mathrm{Tr}(AB)}{n}\right)^n.$$

Moreover we have

$$\begin{aligned}
f^{1/n}(p)[\mathrm{Tr}\Sigma(p)]^{-1/n} &= \left[2\frac{(1 + |p|^2)^{\frac{n+2}{2n}}}{(1 + p_N^2)^{1/n}}\right]\left[\frac{(1 + p_N^2)^{1/n}}{(2n + |p|^2 + (2n - 1)p_N^2)^{1/n}}\right] \\
&\geq 2\frac{(1 + |p|^2)^{\frac{n+2}{2n}}}{(2^n(1 + |p|^2))^{1/n}} \geq (1 + |p|^2)^{1/2}.
\end{aligned} \tag{5.54}$$

We set

$$I(x, u, p) = \frac{D_xk \cdot p + D_uk|p|^2}{(1 + |p|^2)^{1/2}} + gnk^{1+1/n}. \tag{5.55}$$

We recall $k \in C^1(\Omega \times \mathbb{R}, [0, +\infty))$ and therefore the map I is a continuous function. By combining the above estimates and using the fact that $f(p)(1 + |p|^2)^{1/2} \geq 1$, for all $(x, u, p, M) \in V(R, L)$ we obtain

$$\begin{aligned}
D_x\tilde{F} \cdot p + D_u\tilde{F}|p|^2 - g_0D_M\tilde{F} \cdot M^2 &= f(p)D_xk(x, u) \cdot p + f(p)D_uk(x, u) \\
&+ g_0n\left(k(x, u)f(p)\right)^{1+1/n}[\mathrm{Tr}\Sigma(p)]^{-1/n} \\
&\geq f(p)(1 + |p|^2)^{1/2}I(x, u, p) \geq \alpha > 0.
\end{aligned}$$

The above estimate holds in a neighborhood of $V(R, L)$ by continuity. □

Proposition 5.1 (Weak Bernstein Method). *Assume that $k \in C^1(\Omega \times \mathbb{R}, [0, +\infty))$ satisfies* **(H5)** *and* **(H8)** *for some $\alpha > 0$. Let u be a continuous solution of (DP) such that*

$$|u(x) - u(w)| \leq K|x - w|, \quad \textit{for all } (x, w) \in \partial(\Omega \times \Omega). \tag{5.56}$$

Then we have

$$|u(x) - u(w)| \leq \bar{C}|x - w|, \quad \textit{for all } (x, w) \in \Omega \times \Omega$$

where $\bar{C} = \max(L, K)$.

Proof. We follow the arguments of [3, Theorem 1]. We consider the function

$$\Phi(x, w) := u(x) - u(w) - C|x - w|. \tag{5.57}$$

We assume by contradiction that for all $C > \max(L, K)$ there is a point (x, w) such that $\Phi(x, w) > 0$. We observe that because of (5.56), if $C > 0$ is large then $(x, w) \in \Omega \times \Omega$. We set $\psi(x-w) = C|x-w|$. There exist $X, Y \in \mathcal{S}(N)$ such that, if $p := D\psi(x - w)$, we have

$$(p, X) \in \mathcal{J}^{2,+}u(x), \quad (p, Y) \in \mathcal{J}^{2,-}u(w),$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} D^2\psi(x - w) & -D^2\psi(x - w) \\ -D^2\psi(x - w) & D^2\psi(x - w) \end{pmatrix} \tag{5.58}$$

and

$$F_*(x, u(x), p, X) \leq 0 \quad \text{and} \quad F^*(w, u(w), p, Y) \geq 0 \tag{5.59}$$

¿From (5.58) it follows that $A(p, Y) \geq A(p, X) \geq 0$. Moreover for all $t \in [0, 1]$ we have $A(p, tX + (1 - t)Y) \geq 0$ as well. Moreover from (5.59) it follows that we have

$$-\det(A(p, X)) + k(x, u(x))f(p) \leq 0 \quad \text{and} \quad -\det(A(p, Y)) + k(w, u(w))f(p) \geq 0.$$

