REAL HYPERSURFACES EVOLVING BY LEVI CURVATURE: SMOOTH REGULARITY OF SOLUTIONS TO THE PARABOLIC LEVI EQUATION

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Abstract

We prove, with a real analysis technique, the smooth regularity of classical solutions to a nonlinear degenerate parabolic PDE with initial data $C^{2,\alpha}$. This equation arises in the study of the geometric properties of the motion by the trace of the Levi form of a real hypersurface in $\mathbb{C}^2$ with Levi curvature different from zero at every point and which is locally the graph of a $C^{2,\alpha}$ function.

1 Introduction

In this note we prove a smooth regularity result for classical solutions to a degenerate parabolic equation which describes the motion by Levi curvature of a real hypersurface M_0 in $\mathbb{C}^2$. If M_0 is smooth we may expect that its evolution is described by a smooth function $\rho = \rho(z,t)$, i.e. $M_t = \{z \in \mathbb{C}^2 : \rho(z,t) = 0\}$. In this situation, we say that $\{M_t\}_{t\geq 0}$ is the evolution of M_0 by *Levi curvature* if for every non negative t the trajectory $z = z(t)$ starting from a point $p \in M_0$

[1]Investigation supported by University of Bologna. Founds for selected research topics.

is the solution of the problem

$$\begin{cases} \dot{z}(t) = k(z,t)\nu(z,t), \quad t > 0 \\ z(0) = p \end{cases} \tag{1}$$

where $\nu = \nu(z,t)$ is the inner unit normal vector field to M_t and

$$k = |\partial\rho|^{-3}(\rho_{z_1\bar{z}_1}\rho_{z_2}\rho_{\bar{z}_2} - \rho_{z_1\bar{z}_2}\rho_{z_2}\rho_{\bar{z}_1} - \rho_{z_2\bar{z}_1}\rho_{z_1}\rho_{\bar{z}_2} + \rho_{z_2\bar{z}_2}\rho_{z_1}\rho_{\bar{z}_1})$$

is the Levi curvature of the hypersurface $\rho = 0$ and $|\partial\rho|^2 = |\rho_{z_1}|^2 + |\rho_{z_2}|^2$.

This problem arises in complex analysis, in the study of several variable functions: indeed solutions to the evolution equation in (1) give us informations about the envelope of holomorphy of subsets of $\mathbb{C}^2$ (see [13], [14]).

For example, if M_0 is a sphere of radius $r(0)$ in $\mathbb{C}^2$, then M_t is a family of concentric spheres of radius

$$r(t) = \sqrt{r^2(0) - t},$$

which smoothly shrink down to a single point after a finite time. It is natural to aspect that this behaviour is typical of the evolution of a real hypersurface in $\mathbb{C}^2$ with positive Levi curvature, since this happened for example for the motion by classical mean curvature (see [6]). However the crucial difference between the parabolic problem which governs the evolution by Levi curvature and the evolution by classical mean curvature is that, even if M_0 is locally the graph of a smooth function, (1) is a "degenerate" parabolic problem

In a recent paper [5] Huisken and Klingenberg proved that if M_0 is smooth and closed, i.e. compact without boundary, then there exists a time interval $[0, t_0)$, $t_0 > 0$, such that M_0 has a smooth regular evolution $\{M_t\}_{0\le t<t_0}$. The smoothing behaviour of their solution heavily depends on the smoothness of the initial data. Moreover, by their proof it appears clear that for initial data in $C^{2,\alpha}$ a solution exists in $C^{2,\alpha}, 0 < \alpha < 1$. As they explicitly write in [5] further regularity of this solution, e.g. for strictly pseudoconvex initial data, is an open problem. In this paper we give a first positive response to this question.

Since $z(t) \in M_t$ for every $t > 0$, by differentiating with respect to t the identity $\rho(z(t), t) = 0$, we deduce that

$$\partial\rho \cdot \dot{z} + \partial_t\rho = 0.$$

If $\partial\rho \neq 0$, then ρ is a solution of the parabolic equation (see [12])

$$\rho_t = |\partial\rho|^{-2}(\rho_{z_1\bar{z}_1}\rho_{z_2}\rho_{\bar{z}_2} - \rho_{z_1\bar{z}_2}\rho_{z_2}\rho_{\bar{z}_1} - \rho_{z_2\bar{z}_1}\rho_{z_1}\rho_{\bar{z}_2} + \rho_{z_2\bar{z}_2}\rho_{z_1}\rho_{\bar{z}_1}).$$

If we introduce the complex coordinates $z_1 = (x_1, x_2)$, $z_2 = (x_3, v)$ and for any t the hypersurface M_t is the graph of a function u of class C^2

$$M_t := \{v - u(x_1, x_2, x_3, t) = 0\},$$

then the Levi curvature of M_t can be written in term of u as follows

$$k(x_1, x_2, x_3, t, u(x_1, x_2, x_3, t)) = -\frac{(1+u_{x_3}^2)\mathcal{L}u}{2(1+|\nabla u|^2)^{\frac{3}{2}}}, \tag{2}$$

where $\mathcal{L}$ is the second order partial differential operator

$$\mathcal{L} := \partial_{x_1x_1} + \partial_{x_2x_2} + 2a\partial_{x_1x_3} + 2b\partial_{x_2x_3} + (a^2+b^2)\partial_{x_3x_3}, \tag{3}$$

whose coefficients are smooth functions of the first derivatives of u with respect to the variables x_1, x_2, x_3

$$a = a(\nabla u) = \frac{u_{x_2} - u_{x_1}u_{x_3}}{1+u_{x_3}^2}, \quad b = b(\nabla u) = -\frac{u_{x_1} + u_{x_2}u_{x_3}}{1+u_{x_3}^2}. \tag{4}$$

Here we have denoted by ∇u the gradient of u with respect to (x_1, x_2, x_3) and by $u_{x_1}, u_{x_2}, u_{x_3}$ the partial derivatives of u with respect to x_1, x_2, x_3 respectively.

If we assume that M_0 is locally the graph of a $C^{2,\alpha}$ function $u_0 : \mathcal{O} \to \mathbb{R}$ and its Levi curvature k is different from zero at every point of the open set $\mathcal{O}$ of $\mathbb{R}^3$, then by the results in ([5],[13]) there exists $T > 0$ such that also $\{M_t\}_{0<t<T}$ is the graph of a $C^{2+\alpha,1+\alpha/2}$ function $u : \mathcal{O} \times [0,T) \to \mathbb{R}$. Here we have denoted by $C^{2+\alpha,1+\alpha/2}$ the set of continuous functions v with partial derivatives v_x, v_{xx}, v_t α-Hölder continuous with respect to x and $\alpha/2$-Hölder continuous with respect to t.

Since our problem is of local nature it is not restrictive to assume $T > 0$ such that $k(\xi, u(\xi)) \neq 0$ for all $\xi \in \mathcal{O} \times (0,T)$. Our main result is the following

Theorem 1.1 *Let $u_0 \in C^{2,\alpha}(\mathcal{O})$ and $\Omega = \mathcal{O} \times (0,T)$. If $u \in C^{2+\alpha,1+\alpha/2}(\mathcal{O} \times [0,T))$ is a classical solution to the parabolic problem*

$$\begin{cases} \partial_t u = \frac{(1+u_{x_3}^2)}{4(1+|\nabla u|^2)}\mathcal{L}u, \text{ in } \Omega \\ u(x,0) = u_0(x), \ x = (x_1,x_2,x_3) \in \mathcal{O}, \end{cases} \tag{5}$$

then $u \in C^\infty(\Omega)$.

Our result shows that the smoothing behaviour of the evolution does not depend on the initial data but on the parabolic equation in (5).

By our proof it will appear evident that the same smoothness result holds for the parabolic equation which governs the motion by Levi mean curvature of a real hypersurface in $\mathbb{C}^{n+1}$; in this case $\mathcal{L}u$ in (5) is the trace of the Levi form (see [2]).

Let us explicitly remark that if M_0 is pseudoconvex, then $k(x, u_0(x)) \geq 0$ for every $x \in \mathcal{O}$. Hence, by the results in [5], the Levi curvature of M_t is strictly positive for every $t > 0$ and we can apply our regularity result.

The operator in (3) is a quasi-linear elliptic operator whose characteristic form has minimum eigenvalue identically zero at every point. Hence the equation in (5) is a "degenerate" parabolic equation. In order to study the regularity properties of its solutions we will use a technique first introduced in [1] for studying the stationary problem. This technique has been refined in [2], where smooth regularity of real hypersurface in $\mathbb{C}^n$ with prescribed Levi mean-curvature is proved. The existence problem of classical solution to the prescribed Levi curvature equation has been recently solved in [3] and [4]. Starting from a result of Slodkowsky and Tomassini, in [3]-[4] authors proved that, if the curvature k is smooth and different from zero at every point, then every *strong viscosity solution* to the stationary Levi equation is smooth. Since Slodkowsky and Tomassini in [11] proved the existence of *strong viscosity solutions* to the Dirichlet problem associated to Levi equation under general hypotheses on the boundary data and on the Levi curvature, then the two works [3] and [4] fill the gap between the results in [11] and [1] in the stationary case.

In order to present our technique we introduce here some notations. We fix $u \in C^{2+\alpha,1+\alpha/2}(\Omega)$ a solution to the parabolic problem (5) and denote by a and b the coefficients defined in (4) in term of ∇u. We then consider the first order vector fields

$$X_1 = \partial_{x_1} + a\partial_{x_3}, \quad X_2 = \partial_{x_2} + b\partial_{x_3}, \tag{6}$$

then, arguing as in [1], we recognize that

$$\mathcal{L}u = X_1^2 u + X_2^2 u - (X_1 a + X_2 b)\partial_{x_3}, \tag{7}$$

and we can compute the bracket of X_1 and X_2 obtaining:

$$[X_1, X_2] = -\frac{\mathcal{L}u}{1 + u_{x_3}^2}\partial_{x_3}. \tag{8}$$

Moreover, by remarking that

$$1 + |\nabla u|^2 = (1 + a^2 + b^2)(1 + u_{x_3}^2) \tag{9}$$

we can write the parabolic equation in (5) as follows:

$$\partial_t u = \frac{1}{4(1 + a^2 + b^2)}(X_1^2 u + X_2^2 u - (X_1 a + X_2 b)\partial_{x_3} u). \tag{10}$$

If the vector fields X_1, X_2 were linear and smooth, then the operator

$$X_1^2 + X_2^2 - (X_1 a + X_2 b)\partial_{x_3} - 4(1 + a^2 + b^2)\partial_t \tag{11}$$

would be hypoelliptic, since it would satisfy the Hörmander condition.

Indeed, by (8) and (2)

$$[X_1, X_2] = -\frac{\mathcal{L}u}{1+u_{x_3}^2}\partial_{x_3} = \frac{2(1+|\nabla u|^2)^{3/2}}{(1+u_{x_3}^2)^2}k(\xi, u(\xi))\partial_{x_3} \tag{12}$$

with $k(\xi, u(\xi)) \neq 0$ for all $\xi \in \Omega$, and

$$X_1, X_2, [X_1, X_2], 4(1+a^2+b^2)\partial_t$$

are linearly independent at every point $\xi \in \Omega$.

In our situation the coefficients a and b are only $C^{1+\alpha,\frac{1+\alpha}{2}}(\Omega)$ since they are smooth functions of the gradient of a solution $u \in C^{2+\alpha,1+\alpha/2}(\Omega)$.

Theorem 1.1 has been announced in [7] and here we give a complete proof of it. The proof of Theorem 1.1 is organized as follows.

In Section 2 we introduce a linear operator H of *step two* naturally associated to the nonlinear equation in (5) with $C^{1+\alpha,\frac{1+\alpha}{2}}$ coefficients. By using a freezing method, we define an hypoelliptic operator H_{ξ_0}, whose prototype is the heat operator on the Heisenberg group. Estimates of the derivatives of its fundamental solution Γ_{ξ_0} are explicitly known (see for example [8], [9], [10]). In Proposition 2.2 we prove a precise estimate of the dependence of Γ_{ξ_0} and of its derivatives on the frozen point ξ_0. Then we write a representation formula for the solution v to the linear equation $Hv = f$ in term of Γ_{ξ_0}.

