Balls defined by nonsmooth vector fields and the Poincaré inequality

Annamaria Montanari and Daniele Morbidelli*

ABSTRACT. We provide a structure theorem for Carnot–Caratéodory balls defined by a family of Lipschitz continuous vector fields. From this result a proof of Poincaré inequality follows.

Boules définies par champs de vecteurs non réguliers et l'inégalité de Poincaré.

RÉSUMÉ. On prouve un théorème de structure pour les boules de Carnot–Caratéodory définies par des champs de vecteurs lipschitziens. Une inégalité de Poincaré est aussi démontrée.

1 Introduction and main results

Given a family of vector fields $X_1, \ldots, X_m$ in $\mathbb{R}^n$, a crucial problem when dealing with the second order operator $\sum X_j^2$ is to give sufficient conditions to ensure the doubling property of the related control balls and the Poincaré inequality. The problem is quite well understood for smooth vector fields satisfying the Hörmander condition: in this case the mentioned properties have been proved respectively by Nagel Stein and Wainger [19] and by Jerison [13]. The techniques in those papers require a C^k−smoothness of the vector fields, for some k greater than expected. The situation is different if we consider diagonal vector fields. In this setting a description of the control balls and Poincaré inequality was proved by Franchi and Lanconelli [7] in a low regularity situation.

In a recent paper Lanconelli and the second author [15] gave a method for the proof of the Poincaré inequality for vector fields. Their proof does not need smoothness, but it requires that the Carnot Charathéodory balls are representable by means of *controllable almost exponential maps*, see [15, Theorem 2.1]. Here we prove that the necessary tools to use this method can be developed, at least in the step 2 case,

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assuming only a Lipschitz condition on the vector fields and on the commutators involved in the statement of the rank condition.

An interesting feature of our result is that the balls are very easy to visualize: they are equivalent to linear images of boxes (see (7)). We also remark that in the present paper we never use the Campbell Hausdorff formula (a powerful tool whose use in analysis of vector fields requires regularity). The relevant properties of the "almost exponential maps" E_I defined in (6) are established in Section 2 by direct computations (see the exact formula in Lemma 2.2). Exploiting the tools of Section 2 for vector fields of higher step, although of considerable technical difficulty, is an open interesting problem, which would clarify what are the minimal regularity assumptions to have a structure theorem for control balls and the Poincaré inequality. Here we give an answer to this problem in the step 2 case.

For reader convenience we recall the notion of control distance (see [5] and [6]). Given a family $X_1, \ldots, X_m$ of locally Lipschitz continuous vector fields on $\mathbb{R}^n$, we say that an absolutely continuous path $\gamma : [0, T] \to \mathbb{R}^n$ is subunit if, for almost all $t \in [0, T]$, $\dot{\gamma}(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t))$, with $\sum_{j=1}^m |a_j(t)| < 1$. Assuming that for every $x, y \in \mathbb{R}^n$ there exists at least one subunit path connecting x and y, define the control distance related to $X_1, \ldots, X_m$ (or Carnot Charathéodory distance) as $d(x, y) = \inf\{T > 0 :$ there is $\gamma : [0, T] \to \mathbb{R}^n$, subunit path such that $\gamma(0) = x$, $\gamma(T) = y\}$. In the sequel we shall denote by $B(x, r)$ the d-ball with center at x and radius r.

We now state our hypotheses on the vector fields $X_j = \sum_{k=1}^n f_j^k \partial_k$, $j = 1, \ldots, m$. We assume that f_j is locally Lipschitz continuous, that for *any* $x \in \mathbb{R}^n$, $j, k = 1, \ldots, m$, the derivatives

$$(X_j f_k - X_k f_j)(x) = \frac{d}{dt}\Big(f_k(e^{tX_j} x) - f_j(e^{tX_k} x) \Big)\Big|_{t=0}$$

exist and that the functions $X_j f_k - X_k f_j$ are continuous for all j, k (here $t \mapsto e^{tX_j} x$ denotes the integral curve of X_j starting at x). Denote by $[X_j, X_k] = \langle X_j f_k - X_k f_j, \nabla \rangle := \sum_{i=1}^n f_{j,k}^i \partial_i$ the commutator. We require that

$$\operatorname{span}\{X_j(x), [X_j, X_k](x) : j, k = 1, \ldots, m\} = \mathbb{R}^n, \quad \text{for any } x \in \mathbb{R}^n. \tag{1}$$

We finally assume that for any compact $K \subset \mathbb{R}^n$ there is $L > 0$ such that

$$|f_{j,k}(x) - f_{j,k}(y)| \leq L d(x, y), \tag{2}$$

for every $x, y \in K$. Note that, by the recent nonsmooth version of Chow's theorem proved by Rampazzo and Sussman [20], the topology defined by d is the Euclidean one. Thus, in view of [11] and [9], the $d-$Lipschitz continuity (2) of $f_{j,k}$ is equivalent to the boundedness of the distributional derivatives along the vector fields , i.e.

$$\operatorname*{ess\,sup}_{x \in K} |X_i f_{j,k}(x)| \leq L, \quad \text{for all } i, j, k = 1, \ldots, m. \tag{3}$$

For every compact set K put

$$L = \sum_{j=1}^{m} \|\nabla f_j\|_{L^\infty(K)} + \sum_{i,j,k=1}^{m} \operatorname*{ess\,sup}_K |X_i f_{j,k}|. \tag{4}$$

All constants in our structure result will depend on (4).

Given a pair of locally Lipschitz continuous vector fields X and Y define for small s the map $\exp^*(s[X,Y])$ as follows

$$\exp^*(s[X,Y])(x) = \begin{cases} e^{-\sqrt{s}Y} e^{-\sqrt{s}X} e^{\sqrt{s}Y} e^{\sqrt{s}X} x & \text{if } s \geq 0, \\ e^{-\sqrt{|s|}X} e^{-\sqrt{|s|}Y} e^{\sqrt{|s|}X} e^{\sqrt{|s|}Y} x & \text{if } s \leq 0. \end{cases} \tag{5}$$

Enumerate the vector fields and their first order brackets as

$$Y_j = X_j, \quad j = 1, \ldots, m, \quad \{Y_{m+1}, \ldots, Y_q\} = \{[X_i, X_j], 1 \leq i < j \leq m\},$$

define the degree of Y_j as its length as a commutator and denote it by $d(Y_j)$. Given an n-tuple $I = (i_1, \ldots, i_n), i_j = 1, \ldots, q$, we define the "almost exponential map" related to I as follows

$$E_I(x,h) = \Big(\prod_{k=1}^{n} \exp^*(h_k Y_{i_k})\Big)(x) = \Big(\exp^*(h_1 Y_{i_1}) \circ \cdots \circ \exp^*(h_n Y_{i_n})\Big)(x) \tag{6}$$

where, if $d(Y_{i_k}) = 1$ then $\exp^*(h_k Y_{i_k})(x) = \exp(h_k Y_{i_k})(x)$, while if $d(Y_{i_k}) = 2$ (for instance $Y_{i_k} = [X_{p_k}, X_{l_k}]$ for some $p_k, l_k \in \{1, \ldots, m\}$, $p_k < l_k$) then $\exp^*(h_k Y_{i_k})(x)$ is defined in (5). The maps E_I have already been studied in the smooth case, see [19], [14], [22], [18]. Here we define and study their properties in a nonsmooth situation.

We shall prove that the control ball is equivalent to the I-box, defined as

$$\mathrm{Box}_I(x,r) = \Big\{x + \sum_{k=1}^{n} \xi_k Y_{i_k}(x), \|\xi\|_I \leq r\Big\}, \quad \|\xi\|_I = \max_{j=1,\ldots,n} |\xi_j|^{1/d(Y_{i_j})}. \tag{7}$$

Our first result is the following.