We set $\tilde{F}(x, u, p, M) = -\det(A(p, M)) + k(x, u)f(p)$. We consider the function g defined by

$$g(t) = \tilde{F}(tx + (1 - t)w, tu(x) + (1 - t)u(w), p, tX + (1 - t)Y).$$

Since $g(t)$ is continuous there are $t_1 \leq t_2$ such that $g(t_1) \geq 0$, $g(t_2) \leq 0$ and $g(t) \in W(R, L)$ for all $t \in [t_1, t_2]$, δ being the constant in Lemma 5.2. We assume $\tilde{F}$ smooth . We set $\gamma = \dfrac{C}{|x - w|}$. By taking into account (5.57) and **(H5)** we get

$$\begin{aligned} g'(t) &= \frac{1}{\gamma}\{D_x\tilde{F} \cdot p + D_u\tilde{F}|p|^2 + \gamma D_M\tilde{F} \cdot (X - Y) + \gamma D_u\tilde{F} \cdot \Phi(x, w)\} \\ &\geq \frac{1}{\gamma}\{D_x\tilde{F} \cdot p + D_u\tilde{F}|p|^2 + \gamma D_M\tilde{F} \cdot (X - Y)\}. \end{aligned} \tag{5.60}$$

By Lemma 2 in [3] we have

$$\gamma(Y - X) \geq g_0(tX + (1 - t)Y)^2$$

Hence from Lemma 5.2 it follows that $g'(t) > 0$ for $t \in [t_1, t_2]$, but this is in contradiction with the fact that $g(t_1) \geq 0$, $g(t_2) \leq 0$. □

Let us mention that when k is only Lipschitz continuous and **(H8)** holds with some $\alpha > 0$ then the above a priori estimate comes from an approximation argument. Precisely, let $\tilde{k}$ the function defined on $\mathbb{R}^N \times \mathbb{R}$ such that $\tilde{k}(x,u) = k(x,u)$ if $x \in \overline{\Omega}$ and $\tilde{k}(x,u) = 2\left(\frac{\alpha}{gn}\right)^{\frac{n}{n+1}}$ if $x \notin \overline{\Omega}$. We also set

$$\tilde{I}(x,u,p) = \frac{D_x\tilde{k}\cdot p + D_u\tilde{k}|p|^2}{(1+|p|^2)^{1/2}} + gn\tilde{k}^{1+1/n}.$$

We denote by k_ε and I_ε the convolution with respect of the variables (x,u) of $\tilde{k}$ and $\tilde{I}$ respectively. with a positive mollifier J_ε and by $(DP)_\varepsilon$ the Dirichlet problem (DP) for k_ε. The following result holds.

Corollary 5.1. *Assume that k satisfies* **(H7)** *and that* **(H8)** *holds for some $\alpha > 0$ and for a.e. $(x,u) \in \Omega \times \mathbb{R}$. Suppose that for any $\varepsilon > 0$ there is a continuous solution u_ε of $(DP)_\varepsilon$. If the family (u_ε) is equibounded in $\overline{\Omega}$ and there is a positive constant K such that*

$$|u_\varepsilon(x) - u_\varepsilon(w)| \leq K|x-w|, \quad \text{for all } (x,w) \in \partial(\Omega\times\Omega) \text{ and for every } \varepsilon\,, \tag{5.61}$$

then, up to a subsequence, u_ε uniformly converges to a Lipschitz continuous solution u of (DP) on compact sets of $\overline{\Omega}$.

Proof. We claim that if k is Lipschitz continuous and satisfies condition **(H8)** a.e. then, for all $\varepsilon > 0$ small, $k_\varepsilon = \tilde{k} * J_\varepsilon$ satisfies condition **(H8)** for some $0 < \bar{\alpha} < \alpha$. The thesis will then follow by applying Proposition 5.1 to u_ε, by remarking that the constant $\tilde{C}$ is independent of ε, and by using the Ascoli-Arzelà Theorem.