In Section 3 we formally differentiate the linear equation $Hv = f$ with respect to the vector fields X_1, X_2 defined by (6) and of

$$X_3 = (X_1 a + X_2 b)\partial_{x_3}, \quad T = (1+a^2+b^2)\partial_t. \tag{13}$$

Hence we introduce some classes of Hölder continuous functions $C_H^{m,\alpha}$, defined in term of the vector fields (6) and (13), in order to build a linear theory for the operator H. Precisely we show the following

Proposition 1.1 *For every $m \geq 2$ if $a, b \in C_{H,loc}^{m,\alpha}(\Omega)$ and $v \in C^{2+\alpha,1+\alpha/2}(\Omega)$ is a solution to*

$$Hv = f \in C_{H,loc}^{m-1,\alpha}(\Omega),$$

then $v \in C_{H,loc}^{m+1,\beta}(\Omega)$, for all $\beta < \alpha$.

In Section 4 we apply Proposition 1.1 to the nonlinear equation in (5) and, by using a bootstrap technique, we prove Theorem 1.1. We explicitly remark that, in order to use a a bootstrap method, we have to apply Proposition 1.1 to all derivatives of a classical solution u to (5).

In the Appendix we prove equalities (7)-(10) and we formally differentiate the linear equation $Hv = f$ with respect to the vector fields defined in (6) and (13). All these computations will be used in the proofs of Proposition 1.1 and of Theorem 1.1.

2 The frozen operator

In this section we fixed a solution $u \in C^{2+\alpha,1+\alpha/2}(\Omega)$ to (5) and we denote by a and b the coefficients defined in (4) in term of u. We then consider the linear parabolic operator defined by

$$H = L - 4T \tag{14}$$

where

$$L = X_1^2 + X_2^2 - X_3,$$

and X_1, X_2, X_3, T defined by (6) and (13). In order to study the regularity properties of the solution to the linear equation $Hv = f$ we define, for all $\xi_0 = (x_1^0, x_2^0, x_3^0, t^0) \in \Omega$, a frozen operator associated to H as follows:

$$H_{\xi_0} = L_{\xi_0} - 4T_{\xi_0}, \tag{15}$$

where

$$L_{\xi_0} = X_{1,\xi_0}^2 + X_{2,\xi_0}^2 - X_{3,\xi_0},$$

$$X_{1,\xi_0} = \partial_{x_1} + (P_{\xi_0}a)\partial_{x_3}, \quad X_{2,\xi_0} = \partial_{x_2} + (P_{\xi_0}b)\partial_{x_3},$$

$$X_{3,\xi_0} = (X_1 a + X_2 b)(\xi_0)\partial_{x_3}, \quad T_{\xi_0} = (1 + a^2 + b^2)(\xi_0)\partial_{x_3}.$$

Here, for a differentiable function f we have denoted by $P_{\xi_0}f$ the following first order polynomial

$$(P_{\xi_0}f)(\xi) = f(\xi_0) + (X_1 f)(\xi_0)(x_1 - x_1^0) + (X_2 f)(\xi_0)(x_2 - x_2^0),$$

where $\xi = (x_1, x_2, x_3, t)$.

We will show that, up to a change of variable, H_{ξ_0} is the heat operator on the Heisenberg group. We now recall some properties of this last operator. Let us call $\mathcal{H}^3$ the group, defined by $\mathbb{R}^3$, with the following composition law:

$$(x_1, x_2, x_3) \cdot (x_1^0, x_2^0, x_3^0) = (x_1 + x_1^0, x_2 + x_2^0, x_3 + x_3^0 + 2(x_1 x_2^0 - x_2 x_1^0)).$$

If

$$X_{\mathcal{H}} = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \quad \textit{and} \quad Y_{\mathcal{H}} = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3},$$

the Kohn Laplacian (or subelliptic Laplacian) is defined by

$$\Delta_{\mathcal{H}} = X_{\mathcal{H}}^2 + Y_{\mathcal{H}}^2.$$

We define the heat operator on the Heisenberg group as

$$H_{\mathcal{H}} = X_{\mathcal{H}}^2 + Y_{\mathcal{H}}^2 - \partial_t.$$

$\mathcal{H}^3 \times \mathbb{R}$ is a group with respect to the composition law

$$\xi_0 \circ \xi = (x_1 + x_1^0, x_2 + x_2^0, x_3 + x_3^0 + 2(x_1 x_2^0 - x_2 x_1^0), t + t_0),$$

and the operator $H_{\mathcal{H}}$ is invariant with respect to the left translations

$$\xi_0^{-1} \circ \xi = (x_1 - x_1^0, x_2 - x_2^0, x_3 - x_3^0 - 2(x_1 x_2^0 - x_2 x_1^0), t - t_0).$$

Moreover, if $\delta_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ is the group of dilations

$$\delta_\lambda(x_1, x_2, x_3, t) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 t), \quad \lambda > 0,$$

$H_{\mathcal{H}}$ is homogeneous of degree 2 with respect to δ_λ. A natural distance is defined in $\mathcal{H}^3 \times \mathbb{R}$:

$$d_{\mathcal{H}}(\xi, \xi_0) = \eta(\xi_0^{-1} \circ \xi),$$

where

$$\eta(\xi) = ((x_1^2 + x_2^2)^2 + x_3^2 + t^2)^{1/4}, \tag{16}$$

and the measure of balls in this metrics is

$$|B_{\mathcal{H}}(\xi_0, r)| = C_0 r^N$$

where $N = 6$. This number N is called homogeneous dimension of $\mathcal{H}^3 \times \mathbb{R}$. The fundamental solution $\Gamma_{\mathcal{H}}$ of $H_{\mathcal{H}}$ has the following estimates in terms of $d_{\mathcal{H}}$ and N (see [8] p. 114, [9], [10]):

$$|\Gamma_{\mathcal{H}}(\xi_0^{-1} \circ \xi, 0)| = |\Gamma_{\mathcal{H}}(\xi, \xi_0)| \le c d_{\mathcal{H}}^{-N+2}(\xi, \xi_0), \tag{17}$$

$$|X_{\mathcal{H}} \Gamma_{\mathcal{H}}(\xi, \xi_0)| \le c d_{\mathcal{H}}^{-N+1}(\xi, \xi_0), \quad |Y_{\mathcal{H}} \Gamma_{\mathcal{H}}(\xi, \xi_0)| \le c d_{\mathcal{H}}^{-N+1}(\xi, \xi_0)$$

$$|\partial_{x_3} \Gamma_{\mathcal{H}}(\xi, \xi_0)| \le c d_{\mathcal{H}}^{-N}(\xi, \xi_0), \quad |\partial_t \Gamma_{\mathcal{H}}(\xi, \xi_0)| \le c d_{\mathcal{H}}^{-N}(\xi, \xi_0).$$

The above estimates hold uniformly for ξ, ξ_0 in compact subsets of $\Omega \times \Omega$. Thus $X_{\mathcal{H}}, Y_{\mathcal{H}}$ are first derivatives, $-4\partial_{x_3} = [X_{\mathcal{H}}, Y_{\mathcal{H}}]$ is a second derivative in the direction of the vector fields and the weight of ∂_t is that of a second derivative.

In the sequel we will also use the following notation. We will call

$$X_{1\mathcal{H}} = X_{\mathcal{H}}, X_{2\mathcal{H}} = Y_{\mathcal{H}}, X_{3\mathcal{H}} = \partial_{x_3}, X_{4\mathcal{H}} = \partial_t.$$

For every multi-index $I = (i_1, \ldots, i_m)$, $i_k \in \{1, 2, 3, 4\}$ we will define the length of I as follows

$$|I| = |i_1| + \ldots + |i_m|$$

with

$$|i_k| = \begin{cases} 1, & i_k = 1, 2, \\ 2, & i_k = 3, 4. \end{cases}$$

and we will write

$$X_{\mathcal{H}}^I = X_{i_1 \mathcal{H}} \cdots X_{i_m \mathcal{H}}.$$

Proposition 2.1 *For every multi-index I there exists a positive constant c such that, if $d_{\mathcal{H}}(\xi,0) > 2d_{\mathcal{H}}(\xi,\xi_0)$, then*

$$|X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\xi,0) - X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\xi_0,0)| \le c\Big(d_{\mathcal{H}}^{-N+1-|I|}(\xi,0)(|x_1-x_1^0| + |x_2-x_2^0|)$$

$$+d_{\mathcal{H}}^{-N-|I|}(\xi,0)(|x_3-x_3^0-2(x_1x_2^0-x_2x_1^0)| + |t-t_0|)\Big).$$

Proof Let $\gamma:[0,1]\to\mathbb{R}^4$ be the integral curve starting at $\xi_0=(x_1^0,x_2^0,x_3^0,t_0)$ and such that

$$\dot\gamma = \Big((x_1-x_1^0)X_{1\mathcal{H}} + (x_2-x_2^0)X_{2\mathcal{H}} + (x_3-x_3^0-2(x_1x_2^0-x_2x_1^0))X_{3\mathcal{H}} + (t-t_0)X_{4\mathcal{H}}\Big)\gamma.$$

In particular, for every $h\in[0,1]$

$$\gamma(h) = (x_1^0+h(x_1-x_1^0), x_2^0+h(x_2-x_2^0), x_3^0+h(x_3-x_3^0), t_0+h(t-t_0)),$$

and $\gamma(1)=(x_1,x_2,x_3,t)=\xi$. Hence

$$|X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\xi,0) - X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\xi_0,0)| = |X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(1),0) - X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(0),0)|$$

$$= \Big|\int_0^1 \frac{d}{dh}(X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)dh\Big|$$

$$= \Big|\int_0^1 \Big((x_1-x_1^0)X_{1\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0) + (x_2-x_2^0)X_{2\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)$$

$$+(x_3-x_3^0-2(x_1x_2^0-x_2x_1^0))X_{3\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0) + (t-t_0)X_{4\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)\Big)dh\Big|$$

$$\le |x_1-x_1^0|\int_0^1 |X_{1\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)|dh + |x_2-x_2^0|\int_0^1 |X_{2\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)|dh$$

$$+|x_3-x_3^0-2(x_1x_2^0-x_2x_1^0)|\int_0^1 |X_{3\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)|dh$$

$$+|t-t_0|\int_0^1 |X_{4\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(h),0)|dh$$

(for an $\tilde h\in(0,1)$)

$$= |x_1-x_1^0||X_{1\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(\tilde h),0)| + |x_2-x_2^0||X_{2\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(\tilde h),0)|$$

$$+|x_3-x_3^0-2(x_1x_2^0-x_2x_1^0)||X_{3\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(\tilde h),0)| + |t-t_0||X_{4\mathcal{H}}X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\gamma(\tilde h),0)|$$

$$\le c|x_1-x_1^0|d_{\mathcal{H}}^{-N+1-|I|}(\gamma(\tilde h),0) + |x_2-x_2^0|d_{\mathcal{H}}^{-N+1-|I|}(\gamma(\tilde h),0)$$

$$+|x_3-x_3^0-2(x_1x_2^0-x_2x_1^0)|d_{\mathcal{H}}^{-N-|I|}(\gamma(\tilde h),0) + |t-t_0|d_{\mathcal{H}}^{-N-|I|}(\gamma(\tilde h),0)$$

(since $d_{\mathcal{H}}(\gamma(\tilde{h}),0)\geq d_{\mathcal{H}}(\xi,0)-d_{\mathcal{H}}(\xi,\gamma(\tilde{h}))\geq d_{\mathcal{H}}(\xi,0)-d_{\mathcal{H}}(\xi,\xi_0)>(1/2)d_{\mathcal{H}}(\xi,0))$

$$\leq c\Big(d_{\mathcal{H}}^{-N+1-|I|}(\xi,0)(|x_1-x_1^0|+|x_2-x_2^0|)$$

$$+d_{\mathcal{H}}^{-N-|I|}(\xi,0)(|x_3-x_3^0-2(x_1x_2^0-x_2x_1^0)|+|t-t_0|)\Big).$$

Let us recall some properties of the coefficients a,b defined in (4), whose proofs can be found in the Appendix.

