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The Levi Monge–Ampère Equation: Smooth Regularity of Strictly Levi Convex Solutions

By *Annamaria Montanari and Francesca Lascialfari*

ABSTRACT. We prove smoothness of strictly Levi convex solutions to the Levi equation in several complex variables. This equation is fully non linear and naturally arises in the study of real hypersurfaces in $\mathbb{C}^{n+1}$, for $n \geq 2$. For a particular choice of the right-hand side, our equation has the meaning of total Levi curvature of a real hypersurface $\mathbb{C}^{n+1}$ and it is the analogous of the equation with prescribed Gauss curvature for the complex structure. However, it is degenerate elliptic also if restricted to strictly Levi convex functions. This basic failure does not allow us to use elliptic techniques such in the classical real and complex Monge–Ampère equations. By taking into account the natural geometry of the problem we prove that first order intrinsic derivatives of strictly Levi convex solutions satisfy a good equation. The smoothness of solutions is then achieved by mean of a bootstrap argument in tangent directions to the hypersurface.

1. Introduction

Let $M = \{z : \rho(z) = 0\}$ be a real hypersurface in $\mathbb{C}^{n+1}$ of class $C^{2,\alpha}$, $0 < \alpha < 1$. Let us denote by D the open set

$$D := \left\{ z \in \mathbb{C}^{n+1} : \rho(z) < 0 \right\} .$$

By following a suggestion implicitly contained in a article by Bedford and Gaveau [1], we define the *total Levi curvature* of M at a point $z \in M$ as

$$k_M(z) = -|\partial\rho|^{-n-2} \det \begin{pmatrix} 0 & \partial_{\bar{1}}\rho & \cdots & \partial_{\overline{n+1}}\rho \\ \partial_1\rho & \partial_{1\bar{1}}\rho & \cdots & \partial_{1\overline{n+1}}\rho \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{n+1}\rho & \partial_{n+1\bar{1}}\rho & \cdots & \partial_{n+1\overline{n+1}}\rho \end{pmatrix} . \tag{1.1}$$

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In (1.1), ∂_j, $\partial_{\bar{j}}$, $\partial_{l\bar{j}}$ denote the derivatives $\frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial \bar{z}_j}$, $\frac{\partial^2}{\partial z_l \partial \bar{z}_j}$, respectively, $\partial\rho = (\partial_1\rho, \ldots, \partial_{n+1}\rho)$ and the derivatives are computed at z. The total Levi curvature is independent of the defining function ρ and can be considered analogous to the Gauss curvature for the classical Monge–Ampère equation (see [16]). For example, if M is the sphere of radius R with center at zero, then we can choose its defining function as $\rho = |z_1|^2 + \cdots + |z_{n+1}|^2 - R^2$ and an easy calculation gets $k \equiv R^{-n}$.

The aim of this article is to show the following theorem.

Theorem 1.1. *If D is a strictly pseudoconvex domain such that*

$$z \longmapsto k_M(z)$$

is of class C^∞, then M is a real submanifold of $\mathbb{C}^{n+1}$ of class C^∞.

We recall that a domain $\{\rho < 0\}$ is strictly pseudoconvex if the Levi form ρ is positive definite on the boundary.

This problem is of local nature and leads to the study of the C^∞ regularity of classical solutions to a fully nonlinear PDE of subelliptic type, in a sense that will be clear below.

In order to explain our problem, we introduce some notations of the theory of several complex variables. Denote by $T_0^{\mathbb{C}}$ the complex tangent hyperplane to M at $z \in M$, and define the Levi form $L(\rho)$ as the restriction to $T_0^{\mathbb{C}}$ of the Hermitian form related to the complex Hessian matrix of ρ

$$\mathrm{Hess}_{\mathbb{C}}\,\rho = \left(\frac{\partial^2\rho}{\partial z_l \partial \bar{z}_p}\right)_{l,p=1}^{n+1} .$$

The Levi form naturally arises in the study of envelopes of holomorphy in the theory of holomorphic functions in $\mathbb{C}^{n+1}$ (see [13, 17, 19, 23] for details on this matter). It is a standard fact that the Levi form is the biholomorphic invariant part of the real Hessian of the defining function; one way to derive it is to seek for a biholomorphic invariant analog of Euclidean convexity (see for example, [19]). Since $L(\rho)$ is obtained from part of the second fundamental form of M, it is not unreasonable that it will have some properties similar to curvatures. Bedford and Gaveau were the first to remark this, and in [1] they bounded in term of the Levi form the domain over which M can be defined as a non parametric surface.

Since our result is local, we can assume for example $\partial_{z_{n+1}}\rho \neq 0$ in a neighborhood of $z \in M$ and choose $\mathcal{U} = \{h_l, l = 1, \ldots, n\}$ a complex basis of $T_0^{\mathbb{C}}$

$$h_l = e_l - \frac{\partial_{z_l}\rho}{\partial_{z_{n+1}}\rho} e_{n+1} ,$$

with $(e_l)_{l=1,\ldots,n+1}$ the canonical basis of $\mathbb{C}^{n+1}$. $L(\rho)$ is a Hermitian form in n variables whose coefficients are

$$A_{l\overline{p}}(\rho) = \left\langle \left(\mathrm{Hess}_{\mathbb{C}}\,\rho\right) h_l, h_p \right\rangle, \quad \forall\, h_l, h_p \in \mathcal{U} .$$

By our choice of h_l,

$$A_{l\overline{p}}(\rho) = \partial_{l\bar{p}}\rho - \frac{\rho_l \rho_{\overline{n+1}}}{|\rho_{n+1}|^2}\partial_{n+1\bar{p}}\rho - \frac{\rho_{n+1}\rho_{\bar{p}}}{|\rho_{n+1}|^2}\partial_{l\overline{n+1}}\rho + \frac{\rho_l \rho_{\bar{p}}}{|\rho_{n+1}|^2}\partial_{n+1\overline{n+1}}\rho , \tag{1.2}$$

where subscripts denote partial derivatives.

Introduce real coordinates $z_l = x_l + i y_l$ for every $l = 1, \dots, n+1$. Since we have assumed $\partial_{z_{n+1}} \rho \neq 0$ at $z \in M$, it is not restrictive to take $\partial_{y_{n+1}} \rho \neq 0$. With this convention, there is a neighborhood U of z such that $M \cap U$ is the graph of a C^2 function $-u : \Omega \to \mathbb{R}$, with Ω an open bounded subset in $\mathbb{R}^{2n+1}$. Then we can choose the defining function of M as $\rho = 4(u + y_{n+1})$, $M = \{y_{n+1} = -u(x_1, y_1, \dots, x_n, y_n, x_{n+1})\}$. The coefficients $A_{l\overline{p}}(u)$ of the Levi form $L(u)$ are quasilinear partial differential operators. Precisely, the real part and the imaginary part of $A_{l\overline{p}}(u)$ are:

$$\begin{aligned} \operatorname{Re}\left(A_{l\bar{p}}(u)\right) = \Big(&\partial_{x_l x_p} u + \partial_{y_l y_p} u + a_l \partial_{x_p x_{n+1}} u + a_p \partial_{x_l x_{n+1}} u \\ &+ b_l \partial_{y_p x_{n+1}} u + b_p \partial_{y_l x_{n+1}} u + \left(a_l a_p + b_l b_p\right) \partial^2_{x_{n+1}} u\Big) \\ \operatorname{Im}\left(A_{l\bar{p}}(u)\right) = \Big(&\partial_{x_l y_p} u - \partial_{x_p y_l} u - a_p \partial_{y_l x_{n+1}} u + a_l \partial_{y_p x_{n+1}} u \\ &+ b_p \partial_{x_l x_{n+1}} u - b_l \partial_{x_p x_{n+1}} u + \left(b_p a_l - b_l a_p\right) \partial^2_{x_{n+1}} u\Big) \end{aligned} \tag{1.3}$$

where

$$a_l = \frac{-\partial_{y_l} u - \partial_{x_l} u \, \partial_{x_{n+1}} u}{1 + \left(\partial_{x_{n+1}} u\right)^2}, \quad b_l = \frac{\partial_{x_l} u - \partial_{y_l} u \, \partial_{x_{n+1}} u}{1 + \left(\partial_{x_{n+1}} u\right)^2} \, . \tag{1.4}$$

In particular, for every $l = 1, \dots, n$, the diagonal coefficient $A_{l\overline{l}}(u)$ is a degenerate elliptic second-order operator, whose characteristic form

$$\xi = \left(\xi_1, \dots, \xi_{2n+1}\right) \longrightarrow \left(\xi_{2l-1} + a_l \xi_{2n+1}\right)^2 + \left(\xi_{2l} + b_l \xi_{2n+1}\right)^2 ,$$

is non negative definite for every $\xi \in \mathbb{R}^{2n+1}$, but has $2n - 1$ eigenvalues identically zero.

Define the Levi Monge–Ampère operator as

$$LMA(u) = \det\left(A_{l\overline{p}}(u)\right) . \tag{1.5}$$

Definition 1.2. We say that a function $u \in C^2(\Omega)$ is Levi convex (strictly Levi convex) at ξ_0 if $L(u)(\xi_0) \geq 0$ (> 0) and Levi convex (strictly Levi convex) in Ω if $L(u)(\xi) \geq 0$ (> 0) for every $\xi \in \Omega$.

Our main theorem, first announced in [20], is the following.

Theorem 1.3. *If $u \in C^{2,\alpha}(\Omega)$ is a strictly Levi convex solution to the Levi Monge–Ampère equation*

$$LMA(u) = q(\cdot, u, Du) \tag{1.6}$$

in an open set $\Omega \subset \mathbb{R}^{2n+1}$ and $q \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{2n+1})$ is positive, then $u \in C^\infty(\Omega)$.

Here we have denoted by $C^{m,\alpha}$ the ordinary Hölder space with respect to the Euclidean metric and by Du the Euclidean gradient of u in $\mathbb{R}^{2n+1}$.

If in (1.6) we choose

$$q(\cdot, u, Du) = 2^n k(\cdot, -u) \frac{\left(1 + |Du|^2\right)^{\frac{n+2}{2}}}{1 + \left(\partial_{x_{n+1}} u\right)^2} \, , \tag{1.7}$$

then k is the total Levi curvature of M defined in (1.1). Hence, Theorem 1.1 follows from Theorem 1.3.