Theorem 1.1 *Given a compact K, for every $x \in K$ and $r < r_0$ there is an n-tuple I such that*

$$B(x, \varepsilon_2 r) \subset \mathrm{Box}_I(x, \varepsilon_1 r) \subset \Big\{E_I(x,h) : \|h\|_I \leq \varepsilon_0 r\Big\} \subset B(x, C\varepsilon_0 r) \tag{8}$$

where E_I is the map defined in (6). The constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, r_0$ are positive and depend on K and on L in (4), while C is an absolute positive constant only depending on the dimension n.

In the recent paper [20], Rampazzo and Sussman define the Lie bracket (at the point x) $[X,Y](x)$ of a pair of Lipschitz continuous vector fields X and Y as the convex hull of the set of all vectors $v = \lim_{x_j \to x}[X,Y](x_j)$ with x_j a point of differentiability of both X and Y. They prove, under the assumption that $\mathbb{R}^n$ is spanned by the set

$$\{X_j(x), j = 1, \dots, m\} \cup \{X_{k,l}(x), 1 \le k < l \le m\}$$

for any $x \in \mathbb{R}^n$ and for each choice of the vector $X_{k,l}(x)$ in the set $[X_k, X_l](x)$, that the control ball of radius r contains the Euclidean ball D of radius r^2. Namely, given a point x, there is a constant $c = c_x > 0$ such that, for small $r > 0$, $B(x,r) \supseteq D(x, c_x r^2)$. Our structure theorem improves this inclusion. In particular, if I satisfies (19), the ball $D(x, c_x r^2)$ can be replaced by $\mathrm{Box}_I(x,r)$. The latter contains $D(x, c_0 r^2)$ for some $c_0 > 0$ which can be chosen uniformly on compact sets (compare Lemma 3.4). To get this sharp result we have to require that the commutators $[X_k, X_l]$ are Lipschitz continuous, at least along the "horizontal directions". This assumption is somewhat reasonable because it ensures that the set $\mathrm{Box}_I(x,r)$ moves continuously with x in the Hausdorff distance in $\mathbb{R}^n$. Moreover, we mention that extra regularity properties of the commutator naturally appears in the analysis of the regularity properties of a real surface in $\mathbb{C}^2$ with smooth nonzero Levi curvature (see the work by Citti, Lanconelli and the first author [1] and the discussion in Section 5).

Theorem 1.1 gives the representation of the Carnot–Caratéodory balls by means of the maps E_I, which are controllable in the sense of [15]. Therefore the doubling property of the Lebesgue measure and the Poincaré inequality hold.

Theorem 1.2 *For any compact set $K \subset \mathbb{R}^n$ there are $c, r_0, Q > 0, \lambda \ge 1$, depending on K and L in (4), such that*

$$|B(x,2r)| \le 2^Q |B(x,r)|, \quad x \in K, r < r_0, \tag{9}$$

and

$$\int_B |u(y) - u_B| dy \le cr \int_{\lambda B} |Xu(x)| dx, \quad \forall u \in C^1(\lambda B) \tag{10}$$

with $B = B(x,r)$ and $\lambda B = B(x, \lambda r)$. Here $u_B = \fint_B u = \frac{1}{|B|}\int_B u$.

It is known that (9) and (10) are the basic tools for a complete study of the Sobolev embedding for Sobolev Spaces of order 1. See the references by Saloff-Coste [21], Maheux and Saloff Coste [16], Franchi Lu and Wheeden [8], Garofalo and Nhieu [10] and Hajłasz and Koskela [12].

Theorem 1.2 improves our previous results [17], where embeddings for first order Sobolev Spaces were proved but under the more restrictive condition that the vector fields are linearly independent at any point. Moreover, all the results in [17] were obtained for compactly supported functions and no properties of the control distance were studied.

Our paper is organized as follows. In section 2 we prove some estimates for the derivatives of the maps E_I. These will enable us to give, in Section 3, the structure theorem for the control balls. In section 4 we show the doubling property of the control distance and the Poincaré inequality. Section 5 is devoted to some examples. In particular we present a situation of Lipschitz continuous vector fields, related to the prescribed Levi curvature equation, which satisfy conditions (1) and (2).

Notation. We denote by C or c positive constants. If $u : \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}^n$ we denote by $uv = u \circ v$ the composition between u and v. Moreover, if $f_1, \dots, f_p$ are diffeomorphism in $\mathbb{R}^n$, we let $\prod_{j=1}^p f_j = f_1 \circ \cdots \circ f_p$. We denote by $e^{tX}x$ the solution of the Cauchy problem $\frac{d}{dt}e^{tX}x = X(e^{tX}x)$, $e^{tX}x\big|_{t=0} = x$. The Jacobian matrix of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by Df. Finally, $\langle \cdot, \cdot \rangle$ indicates the standard inner product in $\mathbb{R}^n$.

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2 Derivatives of the map E_I

In this section we assume the vector fields of class C^∞ and differentiate the map $E_I(x, h)$ defined in (6) with respect to h. We will discuss how to regularize non smooth vector fields in the next section. Although the vector fields are smooth all the constants appearing in this section depend on L in (4).

Proposition 2.1 *Let $I = (i_1, \dots, i_n)$, $Y_{i_k} = U_k$, $k = 1, \dots, n$. Then, for all $j = 1, \dots, n$,*

$$\frac{\partial}{\partial h_j} E_I(x, h) = U_j(x) + R_j(x, h),$$

where, given a compact set K, there is a neighborhood V of the origin in $\mathbb{R}^n$ such that for all $h \in V$ and $j = 1, \dots, n$, the remainder R_j satisfies the estimate

$$\sup_{x \in K} |R_j(x, h)| \le C\|h\|_I. \tag{11}$$

The proof of Proposition 2.1 relies on the computation of the derivative of the "approximate commutator" defined in (5). This will be done in Lemma 2.2. We shall use the following standard formulas:

$$\frac{d}{dt} Y(ue^{-tX})(e^{tX}x) = [X, Y](ue^{-tX})(e^{tX}x) \quad \text{and} \tag{12}$$

$$Y(ue^{-tX})(e^{tX}x) = Y(u)(x), \quad \text{if } [X, Y] = 0. \tag{13}$$

The following exact formula holds:

Lemma 2.2 *Given a pair X and Y of smooth vector fields, for small $s > 0$ the following formula holds for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$*

$$
\begin{aligned}
&\frac{d}{ds}u(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x) \\
&\quad = \frac{1}{2}[X,Y]\big(ue^{-\sqrt{s}Y}\big)\big(e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) + \frac{1}{2}[X,Y]\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}\big)\big(e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) \\
&\quad + \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}}\int_0^t [X,[X,Y]]\Big(ue^{-\sqrt{s}Y}e^{-\tau X}\Big)\Big(e^{(\tau-\sqrt{s})X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\Big)d\tau dt \\
&\quad + \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}}\int_0^t [[X,Y],Y]\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\tau Y}\big)\big(e^{(-\tau+\sqrt{s})Y}e^{\sqrt{s}X}x\big)d\tau dt.
\end{aligned}
$$

Proof. Write

$$
\begin{aligned}
&\frac{d}{ds}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) \\
&\qquad = \frac{d}{dt}u\big(e^{-\sqrt{t}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big)\big|_{t=s} + \frac{d}{dt}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{t}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big)\big|_{t=s} \\
&\qquad + \frac{d}{dt}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{t}Y}e^{\sqrt{s}X}x\big)\big|_{t=s} + \frac{d}{dt}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{t}X}x\big)\big|_{t=s}.
\end{aligned}
$$