Remark 2.1 *With the previous notations*

$$a=X_2u,\quad b=-X_1u.$$

Moreover, if $u\in C^{2+\alpha,1+\alpha/2}(\Omega)$ is a solution to (5) then

$$(X_2a-X_1b)(\xi)=-2k(\xi,u(\xi))\frac{(1+a^2+b^2)^{3/2}(\xi)}{(1+u_{x_3}^2)^{1/2}(\xi)}=4(1+|\nabla u|^2)(\xi)\partial_t u(\xi),$$

$$(X_1a+X_2b)(\xi)=([X_1,X_2]u)(\xi)=(X_1b-X_2a)(\xi)(\partial_{x_3}u)(\xi).$$

For all $\xi\in\Omega$ the change of variable

$$\phi_{\xi_0}(\xi)=\Big(x_1,x_2,\frac{\varphi_{\xi_0}(\xi)}{(X_2a-X_1b)(\xi_0)},\frac{t}{4(1+a^2+b^2)(\xi_0)}\Big)\tag{18}$$

with

$$\varphi_{\xi_0}(\xi)=2\Big(2x_3-X_1a(\xi_0)x_1^2-X_2b(\xi_0)x_2^2-(X_2a(\xi_0)+X_1b(\xi_0))x_1x_2\Big)$$

$$-4\Big(a(\xi_0)-X_1a(\xi_0)x_1^0-X_2a(\xi_0)x_2^0\Big)x_1-4\Big(b(\xi_0)-X_1b(\xi_0)x_1^0-X_2b(\xi_0)x_2^0\Big)x_2$$

$$-\frac{(X_1a+X_2b)(\xi_0)}{(1+a^2+b^2)(\xi_0)}t$$

changes H_{ξ_0} (defined in (15)) into the heat operator on the Heisenberg group. Indeed, for every $u:\mathbb{R}^4\to\mathbb{R}$, if

$$u_{\mathcal{H}}=u\circ\phi_{\xi_0}^{-1}$$

then

$$X_{1,\xi_0}u(\xi)=X_{\mathcal{H}}u_{\mathcal{H}}(\phi_{\xi_0}(\xi))\quad X_{2,\xi_0}u(\xi)=Y_{\mathcal{H}}u_{\mathcal{H}}(\phi_{\xi_0}(\xi)),$$

$$(X_{3,\xi_0}+4T_{\xi_0})u(\xi)=\partial_t u_{\mathcal{H}}(\phi_{\xi_0}(\xi)).$$

Moreover

$$H_{\xi_0} u(\xi) = H_{\mathcal{H}} u_{\mathcal{H}}(\phi_{\xi_0}(\xi))$$

and the fundamental solution of H_{ξ_0} is $\Gamma_{\xi_0}(\cdot,\cdot) = c(\xi_0)\Gamma_{\mathcal{H}}(\phi_{\xi_0}(\cdot),\phi_{\xi_0}(\cdot))$, with

$$\frac{1}{c(\xi_0)} = (1+a^2+b^2)(\xi_0)(X_2 a - X_1 b)(\xi_0).$$

The estimates of Γ_{ξ_0} and of its derivatives can be written in term of its natural distance:

$$d_{\xi_0}(\xi,\zeta) = d_{\mathcal{H}}(\phi_{\xi_0}(\xi),\phi_{\xi_0}(\zeta)) = \eta((\phi_{\xi_0}(\zeta))^{-1}\circ\phi_{\xi_0}(\xi)).$$

In particular

$$d_\xi(\xi,\zeta) = \eta(e_1,e_2,e_3,e_4)$$

with

$$e_1 = e_1(\xi,\zeta) = x_1 - z_1, \quad e_2 = e_2(\xi,\zeta) = x_2 - z_2,$$
$$e_3 = e_3(\xi,\zeta) = c_1(\xi)\Big(x_3 - z_3 - a(\xi)(x_1-z_1) - b(\xi)(x_2-z_2)$$
$$+\sum_{i,j=1}^{2} c_{ij}(\xi)(x_i-z_i)(x_j-z_j) - c_2(\xi)(t-\tau)\Big),$$
$$e_4 = e_4(\xi,\zeta) = c_3(\xi)(t-\tau),$$

where $\zeta = (z_1,z_2,z_3,\tau)$, $\xi = (x_1,x_2,x_3,t)$, $c_1,c_2 \in C^{\alpha,\alpha/2}$, $c_3 \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Omega)$, $c_1,c_3>0$ in Ω, and $c_{ij}\in C^{\alpha,\alpha/2}(\Omega)$ for every $i,j=1,2$. Precisely

$$c_1(\xi) = \frac{4}{(X_2 a - X_1 b)(\xi)}, \quad c_2(\xi) = \frac{(X_1 a + X_2 b)(\xi)}{4(1+a^2+b^2)(\xi)},$$
$$c_3(\xi) = \frac{1}{4(1+a^2+b^2)(\xi)}.$$

Then, if we define

$$\tilde{d}_\xi(\xi,\zeta) = \eta(\tilde{e}_1,\tilde{e}_2,\tilde{e}_3,\tilde{e}_4), \tag{19}$$

with

$$\tilde{e}_1 = \tilde{e}_1(\xi,\zeta) = x_1 - z_1, \quad \tilde{e}_2 = \tilde{e}_2(\xi,\zeta) = x_2 - z_2,$$
$$\tilde{e}_3 = \tilde{e}_3(\xi,\zeta) = x_3 - z_3 - a(\xi)(x_1-z_1) - b(\xi)(x_2-z_2), \quad \tilde{e}_4 = \tilde{e}_4(\xi,\zeta) = t-\tau,$$

we have that $\tilde{d}_\xi$ is locally equivalent to d_ξ. Moreover, for all $K \subset\subset \Omega$ we have

$$\tilde{d}_\xi(\xi,\zeta) \simeq \tilde{d}_\zeta(\xi,\zeta) \tag{20}$$

for every $\xi,\zeta\in K$. Indeed

$$|\tilde{e}_3(\xi,\zeta) - \tilde{e}_3(\zeta,\xi)| \le |a(\zeta)-a(\xi)||x_1-z_1| + |b(\zeta)-b(\xi)||x_2-z_2|$$

(by Remark 2.1 and using the hypothesis $a, b \in C^{1+\alpha,\frac{1+\alpha}{2}}$)

$$\leq c||\xi - \zeta||^2,$$

with $||\xi|| = \eta(\xi)$ and η as in (16). The thesis now follows since there exist two positive constants c_1, c_2 such that

$$c_1||\xi - \zeta|| \leq d_\xi(\xi, \zeta) \leq c_2||\xi - \zeta||^{1/2}$$

for every $\xi, \zeta \in K$.

In the sequel we will also use the following notation. We will call

$$X_{4,\xi_0} = T_{\xi_0}.$$

For every multi-index $I = (i_1, \dots, i_m)$, $i_k \in \{1, 2, 3, 4\}$ we will write

$$X^I_{\xi_0} = X_{i_1,\xi_0} \cdots X_{i_m,\xi_0}.$$

Let us fix two positive constants M_0, M such that the following inclusion holds:

$$S := \{\zeta \in \Omega : M_0 d_\xi(\xi, \xi_0) < d_\xi(\xi, \zeta) < M\}$$
$$\subseteq \{\zeta \in \Omega : 2d_{\mathcal{H}}(\phi_\xi(\zeta)^{-1} \circ \phi_\xi(\xi), \phi_{\xi_0}(\zeta)^{-1} \circ \phi_{\xi_0}(\xi_0)) < d_\xi(\xi, \zeta)\}.$$

The constants M_0, M such that the previous inclusion holds can be fixed with an easy computation.

Proposition 2.2 *For every multi-index I there exists a positive constant c such that*

$$|X^I_\xi \Gamma_\xi(\xi, \zeta) - X^I_{\xi_0}\Gamma_{\xi_0}(\xi_0, \zeta)| \leq c d_\xi^{-N+2+\alpha-|I|}(\xi, \zeta)||\xi - \xi_0||$$

$$c d_\xi^{-N+1-|I|}(\xi, \zeta)(|x_1 - x_1^0| + |x_2 - x_2^0|) + c d_\xi^{-N-|I|}|e_4(\xi, \zeta) - e_4(\xi_0, \zeta)|$$

$$+c d_\xi^{-N-|I|}(\xi, \zeta)\Big|e_3(\xi, \zeta) - e_3(\xi_0, \zeta) - 2((x_1 - z_1)(x_2^0 - z_2) - (x_2 - z_2)(x_1^0 - z_1))\Big|,$$

for every $\zeta \in S$. Here we have denoted by $\zeta = (z_1, z_2, z_3, \tau)$, $\xi = (x_1, x_2, x_3, t)$, $\xi_0 = (x_1^0, x_2^0, x_3^0, t_0)$.

Proof By (18)

$$|X^I_\xi \Gamma_\xi(\xi, \zeta) - X^I_{\xi_0}\Gamma_{\xi_0}(\xi_0, \zeta)|$$
$$= |c(\xi)X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\phi_\xi(\zeta)^{-1} \circ \phi_\xi(\xi), 0) - c(\xi_0)X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\phi_{\xi_0}(\zeta)^{-1} \circ \phi_{\xi_0}(\xi_0), 0)|$$
$$\leq |c(\xi) - c(\xi_0)||X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\phi_\xi(\zeta)^{-1} \circ \phi_\xi(\xi), 0)|$$
$$+|c(\xi_0)||X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\phi_\xi(\zeta)^{-1} \circ \phi_\xi(\xi), 0) - X^I_{\mathcal{H}}\Gamma_{\mathcal{H}}(\phi_{\xi_0}(\zeta)^{-1} \circ \phi_{\xi_0}(\xi_0), 0)|$$

(by (17) and Proposition 2.1)

$$\le c||\xi-\xi_0||^\alpha d_{\mathcal{H}}^{-N+2-|I|}(\phi_\xi(\zeta)^{-1}\circ\phi_\xi(\xi),0)$$

$$+c\Big(d_{\mathcal{H}}^{-N+1-|I|}(\phi_\xi(\zeta)^{-1}\circ\phi_\xi(\xi),0)(|e_1(\xi,\zeta)-e_1(\xi_0,\zeta)|+|e_2(\xi,\zeta)-e_2(\xi_0,z)|)$$

$$+d_{\mathcal{H}}^{-N-|I|}(\phi_\xi(\zeta)^{-1}\circ\phi_\xi(\xi),0)(|e_3(x,\zeta)-e_3(\xi_0,\zeta)-2(e_1(\xi,\zeta)e_2(\xi_0,z)$$

$$-e_2(\xi,\zeta)e_1(\xi_0,\zeta))|+|e_4(x,\zeta)-e_4(\xi_0,\zeta)|)\Big)$$

$$=c||\xi-\xi_0||^\alpha d_\xi^{-N+2-|I|}(\xi,\zeta)+c\Big(d_\xi^{-N+1-|I|}(\xi,\zeta)(|x_1-x_1^0|+|x_2-x_2^0|)$$

$$+d_\xi^{-N-|I|}(\xi,\zeta)(|e_3(\xi,\zeta)-e_3(\xi_0,\zeta)-2((x_1-z_1)(x_2^0-z_2)$$

$$-(x_2-z_2)(x_1^0-z_1))|+|e_4(\xi,\zeta)-e_4(\xi_0,\zeta)|)\Big).$$

Remark 2.2 *Since $u\in C^{2+\alpha,1+\alpha/2}(\Omega)$ then by Remark 2.1 $a,b\in C^{1+\alpha,\frac{1+\alpha}{2}}(\Omega)$ and we get*

$$|e_3(\xi,\zeta)-e_3(\xi_0,\zeta)-2((x_1-z_1)(x_2^0-x_2)-(x_2-z_2)(x_1^0-x_1))|$$

$$\le c(||\xi-\xi_0||d_\xi(\xi,\zeta)+||\xi-\xi_0||^\alpha d_\xi^2(\xi,\zeta)+d_\xi^2(\xi,\xi_0)+||\xi-\xi_0|||t-\tau|+|t-t_0|)$$

$$\le c(d_\xi(\xi,\xi_0)d_\xi(\xi,\zeta)+d_\xi^\alpha(\xi,\xi_0)d_\xi^2(\xi,\zeta)+d_\xi(\xi,\xi_0)d_\xi^2(\xi,\zeta)+d_\xi^2(\xi,\xi_0)),$$

$$|e_4(\xi,\zeta)-e_4(\xi_0,\zeta)|\le c(||\xi-\xi_0|||t-\tau|+|t-t_0|)\le c(d_\xi(\xi,\xi_0)d_\xi^2(\xi,\zeta)+d_\xi^2(\xi,\xi_0)).$$

We are now ready to write a representation formula for a function $v\in C^{2+\alpha,1+\alpha/2}(\Omega)$ in term of Hv. This representation formula will be the main tool in the proof of Proposition 1.1. Since we are looking for a local result we can fix two open sets Ω_1,Ω_2 such that $\overline{\Omega}_2\subset\subset\Omega_1,\overline{\Omega}_1\subset\subset\Omega$ and a function $\phi\in C_0^\infty(\Omega_1)$ such that $\phi\equiv 1$ in Ω_2.