Since the Levi Monge–Ampère equation presents formal similarities with the real and complex Monge–Ampère equations, which are elliptic PDE's if evaluated on strictly convex and plurisubharmonic functions, respectively (see [16, 2]), we would like to briefly recall how the smoothness follows from the classical Schauder theory for the real Monge–Ampère equation. The real Monge–Ampère equation in a domain $\Omega \subset \mathbb{R}^n$ is of the form $\det(D^2u) = \psi(x, u, Du)$. If $u \in C^{2,\alpha}(\Omega)$ is a strictly convex solution to this equation, then the linearized operator L (at u) is elliptic with C^α coefficients, and Du satisfies a linear uniformly elliptic equation of the form $L(Du) = f \in C^\alpha(\Omega)$. By the Schauder theory, $Du \in C^{2,\alpha}(\Omega)$. Repeating this one proves $u \in C^\infty(\Omega)$. In our case it is not possible to argue in the same way, because the Levi Monge–Ampère operator $LMA(u)$ is not elliptic at any point, also when restricted to the class of strictly Levi convex functions. Indeed, as we prove in Lemma 2.1, if we call D^2u the Euclidean Hessian matrix of u, then by (1.3) there exists a smooth function $F = F(p, r)$, with $p = Du$ and $r = D^2u$, such that

$$LMA(u) = F\big(Du, D^2u\big)$$

and

$$\frac{\partial F}{\partial r_{ij}}\big(Du, D^2u\big) \geq 0\,.$$

Moreover, in Lemma 2.1 we prove that the minimum eigenvalue of the real matrix $\big(\frac{\partial F}{\partial r_{ij}}\big)_{i,j=1}^{2n+1}$ is identically zero. Hence, we are forced to develop a new technique, which takes into account the CR structure of the hypersurface M.

The analogy of the prescribed mean curvature equation for a real hypersurface in $\mathbb{C}^{n+1}$ has been studied in [11], where smooth regularity of classical solutions is proved. In the case $n = 1$, the operator $LMA(u)$ defined in (1.5) coincides with the coefficient $A_{1,\bar{1}}(u)$ of the Levi form $L(u)$ and Equation (1.6) becomes a quasilinear PDE (see for instance, [27]). Regularity properties of its solutions have been studied in [4]–[12], [27, 28]. Starting from a previous weak existence result [27] by Slodkowski and Tomassini, in [7, Corollary 1.1] Citti, Lanconelli, and the first author proved the existence of a smooth solution to the Dirichlet problem

$$\begin{cases} A_{1,\bar{1}}(u) = 2k(\cdot, -u)\dfrac{\big(1+|Du|^2\big)^{\frac{3}{2}}}{1+\big(\partial_{x_{n+1}}u\big)^2} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $g \in C^{2,\alpha}(\partial\Omega)$, $k \in C^\infty(\Omega \times \mathbb{R})$ and $k \neq 0$. Here Ω is a bounded open set in $\mathbb{R}^{2n+1}$ with smooth boundary and which satisfies the hypotheses of [27, Theorem 4].

For the Levi Monge–Ampère equation the technique used in [7] seems to fail, because of the fully nonlinearity of the equation. Moreover, for the Levi Monge–Ampère equation with the right-hand side in (1.7), the existence of viscosity solutions is also an open problem. A first result in this direction is due to Slodkowski and Tomassini, who in [26] generalized the Definition 1.2 to continuous functions and proved the existence of a viscosity solution $u \in \mathrm{Lip}(\bar{\Omega})$ to the Dirichlet problem

$$\begin{cases} LMA(u) = k(\cdot, -u)\big(1 + |Du|^2\big)^{\frac{3n}{2}} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ u \quad \text{is Levi convex} & \end{cases} \tag{1.8}$$

where $g \in C(\partial\Omega)$, $k \in C(\Omega \times \mathbb{R})$ and $k \geq 0$. Here Ω is a bounded open set in $\mathbb{R}^{2n+1}$ with smooth boundary and which satisfies the hypotheses of [26, Theorem 2.4]. Even if the right-hand side in the PDE in (1.8) has not the meaning of total Levi curvature, further regularity of its Lipschitz continuous viscosity solution is another very interesting open problem.

In order to present our technique we introduce here some notations. Let $u \in C^{2,\alpha}(\Omega)$ be a strictly Levi convex solution to (1.6) and define for every $l = 1, \ldots, n$ the first order vector fields

$$X_l = \partial_{x_l} + a_l \partial_{x_{n+1}}, \quad Y_l = \partial_{y_l} + b_l \partial_{x_{n+1}} , \tag{1.9}$$

whose coefficients a_l and b_l are the smooth functions of the gradient of u given by (1.4).

Since the fixed solution u belongs to $C^{2,\alpha}(\Omega)$, then the coefficients a_l, b_l are $C^{1,\alpha}(\Omega)$ functions. The first important remark we will prove is Lemma 2.2, where we write the coefficients of the Levi form $A_{l\bar{p}}$ in terms of the vector fields in (1.9):

$$A_{l\bar{l}}(u) = \Big(1 + \big(\partial_{x_{n+1}} u\big)^2\Big)\Big(X_l^2 u + Y_l^2 u\Big) , \tag{1.10}$$

$$\begin{aligned} \mathrm{Re}\,\big(A_{l\overline{p}}(u)\big) &= \frac{\Big(1 + \big(\partial_{x_{n+1}} u\big)^2\Big)}{2}\big(X_l X_p u + X_p X_l u + Y_l Y_p u + Y_p Y_l u\big) , \\ \mathrm{Im}\,\big(A_{l\overline{p}}(u)\big) &= \frac{\Big(1 + \big(\partial_{x_{n+1}} u\big)^2\Big)}{2}\big(X_l Y_p u + Y_p X_l u - Y_l X_p u - X_p Y_l u\big) . \end{aligned} \tag{1.11}$$

For every $l = 1, \ldots, n$ we put

$$\begin{aligned} Z_{2l} = Y_l, \qquad Z_{2l-1} = X_l , \\ Z = \big(Z_1, Z_2, \ldots, Z_{2n}\big), \qquad Z^2 u = \big(Z_l Z_p u\big)_{l,p=1}^{2n} . \end{aligned} \tag{1.12}$$

For every $z = (z_{m,j})$ in the space of $2n \times 2n$ real matrices, we define for $l \leq p$

$$\begin{aligned} 2\mathcal{H}_{l\bar{p}}(z) = &\Big(z_{2l-1,2p-1} + z_{2p-1,2l-1} + z_{2l,2p} + z_{2p,2l}\Big) \\ &+ i\Big(z_{2l-1,2p} + z_{2p,2l-1} - z_{2l,2p-1} - z_{2p-1,2l}\Big) , \end{aligned} \tag{1.13}$$

and $\mathcal{H}_{l\bar{p}} = \overline{\mathcal{H}_{p\bar{l}}}$ for $l \geq p$. Then

$$\mathcal{H}_{l\bar{p}}\big(Z^2 u\big) = \frac{A_{l\bar{p}}(u)}{1 + \big(\partial_{x_{n+1}} u\big)^2} ,$$

and

$$\begin{aligned} 2\mathcal{H}_{l\bar{p}}\big(Z^2 u\big) = &\big(Z_{2l-1} Z_{2p-1} u + Z_{2p-1} Z_{2l-1} u + Z_{2l} Z_{2p} u + Z_{2p} Z_{2l} u\big) \\ &+ i\big(Z_{2l-1} Z_{2p} u + Z_{2p} Z_{2l-1} u - Z_{2l} Z_{2p-1} u - Z_{2p-1} Z_{2l} u\big) . \end{aligned} \tag{1.14}$$

Define

$$\mathcal{H}\big(Z^2 u\big) = \det\Big(\mathcal{H}_{l\bar{p}}\big(Z^2 u\big)\Big) . \tag{1.15}$$

$\mathcal{H}$ is a smooth function of the second derivatives of u with respect to the vector field Z_j, $j = 1, \ldots, n$, and

$$\frac{LMA(u)}{\Big(1 + (\partial_{x_{n+1}} u)^2\Big)^n} = \mathcal{H}\big(Z^2 u\big) .$$

Moreover, in Lemma 2.3 we prove there exists a smooth positive function K such that

$$\frac{q(\cdot, u, Du)}{\left(1+\left(\partial_{x_{n+1}} u\right)^2\right)^n} = K(\cdot, u, Zu, \partial_{x_{n+1}} u) . \tag{1.16}$$

For example, if $q = k(\cdot, -u)2^n \frac{(1+|Du|^2)^{\frac{n+2}{2}}}{1+(\partial_{x_{n+1}} u)^2}$ as in (1.7), then

$$K = k(\cdot, -u)2^n \left(1+|Zu|^2\right)^{\frac{n+2}{2}} \left(1+\left(\partial_{x_{n+1}} u\right)^2\right)^{-\frac{n}{2}} . \tag{1.17}$$

Hence, we write the fully nonlinear equation in (1.6) as

$$\mathcal{H}\big(Z^2 u\big) = K\big(\cdot, u, Zu, \partial_{x_{n+1}} u\big) . \tag{1.18}$$

To the best of our knowledge, no regularity result has been published about fully nonlinear PDE's of the type (1.18), even in the case of smooth linear vector fields Z. Some authors studied quasilinear equations associated to smooth vector fields of Hörmander's type (see [3, 29]). In particular, in [3] Capogna, Danielli, and Garofalo studied the local behavior of singular solutions of a class of subelliptic equations of the type

$$\sum_{j=1}^{m} X_j^* A_j(x, u, Xu) = f(x, u, Xu)$$

where A_j, f are measurable functions satisfying some grow up conditions with respect to u and $Xu = (X_1 u, \ldots, X_m u)$, and the linear vector fields $X_1, \ldots, X_m$ are smooth and satisfy Hörmander's finite rank condition (see [18]). However, their result does not seem to be useful in the present situation.

In Lemma 2.4 we prove that, if u is strictly Levi convex in Ω, then there exists a positive constant M such that

$$\sum_{m,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{mj}}\big(Z^2 u\big)\eta_m \eta_j \geq M \sum_{j=1}^{2n} \eta_j^2, \quad \forall \eta = \big(\eta_1, \ldots, \eta_{2n}\big) \in \mathbb{R}^{2n} .$$

Roughly speaking, this means that the equation is elliptic in $2n$ directions of the tangent space to the hypersurface, which has real dimension $2n+1$. In order to generate the missing direction we compute the commutators of the vector fields and in Lemma 2.5 we prove that

$$\big[Z_{2l-1}, Z_{2l}\big] = \big[X_l, Y_l\big] = \big(X_l^2 u + Y_l^2 u\big)\partial_{x_{n+1}}, \qquad l = 1, \ldots, n , \tag{1.19}$$

and

$$Z_1, \ldots, Z_{2n}, \big[Z_1, Z_2\big] \tag{1.20}$$

are linearly independent at every point and span $\mathbb{R}^{2n+1}$.

Moreover, in Lemma 2.4 we write the fully nonlinear equation in (1.18) as

$$\mathcal{H}\big(Z^2 u\big) = \frac{1}{n} \sum_{m,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{mj}}\big(Z^2 u\big) Z_m Z_j u = K\big(\cdot, u, Zu, \partial_{x_{n+1}} u\big) . \tag{1.21}$$

For a fixed strictly Levi convex solution $u \in C^{2,\alpha}$ of (1.6) the coefficients

$$h_{mj} = \frac{\partial \mathcal{H}}{\partial z_{mj}}\big(Z^2 u\big) \in C^\alpha \ ,$$

and the coefficients

$$a = (a_1, \dots, a_n), \qquad b = (b_1, \dots, b_n) \tag{1.22}$$

of Z are $C^{1,\alpha}$. Hence, in the light of (1.21), we define a linear subelliptic operator

$$H = \sum_{m,j=1}^{2n} h_{mj} Z_m Z_j - \lambda \partial_{x_{n+1}} \ , \tag{1.23}$$

where the coefficients λ, h_{mj} are α-Hölder continuous and such that $h_{mj} = h_{jm}$, for every $m, j = 1, \dots, 2n$,

$$\sum_{m,j=1}^{2n} h_{mj} \eta_m \eta_j \geq M \sum_{j=1}^{2n} \eta_j^2, \quad \forall \eta = \big(\eta_1, \dots, \eta_{2n}\big) \in \mathbb{R}^{2n} \tag{1.24}$$

for a positive constant M. Here the vector fields Z_j are defined as in (1.12) [see also (1.9)] with coefficients a, b defined in (1.22) of class $C^{1,\alpha}$ and we assume that they are linearly independent at every point together with their first order brackets.