To prove the claim, we observe that by **(H8)** and by the construction of $\tilde{I}$ we have $\tilde{I}(x,u,p) \geq \alpha$, for almost every $x \in \mathbb{R}^N$, $|u| \leq R$, $|p| \geq L$. Moreover since k_ε converge uniformly to k for all $x \in \overline{\Omega}$, $|u| \leq R$ then for all $x \in \overline{\Omega}$, $|u| \leq R$, $|p| \geq L$ we have

$$\begin{aligned}
\alpha \leq I_\varepsilon(x,u,p) =& \frac{D_xk_\varepsilon\cdot p + D_uk_\varepsilon|p|^2}{(1+|p|^2)^{1/2}} + gn(k^{1+1/n} * J_\varepsilon)(x,u) \\
=& \frac{D_xk_\varepsilon\cdot p + D_uk_\varepsilon|p|^2}{(1+|p|^2)^{1/2}} + gnk_\varepsilon^{1+1/n} \\
&+ gn\left((k^{1+1/n} * J_\varepsilon)(x,u) - k_\varepsilon^{1+1/n}(x,u)\right) \\
\leq& \frac{D_xk_\varepsilon\cdot p + D_uk_\varepsilon|p|^2}{(1+|p|^2)^{1/2}} + gnk_\varepsilon^{1+1/n} + gnC\varepsilon
\end{aligned}$$

for some positive constant C depending on the Lipschitz constant of k in $\overline{\Omega} \times [-R,R]$. Then, for ε small, we have $\alpha - gnC\varepsilon > \frac{\alpha}{2}$ and we conclude. □

We explicitly remark that if k is Lipschitz continuous and satisfies **(H1)** then **(H8)** is clearly satisfied a.e. for some $\alpha > 0$. Moreover in this case the existence and uniqueness of a continuous solution follows from Corollary 4.1. Thus in view of Corollary 5.1 in order to prove the Lipschitz regularity of the solution it is enough to verify that (u_ε) is equibounded and the condition (5.61) holds. This is the purpose of the following

Theorem 5.2. [The strict monotone case] *Assume* **(H1)**–**(H4)**, **(H7)**, $\varphi \in C^{1,1}(\partial\Omega)$. *Then there exists a unique Lipschitz continuous viscosity solution u of (DP).*

Proof. For all $\varepsilon > 0$ small, let u_ε be a solution of $(DP)_\varepsilon$. The existence of a continuous solution u_ε of $(DP)_\varepsilon$ follows from Corollary 4.1, because k_ε satisfies **(H1)–(H4)** for ε small. The family (u_ε) is equibounded in $\overline{\Omega}$ by a positive constant M_0 because for ε small we have $\underline{v} \le u_\varepsilon \le \overline{v}$, $\underline{v}, \overline{v}$ being the functions defined in Corollary 4.1 . Because of Corollary 5.1 in order to prove that the family (u_ε) is equicontinuous, it is enough to show that the condition (5.61) is satisfied for some $C > 0$ independent of ε. Let us consider the function $\overline{u}$ and $\underline{u}$ defined in Lemma 5.1. In Lemma 5.1 it is shown that for λ large $\overline{u}$ is a supersolution of (DP) and $\underline{u}$ is a strict super subsolution of (DP) in Ω_γ for some $\gamma > 0$ small. Moreover they are Lipschitz continuous in Ω_γ with $||D\overline{u}||_\infty, ||D\underline{u}||_\infty \le \bar{C}$, with $\bar{C} > 0$ depending on λ. One can readily see that if $\varepsilon > 0$ is small then $\overline{u}$ and $\underline{u}$ are super and subsolutions of $(DP)_\varepsilon$ as well. If we take $\lambda > \dfrac{(M_0 + ||\varphi(x)||_\infty)}{\gamma}$, then $\underline{u}(x) \le u_\varepsilon(x) \le \overline{u}(x)$ for all x such that $d(x, \partial\Omega) = \gamma$. Thus by Theorem 4.1 we have $\underline{u}(x) \le u_\varepsilon(x) \le \overline{u}(x)$ in $\overline{\Omega}_\gamma$.