The first term in the previous equality is

$$
\begin{aligned}
\frac{d}{dt}u\big(e^{-\sqrt{t}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big)\big|_{t=s} &= -\frac{1}{2\sqrt{s}}(Yu)\big(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) \\
&= -\frac{1}{2\sqrt{s}}Y\big(ue^{-\sqrt{s}Y}\big)\big(e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) := A_1,
\end{aligned}
$$

where we used (13) in the second equality. Analogously

$$
\begin{aligned}
\frac{d}{dt}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{t}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big)\big|_{t=s} &= -\frac{1}{2\sqrt{s}}X\big(ue^{-\sqrt{s}Y}\big)\big(e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) \\
&= -\frac{1}{2\sqrt{s}}X\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}\big)\big(e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) := A_2, \\
\frac{d}{dt}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{t}Y}e^{\sqrt{s}X}x\big)\big|_{t=s} &= \frac{1}{2\sqrt{s}}Y\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}\big)\big(e^{\sqrt{s}Y}e^{\sqrt{s}X}x\big) := A_3, \\
\frac{d}{dt}u\big(e^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{t}X}x\big)\big|_{t=s} &= \frac{1}{2\sqrt{s}}X\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}\big)\big(e^{\sqrt{s}X}x\big) := A_4.
\end{aligned}
$$

Let $y = e^{-\sqrt{s}X}e^{\sqrt{s}Y}e^{\sqrt{s}X}x$ and take (12) into account. Thus

$$
\begin{aligned}
A_3 + A_1 =& \frac{1}{2\sqrt{s}}\Big\{Y\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}\big)\big(e^{\sqrt{s}X}y\big) - Y\big(ue^{-\sqrt{s}Y}\big)(y)\Big\} \\
=& \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}} \frac{d}{dt}Y\big(ue^{-\sqrt{s}Y}e^{-tX}\big)\big(e^{tX}y\big)dt \\
=& \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}} [X,Y]\big(ue^{-\sqrt{s}Y}e^{-tX}\big)\big(e^{tX}y\big)dt \\
=& \frac{1}{2}[X,Y]\big(ue^{-\sqrt{s}Y}\big)(y) + \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}}\int_0^t [X,[X,Y]]\big(ue^{-\sqrt{s}Y}e^{-\tau X}\big)\big(e^{\tau X}y\big)d\tau dt.
\end{aligned}
$$

Analogously, letting $y' = e^{\sqrt{s}Y}e^{\sqrt{s}X}x$ we can write

$$
\begin{aligned}
A_4 + A_2 &= \frac{1}{2\sqrt{s}}X\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\sqrt{s}Y}\big)\big(e^{-\sqrt{s}Y}y'\big) - \frac{1}{2\sqrt{s}}X\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}\big)(y') \\
&= \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}} \frac{d}{dt}X\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{tY}\big)\big(e^{-tY}y'\big)dt \\
&= \frac{1}{2}[X,Y]\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}\big)(y')dt \\
&\quad + \frac{1}{2\sqrt{s}}\int_0^{\sqrt{s}}\int_0^t [-Y,[X,Y]]\big(ue^{-\sqrt{s}Y}e^{-\sqrt{s}X}e^{\tau Y}\big)\big(e^{-\tau Y}y'\big)d\tau dt.
\end{aligned}
$$

Now the proof can be easily concluded summing up A_1, A_2, A_3 and A_4. □

To estimate the terms of the exact formula in Lemma 2.2 we shall use the following lemma.

Lemma 2.3 *Let $X_1 \dots, X_p$ and Y be smooth vector fields. Let also $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then, if $t_1, \dots, t_p$ are small numbers and x belong to a compact set K,*

$$
\big|Yu(e^{t_1X_1}e^{t_2X_2}\cdots e^{t_pX_p}x) - Yu(x)\big| \le \sum_{j=1}^{p} |t_j|\,\|X_jYu\|_\infty, \tag{14}
$$

where $\|X_jYu\|_\infty$ denotes the supremum norm in some neighborhood of K. Let $\pi_k(x) = x_k$. Then for all $k = 1, \dots, n$

$$
\big|Y(\pi_k e^{t_1X_1}e^{t_2X_2}\cdots e^{t_pX_p})(x) - Y\pi_k(x)\big| \le C\sum_{j=1}^{p} |t_j|, \tag{15}
$$

where, if $x \in K$ and $\sum |t_j|$ is small enough, the constant C depends on the Lipschitz constants of the X_j's in some neighborhood of K.

Proof. Both the estimates are standard. In order to control carefully the constants in their right hand sides, we recapitulate their proofs.

Estimate (14) is an easy consequence of the fundamental theorem of calculus.

$$\begin{aligned}
&\big|Yu(e^{t_1X_1}e^{t_2X_2}\cdots e^{t_pX_p}x) - Yu(x)\big| \\
&\le \big|Yu(e^{t_1X_1}e^{t_2X_2}\cdots e^{t_pX_p}x) - Yu(e^{t_2X_2}\cdots e^{t_pX_p}x)\big| + \cdots + \big|Yu(e^{t_pX_p}x) - Yu(x)\big| \\
&\le \int_0^{t_1} \big|X_1Yu(e^{tX_1}e^{t_2X_2}\cdots e^{t_pX_p}x)\big|dt + \cdots + \int_0^{t_p} \big|X_pYu(e^{tX_p}x)\big|dt \le \sum_{j=1}^p |t_j|\, \|X_jYu\|_\infty.
\end{aligned}$$

To prove (15) it suffices to show that

$$\big|Y(\pi_k e^{t_1X_1}v)(x) - Y(\pi_k v)(x)\big| \le C_1 |Y(\pi_k v)(x)||t_1|. \tag{16}$$

with $v = e^{t_2X_2}\cdots e^{t_pX_p}$. (15) will follow iterating (16). Let $Y(\pi_k e^{tX_1}v)(x) = \xi_k(t)$, $Y = \sum a_j\partial_j$, $X_1 = \sum b_j\partial_j$. Then (we omit the summation on repeated indices)

$$\begin{aligned}
\xi_k(t) - \xi_k(0) &= \int_0^t \frac{d}{ds} Y(\pi_k e^{sX_1}v)(x)ds = \int_0^t a_j(x)\frac{d}{ds}\frac{\partial}{\partial x_j}(\pi_k e^{sX_1}v)(x)ds \\
&= \int_0^t a_j(x)\frac{\partial}{\partial x_j}\frac{d}{ds}(\pi_k e^{sX_1}v)(x)ds = \int_0^t a_j(x)\frac{\partial}{\partial x_j} b_k(e^{sX_1}v)(x)ds \\
&= \int_0^t a_j(x)(\partial_i b_k)(e^{sX_1}v)(x)\frac{\partial}{\partial x_j}\pi_i(e^{sX_1}v)(x)ds \\
&= \int_0^t (\partial_i b_k)(e^{sX_1}v)(x) Y(\pi_i e^{sX_1}v)(x)ds = \int_0^t (\partial_i b_k)(e^{sX_1}v)(x)\xi_i(s)ds.
\end{aligned}$$

Thus we have $|\xi(t) - \xi(0)| \le C\int_0^t |\xi|$, where we used the boundedness of $(\partial_i b_k)$ in a neighborhood of K. The proof follows from Gronwall inequality. □

Proof of Proposition 2.1. In the proof u is any of the π_k's. Let $u_j = u\prod_{k=1}^{j-1}\exp^*(h_kU_k)$ and $\xi = \prod_{k=j+1}^n \exp^*(h_kU_k)(x)$. We have

$$\frac{\partial}{\partial h_j} u(E_I(x,h)) = \frac{\partial}{\partial h_j} u_j\big(\exp^*(h_jU_j)\xi\big).$$

We distinguish two cases. If $d(U_j) = 1$, then

$$\frac{\partial}{\partial h_j} u(E_I(x,h)) = U_j u_j(e^{h_jU_j}\xi) = U_j\Big(u\prod_{k=1}^{j-1}\exp^*(h_kU_k)\Big)\Big(\prod_{k=j}^{n}\exp^*(h_kU_k(x)\Big).$$

By the inequality (15) we get

$$\Big|\frac{\partial}{\partial h_j} u(E_I(x,h)) - (U_ju)\Big(\prod_{k=j}^n \exp^*(h_kU_k)x\Big)\Big| \le C\sum_{k=1}^{j-1}|h_k|^{1/d(U_k)} \le C\|h\|_I,$$

where the constant C depends on the Lipschitz norm of the original vector fields X_i, $i = 1, \dots, m$. The proof can be concluded by estimating $(U_j u)\big(\prod_{k=j}^n \exp^*(h_k U_k)(x)\big) - (U_j u)(x)$ by means of (14).