We explicitly remark that we can not apply to our operator H the regularity theory developed in [14, 15], and [24], since in those works the smoothness hypothesis on the coefficients of the vector fields is crucial.

However, a regularity theory for sum of squares of $C^{1,\alpha}$ vector fields has been recently established by Citti in [4, 5] and by Citti, and the first author in [11, 12].

In particular, by using the techniques developed in [11, Theorem 4.1], one can prove the following.

Proposition 1.4. *Let $h_{ij}, \lambda \in C^{m-1,\alpha}_{Z,\mathrm{loc}}(\Omega)$, $a, b \in C^{m,\alpha}_{Z,\mathrm{loc}}(\Omega)$, $m \geq 2$ and let $v \in C^{2,\alpha}_{Z,\mathrm{loc}}(\Omega)$ be a solution of equation $Hv = f$ with H as in* (1.23) *and $f \in C^{m-1,\alpha}_{Z,\mathrm{loc}}(\Omega)$. Then the solution v belongs to $C^{m+1,\beta}_{Z,\mathrm{loc}}(\Omega)$ for every $\beta \in (0, \alpha)$.*

Here $C^{m,\alpha}_Z$ denotes the class of functions whose tangent derivatives of order m are α-Hölder continuous with respect to a distance naturally associated to the vector fields Z_j (see (3.1) and (3.2) for precise definitions). Proposition 1.4 requires the following initial regularity of the coefficients: $h_{ij}, \lambda \in C^{1,\alpha}_{Z,\mathrm{loc}}(\Omega)$, $a, b \in C^{2,\alpha}_{Z,\mathrm{loc}}(\Omega)$. This property does not hold for a fixed $u \in C^{2,\alpha}(\Omega)$ solution of (1.6), while it would hold if $u \in C^{3,\alpha}_{Z,\mathrm{loc}}(\Omega)$.

However, in [21] the first author proved interior Schauder-type estimates for solutions of $Hv = f$ with H as in (1.23).

In Section 3 we apply the a priori estimates in [21] to first order Euclidean difference quotients of a strictly Levi convex solution u of (1.6), in order to prove that the function Du is $C^{2,\beta}_{Z,\mathrm{loc}}(\Omega)$.

At this point another problem arises, because it is not possible to apply a classical bootstrap argument to the solutions of (1.21). Indeed, on the right-hand side of (1.21) it appears a function of $\partial_{x_{n+1}} u$, and by (1.19) $\partial_{x_{n+1}}$ is a second derivatives in terms of the vector fields. In particular, if $u \in C_Z^{m,\alpha}(\Omega)$ we can only deduce that $\partial_{x_{n+1}} u \in C_{Z,\mathrm{loc}}^{m-2,\alpha}(\Omega)$ and by applying Proposition 1.4 to a solution $u \in C_Z^{m,\alpha}(\Omega)$ of Equation (1.21) we get only $u \in C_Z^{m,\beta}(\Omega)$ for every $\beta \in (0, \alpha)$. However, by formally differentiating the fully nonlinear Equation (1.6) with respect to the vector fields Z_j, $j = 1, \dots, 2n$ and with respect to $\partial_{x_{n+1}}$, we discover the following.

Proposition 1.5. *If u is a smooth solution of (1.6), then the function*

$$v = \big(v_1, \dots, v_{2n}, v_{2n+1}\big) = \big(Z_1 u, \dots, Z_{2n} u, \arctan u_{x_{n+1}}\big) \tag{1.25}$$

is a solution of

$$\left(\sum_{m,j=1}^{2n} h_{mj} Z_m Z_j - \lambda \partial_{x_{n+1}} \right) v = f(\cdot, u, v, Zv) \ , \tag{1.26}$$

with $f = (f_1, \dots, f_{2n}, f_{2n+1})$ a smooth function of its arguments. Here the coefficients h_{ij}, λ depend on the fixed function u and precisely

$$\begin{aligned} h_{mj} &= \frac{\partial \mathcal{H}}{\partial z_{mj}} \big(Z^2 u\big) \ , \\ \lambda &= nK \partial_{x_{n+1}} u + \frac{\partial K}{\partial u_{x_{n+1}}} \Big(1 + \big(\partial_{x_{n+1}} u\big)^2\Big) \ , \end{aligned} \tag{1.27}$$

with $\mathcal{H}(Z^2 u)$ and K defined as in (1.15) and (1.16), respectively.

This result is crucial in our regularity proceeding and we prove it in Section 4. Then, we apply Proposition 1.4 to v in (1.25) and we prove Theorem 1.3 with a non standard bootstrap argument.

2. Structure of the LMA

Let us fix a strictly Levi convex $C^{2,\alpha}(\Omega)$ solution u to Equation (1.6).

Lemma 2.1. *For every $n \in \mathbb{N}$ denote by $r_{mj} = D_m D_j u$ for all $i, j = 1, \dots, 2n+1$, and by $F(Du, D^2 u) = LMA(u)$. Then*

$$\sum_{m,j=1}^{2n+1} \frac{\partial F}{\partial r_{mj}} \big(Du, D^2 u\big) \xi_m \xi_j \geq 0, \quad \forall \xi = (\xi_1, \dots, \xi_{2n+1}) \in \mathbb{R}^{2n+1} \ ,$$

and the minimum eigenvalue of the real matrix $\big(\frac{\partial F}{\partial r_{mj}}(Du, D^2 u)\big)_{m,j=1}^{2n+1}$ is identically zero.

Proof. The proof will be a consequence of the following statement:

Denote by $\rho_{l\bar{p}} = \partial_l \partial_{\bar{p}} \rho$, $\partial \rho = (\rho_1, \dots, \rho_{n+1})$, $\bar{\partial} \rho = (\rho_{\bar{1}}, \dots, \rho_{\overline{n+1}})$, $\partial \bar{\partial} \rho = (\rho_{l\bar{p}})_{l,p=1}^{2n+1}$ and by $G(\partial\rho, \bar{\partial}\rho, \partial\bar{\partial}\rho) = \det(A_{l\bar{p}}(\rho))$. If the Levi form $L(\rho)$ is positive definite, then

$$\sum_{h,\bar{k}} \frac{\partial G}{\partial \rho_{h\bar{k}}} \big(\partial\rho, \bar{\partial}\rho, \partial\bar{\partial}\rho\big) \eta_h \eta_{\bar{k}} \geq 0, \quad \forall \eta = \big(\eta_1, \dots, \eta_{n+1}\big) \in \mathbb{C}^{n+1} \ , \tag{2.1}$$

and the minimum eigenvalue of the complex matrix

$$\left(\frac{\partial G}{\partial \rho_{h\bar{k}}}\big(\partial\rho, \bar\partial\rho, \partial\bar\partial\rho\big)\right)_{h,k=1}^{n+1}$$

is identically zero.

Indeed, for every $l, p = 1, \dots, n+1, m, j = 1, \dots, 2n+1$ and $\rho = 4(u - y_{n+1})$

$$\frac{\partial \rho_{l\bar{p}}}{\partial r_{mj}} = \begin{cases} 1 & m = 2l-1,\ j = 2p-1 \\ 1 & m = 2l,\ j = 2p \\ i & m = 2l-1,\ j = 2p \\ -i & m = 2l,\ j = 2p-1 \\ 0 & \text{otherwise}\ . \end{cases}$$

Then, if we put $\eta_l = \xi_{2l-1} + i\xi_{2l}$ for every $l = 1, \dots, n$ by (2.1) we get the thesis.

Let us prove (2.1). By (1.2), for every $h, k = 1, \dots, n+1$ and $l, p = 1, \dots, n$

$$\frac{\partial A_{l\bar{p}}(\rho)}{\partial \rho_{h\bar{k}}} = \begin{cases} 1 & (h,k) = (l,p)\ , \\ -\frac{\rho_{n+1}\rho_{\bar{p}}}{|\rho_{n+1}|^2} & (h,k) = (l, n+1)\ , \\ -\frac{\rho_l \rho_{\overline{n+1}}}{|\rho_{n+1}|^2} & (h,k) = (n+1, p)\ , \\ \frac{\rho_l \rho_{\bar{p}}}{|\rho_{n+1}|^2} & (h,k) = (n+1, n+1)\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Denote by $C_{l\bar{p}}$ the cofactor of $A_{l\bar{p}}$ and by $A^{l\bar{p}}$ the inverse matrix of $A_{l\bar{p}}$. Hence, for every $h, k = 1, \dots, n$

$$\begin{aligned}
\frac{\partial G}{\partial \rho_{h\bar{k}}}\big(\partial\rho, \bar\partial\rho, \partial\bar\partial\rho\big) &= \sum_{l,p=1}^{n} \frac{\partial A_{l\bar{p}}(\rho)}{\partial \rho_{h\bar{k}}} C_{l\bar{p}} = C_{h\bar{k}} \\
&= G\big(\partial\rho, \bar\partial\rho, \partial\bar\partial\rho\big)\, A^{k\bar{h}} > 0, \\
\frac{\partial G}{\partial \rho_{h\overline{n+1}}}\big(\partial\rho, \bar\partial\rho, \partial\bar\partial\rho\big) &= \sum_{l,p=1}^{n} \frac{\partial A_{l\bar{p}}(\rho)}{\partial \rho_{h\overline{n+1}}} C_{l\bar{p}} = \sum_{p=1}^{n} \frac{-\rho_{n+1}\rho_{\bar{p}}}{|\rho_{n+1}|^2} C_{h\bar{p}}\ , \\
\frac{\partial G}{\partial \rho_{n+1\overline{n+1}}}\big(\partial\rho, \bar\partial\rho, \partial\bar\partial\rho\big) &= \sum_{l,p=1}^{n} \frac{\partial A_{l\bar{p}}(\rho)}{\partial \rho_{n+1\overline{n+1}}} C_{l\bar{p}} = \sum_{l,p=1}^{n} \frac{\rho_l \rho_{\bar{p}}}{|\rho_{n+1}|^2} C_{l\bar{p}} \\
&= \sum_{l=1}^{n} \frac{-\rho_l \rho_{\overline{n+1}}}{|\rho_{n+1}|^2} \left(\sum_{p=1}^{n} \frac{-\rho_{n+1}\rho_{\bar{p}}}{|\rho_{n+1}|^2} C_{l\bar{p}} \right) \\
&= \sum_{l=1}^{n} \frac{-\rho_l \rho_{\overline{n+1}}}{|\rho_{n+1}|^2} \frac{\partial G}{\partial \rho_{l\overline{n+1}}} \big(\partial\rho, \bar\partial\rho, \partial\bar\partial\rho\big)\ .
\end{aligned}$$

In particular, the last column of the complex matrix $\left(\frac{\partial G}{\partial \rho_{h\bar{k}}}\right)_{h,k=1}^{n+1}$ is a linear combination of the first n columns and statement (2.1) follows. □

Lemma 2.2. *The coefficients of the Levi form $A_{l\bar{p}}$ can be written in terms of the vector fields in* (1.9) *as in* (1.10), *and* (1.11).