Take $(x, w) \in \partial(\Omega \times \Omega)$ and suppose that $x \in \partial\Omega$. There are two possibilities: either $d(w, \partial\Omega) \le \gamma$ or $d(w, \partial\Omega) > \gamma$. If $d(w, \partial\Omega) \le \gamma$ then

$$u_\varepsilon(x) - u_\varepsilon(w) \le \underline{u}(x) - \underline{u}(w) \le \bar{C}|x - w|.$$

Instead if $d(w, \partial\Omega) > \gamma$ then $|x - w| > \gamma$ and therefore

$$u_\varepsilon(x) - u_\varepsilon(w) \le 2||u_\varepsilon||_\infty \le 2||u_\varepsilon||_\infty \frac{|x - w|}{\gamma}.$$

We observe that the Lipschitz constant of the solution u_ε depends on the Lipschitz constant of the barriers $\underline{u}$, $\overline{u}$, on the function k and on the L^∞ norm of the solution u_ε. Thus if we choose $C = \max(\bar{C}, \dfrac{2M_0}{\gamma})$ then the condition (5.61) holds. Since **(H8)** hold a.e., the conclusion follows from Corollary 5.1 and by the uniqueness of a viscosity solution of (DP) proved in Corollary 4.1. □

Proof of Theorem 1.1. The proof is contained in Corollary 4.1 and in Theorem 5.2. □

In the general case, namely when k is not strictly monotone with respect to the u variable, we prove the existence by approximating the operator F by a sequence of operators F^ε which are strictly monotone with respect to u. More precisely for all $\varepsilon > 0$ we define

$$k^\varepsilon(x, u) = \varepsilon q(u) + k(x, u);$$

where $q \colon \mathbb{R} \to [0, +\infty)$ is a bounded function of class C^1 such that $q' > 0$. We consider

$$F^\varepsilon(x, u, p, X) := \begin{cases} (k^\varepsilon(x, u))^{1/n} f^{1/n}(p) - (\det A(p, X))^{1/n}, & \text{if } A(p, X) \ge 0 \\ +\infty, & \text{otherwise} \end{cases} \tag{5.62}$$

and the Dirichlet problem

$$(DP)^\varepsilon \quad \begin{cases} F^\varepsilon(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u(x) = \varphi(x), & \text{in } \partial\Omega, \end{cases}$$

¿From Theorem 4.1 it follows that for all $\varepsilon > 0$ there is a unique viscosity solution of $(DP)^\varepsilon$ which is Lipschitz continuous in $\overline{\Omega}$ by Theorem 5.2. The main goal is to show that the family $(u^\varepsilon)_\varepsilon$ is equibounded and equicontinuous in $\overline{\Omega}$. We denote

$$\tilde{F}^\varepsilon(x, u, p, X) = -\det(A(p, X)) + k^\varepsilon(x, p) f(p).$$

We start by the following Lemma which is the analogous of Lemma 5.2.

Lemma 5.3. *Assume $k \in C^1$ satisfies* **(H8)**. *Then, for all $\varepsilon > 0$ small, k^ε satisfies the condition* (1.8) *for some positive α_ε depending on ε.*

Proof. We set

$$\begin{aligned} I^\varepsilon(x,u,p) &= \frac{D_x k^\varepsilon \cdot p + D_u k^\varepsilon |p|^2}{(1+|p|^2)^{1/2}} + gn(k^\varepsilon)^{1+1/n} \\ &= \frac{D_x k \cdot p + [D_u k + \varepsilon q'(u)]|p|^2}{(1+|p|^2)^{1/2}} + gn(k^\varepsilon)^{1+1/n} \end{aligned}$$

It is enough to observe that since $q, q' > 0$, if $L > 1$ is large enough we have

$$I_\varepsilon(x,u,p) \geq I(x,u,p) + \varepsilon \inf q'/2 + gn(\varepsilon \inf q)^{1+1/n} \geq \varepsilon \inf q'/2 + gn(\varepsilon \inf q)^{1+1/n}.$$

Then the conclusion follows by choosing for instance $\alpha_\varepsilon = \varepsilon \inf(q')/4$ and by arguing exactly as in the proof of Lemma 5.2. □

Next we show that the family u^ε is equibounded and verify (5.56) with some constant independent of ε.