Proposition 2.3 *If $v\in C^{2+\alpha,1+\alpha/2}(\Omega)$ then for all $\xi\in\Omega_2$ we have*

$$v(\xi)=v_1(\xi,\xi_0)+4v_2(\xi,\xi_0),$$

where

$$v_1(\xi,\xi_0)=\int\Gamma_{\xi_0}(\xi,\zeta)Hv(\zeta)\phi(\zeta)d\zeta+\int\Gamma_{\xi_0}(\xi,\zeta)(L_{\xi_0}-L)v(\zeta)\phi(\zeta)d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta+2\sum_{i=1}^{2}\int\Gamma_{\xi_0}(\xi,\zeta)X_{i,\xi_0}v(\zeta)X_{i,\xi_0}\phi(\zeta)d\zeta,$$

$$v_2(\xi,\xi_0)=\int\Gamma_{\xi_0}(\xi,\zeta)((1+a^2+b^2)(\zeta)-(1+a^2+b^2)(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta$$

$$+\partial_t v(\xi_0)\int\Gamma_{\xi_0}(\xi,\zeta)((1+a^2+b^2)(\zeta)-P_{\xi_0}(1+a^2+b^2)(\zeta))\phi(\zeta)d\zeta$$

$$+\partial_t v(\xi_0)\int\Gamma_{\xi_0}(\xi,\zeta)P_{\xi_0}(1+a^2+b^2)(\zeta)\phi(\zeta)d\zeta-\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)T_{\xi_0}\phi(\zeta)d\zeta.$$

Proof For all $\xi\in\Omega_2$ we have

$$v(\xi)=v(\xi)\phi(\xi)=\int\Gamma_{\xi_0}(\xi,\zeta)H_{\xi_0}(v\phi)(\zeta)d\zeta$$

$$=\int\Gamma_{\xi_0}(\xi,\zeta)(H_{\xi_0}-H)v(\zeta)\phi(\zeta)d\zeta+\int\Gamma_{\xi_0}(\xi,\zeta)Hv(\zeta)\phi(\zeta)d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)H_{\xi_0}\phi(\zeta)d\zeta+2\sum_{i=1}^{2}\int\Gamma_{\xi_0}(\xi,\zeta)X_{i,\xi_0}v(\zeta)X_{i,\xi_0}\phi(\zeta)d\zeta$$

$$=\int\Gamma_{\xi_0}(\xi,\zeta)(L_{\xi_0}-L)v(\zeta)\phi(\zeta)d\zeta+4\int\Gamma_{\xi_0}(\xi,\zeta)(T-T_{\xi_0})v(\zeta)\phi(\zeta)d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)Hv(\zeta)\phi(\zeta)d\zeta+\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta$$

$$-4\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)T_{\xi_0}\phi(\zeta)d\zeta+2\sum_{i=1}^{2}\int\Gamma_{\xi_0}(\xi,\zeta)X_{i,\xi_0}v(\zeta)X_{i,\xi_0}\phi(\zeta)d\zeta$$

$$=\int\Gamma_{\xi_0}(\xi,\zeta)(L_{\xi_0}-L)v(\zeta)\phi(\zeta)d\zeta d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)Hv(\zeta)\phi(\zeta)d\zeta+\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta$$

$$+2\sum_{i=1}^{2}\int\Gamma_{\xi_0}(\xi,\zeta)X_{i,\xi_0}v(\zeta)X_{i,\xi_0}\phi(\zeta)d\zeta-4\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)T_{\xi_0}\phi(\zeta)d\zeta$$

$$+4\int\Gamma_{\xi_0}(\xi,\zeta)((1+a^2+b^2)(\zeta)-(1+a^2+b^2)(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta$$

$$+4\partial_t v(\xi_0)\int\Gamma_{\xi_0}(\xi,\zeta)((1+a^2+b^2)(\zeta)-(1+a^2+b^2)(\xi_0))\phi(\zeta)d\zeta$$

$$=\int\Gamma_{\xi_0}(\xi,\zeta)(L_{\xi_0}-L)v(\zeta)\phi(\zeta)d\zeta d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)Hv(\zeta)\phi(\zeta)d\zeta+\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta$$

$$+2\sum_{i=1}^{2}\int\Gamma_{\xi_0}(\xi,\zeta)X_{i,\xi_0}v(\zeta)X_{i,\xi_0}\phi(\zeta)d\zeta-4\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)T_{\xi_0}\phi(\zeta)d\zeta$$

$$+4\int\Gamma_{\xi_0}(\xi,\zeta)((1+a^2+b^2)(\zeta)-(1+a^2+b^2)(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta$$

$$+4\partial_t v(\xi_0)\int\Gamma_{\xi_0}(\xi,\zeta)((1+a^2+b^2)(\zeta)-P_{\xi_0}(1+a^2+b^2)(\zeta))\phi(\zeta)d\zeta$$

$$+4\partial_t v(\xi_0)\int\Gamma_{\xi_0}(\xi,\zeta)P_{\xi_0}(1+a^2+b^2)(\zeta)\phi(\zeta)d\zeta.$$

3 Hölder estimates

In this section we differentiate the representation formula proved in Proposition 2.3 and we prove some regularity results for the solution of the linear equation $Hv=f$, with H defined as in (14), in some Hölder continuous classes naturally associated to the geometry of the problem.

For all $\xi=(x_1,x_2,x_3,t)$ and $\zeta=(z_1,z_2,z_3,\tau)$ let us define a pseudo-distance d as follows

$$d(\xi,\zeta)=\frac{\tilde{d}_\xi(\xi,\zeta)+\tilde{d}_\zeta(\xi,\zeta)}{2} \tag{21}$$

with $\tilde{d}_\xi$ defined as in (19). As a consequence of (20) we get

Corollary 3.1 *For every compact set $K\subset\subset\Omega$ there exist c_1, $c_2>0$ such that*

$$c_1 d_\xi(\xi,\zeta)\le d(\xi,\zeta)\le c_2 d_\xi(\xi,\zeta)$$

for all $\xi,\zeta\in K$.

We now define a class of Hölder continuous functions in term of d.

Definition 3.1 *For every $\alpha\in(0,1)$*

$$C^\alpha_H(\Omega)=\{f:\Omega\to\mathbb{R}\text{ such that }\ \frac{|f(\xi)-f(\zeta)|}{d^\alpha(\xi,\zeta)}\, is\, bounded\, for\, all\, \xi,\zeta\in\Omega\}.$$

We say that $f\in C^\alpha_{H,loc}(\Omega)$ if for every compact set $K\subset\subset\Omega$ there exists a positive constant c such that

$$\frac{|f(\xi)-f(\zeta)|}{d^\alpha(\xi,\zeta)}\le c,$$

for every $\xi,\zeta\in K$.

Let X_1, X_2, X_3, T be the vector fields defined in (6) and (13).

Definition 3.2 *We say that $f \in C_H^{1,\alpha}(\Omega)$ if $f \in C_H^{\alpha}(\Omega)$, $X_i f \in C_H^{\alpha}(\Omega)$ for every $i = 1, 2$, and*

$$\frac{|f(x,t) - f(x,\tau)|}{|t-\tau|^{\frac{1+\alpha}{2}}}$$

is bounded for all $(x,t), (x,\tau) \in \Omega$.

Remark 3.1 *If $f \in C_{H,loc}^{1,\alpha}(\Omega)$ then for every compact set $K \subset\subset \Omega$ there exists a positive constant c such that*

$$|f(\zeta) - P_\xi f(\zeta)| \le c d^{1+\alpha}(\xi, \zeta),$$

for every $\xi, \zeta \in K$.

Indeed, if we denote by $\xi = (x,t), \zeta = (z,\tau)$

$$|f(\zeta) - P_\xi f(\zeta)| \le |f(z,\tau) - f(z,t)| + |f(z,t) - P_\xi f(z,t)|$$

(by Definition 3.2 and arguing as in [1] for the second term on the right hand side)

$$\le c|t-\tau|^{\frac{1+\alpha}{2}} + c d^{1+\alpha}(\xi, (z,t)) \le c d^{1+\alpha}(\xi, \zeta).$$

Definition 3.3 *For every $\alpha \in (0,1)$*

$$C_H^{2,\alpha}(\Omega) = \{f \in C_H^{1,\alpha}(\Omega) : \quad X_i f \in C_H^{1,\alpha}(\Omega),\ i = 1,2, \quad Tf \in C_H^{\alpha}(\Omega)\}.$$

If $m \ge 3$ and the coefficients a, b of H are of class $C_H^{m-1,\alpha}(\Omega)$ we define

$$C_H^{m,\alpha}(\Omega) = \{f \in C_H^{m-1,\alpha}(\Omega) : \quad X_i f \in C_H^{m-1,\alpha}(\Omega),\ i = 1,2,\ Tf \in C_H^{m-2,\alpha}(\Omega)\}.$$

Remark 3.2 *We explicitly note that*

$$C_H^{2m,2\alpha}(\Omega) \subset C^{m+\alpha, \frac{m+\alpha}{2}}(\Omega) \subset C_H^{m,\alpha/2}(\Omega).$$

In the sequel we will also use the following notation. We will call

$$X_4 = T.$$

For every multi-index $I = (i_1, \ldots, i_m)$, $i_k \in \{1,2,3,4\}$ we will write

$$X^I = X_{i_1} \cdots X_{i_m}.$$

In next lemmas we differentiate the representation formula stated in Proposition 2.3.

Lemma 3.1 *Let $a, b \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Omega)$ and $v \in C_H^{2,\alpha}$ be a solution to the equation*

$$Hv = f \in C_H^{1,\alpha}(\Omega),$$

then for all $\xi_0 \in \Omega$ and every multi-index I such that $|I| = 2$

$$\xi \to X_{\xi_0}^I v(\xi)$$

is differentiable at ξ_0 in the direction of the vector fields X_1 and X_2.

Proof For all $\xi \in \Omega_2$ we can represent v as in Proposition 2.3 by writing $v_1(\xi, \xi_0)$ as in [1], but in term of the fundamental solution Γ_{ξ_0} of the parabolic operator H_{ξ_0}. Then, arguing as in [1], for every multi-index I such that $|I| = 2$

$$\xi \to X_{\xi_0}^I v_1(\xi, \xi_0)$$

is differentiable at ξ_0 in the direction of the vector fields X_1 and X_2. Hence we have only to study $v_2(\xi, \xi_0)$.

Let us recall that

$$\xi \to \int \Gamma_{\xi_0}(\xi, \zeta) v(\zeta) T_{\xi_0}\phi(\zeta) d\zeta$$

is of class $C^\infty(\Omega_2)$ since $T_{\xi_0}\phi(\zeta) = 0$ for all ζ in a neighborhood of ξ and the integral has no singularities.

The term

$$\xi \to \int \Gamma_{\xi_0}(\xi, \zeta) P_{\xi_0}(1 + a^2 + b^2)(\zeta)\phi(\zeta) d\zeta$$

is of class $C^\infty(\Omega_2)$ since the function $\zeta \to P_{\xi_0}(1 + a^2 + b^2)(\zeta)\phi(\zeta)$ is smooth.

Now let us define

$$\omega(\xi) = \int \Gamma_{\xi_0}(\xi, \zeta) g(\zeta, \xi_0)\phi(\zeta) d\zeta, \tag{22}$$

with the function g satisfying the following conditions:

(i) $g(\xi_0, \xi_0) = 0$ and for all $\xi_0 \in \Omega$

(ii) for every compact set $K \subset\subset \Omega$ there exists a positive constant c such that

$$|g(\xi, \xi_0)| \le c d_{\xi_0}^{1+\alpha}(\xi, \xi_0)$$

for all $\xi, \xi_0 \in K$

(iii) for every compact set $K \subset\subset \Omega$ there exist three positive constants c_1, c_2, c_3 such that

$$|g(\zeta, \xi_0) - g(\xi, \xi_0)| \le c_1 d_{\xi_0}^{1+\alpha}(\xi, \zeta) + c_2 d_{\xi_0}^{\alpha}(\xi, \zeta) d_{\xi_0}(\xi, \xi_0)$$
$$+ c_3 d_{\xi_0}(\xi, \zeta) d_{\xi_0}^{\alpha}(\xi, \xi_0)$$

for every $\xi, \zeta \in K$.