Proof. By arguing as in [5],

$$a_l = -Y_l u, \quad b_l = X_l u \,. \tag{2.2}$$

By substituting the vector fields (1.9) in the right-hand side of (1.10) and by using (2.2) twice, we get

$$\begin{aligned}
X_l^2 u + Y_l^2 u &= \left(\partial_{x_l} + a_l \partial_{x_{n+1}}\right)\left(\partial_{x_l} + a_l \partial_{x_{n+1}}\right) u \\
&\quad + \left(\partial_{y_l} + b_l \partial_{x_{n+1}}\right)\left(\partial_{y_l} + b_l \partial_{x_{n+1}}\right) u \\
&= A_{l\bar{l}}(u) + (X_l a_l + Y_l b_l)\,\partial_{x_{n+1}} u \\
&= A_{l\bar{l}}(u) - (X_l Y_l u - Y_l X_l u)\,\partial_{x_{n+1}} u \\
&= A_{l\bar{l}}(u) - [X_l, Y_l]\, u \partial_{x_{n+1}} u \\
&= A_{l\bar{l}}(u) - (X_l b_l - Y_l a_l)\left(\partial_{x_{n+1}} u\right)^2 \\
&= A_{l\bar{l}}(u) - \left(X_l X_l u + Y_l Y_l u)(\partial_{x_{n+1}} u\right)^2 \,,
\end{aligned}$$

and by putting the last term in the left-hand side we get (1.10). By arguing in the same way we now prove (1.11)

$$\begin{aligned}
X_l X_p u + X_p X_l u + Y_l Y_p u + Y_p Y_l u &= \left(\partial_{x_l} + a_l \partial_{x_{n+1}}\right)\left(\partial_{x_p} + a_p \partial_{x_{n+1}}\right) u \\
&\quad + \left(\partial_{x_p} + a_p \partial_{x_{n+1}}\right)\left(\partial_{x_l} + a_l \partial_{x_{n+1}}\right) u + \left(\partial_{y_l} + b_l \partial_{x_{n+1}}\right) \cdot \\
&\quad \cdot \left(\partial_{y_p} + b_p \partial_{x_{n+1}}\right) u + \left(\partial_{y_p} + b_p \partial_{x_{n+1}}\right)\left(\partial_{y_l} + b_l \partial_{x_{n+1}}\right) u \\
&= 2\,\mathrm{Re}\left(A_{l\overline{p}}(u)\right) + \left(X_l a_p + X_p a_l + Y_l b_p + Y_p b_l\right)\partial_{x_{n+1}} u \\
&= 2\,\mathrm{Re}\left(A_{l\overline{p}}(u)\right) - \left(X_l Y_p u + X_p Y_l u - Y_l X_p u - Y_p X_l u\right)\partial_{x_{n+1}} u \\
&= 2\,\mathrm{Re}\left(A_{l\overline{p}}(u)\right) - \left(\left[X_l, Y_p\right] u + \left[X_p, Y_l\right] u\right)\partial_{x_{n+1}} u \\
&= 2\,\mathrm{Re}\left(A_{l\overline{p}}(u)\right) - \left(X_l b_p - Y_p a_l + X_p b_l - Y_l a_p\right)\left(\partial_{x_{n+1}} u\right)^2 \\
&= 2\,\mathrm{Re}\left(A_{l\overline{p}}(u)\right) - \left(X_l X_p u + Y_p Y_l u + X_p X_l u + Y_l Y_p u\right)\left(\partial_{x_{n+1}} u\right)^2 \,,
\end{aligned}$$

$$\begin{aligned}
X_l Y_p u + Y_p X_l u - Y_l X_p u - X_p Y_l u &= \left(\partial_{x_l} + a_l \partial_{x_{n+1}}\right)\left(\partial_{y_p} + b_p \partial_{x_{n+1}}\right) u \\
&\quad + \left(\partial_{y_p} + b_p \partial_{x_{n+1}}\right)\left(\partial_{x_l} + a_l \partial_{x_{n+1}}\right) u - \left(\partial_{y_l} + b_l \partial_{x_{n+1}}\right) \cdot \\
&\quad \cdot \left(\partial_{x_p} + a_p \partial_{x_{n+1}}\right) u - \left(\partial_{x_p} + a_p \partial_{x_{n+1}}\right)\left(\partial_{y_l} - b_l \partial_{x_{n+1}}\right) u \\
&= 2\,\mathrm{Im}\left(A_{l\overline{p}}(u)\right) + \left(X_l b_p + Y_p a_l - Y_l a_p - X_p b_l +\right)\partial_{x_{n+1}} u \\
&= 2\,\mathrm{Im}\left(A_{l\overline{p}}(u)\right) - \left(-X_l X_p u + Y_p Y_l u - Y_l Y_p u + X_p X_l u\right)\partial_{x_{n+1}} u \\
&= 2\,\mathrm{Im}\left(A_{l\overline{p}}(u)\right) - \left(\left[X_p, X_l\right] u + \left[Y_p, Y_l\right] u\right)\partial_{x_{n+1}} u \\
&= 2\,\mathrm{Im}\left(A_{l\overline{p}}(u)\right) - \left(X_p a_l - X_l a_p + Y_p b_l - Y_l b_p\right)\left(\partial_{x_{n+1}} u\right)^2 \\
&= 2\,\mathrm{Im}\left(A_{l\overline{p}}(u)\right) - \left(X_l Y_p u + Y_p X_l u - X_p Y_l u - Y_l X_p u\right)\left(\partial_{x_{n+1}} u\right)^2 \,. \qquad \square
\end{aligned}$$

In order to handle the right-hand side in (1.6), we prove the following.

Lemma 2.3. *There exists a smooth positive function K such that* (1.16) *holds.*

Proof. By (2.2) and (1.9)

$$\partial_{x_l} u = X_l u - a_l \partial_{x_{n+1}} u = X_l u + \left(\partial_{x_{n+1}} u\right) Y_l u \,,$$
$$\partial_{y_l} u = Y_l u - b_l \partial_{x_{n+1}} u = Y_l u - \left(\partial_{x_{n+1}} u\right) X_l u \,.$$

Hence, it is enough to take

$$K\left(\cdot, u, Zu, \partial_{x_{n+1}} u\right) = \frac{q\left(\cdot, u, Xu + u_{x_{n+1}} Yu, Yu - u_{x_{n+1}} Xu, \partial_{x_{n+1}} u\right)}{\left(1 + \left(\partial_{x_{n+1}} u\right)^2\right)^n} \,. \qquad \square$$

Write Equation (1.6) as in (1.18), and define the vector fields Z as in (1.12) with $a = a(Du)$, $b = b(Du)$ as in (1.4).

Lemma 2.4. *For every $n \in \mathbb{N}$*

$$\mathcal{H}\left(Z^2 u\right) = \frac{1}{n} \sum_{m,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{mj}} \left(Z^2 u\right) Z_m Z_j u \,, \tag{2.3}$$

and if u is strictly Levi convex

$$\left(\frac{\partial \mathcal{H}}{\partial z_{mj}} \left(Z^2 u\right)\right)_{m,j} > 0 \,.$$

Proof. Denote by $B_{l\bar{p}}$ the cofactor of the element $\mathcal{H}_{l\bar{p}}$ of the matrix $(\mathcal{H}_{l\bar{p}})_{l,p=1}^n$. Then, by the determinant's derivative formula

$$\frac{\partial \mathcal{H}}{\partial z_{mj}} \left(Z^2 u\right) = \sum_{l,p=1}^{n} \frac{\partial \mathcal{H}_{l\bar{p}}}{\partial z_{mj}} \left(Z^2 u\right) B_{l\bar{p}} \left(Z^2 u\right)$$

and by (1.13) for $l \le p$,

$$2 \frac{\partial \mathcal{H}_{l\bar{p}}}{\partial z_{mj}} = \begin{cases} 1 & m = 2l-1,\ j = 2p-1 \\ 1 & m = 2l,\ j = 2p \\ i & m = 2l-1,\ j = 2p \\ -i & m = 2l,\ j = 2p-1 \\ 0 & \text{otherwise} \,. \end{cases}$$

Hence,

$$\begin{aligned} \sum_{m,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{mj}} \left(Z^2 u\right) Z_m Z_j u = {} & \sum_{l,p=1}^{n} \frac{1}{2} \Big(Z_{2l-1} Z_{2p-1} u + Z_{2p-1} Z_{2l-1} u \\ & + Z_{2l} Z_{2p} u + Z_{2p} Z_{2l} u + i \Big(Z_{2l-1} Z_{2p} u + Z_{2p} Z_{2l-1} u \\ & - Z_{2l} Z_{2p-1} u - Z_{2p-1} Z_{2l} u \Big)\Big) B_{l\bar{p}} \left(Z^2 u\right) \\ = {} & \sum_{l,p=1}^{n} \mathcal{H}_{l\bar{p}} \left(Z^2 u\right) B_{l\bar{p}} \left(Z^2 u\right) = n \mathcal{H}\left(Z^2 u\right) \,. \end{aligned} \tag{2.4}$$

Moreover, for every $\xi = (\xi_1, \dots, \xi_{2n}) \in \mathbb{R}^{2n} \setminus \{0\}$ set $\eta = (\eta_1, \dots, \eta_n)$ with $\eta_l = \xi_{2l-1} + i\xi_{2l}$ for every $l = 1, \dots, n$. Hence,

$$\begin{aligned}\sum_{m,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{mj}}(Z^2u)\xi_m\xi_j &= \sum_{l,p=1}^{n} B_{l\bar{p}}(Z^2u)\Big(\xi_{2l-1}\xi_{2p-1} + \xi_{2l}\xi_{2p} \\ &\quad + i\Big(\xi_{2l-1}\xi_{2p} - \xi_{2l}\xi_{2p-1}\Big)\Big) \\ &= \sum_{l,p=1}^{n} B_{l\bar{p}}(Z^2u)\bar{\eta}_l\eta_p \,.\end{aligned}$$

If u is strictly Levi convex then the hermitian form $(\mathcal{H}_{l\bar{p}})_{l,p=1}^n$ is positive definite and, by denoting $(\mathcal{H}^{l\bar{p}})_{l,p=1}^n$ its inverse matrix, we get

$$\sum_{l,p=1}^{n} B_{l\bar{p}}(Z^2u)\bar{\eta}_l\eta_p = \mathcal{H}(Z^2u)\sum_{l,p=1}^{n} \mathcal{H}^{l\bar{p}}(Z^2u)\eta_l\bar{\eta}_p \,.$$

Hence, the real form

$$\sum_{m,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{mj}}(Z^2u)\xi_m\xi_j > 0$$

for every $\xi \in \mathbb{R}^{2n} \setminus \{0\}$. □

Lemma 2.5. *If u is strictly Levi convex in Ω then the vector fields*

$$Z_1, \dots, Z_{2n}, \big[Z_1, Z_2\big]$$

are linearly independent at every point.