Proposition 5.2. *Assume* **(H3)**–**(H5)**, **(H7)**, $\varphi \in C^{1,1}(\partial\Omega)$. *Then there is a constant $K > 0$ such that for all ε small we have*

$$\begin{array}{ll} (i) & ||u^\varepsilon||_\infty \leq K; \\ (ii) & |u^\varepsilon(x) - u^\varepsilon(y)| \leq K|x-y|, \quad \textit{for all } (x,y) \in \partial(\Omega \times \Omega). \end{array}$$

Proof. To show (i) it is enough to observe that the functions considered in Corollary 4.1 are still sub and supersolutions of $(DP)^\varepsilon$. Indeed as far as the function $\underline{v}$ is concerned one observes that by **(H4)** it is a strict subsolution of $F = 0$, thus since q is bounded, for all ε small enough we have $F^\varepsilon \leq 0$. Instead the function $\overline{v}$ is still a supersolution $F^\varepsilon = 0$, since q is positive.

The second property (ii) is a consequence of the fact that the functions built in Lemma 5.1 are still local barriers for $(DP)^\varepsilon$. The proof of this claim follows again from the facts that the function $\underline{u}$ built in Lemma 5.1 is a strict subsolution of $F = 0$ and the function q is positive and bounded. Thus the conclusion follows by the same arguments of the proof of Theorem 5.2. □

By combining Corollary 5.1, Lemma 5.3 and Proposition 5.2 it follows

Corollary 5.2. *Assume* **(H3)**–**(H5)**, **(H7)**–**(H8)**, $\varphi \in C^{1,1}(\partial\Omega)$. *For all $\varepsilon > 0$ let u^ε be the unique viscosity solution of $(DP)^\varepsilon$. Then there is a constant $C > 0$ depending on $K, L, ||k||_\infty$, (K, L being the constant appearing respectively in Proposition 5.2 and in condition* **(H8)**) *such that for ε small we have*

$$|u^\varepsilon(x) - u^\varepsilon(y)| \leq C|x-y|, \quad \textit{for all } (x,y) \in \Omega \times \Omega.$$

From Proposition 5.2 and Corollary 5.2 it follows that the family u^ε is equicontinuous and equibounded in $\overline{\Omega}$. Thus by applying Ascoli-Arzelà Theorem we get the existence of a Lipschitz continuous solution of (DP) also in the case that k satisfies **(H5)**. More precisely we have

Theorem 5.3. *Assume* **(H3)**–**(H5)** *and* **(H7)**–**(H8)** $\varphi \in C^{1,1}(\partial\Omega)$. *Then there exits a Lipschitz continuous solution of* (DP).

Proof. For all $\varepsilon > 0$ let u^ε be the unique viscosity solution of $(DP)^\varepsilon$. From Proposition 5.2 and Corollary 5.2 it follows that the family u^ε is equicontinuous and equibounded in $\overline{\Omega}$. Thus by applying Ascoli-Arzelà Theorem there is a subsequence u_{ε_j} which converge uniformly as $j \to \infty$ to a function u which is Lipschitz continuous in $\overline{\Omega}$. Since F^ε converges locally uniformly to F, by the stability of viscosity solutions with respect to the uniformly convergence of F^ε to F, we get that u is a viscosity solution of (DP) and we conclude. □

Proof of Theorem 1.3. The proof is contained in Theorems 5.3 and 4.3. □

6 Non existence results on balls

In this section we present some non existence results which show that condition **(H4)** cannot be significantly relaxed when the domain Ω is a ball. We will denote by $\nu(x)$ the inner normal vector to $\partial\Omega$ at $x \in \partial\Omega$. First of all, by following the argument in [6, Theorem 1] and in [20, Corollary 1.1], we easily have

Proposition 6.1. *Let $B = B(R) \subset \mathbb{R}^N$ be a ball of radius R and let $u \in Lip(\overline{B})$ be a viscosity solution of $F = 0$. Then necessarily*

$$r \le \sup_{\Omega\times\mathbb{R}} (1/k)^{1/n} . \tag{6.63}$$

Proof. For $0 < r < R$ we have that

$$\phi(x) = C - (r^2 - |x|^2)^{1/2}$$

is in $C^2(B(r))$, and $\frac{\partial\phi}{\partial\nu}$ is $-\infty$ on the boundary. Since $u \in Lip(\overline{B})$ then $u - \phi$ has a maximum point at an interior point $x_0 \in B(r)$. By definition of a viscosity solution of $F = 0$, we have that $F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \le 0$, i.e.

$$k(x_0, u(x_0)) \le \frac{\det A(D\phi, D^2\phi)}{f(D\phi)} = \frac{1}{r^n},$$

for all $r < R$. By letting $r \to R$ we get (6.63). □

The estimate (6.63) can be obviously regarded as a first non existence result: *If* (6.63) *does not hold, then we cannot find a viscosity solution $u \in Lip(\Omega)$ of $F = 0$.*

We shall prove a stronger result when Ω is a ball. Our main tool is the following variant of the comparison principle.