It is standard to show that for all multi-index $I=(i,j)$ such that $|I|=2$

$$X^I_{\xi_0}\omega(\xi)=\int X^I_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(g(\zeta,\xi_0)\phi(\zeta)-g(\xi,\xi_0)\phi(\xi))d\zeta+c_{ij}(\xi_0)g(\xi,\xi_0)\phi(\xi),$$

with $\xi_0\to c_{ij}(\xi_0)$ of class $C^{\alpha,\alpha/2}$.

In order to take another derivative of $X^I_{\xi_0}\omega$ we fix a function $\vartheta\in C^\infty(\mathbb{R})$ such that $0\le\vartheta\le 1$,

$$\vartheta(s)=\begin{cases}0, & s\le 1\\ 1, & s\ge 2\end{cases}$$

and define for all $\varepsilon>0$

$$\omega^I_\varepsilon(\xi)=\int X^I_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\vartheta\Big(\frac{d_{\xi_0}(\xi,\zeta)}{\varepsilon}\Big)\Big(g(\zeta,\xi_0)\phi(\zeta)-g(\xi,\xi_0)\phi(\xi)\Big)d\zeta.$$

Now we will show that

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2}|\omega^I_\varepsilon(\xi)-X^I_{\xi_0}\omega(\xi)|=O(\varepsilon^{1+\alpha}) \tag{23}$$

as $\varepsilon\to 0$. Indeed, for every $\xi\in\Omega_2$ such that $d_{\xi_0}(\xi,\xi_0)<\varepsilon/2$ we get by (ii)

$$|\omega^I_\varepsilon(\xi)-X^I_{\xi_0}\omega(\xi)|$$

$$\le\Big|\int_{\varepsilon<d_{\xi_0}(\xi,\zeta)<2\varepsilon}X^I_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(\vartheta-1)\Big(g(\zeta,\xi_0)\phi(\zeta)-g(\xi,\xi_0)\phi(\xi)\Big)d\zeta\Big|$$

$$+\Big|\int_{d_{\xi_0}(\xi,\zeta)<\varepsilon}X^I_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\Big(g(\zeta,\xi_0)\phi(\zeta)-g(\xi,\xi_0)\phi(\xi)\Big)d\zeta\Big|+c\varepsilon^{1+\alpha}$$

(by (iii))

$$\le c\int_\varepsilon^{2\varepsilon}(\rho^\alpha+\varepsilon\rho^{\alpha-1}+\varepsilon^\alpha)d\rho+c\int_0^\varepsilon(\rho^\alpha+\varepsilon\rho^{\alpha-1}+\varepsilon^\alpha)d\rho+c\varepsilon^{1+\alpha}\le C\varepsilon^{1+\alpha}.$$

For every $i=1,2$ let us call

$$w^{i,I}(\xi_0)=\int X_iX^I_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)g(\zeta,\xi_0)\phi(\zeta)d\zeta. \tag{24}$$

Then, arguing as in (23), we get

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2}|X_i\omega^I_\varepsilon(\xi)-w^{i,I}(\xi_0)|=O(\varepsilon^\alpha) \tag{25}$$

as $\varepsilon\to 0$. Now we denote by $\exp(hX_i)(\xi_0)$ the exponential map with starting point at ξ_0 and we compute $X_iX^I_{\xi_0}$ as follows

$$\lim_{h\to 0}\frac{X^I_{\xi_0}\omega(\exp(hX_i)(\xi_0))-X^I_{\xi_0}\omega(\xi_0)}{h}$$

(by (23))

$$\lim_{h\to 0}\frac{\omega^I_{|h|}(\exp(hX_i)(\xi_0))-\omega^I_{|h|}(\xi_0)+O(|h|^{1+\alpha})}{h}$$

(for an $\tilde{h}\in(0,h)$)

$$=\lim_{h\to 0}X_i\omega^I_{|h|}(\exp(\tilde{h}X_i)(\xi_0))+O(|h|^{\alpha}).$$

Hence, by (25),

$$\lim_{h\to 0}\frac{X^I_{\xi_0}\omega(\exp(hX_i)(\xi_0))-X^I_{\xi_0}\omega(\xi_0)}{h}=w^{i,I}(\xi_0).$$

To end the proof it is enough to apply the previous result to the functions

$$g(\zeta,\xi_0)=((1+a^2+b^2)(\zeta)-(1+a^2+b^2)(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0)),$$

$$g(\zeta,\xi_0)=((1+a^2+b^2)(\zeta)-P_{\xi_0}(1+a^2+b^2)(\zeta)).$$

Lemma 3.2 *Let $v\in C^{2,\alpha}_H(\Omega)$ be a solution to the equation*

$$Hv=f\in C^{1,\alpha}_H(\Omega),$$

then for every multi-index I, such that $|I|=2$, and for every $i=1,2$ the application

$$\xi_0\to X_iX^I_{\xi_0}v(\xi_0)$$

is of class $C^{\beta}_{H,loc}(\Omega)$ for all $\beta<\alpha$.

Moreover, if $a,b\in C^{2,\alpha}_{H,loc}(\Omega)$ and $\partial_t v,\partial_{x_3}v\in C^{1,\alpha}_{H,loc}$ then for every compact K, $K\subset\subset\Omega$, there exists a positive constant c such that

$$|X^I_{(x,t_0)}v(x,t_0)-X^I_{(x,t)}v(x,t)|\le c|t-t_0|^{\frac{1+\beta}{2}},$$

for every $(x,t),(x_0,t)\in K$, and for all $\beta<\alpha$.

Proof Let $w^{i,I}$ be the function defined by (24), with the function g satisfying the properties (i),(ii),(iii). Moreover, let us assume that

(iv) for every compact K, $K\subset\subset\Omega$, there exist three positive constants c_1,c_2,c_3 such that

$$|g(\zeta,\xi_0)-g(\zeta,\xi)|\le c_1d^{1+\alpha}(\xi,\xi_0)+c_2d^{\alpha}(\xi,\xi_0)d(\xi,\zeta)+c_3d(\xi,\xi_0)d^{\alpha}_{\xi_0}(\xi,\zeta)$$

for every $\xi,\zeta\in K$.

If we call $\varepsilon = M_0 d(\xi,\xi_0)$ and by $B = \{\zeta : \ d(\zeta,\xi) < \varepsilon\}$, then it is not restrictive to assume that

$$\Omega_1 \setminus B \subseteq \{\xi \in \Omega : \varepsilon < d_\xi(\xi,\zeta) < M\},$$

with M_0, M as in Proposition 2.2. Hence

$$|w^{i,I}(\xi_0) - w_\xi^{i,I}(\xi)| \le I_1 + I_2 + I_3 + I_4,$$

with

$$I_1 = \Big| \int_{\Omega_1\setminus B} \Big(X_{\xi_0}^{i,I}\Gamma_{\xi_0}(\xi_0,\zeta) - X_\xi^{i,I}\Gamma_\xi(\xi,\zeta)\Big) g(\zeta,\xi_0)\phi(\zeta)d\zeta\Big|,$$

$$I_2 = \Big| \int_{\Omega_1\setminus B} X_\xi^{i,I}\Gamma_\xi(\xi,\zeta)\Big(g(\zeta,\xi_0) - g(\zeta,\xi)\Big)\phi(\zeta)d\zeta\Big|,$$

$$I_3 = \Big| \int_{B} \Big(X_{\xi_0}^{i,I}\Gamma_{\xi_0}(\xi_0,\zeta) - X_\xi^{i,I}\Gamma_\xi(\xi,\zeta)\Big) g(\zeta,\xi_0)\phi(\zeta)d\zeta\Big|,$$

$$I_4 = \Big| \int_{B} X_\xi^{i,I}\Gamma_\xi(\xi,\zeta)\Big(g(\zeta,\xi_0) - g(\zeta,\xi)\Big)\phi(\zeta)d\zeta\Big|.$$

By Proposition 2.2 and Remark 2.2

$$I_1 \le c\int_{\Omega_1\setminus B}\Big(d(\xi,\xi_0)d^{-N-2}(\xi,\zeta) + d(\xi,\xi_0)^\alpha d^{-N-1}(\xi,\zeta)$$

$$+d^2(\xi,\xi_0)d^{-N-3}(\xi,\zeta)\Big)|g(\zeta,\xi_0)\phi(\zeta)|d\zeta$$

(by (ii) and by the inequality $c_1 d(\xi_0,\zeta) \le d(\xi,\zeta)) \le c_2 d(\xi_0,\zeta)$ for every $\zeta \in \Omega_1 \setminus B$)

$$\le cd(\xi,\xi_0)\int_{\Omega_1\setminus B} d^{-N-1+\alpha}(\xi,\zeta)d\zeta$$

$$+cd^\alpha(\xi,\xi_0)\int_{\Omega_1\setminus B} d^{-N+\alpha}(\xi,\zeta)d\zeta + cd^2(\xi,\xi_0)\int_{\Omega_1\setminus B} d^{-N-2+\alpha}(\xi,\zeta)d\zeta$$

$$\le cd(\xi,\xi_0)\int_\varepsilon^M \rho^{-2+\alpha}d\rho + cd(\xi,\xi_0)^\alpha\int_\varepsilon^M \rho^{-1+\alpha}d\rho + cd^2(\xi,\xi_0)\int_\varepsilon^M \rho^{-3+\alpha}d\rho$$

$$\le c\varepsilon(\varepsilon^{-1+\alpha} - M^{-1+\alpha}) + c\varepsilon^\alpha(M^\alpha - \varepsilon^\alpha) + c\varepsilon^2(\varepsilon^{-2+\alpha} - M^{-2+\alpha}) \le c\varepsilon^\alpha.$$

By (iv) we get

$$I_2 \le c\int_{\Omega_1\setminus B} d^{-N-1}(\xi,\zeta)\Big(c_1 d^{1+\alpha}(\xi,\xi_0) + c_2 d^\alpha(\xi,\xi_0)d(\xi,\zeta)$$

$$+c_3 d(\xi,\xi_0)d_{\xi_0}^\alpha(\xi,\zeta)\Big)d\zeta$$

$$\leq C_1\varepsilon^{1+\alpha}\int_{\Omega_1\setminus B} d^{-N-1}(\xi,\zeta)d\zeta + C_2\varepsilon^{\alpha}\int_{\Omega_1\setminus B} d^{-N}(\xi,\zeta)d\zeta$$
$$+C_3\varepsilon\int_{\Omega_1\setminus B} d^{-N-1+\alpha}(\xi,\zeta)d\zeta$$
$$\leq C_1\varepsilon^{1+\alpha}\int_{\varepsilon}^{M}\rho^{-2}d\rho + C_2\varepsilon^{\alpha}\int_{\varepsilon}^{M}\rho^{-1}d\rho + C_3\varepsilon\int_{\varepsilon}^{M}\rho^{-2+\alpha}d\rho$$
$$\leq C_1\varepsilon^{1+\alpha}(\varepsilon^{-1}-M^{-1}) + C_2\varepsilon^{\alpha}(\ln M - \ln\varepsilon) + C_3\varepsilon(\varepsilon^{-1+\alpha}-M^{-1+\alpha}) \leq C\varepsilon^{\beta}$$

for every $\beta<\alpha$. By arguing in the same way we recognize that

$$I_3 + I_4 \leq d^{\beta}(\xi,\xi_0),$$

for every $\beta<\alpha$. Since all the terms of $X_iX^I_{\xi_0}v(\xi_0)$ may be written as (24) with functions g satisfying (i), (ii), (iii) and (iv), then the first statement is proved.

To prove the second statement, let us call $\varepsilon = M_0|t-t_0|^{1/2}$, $\xi=(x,t)$ and denote by $B=\{\zeta=(z,\tau):\ d(\zeta,\xi)<\varepsilon\}$. Hence, if ω is the function defined in (22) with g satisfying (i),(ii),(iii) and the following condition

(v) for every compact set $K\subset\subset\Omega$ there exist two positive constants c_1,c_2 such that

$$|g(\zeta,\xi)-g(\zeta,\xi_0)| \leq c_1|t-t_0|^{\frac{1+\alpha}{2}}d(\xi,\zeta) + c_2|t-t_0|^{\frac{1+\alpha}{2}}$$

for every $\xi=(x,t),\xi_0=(x,t_0),\zeta\in K$.