Proof. We have

$$\big[Z_{2l-1}, Z_{2l}\big] = \big[X_l, Y_l\big] = \big(X_l b_l - Y_l a_l\big)\partial_{x_{n+1}}, \qquad l = 1, \dots, n \,. \tag{2.5}$$

By (2.2)

$$\big(X_l b_l - Y_l a_l\big) = \big(X_l^2 u + Y_l^2 u\big) \,.$$

Since u is strictly Levi convex in Ω, then

$$A_{l\bar{l}}(u) > 0, \quad l = 1, \dots, n \tag{2.6}$$

and by (1.10) $X_l^2 u + Y_l^2 u > 0$. We can compute the determinant of the $(2n+1) \times (2n+1)$ real matrix whose columns are the coefficients of the vector fields $Z_1, \dots, Z_{2n}, [Z_1, Z_2]$ and by (1.9), (1.10), (2.6), and (2.5), we get

$$\det\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ a_1 & b_1 & \dots & a_n & b_n & \big(X_1^2u + Y_1^2u\big) \end{pmatrix} = \big(X_1^2u + Y_1^2u\big) \neq 0 \,. \qquad \square$$

3. $C_Z^{2,\alpha}$ regularity of Du

In this section we first introduce some classes $C_Z^{m,\alpha}$ of Hölder continuous functions naturally arising from the geometry of the problem. We then prove that the Euclidean gradient of a strictly Levi convex solution is in $C^{2,\beta}_{Z,\mathrm{loc}}$ by also using the a priori estimates proved by the authors in [21].

Let us introduce some notations. For every $l = 1, \dots, n$ let us define the first-order vector fields Z_l as in (1.12) with coefficients $a, b \in C^{1,\alpha}(\Omega)$. Moreover, let us assume that the vector fields $Z_1, \dots, Z_{2n}, [Z_1, Z_2]$ are linearly independent at every point and span $\mathbb{R}^{2n+1}$.

If the coefficients of the vector fields were smooth, then the linear operator H in (1.23) would satisfy Hörmander's condition of hypoellipticity. In our context, the coefficients are only $C^{1,\alpha}(\Omega)$. However, for every $\xi, \xi_0 \in \Omega$ there exists an absolutely continuous mapping $\gamma : [0, 1] \to \mathbb{R}^{2n+1}$, which is a piecewise integral curve of the vector fields Z introduced in (1.12), which connects ξ_0 and ξ. Then there exists a Carnot–Carathéodory distance $d_Z(\xi, \xi_0)$ naturally associated to the geometry of the problem (see for example the distance ϱ_4 defined in [22, p. 113]). Precisely, if $C(\delta)$ denotes the class of absolutely continuous mappings $\varphi : [0, 1] \to \Omega$ which almost everywhere satisfy $\varphi'(t) = \sum_{j=1}^{2n} \alpha_j(t) Z_j(\varphi(t))$ with $|\alpha_j(t)| < \delta$, define

$$d_Z(\xi_0, \xi) = \inf \left\{ \delta > 0 \,:\, \exists\, \varphi \in C(\delta) \text{ such that } \varphi(0) = \xi_0,\ \varphi(1) = \xi \right\}. \tag{3.1}$$

The fact that d_Z is finite follows because the commutators of the vector fields Z span $\mathbb{R}^{2n+1}$ at every point. This was first proved by Carathéodory for smooth vector fields; for vector fields with $C^{1,\alpha}$ coefficients, the proof is contained in [4].

We now define the class of Hölder continuous functions in terms of d_Z: for $0 < \alpha < 1$

$$C_Z^{\alpha}(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \text{ s.t. there exists a constant } c > 0 : \\ |v(\xi) - v(\xi_0)| \le c\, d_Z^{\alpha}(\xi, \xi_0) \text{ for all } \xi, \xi_0 \in \Omega \Big\}$$

and

$$C_Z^{1,\alpha}(\Omega) = \left\{ v \in C_Z^{\alpha}(\Omega) \,:\, \exists\, Z_j v \in C_Z^{\alpha}(\Omega) \quad \forall\, j = 1, \dots, 2n \right\}.$$

If the coefficients $a, b \in C_Z^{m-1,\alpha}(\Omega)$, $m \ge 2$, we define

$$C_Z^{m,\alpha}(\Omega) = \left\{ v \in C_Z^{m-1,\alpha}(\Omega) \,:\, Z_j v \in C_Z^{m-1,\alpha}(\Omega) \quad \forall\, j = 1, \dots, 2n \right\}. \tag{3.2}$$

Obviously (see [11]),

$$C^{m,\alpha}(\Omega) \subset C_Z^{m,\alpha}(\Omega) \subset C^{m/2,\alpha/2}(\Omega).$$

For every $m \ge 0$ we also define spaces of locally Hölder continuous functions:

$$C^{m,\alpha}_{Z,\mathrm{loc}}(\Omega) = \left\{ v : \Omega \to \mathbb{R} \,:\, v \in C_Z^{m,\alpha}(\Omega') \quad \forall\, \Omega' \subset\subset \Omega \right\}.$$

If $v \in C_Z^{\alpha}(\Omega)$ we define

$$[v]^Z_{\alpha;\Omega} = \sup_{\xi,\zeta \in \Omega} \frac{|v(\xi) - v(\zeta)|}{d_Z^{\alpha}(\xi, \zeta)}.$$

Denote by

$$Z^I = Z_{i_1} Z_{i_2} \cdots Z_{i_m},$$

where

$$I = (i_1, \dots, i_m) \tag{3.3}$$

is a multi-index of length $|I| = m$. If $v \in C_Z^{m,\alpha}(\Omega)$, with $m = 0, 1, 2, \dots$, and $0 < \alpha < 1$ we define the seminorm

$$[v]^Z_{m,\alpha;\Omega} = \sup_{|I|=m} \left[Z^I v\right]^Z_{\alpha;\Omega} ,$$

and the norms

$$|v|^Z_{m;\Omega} = \sum_{j=0}^{m} \left(\sup_{|I|=j} \sup_{\Omega} \left|Z^I v\right| \right) ,$$

$$|v|^Z_{m,\alpha;\Omega} = |v|^Z_{m;\Omega} + [v]^Z_{m,\alpha;\Omega} .$$

The following a priori estimates have been proved in [21] for the linear operator in (1.23).

Proposition 3.1. *Let $h_{ij}, \lambda \in C_Z^{\alpha}(\Omega)$, $a, b \in C_Z^{1,\alpha}(\Omega)$ and $v \in C_Z^{2,\alpha}(\Omega)$ be a solution of equation $Hv = f \in C_Z^{\alpha}(\Omega)$ with H as in (1.23). For every $\Omega' \subset\subset \Omega$ with $d_Z(\Omega', \partial\Omega) \geq \delta > 0$, there is a positive constant c such that for every $\beta \in (0, \alpha)$*

$$\delta |Zv|^Z_{0;\Omega'} + \delta^2 \left|Z^2 v\right|^Z_{0;\Omega'} + \delta^{2+\beta} \left[Z^2 v\right]^Z_{\beta;\Omega'} \leq c \left(\sup_{\Omega} |v| + |f|^Z_{0,\alpha;\Omega} \right) \tag{3.4}$$

where c depends only on the constant M in (1.24), on $|h_{ij}|^Z_{0,\alpha;\Omega}$, $|\lambda|^Z_{0,\alpha;\Omega}$, $|a|^Z_{1,\alpha;\Omega}$, $|b|^Z_{1,\alpha;\Omega}$ as well as on $n, \alpha, \delta, \Omega$.

For all B and B' in Ω such that $B' \subset B \subset\subset \Omega$ we define $h_0 = d_Z(B', \partial B) > 0$, and for every $h \in \mathbb{R}$ such that $0 < |h|^{1/2} < h_0$ we define

$$w_h(\xi) = \Delta_h^j u = \frac{u(\xi + he_j) - u(\xi)}{h}$$

with e_j the unit coordinate vector in $\mathbb{R}^{2n+1}$ in the j direction, $j = 1, \dots, 2n + 1$.

Lemma 3.2. *The function w_h is a solution of*

$$H_u w_h = F_h$$

with

$$H_u = \sum_{m,j=1}^{2n} a_{mj} Z_m Z_j - \chi \, \partial_{2n+1}$$

for suitable Hölder-continuous coefficients a_{mj}, χ.

Proof. By applying the difference quotient Δ_h^j to both sides of Equation (1.18) we get

$$\begin{aligned}
\Delta_h^j K(\xi, u(\xi), Zu(\xi), \partial_{2n+1} u(\xi)) &= \frac{\mathcal{H}\big(Z^2 u\big)(\xi + he_j) - \mathcal{H}\big(Z^2 u\big)(\xi)}{h} \\
&= \frac{1}{h} \int_0^1 \frac{d}{d\theta} \Big(\mathcal{H}\big(\theta\big(Z^2 u\big)(\xi + he_j) + (1 - \theta)\big(Z^2 u\big)(\xi)\big) \Big) \, d\theta \\
&= \sum_{m,k=1}^{2n} \int_0^1 \frac{\partial \mathcal{H}}{\partial z_{mk}} \Big(\theta\big(Z^2 u\big)(\xi + he_j) + (1 - \theta)\big(Z^2 u\big)(\xi) \Big) \, d\theta \\
&\quad \cdot \frac{1}{h} \Big((Z_m Z_k u)(\xi + he_j) - (Z_m Z_k u)(\xi) \Big) \\
&= \sum_{l,p=1}^{n} \int_0^1 B_{l\bar{p}} \Big(\theta\big(Z^2 u\big)(\xi + he_j) + (1 - \theta)\big(Z^2 u\big)(\xi) \Big) \, d\theta \, \Delta_h^j \mathcal{H}_{l\bar{p}}(\xi) .
\end{aligned}$$