If instead $d(U_j) = 2$, say $U_j = [X_{p_j}, X_{l_j}]$, then by Lemma 2.2

$$
\begin{aligned}
&\frac{\partial}{\partial h_j} u(E_I(x,h)) \\
&= \frac{1}{2} U_j \Big(u_j e^{-\sqrt{h_j} X_{l_j}} \Big)(\zeta) + \frac{1}{2} U_j \Big(u_j e^{-\sqrt{h_j} X_{l_j}} e^{-\sqrt{h_j} X_{p_j}} \Big)(\zeta') \\
&\quad + \frac{1}{2\sqrt{h_j}} \int_0^{\sqrt{h_j}} \Big(\int_0^t [X_{p_j}, U_j] \Big(u_j e^{-\sqrt{h_j} X_{l_j}} e^{-\tau X_{p_j}} \Big) \big(e^{\tau X_{p_j}} \zeta \big) d\tau \Big) dt \\
&\quad + \frac{1}{2\sqrt{h_j}} \int_0^{\sqrt{h_j}} \Big(\int_0^t [U_j, X_{l_j}] \Big(u_j e^{-\sqrt{h_j} X_{l_j}} e^{-\sqrt{h_j} X_{p_j}} e^{\tau X_{l_j}} \Big) \Big(e^{-\tau X_{l_j}} \zeta' \Big) d\tau \Big) dt,
\end{aligned}
\tag{17}
$$

where we let $\zeta = e^{-\sqrt{h_j} X_{p_j}} e^{\sqrt{h_j} X_{l_j}} e^{\sqrt{h_j} X_{p_j}} \xi$ and $\zeta' = e^{\sqrt{h_j} X_{l_j}} e^{\sqrt{h_j} X_{p_j}} \xi$. We use again (15) and (14) to estimate

$$
\Big| \frac{1}{2} U_j \Big(u_j e^{-\sqrt{h_j} X_{l_j}} \Big)(\zeta) + \frac{1}{2} U_j \Big(u_j e^{-\sqrt{h_j} X_{l_j}} e^{-\sqrt{h_j} X_{p_j}} \Big)(\zeta') - U_j(x) \Big| \le C \|h\|_I
$$

where the constant C depends on L in (4).

To conclude the proof of the Proposition, note that both the terms in the last two lines of in (17) can be estimated by a sum of terms of the form $\|h\|_I \|[X_i, [X_j, X_k]]\|_\infty$, where $i, j, k = 1, \dots, m$. All these suprema can be estimated by L, the constant appearing in (4). □

3 $I-$Boxes and structure of balls

In this section we take Lipschitz continous vector fields satisfying (1) and (2) and we shall prove Theorem 1.1. We start with the following remark.

Remark 3.1 *The last inclusion in the right hand side of* (8) *can be proved rather easily. Indeed, since* $t \mapsto \exp(tX_j)(x)$ *is a subunit path, then* $d(x, \exp(tX_j)(x)) \le |t|$, *for every* $j = 1, \dots, m$. *By the definition of* E_I *in* (6), *for every* n*-tuple* $I = (i_1, \dots, i_n)$ *we have the inequality*

$$
d(x, E_I(x,h)) \le \sum_{k=1}^n 4^{d(Y_{i_k})-1} |h_k|^{1/d(Y_{i_k})} \le C \|h\|_I,
$$

with an absolute constant $C = C(n)$, *and the inclusion* $\{E_I(x,h) : \|h\|_I \le r\} \subset B(x, Cr)$ *follows.*

In order to prove the remaining part of the theorem, given an n-tuple $I = (i_1, \dots, i_n) \in \{1, \dots, q\}^n$, we shall denote $d(I) = \sum_{j=1}^n d(Y_{i_j})$ and

$$\lambda_I(x) = \det[Y_{i_1}(x), \dots, Y_{i_n}(x)]. \tag{18}$$

By condition (1) at every point x there is an n-tuple I such that $\lambda_I(x) \neq 0$.

Theorem 1.1 will be an immediate consequence of Remark 3.1 and of the following theorem.

Theorem 3.2 *Given a compact set K, there is a neighborhood V of the origin in $\mathbb{R}^n$ such that the map $E_I(x,\cdot)$ is Lipschitz continuous in V for all I and for any $x \in K$. Moreover there are $r_0 > 0$, $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$ such that if $x \in K$, $r < r_0$, and I satisfy*

$$|\lambda_I(x)| r^{d(I)} > \frac{1}{2} \max_{J \in \{1,\dots,q\}^n} |\lambda_J(x)| r^{d(J)}, \tag{19}$$

then

- *(i)* $\frac{1}{2}|\lambda_I(x)| \leq \left|\det \frac{\partial E_I}{\partial h}(x,h)\right| \leq 2|\lambda_I(x)|$ *for almost all* $h \in \mathbb{R}^n$, $\|h\|_I \leq \varepsilon_0 r$.
- *(ii) The map* $E_I(x,\cdot)$ *is one–to–one on* $\{\|h\|_I \leq \varepsilon_0 r\}$.
- *(iii)* $\mathrm{Box}_I(x, \varepsilon_1 r) \subset \{E_I(x,h) : \|h\|_I \leq \varepsilon_0 r\}$.
- *(iv)* $B(x, \varepsilon_2 r) \subset \mathrm{Box}_I(x, \varepsilon_1 r)$.

We start by proving the Lipschitz continuity of $E_I(x,\cdot)$ and some estimates on derivatives. The following lemma is the nonsmooth version of Lemma 2.1.

Lemma 3.3 *Fix a compact set $K \subset \mathbb{R}^n$. Then there exist a neighborhood V of the origin in $\mathbb{R}^n$ and a constant $C > 0$ such that, for all I and for any $x \in K$, the map $E_I(x,\cdot)$ is Lipschitz continuous on V and satisfies for $j = 1, \dots, n$*

$$\frac{\partial}{\partial h_j} E_I(x,h) = U_j(x) + R_j(x,h), \qquad \textit{for a.e. } h \in V, \tag{20}$$

where

$$|R_j(x,h)| \leq C\|h\|_I, \quad \textit{for any } h \in V. \tag{21}$$

Proof. The proof relies on the smooth version of the lemma, proved in the previous section, and on an approximation argument. Denote by $X_j^{(\varepsilon)} = \langle f_j^{(\varepsilon)}, \nabla\rangle$, where $f_j^{(\varepsilon)}(x) = \int_{\mathbb{R}^n} \rho(h) f_j(x + \varepsilon h) dh$ is the usual mollifier. We have $[X_j^{(\varepsilon)}, X_k^{\varepsilon)}] = \langle (Df_k^{(\varepsilon)}) f_j^{(\varepsilon)} - (Df_j^{(\varepsilon)}) f_k^{(\varepsilon)}, \nabla\rangle$ and it is easy to realize that

$$\left|(Df_k^{(\varepsilon)}) f_j^{(\varepsilon)} - (Df_j^{(\varepsilon)}) f_k^{(\varepsilon)} - \big((Df_k) f_j - (Df_j) f_k\big)^{(\varepsilon)}\right| \leq C\varepsilon$$

with C depending on the Lipschitz constants of the coefficients of the vector fields. Hence, for ε small, the vector fields $X_j^{(\varepsilon)}$ satisfy Hörmander condition (1) at every point. We now use the results of the previous section for the mollified vector fields. In particular Proposition 2.1 holds for the vector fields $X_j^{(\varepsilon)}$ with constants independent of ε. Namely, denoting by $E_I^{(\varepsilon)}$ the (smooth) map arising from the $X_j^{(\varepsilon)}$'s,

$$\frac{\partial}{\partial h_j} E_I^{(\varepsilon)}(x,h) = U_j^{(\varepsilon)}(x) + R_j^{(\varepsilon)}(x,h), \tag{22}$$

$$|R_j^{(\varepsilon)}(x,h)| \le C\|h\|_I, \quad h \in V, j = 1,\dots,n \tag{23}$$

where C and V are independent of ε. By the form (6) of E_I we immediately recognize that for any x, I, h, the limit as ε goes to 0 of $E_I^{(\varepsilon)}(x,h)$ exists and we put $E_I(x,h) := \lim_{\varepsilon\to 0} E_I^{(\varepsilon)}(x,h)$.