Proposition 6.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\Gamma \subseteq \partial\Omega$ be relatively open and of class C^1. If $u \in C(\bar\Omega) \cap Lip(\Omega \cup \Gamma)$ is a viscosity solution of $F \ge 0$ and $v \in C(\bar\Omega) \cap Lip(\Omega)$ is a viscosity solution of $F < 0$ such that for all $x \in \Gamma$ $\liminf_{t\to 0^+} \frac{v(x+t\nu(x))-v(x)}{t} = -\infty$ on Γ and $u \le v$ in $\partial\Omega\setminus\Gamma$, then $u \le v$ in Ω.*

Proof. By the comparison principle we have $\sup_\Omega(u - v) \le \sup_\Gamma(u - v)^+$, but on Γ we have $\liminf_{t\to 0^+} \frac{v(x+t\nu(x))-v(x)}{t} = -\infty$. Hence, $u - v$ cannot achieve a maximum value on Γ. Then $u \le v$ on $\partial\Omega$. □

Now assume that $k > 0$ satisfies condition **(H6)** and $u \in Lip(\overline{B}(R))$ is a viscosity solution of $F = 0$ in B. Assume that there is a point $\xi_0 \in \partial B$ such that

$$k(\xi_0) > \frac{1}{R^n}. \tag{6.64}$$

It is not restrictive to assume $\xi_0 = 0$ and that the interior unit normal to ∂B at ξ_0 is $(0_n, 0_n, 1)$. We shall show that the boundary value of the function u cannot be arbitrarily on ξ_0.

By the continuity of k we can assume that (6.64) holds in a neighborhood of ξ_0. In particular there is a positive $a < R$ such that (6.64) holds in $B_a = \{(x, y, t) \in B : t < a\}$. Next define the function

$$w(x, y, t) = m + \psi(t),$$

where $m = \sup_{\partial B \setminus B_a} u$, $\psi \in C^2(a, 2R)$ is such that $\psi(2R) = 0, \psi' \leq 0, \psi'(a) = -\infty$.

In $B \setminus \overline{B_a}$ we have that $F(\xi, w, Dw, D^2 w) = k(\xi) > 0$. By Proposition 6.2 we then have

$$\sup_{B \setminus B_a} u \leq m + \psi(a). \tag{6.65}$$

Now we consider

$$w_a(\xi) = m_a - (R^2 - |\xi - \xi'|^2)^{1/2} + M_a$$

with $m_a = \sup_{B \cap \{t=a\}} u, M_a = \sup_{B \cap \{t=a\}} (R^2 - |\xi - \xi'|^2)^{1/2}$ and ξ' is the center of the ball B. In B_a we have $F(\xi, w_a, Dw_a, D^2 w_a) = k(\xi) - R^{-n} > 0$. By Proposition 6.2 we get

$$\sup_{B_a} u \leq m_a + M_a. \tag{6.66}$$

Finally, by applying estimate (6.65) to m_a in (6.66) we obtain

$$u(\xi_0) \leq m + \psi(a) + M_a = \sup_{\partial B \setminus B_a} u + \psi(a) + M_a. \tag{6.67}$$

We remark that $\lim_{a \to 0} M_a = 0$ and that we can choose ψ such that $\lim_{a \to 0} \psi(a) = 0$ ([14, Equation (14.67) p. 348]). Hence the estimate (6.67) shows that u cannot be prescribed arbitrarily on $\partial \Omega$. Thus we have proved the following non existence theorem.

Theorem 6.1 (Non existence result on balls). *Assume $k > 0$ satisfies condition* **(H6)** *and there is a point $\xi_0 \in \partial B$ such that* (6.64) *holds. Then there is $\varphi \in C^\infty(\bar{B})$ such that the Dirichlet problem* (DP) *(or* (1.2)*) is not solvable in the class of Lipschitz continuous viscosity solution.*

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