Hence

$$|X^I_{(x,t_0)}\omega(x,t_0) - X^I_{(x,t)}\omega(x,t)| \leq I_1+I_2+I_3+I_4,$$

with

$$I_1 = \Big|\int_{\Omega_1\setminus B}\Big(X^I_{(x,t_0)}\Gamma_{(x,t_0)}((x,t_0),\zeta) - X^I_{(x,t)}\Gamma_{(x,t)}((x,t),\zeta)\Big)g(\zeta,\xi_0)\phi(\zeta)d\zeta\Big|,$$

$$I_2 = \Big|\int_{\Omega_1\setminus B}X^I_{\xi}\Gamma_{\xi}(\xi,\zeta)\Big(g(\zeta,(x,t_0)) - g(\zeta,(x,t))\Big)\phi(\zeta)d\zeta\Big|,$$

$$I_3 = \Big|\int_{B}\Big(X^I_{(x_0,t)}\Gamma_{(x_0,t)}((x,t_0),\zeta) - X^I_{(x,t)}\Gamma_{(x,t)}((x,t),\zeta)\Big)g(\zeta,\xi_0)\phi(\zeta)d\zeta\Big|,$$

$$I_4 = \Big|\int_{B}X^I_{\xi}\Gamma_{\xi}(\xi,\zeta)\Big(g(\zeta,(x,t_0)) - g(\zeta,(x,t))\Big)\phi(\zeta)d\zeta\Big|.$$

By Proposition 2.2 and Remark 2.2

$$I_1 \leq c\int_{\varepsilon\leq d_{\xi}(\xi,\zeta)\leq cM}(|t-t_0|d^{-N}(\xi,\zeta) + |t-t_0|d^{-N-2}(\xi,\zeta))|g(\zeta,\xi_0)\phi(\zeta)|d\zeta$$

(by (ii) and by the inequality $c_1 d((x,t_0),\zeta) \le d(\xi,\zeta)) \le c_2 d((x,t_0),\zeta)$ for every $\zeta \in \Omega_1 \setminus B$)

$$\le c|t-t_0| \int_{\Omega_1\setminus B} d^{-N+1+\alpha}(\xi,\zeta)d\zeta + c|t-t_0| \int_{\Omega_1\setminus B} d^{-N-1+\alpha}(\xi,\zeta)d\zeta$$

$$\le c|t-t_0| \int_{\varepsilon}^{M} \rho^{\alpha} d\rho + c|t-t_0| \int_{\varepsilon}^{M} \rho^{-2+\alpha} d\rho$$

$$\le c\varepsilon^2 (M^{1+\alpha} - \varepsilon^{1+\alpha}) + c\varepsilon^2(\varepsilon^{\alpha-1} - M^{\alpha-1}) \le c\varepsilon^{1+\alpha}.$$

By (v) with $\xi = (x,t), \xi_0 = (x,t_0)$ we get

$$I_2 \le c_1 \int_{\Omega_1\setminus B} d^{-N}(\xi,\zeta) d^{1+\alpha}(\xi,\xi_0) d(\xi,\zeta) d\zeta + c_2 \int_{\Omega_1\setminus B} d^{-N}(\xi,\zeta) d^{1+\alpha}(\xi,\xi_0) d\zeta$$

$$\le c_1 \varepsilon^{1+\alpha} \int_{\Omega_1\setminus B} d^{-N+1}(\xi,\zeta)d\zeta + c_2\varepsilon^{1+\alpha} \int_{\Omega_1\setminus B} d^{-N}(\xi,\zeta)d\zeta$$

$$\le C_1 \varepsilon^{1+\alpha}(M-\varepsilon) + C_2\varepsilon^{1+\alpha}(\ln M - \ln\varepsilon) \le c\varepsilon^{1+\beta},$$

for every $\beta < \alpha$. By arguing in the same way we recognize that

$$I_3 + I_4 \le c d^{1+\beta}(\xi,\xi_0).$$

Since $a, b \in C^{2,\alpha}_{H,loc}(\Omega)$ and $\partial_t v, \partial_{x_3} v \in C^{1,\alpha}_{H,loc}$ then all the terms of $X^I_{\xi_0} v(\xi_0)$ may be written as $X^I_{\xi_0}\omega(\xi_0)$ with functions g satisfying (i), (ii), (iii) and (v), this completes the proof.

Proposition 3.1 *Let $a, b \in C^{2,\alpha}_{H,loc}(\Omega)$ and $v \in C^{2,\alpha}_{H}(\Omega)$ be a solution to the equation*

$$Hv = f \in C^{1,\alpha}_{H}(\Omega),$$

then $X^J v \in C^{\beta}_{H,loc}(\Omega)$ for every multi-index J such that $|J| = 3$ and for every $\beta < \alpha$.

Moreover, if $\partial_t v, \partial_{x_3} v \in C^{1,\alpha}_{H}(\Omega)$, then $v \in C^{3,\beta}_{H,loc}(\Omega)$, for every $\beta < \alpha$.

Proof Since

$$X^I v = (X^I - X^I_{\xi_0})v + X^I_{\xi_0} v,$$

then for all $i = 1, 2$, we have

$$X_i X^I v(\xi_0) = (X_i(X^I - X^I_{\xi_0})v)(\xi_0) + X_i X^I_{\xi_0} v(\xi_0), \tag{26}$$

and since the coefficient of X^I is of class $C^{1,\alpha}_{H,loc}(\Omega)$ for all I such that $|I| = 2$, then by applying Lemma 3.2 to right hand side of (26) we get the thesis.

Corollary 3.2 *Let $a, b \in C^{1,\alpha}_{H,loc}(\Omega)$ and assume that for every multi-index I such that $|I| = 2$, $X^I v \in C^{\alpha}_H(\Omega)$ with v a solution to the equation*

$$Hv = f \in C^{\alpha}_H(\Omega),$$

then $v \in C^{2,\beta}_{H,loc}(\Omega)$ for every $\beta < \alpha$.

In order to prove Proposition 1.1 we also need the following *weak* version of Proposition 3.1.

Proposition 3.2 *Let $a, b \in C^{1+\alpha, \frac{1+\alpha}{2}}(\Omega)$, $f \in C^{1,\alpha}_H(\Omega)$, $\partial_{x_3} f, \partial_t f \in C^{\alpha}_H(\Omega)$. If $v \in C^{2+\alpha,1+\alpha/2}(\Omega)$ is a solution to $Hv = f$ and $\partial_t v \in C^{1,\alpha}_{H,loc}(\Omega)$, then $\partial_t v, \partial_{x_3} v \in C^{2,\beta}_{H,loc}(\Omega)$, for every $\beta < \alpha$.*

Proof We will apply Proposition 2.3 to the function

$$w(\xi) = v(\xi) - x_3(P_{\xi_0}\partial_{x_3} v(\xi) - \partial_{x_3} v(\xi_0)) - t(P_{\xi_0}\partial_t v(\xi) - \partial_t v(\xi_0)),$$

for all $\xi = (x_1, x_2, x_3, t) \in \Omega$. Then

$$\partial_t w(\xi) = \partial_t v(\xi) - (P_{\xi_0}\partial_t v(\xi) - \partial_t v(\xi_0)),$$

and

$$Hw(\xi) = Lw(\xi) - 4Tw(\xi)$$
$$= Hv(\xi) + (Lw - Lv)(\xi) + 4(1 + a^2 + b^2)(\xi)(P_{\xi_0}\partial_t v(\xi) - \partial_t v(\xi_0)).$$

Hence we obtain a new representation formula for v which use the fact that $\partial_t v, \partial_{x_3} v \in C^{1,\alpha}_H(\Omega)$. By differentiating it with respect to $X^I \partial_t$ and $X^I \partial_{x_3}$ for every multi-index I such that $|I| = 2$ and arguing as in Proposition 3.1 we get the thesis.

Now we will apply Proposition 3.1 and Proposition 3.2 to the derivatives of a solution v to the linear equation $Hv = f$, and prove Proposition 1.1.

Proof of Proposition 1.1 Since $a, b \in C^{2,\alpha}_{H,loc}(\Omega)$ and $v \in C^{2+\alpha,1+\alpha/2}(\Omega)$, in particular $v \in C^{2,\alpha}_H(\Omega)$ and $\partial_{x_3} v \in C^{1,\alpha}_H(\Omega)$. By the first assertion of Proposition 3.1 $X^J v \in C^{\beta}_{H,loc}(\Omega)$ for every multi-index J such that $|J| = 3$ and for every $\beta < \alpha$. By Lemma 5.3, Lemma 5.4 $HX_1 v, HX_2 v \in C^{\beta}_{H,loc}(\Omega)$, hence by Corollary 3.2 $X_1 v, X_2 v \in C^{2,\beta}_{H,loc}(\Omega)$ for every $\beta < \alpha$. Moreover, since $Hv = f \in C^{1,\alpha}_H(\Omega)$, we can deduce that $\partial_t v \in C^{1,\alpha}_H(\Omega)$. Hence $v \in C^{3,\beta}_{H,loc}(\Omega)$ for every $\beta < \alpha$.

Now let us prove the assertion for $m = 3$. Since $a, b \in C^{3,\alpha}_H(\Omega)$, then $a, b \in C^{1+\alpha, \frac{1+\alpha}{2}}(\Omega)$ and we can apply Proposition 3.2 to get $\partial_t v, \partial_{x_3} v \in C^{2,\beta}_{H,loc}(\Omega)$. By Lemma 5.3, Lemma 5.4, the functions $\omega = X_1 v, X_2 v$ satisfy $H\omega \in C^{1,\beta}_{H,loc}(\Omega)$, hence by Proposition 3.1 $X_1 v, X_2 v \in C^{3,\beta}_{H,loc}(\Omega)$, and $v \in C^{4,\beta}_{H,loc}(\Omega)$. Now we conclude the proof by induction. We assume the thesis true for $m-1, m-2$ and we will prove that if $a, b \in C^{m,\beta}_{H,loc}(\Omega)$, $f \in C^{m-1,\beta}_{H,loc}(\Omega)$ for every $\beta < \alpha$, then $v \in C^{m+1,\beta}_{H,loc}(\Omega)$. By Lemma 5.1, Lemma 5.2, $\partial_{x_3} v, \partial_t v \in C^{m-2,\beta}_{H,loc}(\Omega)$ satisfy $H\omega \in C^{m-3,\beta}_{H,loc}(\Omega)$, then by the induction hypothesis $\partial_{x_3} v, \partial_t v \in C^{m-1,\beta}_{H,loc}(\Omega)$. By Lemma 5.3, Lemma 5.4, $X_1 v, X_2 v \in C^{m-1,\beta}_{H,loc}(\Omega)$ satisfy $H\omega \in C^{m-2,\beta}_{H,loc}(\Omega)$, then by the induction hypothesis $X_1 v, X_2 v \in C^{m,\beta}_{H,loc}(\Omega)$. In particular $v \in C^{m+1,\beta}_{H,loc}(\Omega)$, for every $\beta < \alpha$.

4 Smooth regularity

In this section we will prove our main result Theorem 1.1 by applying Proposition 3.1 and Proposition 1.1 to a solution u of (5). We have already noticed in the Introduction that the parabolic equation in (5) may be written as in (7), hence if $u \in C^{2+\alpha,1+\alpha/2}(\Omega)$ is a solution to (5) then $Hu = 0$, with H the second order parabolic operator defined by (14) and with coefficients $a = a(\nabla u), b = b(\nabla u) \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Omega)$. In particular $a = X_2 u, b = -X_1 u \in C^{1,\alpha}_{H,loc}(\Omega)$. In the sequel we will always denote by β a real number such that $0 < \beta < \alpha$.

Before applying the second part of Proposition 3.1 to our solution u we have to regularize a little more the coefficients a, b. This will be done in next Proposition 4.1 by using Lemma 3.2 and Proposition 3.2.