The expression of $\Delta_h^j \mathcal{H}_{l\bar{p}}(\xi)$ is very complicated because of the fully non linearity of the coefficients $\mathcal{H}_{l\bar{p}}$. By (1.10) and (1.3)

$$\begin{aligned}
\Delta_h^j \mathcal{H}_{l\bar{p}}(\xi) &= \Delta_h^j\big(\big(1+u_{2n+1}^2\big)^{-2} A_{l\bar{p}}(u)\big)(\xi)\\
&= \Delta_h^j\Big(\big(1+u_{2n+1}^2\big)^{-1}\Big(\partial_{x_l x_p}u+\partial_{y_l y_p}u+a_l\partial_{x_p x_{n+1}}u+a_p\partial_{x_l x_{n+1}}u\\
&\quad+b_l\partial_{y_p x_{n+1}}u+b_p\partial_{y_l x_{n+1}}u+\big(a_l a_p+b_l b_p\big)\partial_{x_{n+1}}^2 u\Big)\\
&\quad+i\Big(\partial_{x_l y_p}u-\partial_{x_p y_l}u-a_p\partial_{y_l x_{n+1}}u+a_l\partial_{y_p x_{n+1}}u\\
&\quad+b_p\partial_{x_l x_{n+1}}u-b_l\partial_{x_p x_{n+1}}u+\big(b_p a_l-b_l a_p\big)\partial_{x_{n+1}}^2 u\Big)\Big)(\xi)\\
&= \Delta_h^j\big(1+u_{2n+1}^2\big)^{-1}(\xi)\Big(\Big(\partial_{x_l x_p}u+\partial_{y_l y_p}u+a_l\partial_{x_p x_{n+1}}u\\
&\quad+a_p\partial_{x_l x_{n+1}}u+b_l\partial_{y_p x_{n+1}}u+b_p\partial_{y_l x_{n+1}}u+\big(a_l a_p+b_l b_p\big)\partial_{x_{n+1}}^2 u\Big)\\
&\quad+i\Big(\partial_{x_l y_p}u-\partial_{x_p y_l}u-a_p\partial_{y_l x_{n+1}}u+a_l\partial_{y_p x_{n+1}}u\\
&\quad+b_p\partial_{x_l x_{n+1}}u-b_l\partial_{x_p x_{n+1}}u+\big(b_p a_l-b_l a_p\big)\partial_{x_{n+1}}^2 u\Big)\Big)(\xi+he_j)\\
&\quad+\big(1+u_{2n+1}^2\big)^{-1}(\xi)\Big(\Big(\Delta_h^j a_l(\xi)\big(\partial_{x_p x_{n+1}}u\big)\big(\xi+he_j\big)\\
&\quad+\Delta_h^j a_p(\xi)\big(\partial_{x_l x_{n+1}}u\big)\big(\xi+he_j\big)+\Delta_h^j b_l(\xi)\big(\partial_{y_p x_{n+1}}u\big)\big(\xi+he_j\big)\\
&\quad+\Delta_h^j b_p(\xi)\big(\partial_{y_l x_{n+1}}u\big)\big(\xi+he_j\big)\\
&\quad+\Delta_h^j\big(a_l a_p+b_l b_p\big)(\xi)\big(\partial_{x_{n+1}}^2 u\big)(\xi+he_j)\Big)\\
&\quad+i\Big(-\Delta_h^j a_p(\xi)\big(\partial_{y_l x_{n+1}}u\big)\big(\xi+he_j\big)+\Delta_h^j a_l(\xi)\big(\partial_{y_p x_{n+1}}u\big)\big(\xi+he_j\big)\\
&\quad+\Delta_h^j b_p(\xi)\big(\partial_{x_l x_{n+1}}u\big)\big(\xi+he_j\big)-\Delta_h^j b_l(\xi)\big(\partial_{x_p x_{n+1}}u\big)\big(\xi+he_j\big)\\
&\quad+\Delta_h^j\big(b_p a_l-b_l a_p\big)(\xi)\big(\partial_{x_{n+1}}^2 u\big)\big(\xi+he_j\big)\Big)\\
&\quad+\big(1+u_{2n+1}^2\big)^{-1}(\xi)\Big(\Big(\partial_{x_l x_p}w_h+\partial_{y_l y_p}w_h+a_l(\nabla u)\partial_{x_p x_{n+1}}w_h\\
&\quad+a_p(\nabla u)\partial_{x_l x_{n+1}}w_h+b_l(\nabla u)\partial_{y_p x_{n+1}}w_h+b_p(\nabla u)\partial_{y_l x_{n+1}}w_h\\
&\quad+\big(a_l a_p+b_l b_p\big)(\nabla u)\partial_{x_{n+1}}^2 w_h\Big)+i\Big(\partial_{x_l y_p}w_h-\partial_{x_p y_l}w_h\\
&\quad-a_p(\nabla u)\partial_{y_l x_{n+1}}w_h+a_l(\nabla u)\partial_{y_p x_{n+1}}w_h+b_p(\nabla u)\partial_{x_l x_{n+1}}w_h\\
&\quad-b_l(\nabla u)\partial_{x_p x_{n+1}}w_h+\big(b_p a_l-b_l a_p\big)(\nabla u)\partial_{x_{n+1}}^2 w_h\Big)\Big)(\xi)\,.
\end{aligned}$$

The last four lines in the previous equality are

$$\begin{aligned}
\frac{A_{l\bar{p}}(w_h)}{\big(1+u_{2n+1}^2\big)}(\xi) &= \frac{1}{2}\big(1+u_{2n+1}^2\big)^{-1}(\xi)\Big(X_l X_p+X_p X_l+Y_l Y_p+Y_l Y_p\\
&\quad-\big(X_l a_p+X_p a_l+Y_l b_p+Y_p b_l\big)\partial_{x_{n+1}}+i\Big(X_l Y_p+Y_p X_l\\
&\quad-X_p Y_l-Y_l X_p-\big(X_l b_p+Y_p a_l-X_p b_l-Y_l a_p\big)\partial_{x_{n+1}}\Big)\Big)w_h(\xi)\,.
\end{aligned}$$

Then, by building again the operator, the thesis follows with

$$\begin{aligned}
a_{mk} &= \big(1+u_{2n+1}^2\big)\int_0^1 \frac{\partial\mathcal{H}}{\partial z_{mk}}\Big(\theta\big(Z^2u\big)(\xi+he_j)+(1-\theta)\big(Z^2u\big)(\xi)\Big)\,d\theta\,,\\
2\chi &= \sum_{l,p=1}^{n}\int_0^1 B_{l\bar{p}}\Big(\theta\big(Z^2u\big)(\xi+he_j)+(1-\theta)\big(Z^2u\big)(\xi)\Big)\,d\theta\cdot\big(1+u_{2n+1}^2\big)^{-1}\\
&\quad\cdot\Big(\big(X_l a_p+X_p a_l+Y_l b_p+Y_p b_l\big)+i\big(X_l b_p+Y_p a_l-X_p b_l-Y_l a_p\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
F_h(\xi) \;=\;& \Delta_h^j K(\xi, u(\xi), Zu(\xi), \partial_{2n+1}u(\xi)) \\
& - \sum_{l,p=1}^{n} \int_0^1 B_{l\bar{p}}\Big(\theta\big(Z^2u\big)\big(\xi + he_j\big) + (1-\theta)\big(Z^2u\big)(\xi)\Big)\, d\theta \\
& \cdot \bigg(\Delta_h^j\big(1 + u_{2n+1}^2\big)^{-1}(\xi)\bigg(\bigg(\partial_{x_l x_p}u + \partial_{y_l y_p}u + a_l\partial_{x_p x_{n+1}}u \\
& + a_p\partial_{x_l x_{n+1}}u + b_l\partial_{y_p x_{n+1}}u + b_p\partial_{y_l x_{n+1}}u + \big(a_l a_p + b_l b_p\big)\partial_{x_{n+1}}^2 u\bigg) \\
& + i\Big(\partial_{x_l y_p}u - \partial_{x_p y_l}u - a_p\partial_{y_l x_{n+1}}u + a_l\partial_{y_p x_{n+1}}u \\
& + b_p\partial_{x_l x_{n+1}}u - b_l\partial_{x_p x_{n+1}}u + \big(b_p a_l - b_l a_p\big)\partial_{x_{n+1}}^2 u\Big)\bigg)\big(\xi + he_j\big) \\
& + \big(1 + u_{2n+1}^2\big)^{-1}(\xi)\Big(\Big(\Delta_h^j a_l(\xi)\big(\partial_{x_p x_{n+1}}u\big)\big(\xi + he_j\big) \\
& + \Delta_h^j a_p(\xi)\big(\partial_{x_l x_{n+1}}u\big)\big(\xi + he_j\big) + \Delta_h^j b_l(\xi)\big(\partial_{y_p x_{n+1}}u\big)\big(\xi + he_j\big) \\
& + \Delta_h^j b_p(\xi)\big(\partial_{y_l x_{n+1}}u\big)\big(\xi + he_j\big) \\
& + \Delta_h^j\big(a_l a_p + b_l b_p\big)(\xi)\big(\partial_{x_{n+1}}^2 u\big)\big(\xi + he_j\big)\Big) \\
& + i\Big(- \Delta_h^j a_p(\xi)\big(\partial_{y_l x_{n+1}}u\big)\big(\xi + he_j\big) + \Delta_h^j a_l(\xi)\big(\partial_{y_p x_{n+1}}u\big)\big(\xi + he_j\big) \\
& + \Delta_h^j b_p(\xi)\big(\partial_{x_l x_{n+1}}u\big)\big(\xi + he_j\big) - \Delta_h^j b_l(\xi)\big(\partial_{x_p x_{n+1}}u\big)\big(\xi + he_j\big) \\
& + \Delta_h^j\big(b_p a_l - b_l a_p\big)(\xi)\big(\partial_{x_{n+1}}^2 u\big)\big(\xi + he_j\big)\Big)\Big)\bigg) . \qquad \square
\end{aligned}
$$

For a strictly Levi convex u, Lemma 2.4 ensures the existence of a positive constant M such that

$$\sum_{i,j=1}^{2n} a_{ij}\eta_i\eta_j \geq M \sum_{i=1}^{2n} \eta_i^2, \quad \forall \eta = \big(\eta_1, \ldots, \eta_{2n}\big) \in \mathbb{R}^{2n} .$$

Moreover, $|F_h|_{0,\alpha;B'}$, $|a_{ij}|_{0,\alpha;B}$ are bounded by a positive constant independent of h. Hence, we can apply Proposition 3.1 to w_h, and we may assert that for all $B'' \subset\subset B'$ there exists a subsequence of $Z_i Z_l w_h$ which uniformly converges in $C_Z^{\beta}(B'')$ to $Z_i Z_l D_j u$ for every $\beta < \alpha$, for all $i, l = 1, \ldots, 2n$, and $j = 1, \ldots, 2n+1$. In particular we get the following.

Proposition 3.3. *If $u \in C^{2,\alpha}(\Omega)$ is a strictly Levi convex solution to* (1.6), *then $Du \in C^{2,\beta}_{Z,\mathrm{loc}}(\Omega)$ for every $\beta \in (0, \alpha)$.*

4. Smooth regularity

In this section we formally differentiate the nonlinear Equation (1.6), which we write as in (1.18). Then, by mean of a bootstrap argument, we prove Theorem 1.3.

Let H as in (1.23), with

$$h_{ij} = \frac{\partial \mathcal{H}}{\partial z_{ij}}\big(Z^2 u\big) ,$$

λ as in (1.27) and K defined as in (1.16).