Fix any $x \in K$. The family $E_I^{(\varepsilon)}(x,\cdot)$ is bounded in $W^{1,\infty}(V)$, by (22) and (23). Thus it is bounded in $W^{1,2}(V)$. We can choose a sequence $\varepsilon_k \downarrow 0$ such that $E_I^{(\varepsilon_k)}(x,\cdot) \to E_I(x,\cdot)$, weakly in $W^{1,2}(V)$, i.e. $E_I^{(\varepsilon_k)}(x,\cdot) \to E_I(x,\cdot)$ and $\partial_{h_j} E_I^{(\varepsilon_k)}(x,\cdot) \to \partial_{h_j} E_I(x,\cdot)$ weakly in $L^2(V)$ as $k \to \infty$. By taking the L^2-weak limit in (22) we get

$$\lim_{k\to\infty} R_I^{(\varepsilon_k)}(x,\cdot) = \lim_{k\to\infty} \partial_{h_j} E_I^{(\varepsilon_k)}(x,\cdot) - U_j^{(\varepsilon_k)}(x) = \partial_{h_j} E_I(x,\cdot) - U_j(x) := R_I^{\infty}(x,\cdot),$$

where the function $R_I^\infty(x,\cdot)$ is defined by the last equality. By standard properties of weak convergence, the estimate (23) preserves under the limit, i.e. $|R_j^\infty(x,h)| \le C\|h\|_I$ for almost any $h \in V$. The function $E_I(x,\cdot)$ is continuous by (6). Moreover, it turns out to belong to $W^{1,\infty}(V)$, because its (distributional) derivative $\partial_{h_j} E_I(x,\cdot)$ is essentially bounded. Then (see [4, Section 4.2.3]) $E_I(x,\cdot)$ is Lipschitz continuous on V. Thus, (20) holds and the derivatives in the left hand side actually are pointwise derivatives calculated at a point of differentiability of $E_I(x,\cdot)$.

Finally, by choosing a suitable representative $R_j(x,\cdot) = R_j^\infty(x,\cdot)$ a.e., we conclude that (21) holds for any $h \in V$. □

To prove Theorem 3.2 we need some more preliminary lemmas.

Lemma 3.4 *For all $\lambda \in \,]0,1]$ there is $\eta = \eta(\lambda) > 0$ such that if $x \in K$, $r < r_0$ and $I \in \{1,\dots,q\}^n$ satisfy (19), then*

$$D(0,\eta r^2) \subset \Big\{ \sum_{j=1}^n \xi_j Y_{i_j}(x) : \|\xi\|_I \le \lambda r \Big\}, \tag{24}$$

with $D(x,r)$ the Euclidean disc of radius r and center at x.

Proof. Denote for brevity $Y_{i_j} = U_j$. Fix $\lambda \in \,]0,1]$. We first prove that we can find $\eta > 0$ such that

$$D(0,\eta r^2) \subset \Big\{ \sum_{i=1}^q \xi_i Y_i(x) : |\xi| \le \frac{\lambda^2}{2q} r^2 \Big\}, \tag{25}$$

with $|\cdot|$ the Euclidean norm. To prove (25) it suffices to take a $n-$tuple $P = (p_1, \dots, p_n)$ such that $|\lambda_P(x)| = \max_J |\lambda_J(x)|$. Then

$$\left\| \left[Y_{p_1}(x), \dots, Y_{p_n}(x) \right]^{-1} \right\| \le \frac{C}{|\lambda_P(x)|} \le \tilde{C}, \tag{26}$$

where the constant C depends on an upper bound on the norm of the Y_j's on the compact K (recall (18)) and the last estimate comes from the fact that $\inf_{x\in K} \sum_J |\lambda_J(x)| > 0$ by (1). Thus (25) follows by choosing $\xi_s = 0$ if $s \notin \{p_1, \dots, p_n\}$, by solving the Cramer system $\sum_{j=1}^n Y_{p_j}(x)\xi_{p_j} = u$ and by using estimate (26).

Since the vector fields $Y_{i_j}(x) = U_j(x)$, $j = 1, \dots, n$ are linearly independent, for any $l = 1, \dots, q$ we can uniquely write

$$Y_l(x) = \sum_{k=1}^n a_{lk}(x) U_k(x). \tag{27}$$

Following [19] we use the Cramer rule

$$a_{lk}(x) = \frac{\det(U_1, \dots, U_{k-1}, Y_l, U_{k+1}, \dots, U_n)(x)}{\det(U_1, \dots, U_{k-1}, U_k, U_{k+1}, \dots, U_n)(x)} \tag{28}$$

and, since we know that $|\lambda_I(x)| r^{d(I)} > \frac{1}{2} \max_J |\lambda_J(x)| r^{d(J)}$ we get

$$|a_{lk}(x)| \le 2 r^{d(U_k) - d(Y_l)}. \tag{29}$$

We are now ready to prove (24). Indeed, for any $u \in D(0, \eta r^2)$, by (25) there is a choice of ξ such that $u = \sum_{l=1}^q \xi_l Y_{i_l}(x)$, with $|\xi| \le \frac{\lambda^2}{2q} r^2$. Then, by (27),

$$u = \sum_{l=1}^q \xi_l \sum_{k=1}^n a_{lk}(x) U_k(x) := \sum_{k=1}^n b_k U_k(x),$$

where $|b_k| \le \sum_{l=1}^q |\xi_l|\, |a_{lk}(x)| \le \sum_{l=1}^q \frac{\lambda^2 r^2}{2q} 2 r^{d(U_k) - d(Y_l)} \le \lambda^2 r^{d(U_k)}$, for any $r < 1$. □

Proposition 3.5 *For all $\chi > 0$ there is $\varepsilon_0(\chi) > 0$ such that if (19) holds for $x \in K$, $r < r_0$ and I, then we can write for almost any h,*

$$\begin{gathered} \frac{\partial E_I(x,h)}{\partial h_j} = U_j(x) + \sum_{k=1}^n b_{j,k} U_k(x), \\ \textit{where } |b_{j,k}| = |b_{j,k}(x,h)| \le \chi r^{d(U_k) - d(U_j)} \ \forall\ h,\ \|h\|_I < \varepsilon_0 r. \end{gathered} \tag{30}$$

Proof. The estimate in the second line of (30) is equivalent to $r^{d(U_j)} |b_{j,k}(x,h)| \le \chi r^{d(U_k)}$ for all k, j. For $\chi < 1$ this will be ensured by the stronger estimate $r^{d(U_j)} |b_{j,k}(x,h)| \le (\chi r)^{d(U_k)}$, i.e.

$$r^{d(U_j)} R_j(x,h) \in \mathrm{Box}_I(x, \chi r), \tag{31}$$

because, in the notation of Lemma 3.3, $R_j(x,h) = \sum_{k=1}^n b_{j,k} U_k(x)$. We have already proved (see (21)) that $|R_j(x,h)| \le C\|h\|_I$. Thus, assuming $\|h\|_I \le \varepsilon_0 r$, we have $|R_j(x,h)| \le C\varepsilon_0 r$. Recall now that we also proved in Lemma 3.4 that, if (19) holds, $\mathrm{Box}_I(x,\chi r) \supset D(0,\eta(\chi)r^2)$. Thus we conclude that (31) is ensured by the choice $C\varepsilon_0 \le \eta(\chi)$. □

Proof of Theorem 3.2. In the following proof the $n-$tuple I satisfying (19) is fixed. We write $Y_{i_j} = U_j$ and $d(U_j) = d_j$, $j = 1,\dots,n$.