Proposition 4.1 *If $u \in C^{2+\alpha,1+\alpha/2}(\Omega)$ is a solution to (5) then $a, b \in C^{2,\beta}_{H,loc}(\Omega)$ for all $\beta < \alpha$.*

Proof By (34), (6), (15)

$$a = X_2 u = (X_2 - X_{2,\xi_0})u + X_{2,\xi_0} u = (b - P_{\xi_0} b)\partial_{x_3} u + X_{2,\xi_0} u. \tag{27}$$

By differentiating (27) with respect to X_1 we get

$$X_1 a = X_1 X_2 u = X_1((b - P_{\xi_0} b)\partial_{x_3} u + X_{2,\xi_0} u) \tag{28}$$

$$= (X_1 b - X_1 b(\xi_0))\partial_{x_3} u + (b - P_{\xi_0} b) X_1 \partial_{x_3} u + (X_1 - X_{1,\xi_0}) X_{2,\xi_0} u + X_{1,\xi_0} X_{2,\xi_0} u$$

$$= (X_1 b - X_1 b(\xi_0))\partial_{x_3} u + (b - P_{\xi_0} b) X_1 \partial_{x_3} u + (a - P_{\xi_0} a)\partial_{x_3} X_{2,\xi_0} u + X_{1,\xi_0} X_{2,\xi_0} u.$$

We are now ready to compute $X_i X_1 a(\xi_0)$, with $i = 1, 2$, by differentiating (28) with respect to X_i at ξ_0 :

$$X_i X_1 a(\xi_0) = X_i X_1 b(\xi_0)\partial_{x_3} u(\xi_0) + X_i X_{1,\xi_0} X_{2,\xi_0} u(\xi_0). \tag{29}$$

Arguing in the same way for $b = -X_1 u$ we get

$$X_i X_1 b(\xi_0) = -X_i X_1 a(\xi_0)\partial_{x_3} u(\xi_0) - X_i X_{1,\xi_0} X_{1,\xi_0} u(\xi_0), \tag{30}$$

and by substituting (30) in (29) we get

$$X_iX_1a(\xi_0) = \frac{X_iX_{1,\xi_0}X_{2,\xi_0}u(\xi_0) - \partial_{x_3}u(\xi_0)X_iX_{1,\xi_0}X_{1,\xi_0}(\xi_0)}{1+(\partial_{x_3}u(\xi_0))^2}.$$

By arguing in the same way we get for all $i,j=1,2$

$$X_iX_ja(\xi_0) = \frac{X_iX_{j,\xi_0}X_{2,\xi_0}u(\xi_0) - \partial_{x_3}u(\xi_0)X_iX_{j,\xi_0}X_{1,\xi_0}(\xi_0)}{1+(\partial_{x_3}u(\xi_0))^2}, \tag{31}$$

$$X_iX_jb(\xi_0) = \frac{-X_iX_{j,\xi_0}X_{1,\xi_0}u(\xi_0) - \partial_{x_3}u(\xi_0)X_iX_{j,\xi_0}X_{2,\xi_0}(\xi_0)}{1+(\partial_{x_3}u(\xi_0))^2}. \tag{32}$$

We can now apply Lemma 3.2 to the right hand sides in (31) and (32) to deduce that the applications

$$\xi_0 \to X_iX_ja(\xi_0), \quad \xi_0 \to X_iX_jb(\xi_0)$$

are of class $C^{\beta}_{H,loc}(\Omega)$.

By Proposition 3.2 the functions $\partial_{x_3}u, \partial_t u \in C^{2,\beta}_{H,loc}(\Omega)$. Now we will show that for every $i=1,2$ the functions $\partial_t X_i a, \partial_t X_i b \in C^{\beta}_{H,loc}(\Omega)$. From this it will follow that for every compact $K \subset\subset \Omega$ there exists a positive constant c such that

$$|X_ia(x,t) - X_ia(x,t_0)| \leq c|t-t_0|^{\frac{1+\beta}{2}}, \quad |X_ib(x,t) - X_ib(x,t_0)| \leq c|t-t_0|^{\frac{1+\beta}{2}}$$

for every $(x,t),(x,t_0) \in K$, and we will conclude that $a,b \in C^{2,\beta}_{H,loc}(\Omega)$. Let us compute for example $\partial_t X_1 a$, by (34)

$$\partial_t X_1 a = \partial_t X_1 X_2 u = [\partial_t, X_1]X_2u + X_1[\partial_t, X_2]u + X_1X_2\partial_t u$$

$$= \partial_t a \partial_{x_3} X_2 u + X_1(\partial_t b \partial_{x_3} u) + X_1X_2\partial_t u$$

$$= \partial_t a \partial_{x_3} X_2 u + \partial_t b X_1 \partial_{x_3} u + X_1\partial_t b \partial_{x_3} u + X_1X_2\partial_t u$$

(by (33))

$$= \partial_t a \partial_{x_3} X_2 u + \partial_t b X_1 \partial_{x_3} u - X_1\partial_t X_1 u \partial_{x_3} u + X_1X_2\partial_t u$$

$$= \partial_t a \partial_{x_3} X_2 u + \partial_t b X_1 \partial_{x_3} u - X_1[\partial_t, X_1] u \partial_{x_3} u - X_1X_1\partial_t u + X_1X_2\partial_t u$$

$$= \partial_t a \partial_{x_3} X_2 u + \partial_t b X_1 \partial_{x_3} u - X_1(\partial_t a \partial_{x_3} u)\partial_{x_3} u - X_1X_1\partial_t u + X_1X_2\partial_t u$$

$$= \partial_t a \partial_{x_3} X_2 u + \partial_t b X_1 \partial_{x_3} u - \partial_t a X_1 \partial_{x_3} u \partial_{x_3} u$$

$$- X_1 \partial_t a \partial_{x_3} u \partial_{x_3} u - X_1X_1\partial_t u + X_1X_2\partial_t u$$

$$= \partial_t a \partial_{x_3} X_2 u + \partial_t b X_1 \partial_{x_3} u - \partial_t a X_1 \partial_{x_3} u \partial_{x_3} u$$

$$-[X_1,\partial_t]a\partial_{x_3}u\partial_{x_3}u-\partial_tX_1a(\partial_{x_3}u)^2-X_1X_1\partial_tu+X_1X_2\partial_tu,$$

hence

$$\partial_tX_1a=\frac{1}{1+(\partial_{x_3}u)^2}\Big(\partial_ta\partial_{x_3}X_2u+\partial_tbX_1\partial_{x_3}u-\partial_taX_1\partial_{x_3}u\partial_{x_3}u$$
$$+\partial_ta\partial_{x_3}a(\partial_{x_3}u)^2-\partial_tX_1a-X_1X_1\partial_tu+X_1X_2\partial_tu\Big)$$

and the right hand side is of class $C^{\beta}_{H,loc}(\Omega)$. Arguing in the same way for the functions ∂_tX_2a, ∂_tX_1b, ∂_tX_2b we get the thesis.

Proof of Theorem 1.1 Since $u\in C^{2+\alpha,1+\alpha/2}(\Omega)$ is a solution to $Hu=0$, if $a,b\in C^{m,\beta}_{H,loc}$, for every $m\in\mathbb{N}$, then by Proposition 1.1 $u\in C^{\infty}(\Omega)$. Hence we have only to show that $a,b\in C^{m,\beta}_{H,loc}$, for every $m\in\mathbb{N}$. By Proposition 4.1 $a,b\in C^{2,\beta}_{H,loc}(\Omega)$. Now we will conclude the proof by induction. We assume that $a,b\in C^{m,\beta}_{H,loc}(\Omega)$ and we will show that $a,b\in C^{m+1,\beta}_{H,loc}(\Omega)$. Indeed if $a,b\in C^{m,\beta}_{H,loc}(\Omega)$ then by Proposition 1.1 $u\in C^{m+1,\beta}_{H,loc}(\Omega)$ and by Lemma 5.1, Lemma 5.2, the functions $\omega=\partial_{x_3}u,\partial_tu$ satisfy $H\omega\in C^{m-2,\beta}_{H,loc}(\Omega)$ since $f=0$. Hence, by Proposition 1.1 $\omega=\partial_{x_3}u,\partial_tu\in C^{m,\beta}_{H,loc}(\Omega)$. By Lemma 5.3, Lemma 5.4, the functions $\omega=X_1u,X_2u$ satisfy $H\omega\in C^{m-1,\beta}_{H,loc}(\Omega)$. Hence, by Proposition 1.1 $a=X_2u,b=-X_1u\in C^{m+1,\beta}_{H,loc}(\Omega)$.

5 Appendix

In this section we will prove some equalities which are useful in the proof of Theorem 1.1. Since all the computations here are formal we can assume that u is a smooth solution to (5) and a,b the coefficients defined by (4) in term of it.

Proof of (7)–(10) Let us begin by computing X_1u,X_2u with X_1,X_2 defined by (6)

$$X_1u=(\partial_{x_1}+a\partial_{x_3})u=\Big(\partial_{x_1}+\frac{u_{x_2}-u_{x_1}u_{x_3}}{1+u_{x_3}^2}\partial_{x_3}\Big)u \tag{33}$$
$$=\frac{u_{x_1}+u_{x_2}u_{x_3}}{1+u_{x_3}^2}=-b.$$

$$X_2u=(\partial_{x_2}+b\partial_{x_3})u=\Big(\partial_{x_2}-\frac{u_{x_1}+u_{x_2}u_{x_3}}{1+u_{x_3}^2}\partial_{x_3}\Big)u \tag{34}$$
$$=\frac{u_{x_2}-u_{x_1}u_{x_3}}{1+u_{x_3}^2}=a.$$

Let us prove (7):

$$X_1^2 u + X_2^2 u = X_1(\partial_{x_1} + a\partial_{x_3})u + X_2(\partial_{x_2} + b\partial_{x_3})u \tag{35}$$

$$= X_1\partial_{x_1}u + aX_1\partial_{x_3}u + X_2\partial_{x_2}u + bX_2\partial_{x_3}u + (X_1a + X_2b)\partial_{x_3}u$$

$$= \partial_{x_1x_1}u + \partial_{x_2x_2}u + 2a\partial_{x_1x_3}u + 2b\partial_{x_2x_3}u + (a^2+b^2)\partial_{x_3x_3}u$$

$$+(X_1a + X_2b)\partial_{x_3}u$$

(by (3))

$$= \mathcal{L}u + (X_1a + X_2b)\partial_{x_3}u.$$

By (33), (34) we also get

$$X_1^2 u + X_2^2 u = \mathcal{L}u + (X_1a + X_2b)\partial_{x_3}u$$

$$= \mathcal{L}u + (X_1X_2 - X_2X_1)u\partial_{x_3}u = \mathcal{L}u + [X_1, X_2]u\partial_{x_3}u.$$

We can compute the previous bracket as follows

$$[X_1, X_2] = X_1(\partial_{x_2} + b\partial_{x_3}) - X_2(\partial_{x_1} + a\partial_{x_3}) = (X_1b - X_2a)\partial_{x_3} \tag{36}$$

(by (33) and (34))

$$= -(X_1^2 + X_2^2)u\partial_{x_3}.$$

In particular by (35) and (36) we get

$$X_1^2 u + X_2^2 u = \mathcal{L}u - (X_1^2 + X_2^2)u(\partial_{x_3}u)^2,$$

hence

$$X_1^2 u + X_2^2 u = \frac{\mathcal{L}u}{1 + u_{x_3}^2}. \tag{37}$$

By replacing (37) in (36) we get (8).

In order to prove (9) we use (33) and (34) to compute

$$1 + a^2 + b^2 = 1 + \Big(\frac{u_{x_2} - u_{x_1}u_{x_3}}{1 + u_{x_3}^2}\Big)^2 + \Big(\frac{u_{x_1} + u_{x_2}u_{x_3}}{1 + u_{x_3}^2}\Big)^2$$

$$= 1 + \frac{u_{x_1}^2 + u_{x_2}^2}{1 + u_{x_3}^2} = \frac{1 + u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2}{1 + u_{x_3}^2} = \frac{1 + |\nabla u|^2}{1 + u_{x_3}^2}.$$

Hence, by (9) and (7) the second order parabolic equation in (5) can be written as in (10). Moreover, if we call H the operator defined in (14) with coefficients a and b, then $Hu = 0$.