Proof of Proposition 1.5. In this proof we denote by $\partial_{2n+1} = \partial_{x_{n+1}}$ and by $v = (v_1, \dots, v_{2n+1})$ the function in (1.25). Let us differentiate Equation (1.18) with respect to Z_m. By the determinant's derivative formula and by using the notations of (2.4)

$$Z_m\big(K\big(\xi, u, Zu, \partial_{2n+1}u\big)\big) = Z_m\big(\mathcal{H}\big(Z^2u\big)\big) = \sum_{l,p=1}^{n} Z_m\mathcal{H}_{l\bar{p}}\big(Z^2u\big)B_{l\bar{p}}\big(Z^2u\big)\ . \tag{4.1}$$

Let us compute separately $Z_qZ_jZ_mu$. Define

$$\omega_{2j} = b_j, \quad \omega_{2j-1} = a_j, \qquad \forall\, j = 1, \dots, n\ ,$$
$$Z_j^* = \begin{cases} -Z_{j+1} & \text{if j is odd}\ ,\\ Z_{j-1} & \text{if j is even}\ ,\end{cases}$$

and $\omega_j^* = -Z_ju$. Then, by taking into account (2.2), we get $\omega_j = Z_j^*u$ and

$$\begin{aligned}
&Z_mZ_qZ_ju + \big(\partial_{2n+1}u\big)Z_m^*Z_qZ_ju = Z_qZ_jv_m + [Z_m, Z_q](Z_ju)\\
&\quad + Z_q([Z_m, Z_j]u) + \big(\partial_{2n+1}u\big)Z_m^*Z_qZ_ju\\
&= Z_qZ_jv_m + \big(Z_m\omega_q - Z_q\omega_m\big)\partial_{2n+1}(Z_ju)\\
&\quad + Z_q\big(\big(Z_m\omega_j - Z_j\omega_m\big)\partial_{2n+1}u\big) + \big(\partial_{2n+1}u\big)Z_m^*Z_qZ_ju\\
&= Z_qZ_jv_m + \big(Z_mZ_q^*u - Z_qZ_m^*u\big)\partial_{2n+1}(Z_ju)\\
&\quad + Z_q\big(\big(Z_mZ_j^*u - Z_jZ_m^*u\big)\partial_{2n+1}u\big) + \big(\partial_{2n+1}u\big)Z_m^*Z_qZ_ju\\
&= Z_qZ_jv_m + \big(Z_mZ_q^*u - Z_qZ_m^*u\big)\partial_{2n+1}(Z_ju) + \big(Z_mZ_j^*u\\
&\quad - Z_jZ_m^*u\big)Z_q\partial_{2n+1}u + \big(Z_qZ_mZ_j^*u + Z_m^*Z_qZ_ju - Z_qZ_jZ_m^*u\big)\partial_{2n+1}u\\
&= Z_qZ_jv_m\Big(1 + \big(\partial_{2n+1}u\big)^2\Big) + g_{m,q,j}(v, Zv) + \Big(Z_qZ_mZ_j^*u\\
&\quad + \big(Z_qZ_m^*Z_j^*u + g^*_{m,q,j}(v, Zv)\big)\big(\partial_{2n+1}u\big)\Big)\partial_{2n+1}u\\
&= Z_qZ_jv_m\Big(1 + \big(\partial_{2n+1}u\big)^2\Big) + g_{m,q,j}(v, Zv) + \Big(Z_mZ_qZ_j^*u\\
&\quad + \big[Z_q, Z_m\big]Z_j^*u + \big(Z_m^*Z_qZ_j^*u + \big[Z_q, Z_m^*\big]Z_j^*u\\
&\quad + g^*_{m,q,j}(v, Zv)\big)\big(\partial_{2n+1}u\big)\Big)\partial_{2n+1}u\\
&= Z_qZ_jv_m\Big(1 + \big(\partial_{2n+1}u\big)^2\Big) + g_{m,q,j}(v, Zv)\\
&\quad + \Big(Z_mZ_qZ_j^*u + \big(Z_qZ_m^*u - Z_mZ_q^*u\big)\partial_{2n+1}Z_j^*u\\
&\quad + \Big(Z_m^*Z_qZ_j^*u + \big(-Z_qZ_mu - Z_m^*Z_q^*u\big)\partial_{2n+1}Z_j^*u\\
&\quad + g^*_{m,q,j}(v, Zv)\Big)\big(\partial_{2n+1}u\big)\Big)\partial_{2n+1}u\ ,
\end{aligned}$$

with

$$g_{m,q,j}(v, Zv) = \big(Z_mZ_q^*u - Z_qZ_m^*u\big)\partial_{2n+1}(Z_ju) + \big(Z_mZ_j^*u - Z_jZ_m^*u\big)Z_q\partial_{2n+1}u\ ,$$

and

$$g^*_{m,q,j}(v, Zv) = \big(Z_m^*Z_q^*u + Z_qZ_mu\big)\partial_{2n+1}\big(Z_ju\big) + \big(Z_m^*Z_j^*u + Z_jZ_mu\big)Z_q\partial_{2n+1}u\ .$$

Hence, for every $m = 1, \dots, 2n$

$$Z_q Z_j v_m = \Big(1 + \big(\partial_{2n+1} u\big)^2\Big)^{-1} \Big(\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big) Z_q Z_j u - \big(\partial_{2n+1} u\big)\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big) Z_q Z_j^* u + \varphi_{m,q,j}(v, Zv)\Big) ,$$

with $v_m = Z_m u$ for every $m = 1, \dots, 2n$,

$$\varphi_{m,q,j}(v, Zv) = -g_{m,q,j}(v, Zv) - \Big(\big(Z_q Z_m^* u - Z_m Z_q^* u\big)\partial_{2n+1} Z_j^* u + \Big(\big(- Z_q Z_m u - Z_m^* Z_q^* u\big)\partial_{2n+1} Z_j^* u + g^*_{m,q,j}(v, Zv)\Big)\big(\partial_{2n+1} u\big)\Big)\partial_{2n+1} u \ .$$

Analogously,

$$Z_j^* Z_q^* v_m = \Big(1 + \big(\partial_{2n+1} u\big)^2\Big)^{-1} \Big(\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big) Z_j^* Z_q^* u + \big(\partial_{2n+1} u\big)\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big) Z_j^* Z_q u + f^*_{m,q,j}(v, Zv)\Big) ,$$

where $f^*_{m,q,j}(v, Zv)$ does not depend on the third derivatives of u. Hence, by (1.13)

$$\begin{aligned}
2\mathcal{H}_{l\bar{p}}\big(Z^2 v_m\big) &= \Big(Z_{2l-1} Z_{2p-1} + Z^*_{2p-1} Z^*_{2l-1} + Z_{2l} Z_{2p} + Z^*_{2p} Z^*_{2l} \\
&\quad + i\big(Z_{2l-1} Z_{2p} + Z^*_{2p} Z^*_{2l-1} - Z_{2l} Z_{2p-1} - Z^*_{2p-1} Z^*_{2l}\big)\Big) v_m \\
&= \Big(1 + \big(\partial_{2n+1} u\big)^2\Big)^{-1} \Big(2\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\mathcal{H}_{l\bar{p}}(u) \\
&\quad + \big(\partial_{2n+1} u\big)\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\Big(\big[Z^*_{2l-1}, Z_{2p-1}\big]u + \big[Z^*_{2l}, Z_{2p}\big]u \\
&\quad + i\big(\big[Z^*_{2l-1}, Z_{2p}\big]u - \big[Z^*_{2l}, Z_{2p-1}\big]u\big)\Big) + \varphi_{l\bar{p}}(v, Zv)\Big) \\
&= \Big(1 + \big(\partial_{2n+1} u\big)^2\Big)^{-1} \Big(2\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\mathcal{H}_{l\bar{p}}(u) \\
&\quad + \big(\partial_{2n+1} u\big)^2\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\Big(Z^*_{2l-1}\omega_{2p-1} - Z_{2p-1}\omega^*_{2l-1} \\
&\quad + Z^*_{2l}\omega_{2p} - Z_{2p}\omega^*_{2l} + i\Big(Z^*_{2l-1}\omega_{2p} - Z_{2p}\omega^*_{2l-1} \\
&\quad - Z^*_{2l}\omega_{2p-1} + Z_{2p-1}\omega^*_{2l}\Big)\Big) + \varphi_{l\bar{p}}(v, Zv)\Big) \\
&= \Big(1 + \big(\partial_{2n+1} u\big)^2\Big)^{-1} \Big(2\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\mathcal{H}_{l\bar{p}}(u) \\
&\quad + \big(\partial_{2n+1} u\big)^2\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\Big(Z^*_{2l-1} Z^*_{2p-1} u + Z_{2p-1} Z_{2l-1} u \\
&\quad + Z^*_{2l} Z^*_{2p} u + Z_{2p} Z_{2l} u + i\Big(Z^*_{2l-1} Z^*_{2p} u + Z_{2p} Z_{2l-1} u \\
&\quad - Z^*_{2l} Z^*_{2p-1} u - Z_{2p-1} Z_{2l} u\Big)\Big) + \varphi_{l\bar{p}}(v, Zv)\Big) \\
&= 2\big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big)\mathcal{H}_{l\bar{p}}(u) + \Big(1 + \big(\partial_{2n+1} u\big)^2\Big)^{-1} \varphi_{l\bar{p}}(v, Zv) \ .
\end{aligned}$$

By using the previous equality in (4.1) we get

$$\sum_{l,p=1}^{n} \mathcal{H}_{l\bar{p}}\big(Z^2 v_m\big) B_{l\bar{p}}\big(Z^2 u\big) = \big(Z_m + \big(\partial_{2n+1} u\big) Z_m^*\big) K\big(\xi, u, Zu, \partial_{2n+1} u\big) + \varphi(v, Zv) \ .$$

Let us now consider the derivative ∂_{2n+1}. We have:

$$\begin{aligned}\partial_{2n+1} Z_q Z_j u &= Z_q Z_j \partial_{2n+1} u + \left[\partial_{2n+1}, Z_q\right] Z_j u + Z_q \left[\partial_{2n+1}, Z_j\right] u \\ &= Z_q Z_j \partial_{2n+1} u + \partial_{2n+1}\omega_q \partial_{2n+1} Z_j u + Z_q \left(\partial_{2n+1}\omega_j \partial_{2n+1} u\right) \\ &= Z_q Z_j \partial_{2n+1} u + \partial_{2n+1} Z_q^* u \partial_{2n+1} Z_j u + Z_q \partial_{2n+1} u \partial_{2n+1} Z_j^* u \\ &\quad + Z_q \left(\partial_{2n+1} Z_j^* u\right) \partial_{2n+1} u \\ &= Z_q Z_j \partial_{2n+1} u + \partial_{2n+1}\left(Z_q Z_j^* u\right) \partial_{2n+1} u + \varphi(u)\end{aligned}$$

where

$$\varphi(u) = \partial_{2n+1} Z_q^* u \partial_{2n+1} Z_j u + Z_q \partial_{2n+1} u \partial_{2n+1} Z_j^* u + \left[Z_q, \partial_{2n+1}\right] Z_j^* u \partial_{2n+1} u\ .$$

In the same way we get:

$$\partial_{2n+1}\left(Z_j^* Z_q^* u\right) = Z_j^* Z_q^* \partial_{2n+1} u - \partial_{2n+1}\left(Z_j^* Z_q u\right) \partial_{2n+1} u + \varphi^*(u)$$

where

$$\varphi^*(u) = -\partial_{2n+1} Z_j u \partial_{2n+1} Z_q^* u - Z_j^* \partial_{2n+1} u \partial_{2n+1} Z_q u - \left[Z_j^*, \partial_{2n+1}\right] Z_q u \partial_{2n+1} u\ .$$