Proof of (i). By Proposition 3.5 we have for a.e. h, $\det \frac{\partial E_I(x,h)}{\partial h_j} = \lambda_I(x) \det(\delta_{j,k} + b_{j,k})$ (here $\delta_{j,k}$ denotes the Kronecker symbol). The proof of (i) can be easily concluded. Indeed, since $\delta_{j,k}$ is a diagonal matrix, we have $\det(\delta_{j,k} + b_{j,k}) = \det(\delta_{j,k} + b_{j,k} r^{d_j - d_k}) \in (1/2, 2)$, if we choose the constant χ smaller than a suitable dimensional constant $\chi_0(n)$.

Proof of (ii). Fix $\chi > 0$ such that (i) holds. The map $E_I(x,\cdot)$ is Lipschitz continuous and, by Rademacher theorem, for any h' there is a subset $\Sigma(h')$ of the $(n-1)$–sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, such that the surface measure of $\mathbb{S}^{n-1} \setminus \Sigma(h')$ is zero and such that for any $v \in \Sigma(h')$ the map $h \mapsto E_I(x,h)$ is differentiable at any point of the set $\{h' + tv \in \mathbb{R}^n : t \ge 0\} \cap \{h \in \mathbb{R}^n : \|h\|_I < \varepsilon_0 r\}$ except a subset of $1-$dimensional measure zero. Thus for any h such that $\frac{h-h'}{|h-h'|} \in \Sigma(h')$ we can use the gradient formula to calculate $\frac{d}{dt} E_I(x, h' + t(h-h'))$ and the fundamental theorem of calculus to get

$$\begin{aligned} E_I(x,h) - E_I(x,h') &= \int_0^1 \sum_{j=1}^n \frac{\partial E_I}{\partial h_j}(x, h' + t(h-h'))\,(h_j - h'_j)dt \\ &= \sum_{j=1}^n \Big(U_j(x) + \sum_{k=1}^n \int_0^1 b_{j,k}(x, h' + t(h-h'))dt U_k(x)\Big)\,(h_j - h'_j) \\ &= U(x)\,(I_n + B^T)\,(h-h'), \end{aligned} \tag{32}$$

where I_n is the identity matrix, $U = [U_1,\dots,U_n]$ and B is the $n \times n$ matrix whose entries $\beta_{j,k} = \int_0^1 b_{j,k}(x, h' + t(h-h'))dt$ satisfy $|\beta_{j,k}| \le \chi r^{d_k - d_j}$. Since

$$\det(U(x)(I + B^T)) = \lambda(x)\det(\delta_{j,k} + \beta_{k,j}) = \lambda(x)\det(\delta_{j,k} + \beta_{k,j} r^{d_k - d_j})$$

with $\lambda(x) \ne 0$, and we have chosen χ such that $\det(\delta_{j,k} + \beta_{k,j} r^{d_k - d_j}) \in (1/2, 2)$ then the matrix $U(x)\,(I + B^T)$ is invertible. Ultimately, we have for any fixed h', $\|h'\|_I \le \varepsilon_0 r$, for almost any h, $\|h\|_I \le \varepsilon_0 r$, the following estimate

$$|h' - h| \le \|(I + B^T)^{-1}\|\,\|U(x)^{-1}\|\,|E_I(x,h) - E_I(x,h')| \le C|E_I(x,h) - E_I(x,h')|, \tag{33}$$

where the constant C does not depend on h and h'. Thus, by continuity, (33) holds for all h, h', with $\|h\|_I, \|h'\|_I \le \varepsilon_0 r$. Therefore the map $h \mapsto E_I(x,h)$ is injective.

Proof of (iii). We shall show that if ε_1 is small enough, for every $\xi \in \mathbb{R}^n$, $\|\xi\|_I \le \varepsilon_1 r$ the equation

$$x + U(x)\xi = E_I(x,h) \tag{34}$$

has a solution $h \in Q_I(\varepsilon_0 r) := \{h \in \mathbb{R}^n : \|h\|_I \le \varepsilon_0 r\}$. Here ε_0 is the constant fixed in the proof of (i) and (ii).

To prove (34) we shall use the homotopic invariance of the topological degree. Let

$$F_\lambda(h) = \lambda U(x)^{-1}\big(E_I(x,h) - x\big) + (1-\lambda)h$$

Recall that the map $h \to E_I(x,h)$ is continuous. Thus F_λ is a continuous map in $(\lambda, h) \in [0,1] \times \overline{Q_I(\varepsilon_0 r)}$. Moreover $F_0(h) = h$ and $F_1(h) = U(x)^{-1}(E_I(x,h) - x)$. Equation (34) is equivalent to $F_1(x,h) = \xi$. The topological degree $\deg(F_0; Q_I(\varepsilon_0 r); \xi)$ of the map $F_0(h) = h$ is clearly 1 as soon as $\varepsilon_1 < \varepsilon_0$. In order to use here the invariance of the degree (see, e.g. [3]) we have to check that for a suitable choice of ε_1 we have

$$\xi \notin F_\lambda(\partial Q_I(\varepsilon_0 r)), \quad \text{for all } \lambda \in [0,1]. \tag{35}$$

If (35) holds, then we can assert that $\deg(F_1; Q_I(\varepsilon_0 r); \xi) = 1$. Thus there is $h \in Q_I(\varepsilon_0 r)$ such that (34) holds.

In order to prove (35) note that the latter is implied by the strict inequality

$$\big\|\lambda U(x)^{-1}\big(E_I(x,h) - x\big) + (1-\lambda)h\big\|_I > \|\xi\|_I, \quad \forall\, h \in \partial Q_I(\varepsilon_0 r). \tag{36}$$

To show (36) we prove that there is an absolute constant C_0 such that

$$\big\|\lambda U(x)^{-1}\big(E_I(x,h) - x\big) + (1-\lambda)h\big\|_I \ge C_0 \varepsilon_0^2 r, \quad \forall\, h \in \partial Q_I(\varepsilon_0 r). \tag{37}$$

Then the choice of any positive ε_1 such that $\varepsilon_1 < C_0 \varepsilon_0^2$ gives (iii).