Proof of Remark 2.1 By (33) and (34)

$$X_2a - X_1b = X_2^2 u + X_1^2 u$$

(by (37) and (5))

$$= \frac{\mathcal{L}u}{1+u_{x_3}^2} = 4(1+|\nabla u|^2)\partial_t u$$

(by (2))

$$= -2\frac{(1+|\nabla u|^2)^{3/2}}{(1+u_{x_3}^2)^2}k(\cdot,u)$$

(by (9))

$$= -2\frac{(1+a^2+b^2)^{3/2}}{(1+u_{x_3}^2)^{1/2}}k(\cdot,u).$$

By (33) and (34)

$$X_1a + X_2b = X_1X_2u - X_2X_1u = [X_1,X_2]u = (X_1b - X_2a)\partial_{x_3}u.$$

Since u is fixed we will now consider H as a linear operator. Now we are interested in differentiating the linear equation

$$Hv = f. \tag{38}$$

Lemma 5.1 *If v is a solution to (38) then the function $\omega = \partial_{x_3}v$ is a solution to*

$$H\omega = \partial_{x_3}f - 2(\partial_{x_3}aX_1\partial_{x_3}v + \partial_{x_3}bX_2\partial_{x_3}v) + 4\partial_{x_3}(1+a^2+b^2)\partial_t v. \tag{39}$$

Proof

$$\partial_{x_3}f = \partial_{x_3}Hv = H\partial_{x_3}v + [\partial_{x_3},H]v$$
$$= H\partial_{x_3}v + [\partial_{x_3},X_1]X_1v + X_1[\partial_{x_3},X_1]v + [\partial_{x_3},X_2]X_2v + X_2[\partial_{x_3},X_2]v$$
$$-[\partial_{x_3},X_3]v - 4[\partial_{x_3},T]v$$

(by (6), (13))

$$= H\omega + \partial_{x_3}a\partial_{x_3}X_1v + X_1(\partial_{x_3}a\partial_{x_3}v) + \partial_{x_3}b\partial_{x_3}X_2v + X_2(\partial_{x_3}b\partial_{x_3}v)$$
$$-\partial_{x_3}(X_1a + X_2b)\partial_{x_3}v - 4\partial_{x_3}(1+a^2+b^2)\partial_t v$$
$$= H\omega + \partial_{x_3}a(\partial_{x_3}X_1v + X_1\partial_{x_3}v) + \partial_{x_3}b(\partial_{x_3}X_2v + X_2\partial_{x_3}v) + (X_1\partial_{x_3}a + X_2\partial_{x_3}b)\partial_{x_3}v$$
$$-\partial_{x_3}(X_1a + X_2b)\partial_{x_3}v - 4\partial_{x_3}(1+a^2+b^2)\partial_t v$$
$$= H\omega + \partial_{x_3}a([\partial_{x_3},X_1]v + 2X_1\partial_{x_3}v) + \partial_{x_3}b([\partial_{x_3},X_2]v + 2X_2\partial_{x_3}v)$$
$$+([X_1,\partial_{x_3}]a + [X_2,\partial_{x_3}]b)\partial_{x_3}v - 4\partial_{x_3}(1+a^2+b^2)\partial_t v$$

(by (6))

$$= H\omega + \partial_{x_3}a(\partial_{x_3}a\partial_{x_3}v + 2X_1\partial_{x_3}v) + \partial_{x_3}b(\partial_{x_3}b\partial_{x_3}v + 2X_2\partial_{x_3}v)$$
$$-((\partial_{x_3}a)^2 + (\partial_{x_3}b)^2)\partial_{x_3}v - 4\partial_{x_3}(1+a^2+b^2)\partial_t v$$
$$= H\omega + 2\partial_{x_3}aX_1\partial_{x_3}v + 2\partial_{x_3}bX_2\partial_{x_3}v - 4\partial_{x_3}(1+a^2+b^2)\partial_t v.$$

Lemma 5.2 *If v is a solution to (38) then the function $\omega = \partial_t v$ is a solution to*

$$H\omega = \partial_t f - 2(\partial_t a X_1 \partial_{x_3} v + \partial_t b X_2 \partial_{x_3} v) + 4\partial_t(1 + a^2 + b^2)\partial_t v. \qquad (40)$$

Proof

$$\partial_t f = \partial_t H v = H\partial_t v + [\partial_t, H]v$$

$$= H\partial_t v + [\partial_t, X_1]X_1 v + X_1[\partial_t, X_1]v + [\partial_t, X_2]X_2 v + X_2[\partial_t, X_2]v$$

$$-[\partial_t, X_3]v - 4[\partial_t, T]v$$

(by (6),(13))

$$= H\omega + \partial_t a\partial_{x_3} X_1 v + X_1(\partial_t a\partial_{x_3} v) + \partial_t b\partial_{x_3} X_2 v + X_2(\partial_t b\partial_{x_3} v)$$

$$-\partial_t(X_1 a + X_2 b)\partial_{x_3} v - 4\partial_t(1 + a^2 + b^2)\partial_t v$$

$$= H\omega + \partial_t a(\partial_{x_3} X_1 v + X_1\partial_{x_3} v) + \partial_t b(\partial_{x_3} X_2 v + X_2\partial_{x_3} v) + (X_1\partial_t a + X_2\partial_t b)\partial_{x_3} v$$

$$-\partial_t(X_1 a + X_2 b)\partial_{x_3} v - 4\partial_t(1 + a^2 + b^2)\partial_t v$$

$$= H\omega + \partial_t a([\partial_{x_3}, X_1]v + 2X_1\partial_{x_3} v) + \partial_t b([\partial_{x_3}, X_2]v + 2X_2\partial_{x_3} v)$$

$$+([X_1, \partial_t]a + [X_2, \partial_t]b)\partial_{x_3} v - 4\partial_t(1 + a^2 + b^2)\partial_t v$$

(by (6))

$$= H\omega + \partial_t a(\partial_{x_3} a\partial_{x_3} v + 2X_1\partial_{x_3} v) + \partial_t b(\partial_{x_3} b\partial_{x_3} v + 2X_2\partial_{x_3} v)$$

$$-(\partial_t a\partial_{x_3} a + \partial_t b\partial_{x_3} b)\partial_{x_3} v - 4\partial_t(1 + a^2 + b^2)\partial_t v$$

$$= H\omega + 2\partial_t a X_1\partial_{x_3} v + 2\partial_t b X_2\partial_{x_3} v - 4\partial_t(1 + a^2 + b^2)\partial_t v.$$

Before differentiating the equation (38) with respect to the vector field X_1, X_2 let us remark that the following equalities hold

$$X_2\partial_{x_3} u = \partial_{x_3} a - \partial_{x_3} u\partial_{x_3} b, \quad X_1\partial_{x_3} u = -\partial_{x_3} b - \partial_{x_3} u\partial_{x_3} a. \qquad (41)$$

Indeed, by (6), (33),(34)

$$X_2\partial_{x_3} u = [X_2, \partial_{x_3}]u + \partial_{x_3} X_2 u = -\partial_{x_3} b\partial_{x_3} u + \partial_{x_3} a,$$

$$X_1\partial_{x_3} u = [X_1, \partial_{x_3}]u + \partial_{x_3} X_1 u = -\partial_{x_3} a\partial_{x_3} u - \partial_{x_3} b.$$

Lemma 5.3 *If v is a solution to (38) then the function $\omega = X_1 v$ is a solution to*

$$H\omega = X_1 f + 2q(X_2\partial_{x_3} v - X_1\partial_{x_3} u\partial_{x_3} v) + (X_2 q - \partial_{x_3} u X_1 q)\partial_{x_3} v \qquad (42)$$

$$+4X_1(1 + a^2 + b^2)\partial_t v - 4\partial_t a(1 + a^2 + b^2)\partial_{x_3} v,$$

with

$$q = 4\frac{1 + a^2 + b^2}{1 + u_{x_3}^2}\partial_t u.$$

Proof By (8), (7) and (10)

$$[X_1, X_2] = -q\partial_{x_3}, \tag{43}$$

and by (33), (34)

$$X_1 a + X_2 b = (X_1 X_2 u - X_2 X_1)u = [X_1, X_2]u = -q\partial_{x_3} u. \tag{44}$$

Let us differentiate (38) with respect to X_1.

$$X_1 f = X_1 H v = H X_1 v + [X_1, H]v$$

$$= H\omega + [X_1, X_2]X_2 v + X_2[X_1, X_2]v - [X_1, X_3]v - 4[X_1, T]v$$

(by (43), (44) and (13))

$$= H\omega - q\partial_{x_3} X_2 v - X_2(q\partial_{x_3} v) + X_1(q\partial_{x_3} u\partial_{x_3} v)$$

$$-q\partial_{x_3} u\partial_{x_3} X_1 v - 4X_1(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t a\partial_{x_3} v$$

$$= H\omega - q[\partial_{x_3}, X_2]v - 2qX_2\partial_{x_3} v - X_2 q\partial_{x_3} v + X_1(q\partial_{x_3} u)\partial_{x_3} v + q\partial_{x_3} u X_1\partial_{x_3} v$$

$$-q\partial_{x_3} u\partial_{x_3} X_1 v - 4X_1(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t a\partial_{x_3} v$$

$$= H\omega - q\partial_{x_3} b\partial_{x_3} v - 2qX_2\partial_{x_3} v - X_2 q\partial_{x_3} v + X_1(q\partial_{x_3} u)\partial_{x_3} v$$

$$-q\partial_{x_3} u[\partial_{x_3}, X_1]v - 4X_1(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t a\partial_{x_3} v$$

$$= H\omega - q\partial_{x_3} b\partial_{x_3} v - 2qX_2\partial_{x_3} v - X_2 q\partial_{x_3} v + qX_1\partial_{x_3} u\partial_{x_3} v + X_1 q\partial_{x_3} u\partial_{x_3} v$$

$$-q\partial_{x_3} u\partial_{x_3} a\partial_{x_3} v - 4X_1(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t a\partial_{x_3} v$$

(by (41))

$$= H\omega - 2qX_2\partial_{x_3} v + 2q(X_1\partial_{x_3} u)\partial_{x_3} v - (X_2 q - \partial_{x_3} u X_1 q)\partial_{x_3} v$$

$$-4X_1(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t a\partial_{x_3} v.$$

With the same notations of Lemma 5.3 we have

Lemma 5.4 *If v is a solution to (38) then the function $\omega = X_2 v$ is a solution to*

$$H\omega = X_2 f - 2q(X_1\partial_{x_3} v + X_2\partial_{x_3} u\partial_{x_3} v) - (X_1 q + \partial_{x_3} u X_2 q)\partial_{x_3} v \tag{45}$$

$$+4X_2(1 + a^2 + b^2)\partial_t v - 4\partial_t b(1 + a^2 + b^2)\partial_{x_3} v.$$

Proof Let us differentiate (38) with respect to X_2.

$$X_2 f = X_2 H v = H X_2 v + [X_2, H] v$$

$$= H\omega - [X_1, X_2] X_1 v - X_1 [X_1, X_2] v - [X_2, X_3] v - 4[X_2, T] v$$

(by (43), (44) and (13))

$$= H\omega + q\partial_{x_3} X_1 v + X_1 (q\partial_{x_3} v) + X_2 (q\partial_{x_3} u \partial_{x_3} v)$$

$$-q\partial_{x_3} u \partial_{x_3} X_2 v - 4X_2(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t b \partial_{x_3} v$$

$$= H\omega + q[\partial_{x_3}, X_1] v + 2qX_1 \partial_{x_3} v + X_1 q \partial_{x_3} v + X_2 (q\partial_{x_3} u) \partial_{x_3} v + q\partial_{x_3} u X_2 \partial_{x_3} v$$

$$-q\partial_{x_3} u \partial_{x_3} X_2 v - 4X_2(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t b \partial_{x_3} v$$

$$= H\omega + q\partial_{x_3} a \partial_{x_3} v + 2qX_1 \partial_{x_3} v + X_1 q \partial_{x_3} v + X_2 (q\partial_{x_3} u) \partial_{x_3} v$$

$$-q\partial_{x_3} u [\partial_{x_3}, X_2] v - 4X_2(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t b \partial_{x_3} v$$

$$= H\omega + q\partial_{x_3} a \partial_{x_3} v + 2qX_1 \partial_{x_3} v + X_1 q \partial_{x_3} v + qX_2 \partial_{x_3} u \partial_{x_3} v + X_2 q \partial_{x_3} u \partial_{x_3} v$$

$$-q\partial_{x_3} u \partial_{x_3} b \partial_{x_3} v - 4X_2(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t b \partial_{x_3} v$$

(by (41))

$$= H\omega + 2qX_1 \partial_{x_3} v + 2q(X_2 \partial_{x_3} u)\partial_{x_3} v + (X_1 q + \partial_{x_3} u X_2 q)\partial_{x_3} v$$

$$-4X_2(1 + a^2 + b^2)\partial_t v + 4(1 + a^2 + b^2)\partial_t b \partial_{x_3} v.$$

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