Then

$$\begin{aligned}&\partial_{2n+1}\left(Z_q Z_j u + Z_j^* Z_q^* u\right) = Z_q Z_j \partial_{2n+1} u + Z_j^* Z_q^* \partial_{2n+1} u + \partial_{2n+1}\left(\left[Z_q, Z_j^*\right] u\right) \\ &\qquad \cdot \partial_{2n+1} u + \varphi(u) + \varphi^*(u) \\ &\quad = \left(Z_q Z_j + Z_j^* Z_q^*\right) \partial_{2n+1} u - \partial_{2n+1}\left(\left(Z_q Z_j u + Z_j^* Z_q^* u\right) \partial_{2n+1} u\right) \\ &\qquad \cdot \partial_{2n+1} u + \varphi(u) + \varphi^*(u) \\ &\quad = \left(1 + \left(\partial_{2n+1} u\right)^2\right)^{-1} \left(\left(Z_q Z_j + Z_j^* Z_q^*\right) \partial_{2n+1} u \right.\\ &\qquad \left. - \left(\left(Z_q Z_j u + Z_j^* Z_q^* u\right) \partial_{2n+1}^2 u\right) \partial_{2n+1} u + \varphi(u) + \varphi^*(u)\right)\ .\end{aligned}$$

Now recall that we put $v_{2n+1} = \arctan(\partial_{2n+1} u)$; then

$$Z_j v_{2n+1} = \frac{Z_j \partial_{2n+1} u}{1 + \left(\partial_{2n+1} u\right)^2}$$

and

$$Z_q Z_j v_{2n+1} = \frac{Z_q Z_j \partial_{2n+1} u}{1 + \left(\partial_{2n+1} u\right)^2} - 2\frac{\partial_{2n+1} u\, Z_j \partial_{2n+1} u\, Z_q \partial_{2n+1} u}{\left(1 + \left(\partial_{2n+1} u\right)^2\right)^2}\ .$$

Since

$$\begin{aligned}Z_j \partial_{2n+1} u &= \partial_{2n+1} Z_j u + \left[Z_j, \partial_{2n+1}\right] u \\ &= \partial_{2n+1} Z_j u - \partial_{2n+1}\omega_j \partial_{2n+1} u \\ &= \partial_{2n+1} Z_j u - \partial_{2n+1} u \partial_{2n+1} Z_j^* u\ , \\ Z_j^* \partial_{2n+1} u &= \partial_{2n+1} Z_j^* u + \partial_{2n+1} u \partial_{2n+1} Z_j^* u\ ,\end{aligned} \tag{4.2}$$

then

$$\begin{aligned}\partial_{2n+1}Z_j u &= \frac{Z_j\partial_{2n+1}u + \partial_{2n+1}u Z_j^*\partial_{2n+1}u}{1+\left(\partial_{2n+1}u\right)^2}\,,\\ \partial_{2n+1}Z_j^* u &= \frac{Z_j^*\partial_{2n+1}u - \partial_{2n+1}u Z_j\partial_{2n+1}u}{1+\left(\partial_{2n+1}u\right)^2}\,,\end{aligned}\tag{4.3}$$

and by using (4.2) and (4.3) we get

$$\begin{aligned}\varphi(u)+\varphi^*(u) &= \partial_{2n+1}Z_q^*u\partial_{2n+1}Z_ju + Z_q\partial_{2n+1}u\partial_{2n+1}Z_j^*u\\ &\quad+\left[Z_q,\partial_{2n+1}\right]Z_j^*u\partial_{2n+1}u - \partial_{2n+1}Z_ju\partial_{2n+1}Z_q^*u\\ &\quad- Z_j^*\partial_{2n+1}u\partial_{2n+1}Z_qu - \left[Z_j^*,\partial_{2n+1}\right]Z_qu\partial_{2n+1}u\\ &= Z_q\partial_{2n+1}u\partial_{2n+1}Z_j^*u - \partial_{2n+1}\omega_q\partial_{2n+1}Z_j^*u\partial_{2n+1}u\\ &\quad- Z_j^*\partial_{2n+1}u\partial_{2n+1}Z_qu + \partial_{2n+1}\omega_j^*\partial_{2n+1}Z_q^*u\partial_{2n+1}u\\ &= \left(Z_q\partial_{2n+1}u - \partial_{2n+1}u\partial_{2n+1}Z_q^*u\right)\partial_{2n+1}Z_j^*u\\ &\quad- \left(Z_j^*\partial_{2n+1}u + \partial_{2n+1}u\partial_{2n+1}Z_ju\right)\partial_{2n+1}Z_qu\\ &= -2\partial_{2n+1}u\left(\partial_{2n+1}Z_q^*u\partial_{2n+1}Z_j^*u + \partial_{2n+1}Z_ju\partial_{2n+1}Z_qu\right)\\ &= -2\partial_{2n+1}u\frac{Z_q^*\partial_{2n+1}uZ_j^*\partial_{2n+1}u + Z_j\partial_{2n+1}uZ_q\partial_{2n+1}u}{1+\left(\partial_{2n+1}u\right)^2}\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\left(Z_qZ_j + Z_j^*Z_q^*\right)v_{2n+1} - \partial_{2n+1}\left(Z_qZ_ju + Z_j^*Z_q^*u\right)\partial_{2n+1}v_{2n+1}\\ &\quad= \partial_{2n+1}\left(Z_qZ_ju + Z_j^*Z_q^*u\right)\end{aligned}$$

and

$$\mathcal{H}_{l\bar{p}}\left(Z^2v_{2n+1}\right) - \partial_{2n+1}u\left(\mathcal{H}_{l\bar{p}}\left(Z^2u\right)\right)\partial_{2n+1}v_{2n+1} = \partial_{2n+1}\left(\mathcal{H}_{l\bar{p}}\left(Z^2u\right)\right)\,.$$

So we get

$$\begin{aligned}&\sum_{l,p=1}^{n}\Big(\mathcal{H}_{l\bar{p}}\left(Z^2v_{2n+1}\right) - \partial_{2n+1}u\left(\mathcal{H}_{l\bar{p}}\left(Z^2u\right)\right)\partial_{2n+1}v_{2n+1}\Big)B_{l\bar{p}}\left(Z^2u\right)\\ &\quad= \partial_{2n+1}\left(\sum_{l,p=1}^{n}\mathcal{H}_{l\bar{p}}\left(Z^2u\right)B_{l\bar{p}}\left(Z^2u\right)\right)\\ &\quad= \partial_{2n+1}\Big(K\left(\cdot,u,Zu,\partial_{2n+1}u\right)\Big)\,.\end{aligned}$$

By (2.3) we get:

$$\begin{aligned}f_{2n+1}(\xi,u,v,Zv) = \sum_{i,j=1}^{2n}\frac{\partial\mathcal{H}}{\partial z_{ij}}\left(Z^2u\right)Z_iZ_j\left(v_{2n+1}\right) - \Big(&nK\partial_{2n+1}u\\ &+\frac{\partial K}{\partial u_{2n+1}}\Big(1+\left(\partial_{2n+1}u\right)^2\Big)\Big)\partial_{2n+1}v_{2n+1}\end{aligned}$$

that is

$$\left(\sum_{i,j=1}^{2n} \frac{\partial \mathcal{H}}{\partial z_{ij}}\big(Z^2 u\big) Z_i Z_j - \lambda \partial_{2n+1}\right) v_{2n+1} = f_{2n+1}(\xi, u, v, Zv) \; . \qquad \square$$

Proof of Theorem 1.3. Let us define H in terms of u as in (1.23), with

$$h_{ij} = \frac{\partial \mathcal{H}}{\partial z_{ij}}\big(Z^2 u\big) \; ,$$

$$\lambda = nK \partial_{x_{n+1}} u + \frac{\partial K}{\partial u_{x_{n+1}}}\Big(1 + \big(\partial_{x_{n+1}} u\big)^2\Big) \; ,$$

and K defined as in (1.16). As an immediate consequence of Proposition 3.3 the coefficients $a, b, \lambda \in C^{2,\beta}_{Z,\mathrm{loc}}(\Omega)$, while $\frac{\partial \mathcal{H}}{\partial z_{ij}}(Z^2 u) \in C^{1,\beta}_{Z,\mathrm{loc}}(\Omega)$. Moreover, by Proposition 1.5 the function

$$v = \big(v_1, \dots, v_{2n}, v_{2n+1}\big) = \big(Z_1 u, \dots, Z_{2n} u, \arctan u_{x_{n+1}}\big)$$

is a $C^{2,\beta}_{Z,\mathrm{loc}}(\Omega)$ solution to $Hv = f(\cdot, u, v, Zv)$ with $f = (f_1, \dots, f_{2n+1})$ a smooth function of its arguments. Since the right-hand side in (1.26) is of class $C^{1,\beta}_{Z,\mathrm{loc}}(\Omega)$, we apply Proposition 1.4 with $m = 2$ to get $v \in C^{3,\gamma}_{Z,\mathrm{loc}}(\Omega)$ for every $\gamma \in (0, \beta)$.

Then we conclude the proof by induction. Let us assume that the function v defined in (1.25) belongs to $C^{m,\alpha}_{Z,\mathrm{loc}}(\Omega)$ and prove that $v \in C^{m+1,\beta}_{Z,\mathrm{loc}}(\Omega)$ for every $\beta \in (0, \alpha)$. Indeed, $a, b, \lambda \in C^{m,\alpha}_{Z,\mathrm{loc}}(\Omega)$, $\frac{\partial \mathcal{H}}{\partial z_{ij}} \in C^{m-1,\alpha}_{Z,\mathrm{loc}}(\Omega)$ and v is a solution to (1.26) with right-hand side of class $C^{m-1,\alpha}_{Z,\mathrm{loc}}(\Omega)$. Hence, by Proposition 1.4, $v \in C^{m+1,\beta}_{Z,\mathrm{loc}}(\Omega)$ for every $\beta \in (0, \alpha)$ and Theorem 1.3 is proved. $\square$

Acknowledgements and comments

We would like to thank the anonymous referee for asking us to clarify the connection between Theorem 1.1 and the sharp boundary regularity of the solution of the Fefferman problem. We recall that, given a strictly pseudoconvex set D, the Fefferman problem

$$\begin{cases} (-1)^{n+1} \det \begin{pmatrix} \varphi & \varphi_{\bar{k}} \\ \varphi_j & \varphi_{j\bar{k}} \end{pmatrix} = 1 & \text{in } D \; , \\ \varphi = 0 & \text{on } \partial D \end{cases}$$

has a unique solution $\varphi \in C^{n+2-\varepsilon}(\overline{D})$, which is not smooth up to the boundary. Obviously, if φ is the solution of the Fefferman problem, then we can take $-\varphi$ as a defining function for ∂D. However, this is not in contrast with Theorem 1.1, which states that if k_M is smooth, then $M = \partial D$ is locally the graph of a smooth function.

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Dipartimento di Matematica, Università di Bologna Piazza Porta San Donato 5, 40126 Bologna, Italy
e-mail: montanar@dm.unibo.it

Dipartimento di Matematica, Università di Bologna Piazza Porta San Donato 5, 40126 Bologna, Italy
e-mail: lascia@dm.unibo.it

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