Now by continuity it is enough to check (37) for any $h \in \partial Q_I(\varepsilon_0 r)$ with $\frac{h}{|h|} \in \Sigma(0)$ (recall that $\Sigma(0)$ has full measure in $\mathbb{S}^{n-1}$). For any of those h's, equation (32) with $h' = 0$ holds and gives $E_I(x,h) = x + U(x)\,(I_n + B^T(x,h))h$. Therefore, (37) is ensured by

$$\begin{aligned}
&\|\lambda(I_n + B^T(x,h))h + (1-\lambda)h\|_I \ge C_0\varepsilon_0^2 r, \quad \forall\, h \in \partial Q_I(\varepsilon_0 r)\\
\Leftrightarrow\quad & \frac{\|(I_n + \lambda B^T(x,h))h\|_I}{r} \ge C_0 \varepsilon_0^2, \quad \forall\, h \in \partial Q_I(\varepsilon_0 r)\\
\Leftrightarrow\quad & \max_{k=1,\dots,n} \big|\big((I_n + \lambda B^T(x,h))h\big)_k r^{-d_k}\big|^{1/d_k} \ge C_0\varepsilon_0^2, \quad \forall\, h \in \partial Q_I(\varepsilon_0 r)
\end{aligned} \tag{38}$$

In order to prove the last inequality, take h such that $\|h\|_I = \varepsilon_0 r$. This means that for some l we have $|h_l|^{1/d_l} = \varepsilon_0 r$. Then write

$$((I_n + \lambda B^T)h)_k r^{-d_k} = \sum_{j=1}^n (I_n + \lambda B^T)_{k,j} r^{d_j - d_k}\big(h_j r^{-d_j}\big) = \sum_{j=1}^n (\delta_{k,j} + \lambda \beta_{j,k} r^{d_j - d_k})(h_j r^{-d_j}),$$

where $\beta_{j,k}$ were introduced after (32). Introduce the matrix M defined by $M_{k,j} = (\delta_{k,j} + \lambda\beta_{j,k} r^{d_j - d_k})$ and denote $v = \sum_{j=1}^n h_j r^{-d_j} e_j$. Thus the inequality in the last line of (38) may be written as

$$\|Mv\|_I \geq C_0 \varepsilon_0^2.$$

By the same argument of the proof of (i), we may assert that the matrix M satisfies $|Mv| \geq C|v|$ for all $v \in \mathbb{R}^n$ (here $|\cdot|$ denotes the Euclidean norm), for every $h \in Q_I(\varepsilon_0 r)$, $\lambda \in [0,1]$. Note also that both v and Mv belong to a fixed compact set (ε_0 has been fixed in (ii)). Thus there are positive constants C_2, C_1, C_0 such that

$$\|Mv\|_I \geq C_1 |Mv| \geq C_2 |v| \geq C_0 \|v\|_I^2 = C_0 \Big(\max_j \frac{|h_j|^{1/d_j}}{r} \Big)^2 \geq C_0 \Big(\frac{|h_l|^{1/d_l}}{r} \Big)^2 = C_0 \varepsilon_0^2.$$

The proof of (iii) is concluded.

Proof of (iv). Now ε_1 has been fixed. We will prove that there is $\varepsilon_2 > 0$ such that for any $y \in B(x, \varepsilon_2 r)$, there is ξ such that $\|\xi\|_I \leq \varepsilon_1 r$ and

$$y = x + \sum_{k=1}^n \xi_k Y_{i_k}(x). \tag{39}$$

Write $y = \gamma(1)$, where $\dot\gamma(t) = \sum_{j=1}^n b_j(t) X_j(\gamma(t))$ and $|b(t)| \leq \varepsilon_2 r$ for a.e t. Thus

$$\begin{aligned} y &= x + \int_0^1 \dot\gamma(t) = x + \sum_{j=1}^m \int_0^1 b_j(t) X_j(\gamma(t)) dt \\ &= x + \sum_{j=1}^m \int_0^1 b_j(t) dt X_j(x) + \sum_{j=1}^m \int_0^1 b_j(t) \big(X_j(\gamma(t)) - X_j(x) \big) dt \\ &:= x + \sum_{j=1}^m \beta_j X_j(x) + G(\gamma), \end{aligned}$$

where we let $\int_0^1 b_j(t) dt = \beta_j$. Note that $|\beta_j| \leq \varepsilon_2 r$. The remainder $G(\gamma)$ satisfies $|G(\gamma)| \leq C\varepsilon_2 r^2$. Indeed $|X_j(\gamma(t)) - X_j(x)| \leq L|\gamma(t) - x| \leq CLd(\gamma(t), x) \leq CL\varepsilon_2 r$ (here L is the Lipschitz constant of the vector fields). Thus, using (27) we conclude

$$y = x + \sum_{j=1}^m \beta_j \sum_{k=1}^n a_{jk}(x) U_k(x) + G(\gamma) = x + \sum_{k=1}^n \Big(\sum_{j=1}^m \beta_j a_{jk}(x) \Big) U_k(x) + G(\gamma).$$

By (29)

$$\big| \big(\sum_{j=1}^m \beta_j a_{jk}(x) \big) \big| \leq C \varepsilon_2 r^{d(U_k)}. \tag{40}$$

Thus the proof will be concluded as soon as we are able to solve

$$\sum_{k=1}^n \Big(\sum_{j=1}^m \beta_j a_{jk}(x) \Big) U_k(x) + G(\gamma) = \sum_{k=1}^n \xi_k U_k(x)$$

with $\|\xi\|_I \le \varepsilon_1 r$. This can be easily done, if ε_2 is small enough, by taking into account estimate (40) and Lemma 3.4. $\square$

4 Doubling property and Poincaré inequality

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. The doubling property (9) will follow from the equivalence $c^{-1}\Lambda(x,r) \le |B(x,r)| \le c\Lambda(x,r)$, where $\Lambda(x,r) = \sum_I |\lambda_I(x)| r^{d(I)}$. Fix x and r and choose I such that (19) holds. By (8) $|B(x,\varepsilon_2 r)| \le |\mathrm{Box}_I(x,\varepsilon_1 r)| = |\lambda_I(x)|(\varepsilon_1 r)^{d(I)} \le c\Lambda(x,\varepsilon_2 r)$.

The estimate from below can be proved by (8) with a similar argument (recall that $|\lambda_I(x)| r^{d(I)} \ge c\Lambda(x,r)$ if (19) holds).

We now prove the Poincaré inequality. Fix a ball $B(x_0,\varepsilon_2 r)$. For any x and I satisfying (19) for the prescribed r, by Proposition 3.2, the map $h \mapsto E_I(x,h)$ is one–to–one on $Q_I(\varepsilon_0 r) = \{\|h\|_I \le \varepsilon_0 r\}$ with $\varepsilon_0 > \varepsilon_1$ and its Jacobian satisfies $\frac{1}{2}|\lambda_I(x)| \le |\det \frac{\partial E_I}{\partial h}(x,h)| \le 2|\lambda_I(x)|$, for almost all h, $\|h\|_I \le \varepsilon_0 r$. Moreover, (8) gives $\mathrm{Box}_I(x,\varepsilon_1 r) \subset \{E_I(x,h) : \|h\|_I \le \varepsilon_0 r\}$. In the language of [15], these facts mean that, letting

$$\Omega_I = \Big\{x \in B(x_0,\varepsilon_2 r) : |\lambda_I(x)| r^{d(I)} > \frac{1}{2}\max_J |\lambda_J(x)| r^{d(J)}\Big\},$$

the map $E_I : \Omega_I \times Q_I(\varepsilon_0 r)$ is an almost exponential map. Moreover, we can choose I such that $|\Omega_I| \ge \frac{1}{N}|B(x_0,\varepsilon_2 r)|$ where N is the total number of n-tuples.

The controllability in [15] requires that there exists $\gamma : \Omega_I \times Q_I(\varepsilon_0 r) \times [0, cr]$ such that :

(C1) For any $(x,h) \in \Omega_I \times \{\|h\|_I \le \varepsilon_0 r\}$, $t \mapsto \gamma(x,h,t)$ is a subunit path connecting x and $E_I(x,h)$, i.e. $\gamma(x,h,0) = x$, $\gamma(x,h,T(x,h)) = E_I(x,h)$ for a suitable $T(x,h) \le cr$.

(C2) For any $(h,t) \in Q_I(\varepsilon_0 r) \times [0,cr]$, $x \mapsto \gamma(x,h,t)$ is a one-to-one map having continuous first derivatives and Jacobian determinant uniformly bounded away from zero, i.e. $\inf_{\Omega_I \times Q_I(\varepsilon_0 r)\times[0,cr]} \left|\det \frac{\partial \gamma}{\partial x}\right| \ge c > 0$.

The points x and $E_I(x,h)$ can be joined by a piecewise integral curve of the X_j's. Hence the map γ can be defined as follows. Denote by $x_j = \prod_{i=j+1}^n \exp^*(h_i U_i)x$. Let $h_j \ge 0$. If $d(U_j) = 1$, then let $\gamma_j(t) = e^{tU_j}x_j$ for $0 \le t \le h_j$. If instead $d(U_j) = 2$, say

$U_j = [X_{p_j}, X_{l_j}]$, then let

$$\gamma_j(t) = \begin{cases} e^{tX_{p_j}} x_j & 0 \le t \le \sqrt{h_j}, \\ e^{(t-\sqrt{h_j})X_{l_j}} e^{\sqrt{h_j}X_{p_j}} x_j & \sqrt{h_j} \le t \le 2\sqrt{h_j}, \\ e^{-(t-2\sqrt{h_j})X_{p_j}} e^{\sqrt{h_j}X_{l_j}} e^{\sqrt{h_j}X_{p_j}} x_j & 2\sqrt{h_j} \le t \le 3\sqrt{h_j}, \\ e^{-(t-3\sqrt{h_j})X_{l_j}} e^{-\sqrt{h_j}X_{p_j}} e^{\sqrt{h_j}X_{l_j}} e^{\sqrt{h_j}X_{p_j}} x_j & 3\sqrt{h_j} \le t \le 4\sqrt{h_j}. \end{cases}$$

If $h_j < 0$ the construction is analogous.

By taking the path $\gamma = \gamma_n + \cdots + \gamma_1$ we see that condition (C1) is satisfied with a $T(x,h) \le cr$, where c is an absolute constant. Concerning condition (C2), although we can not expect that the map $x \mapsto \gamma(x,h,t)$ is C^1, it is known that it is a Lipschitz continuous change of variables and $\det \frac{\partial \gamma}{\partial x}(x,h,t) = 1 + \varphi(x,h,t)$, with $|\varphi(x,h,t)| \le cr$ a.e. on $\Omega_I \times Q_I(\varepsilon_0 r) \times [0, cr]$ (this is a quite standard fact in ODE's, see the discussions in [9, pp. 99–101] and [11, Lemma 2.2]). This property is actually sufficient in the proof of the Poincaré inequality in [15, p. 332, ll. 6–9], where condition (C2) is used to make a change of variable in a Lebesgue integral.

We conclude that all the hypotheses of [15, Theorem 2.1] are satisfied and the Poincaré inequality (10) holds on the ball $B(x, \varepsilon_2 r)$. □

5 Some examples

In this section we show some applications of Theorem 1.2.

Example 5.1 (Levi vector fields) Here we precise the Example in [15, Section 5]. Given a real valued function $u \in C^2(\Omega)$, $\Omega \subseteq \mathbb{R}^3$, define the first order operators in $\mathbb{R}^3$

$$X_1 = \partial_{x_1} + a_1(\nabla u)\partial_{x_3}, \quad X_2 = \partial_{x_2} + a_2(\nabla u)\partial_{x_3}, \tag{41}$$

where, for any $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $a_1(p) = \frac{p_2 - p_1 p_3}{1+p_3^2}$, $a_2(p) = \frac{-p_1 - p_2 p_3}{1+p_3^2}$. In particular $x \mapsto a_1(\nabla u)(x)$ and $x \mapsto a_2(\nabla u)(x)$ are C^1 functions. However, this regularity assumption does not seem to be enough to get the Poincaré inequality (10). Here we add to that condition the following: assume that *u is a solution of the prescribed Levi curvature equation*

$$X_1^2 u + X_2^2 u + q(x, u, \nabla u) = 0, \quad q(x,u,p) = k(x,u)\frac{(1+|p|^2)^{3/2}}{(1+p_3^2)^2}, \tag{42}$$

and assume that the Levi curvature k is Lipschitz continuous and different from zero at any point. This assumption provides both the rank condition and the "horizontal Lipschitz continuity" of the commutator, which are required in our main theorem. Indeed, in [2] it has been proved that if u is a solution of (42) then $a_1(\nabla u) = X_2 u, a_2(\nabla u) = -X_1 u$ and

$$[X_1, X_2] = q(x, u, \nabla u)\partial_{x_3}. \tag{43}$$

Since k is different from zero at any point, then condition (1) is satisfied. We now show that $X_j(q(\cdot, u, \nabla u))$ is bounded for $j = 1, 2$. An easy calculation shows

$$\begin{aligned} X_j(q(x, u,\nabla u)) &= X_j(k(x, u))\frac{(1 + |\nabla u|^2)^{3/2}}{(1 + u_{x_3}^2)^2} \\ &\quad + k(x, u)\frac{(1 + |\nabla u|^2)^{1/2}}{(1 + u_{x_3}^2)^2}\Big(3\sum_{i=1}^{2} u_{x_i} X_j u_{x_i} - \Big(1 + 4\frac{u_{x_1}^2 + u_{x_2}^2}{1 + u_{x_3}^2}\Big)u_{x_3} X_j u_{x_3}\Big). \end{aligned}$$

Remark that, for $i = 1, \ldots, 3$, $X_1 u_{x_i} = [X_1, \partial_{x_i}]u + \partial_{x_i} X_1 u = -\partial_{x_i}(a_1(\nabla u))u_{x_3} - \partial_{x_i}(a_2(\nabla u))$, and analogously $X_2 u_{x_i} = -\partial_{x_i}(a_2(\nabla u))u_{x_3} + \partial_{x_i}(a_1(\nabla u))$. Therefore

$$\|X_1 \nabla u\|_\infty + \|X_2 \nabla u\|_\infty \le (1 + \|\nabla u\|_\infty)(\|\nabla(a_1(\nabla u))\|_\infty + \|\nabla(a_2(\nabla u))\|_\infty). \tag{44}$$

By (43) and (44) $\|X_1(q(\cdot, u, \nabla u))\|_\infty + \|X_2(q(\cdot, u, \nabla u))\|_\infty$ is bounded by a positive constant which only depends on $\|u\|_\infty + \|\nabla u\|_\infty + \|\nabla(a_1(\nabla u))\|_\infty + \|\nabla(a_2(\nabla u))\|_\infty + \|\nabla(k(\cdot, u))\|_\infty$. Thus, q satisfies (2). Hence, the vector fields in (41) satisfy the hypotheses of Theorem 1.2, which is the main tool in the Moser iteration technique for the study of regularity of solutions. In particular, we believe that this tool will enable us to improve Theorem 1.1 in [1], where, in order to prove $C^{2,\alpha}$ estimates of a viscosity solution, it was required the smoothness of k.

We end this section by exhibiting another example of Lipschitz continuous vector fields for which Poincaré inequality (10) holds.

Example 5.2 Take in $\mathbb{R}^3$ the two vector fields

$$X_1 = \partial_{x_1} - x_2\varphi(x_1, x_3)\partial_{x_3}, \quad X_2 = \partial_{x_2}$$

with φ a Lipschitz continuous function such that $|\varphi| \ge c > 0$. At every point there exists $[X_1, X_2] = \varphi(x_1, x_3)\partial_{x_3}$ and it is Lipschitz continuous. Hence, both conditions (1) and (2) are satisfied.

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Annamaria Montanari
Dipartimento di Matematica, Università di Bologna,
Piazza di Porta S. Donato, 5.
40127 Bologna (Italy).
E-mail: `montanar@dm.unibo.it`

Daniele Morbidelli
Dipartimento di Matematica, Università di Bologna,
Piazza di Porta S. Donato, 5.
40127 Bologna (Italy).
E-mail: `morbidel@dm.unibo.it`