

Step- s involutive families of vector fields, their orbits and the Poincaré inequality *

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Abstract

We consider a family $\mathcal{H} := \{X_1, \dots, X_m\}$ of vector fields in \mathbb{R}^n . Under a suitable *s-involutivity* assumption on commutators of order at most s , we show a ball-box theorem for Carnot–Carathéodory balls of the family \mathcal{H} and we prove the related Poincaré inequality. Each control ball is contained in a suitable *Sussmann’s orbit* of which we discuss some regularity properties. Our main tool is a class of *almost exponential maps* which we discuss carefully under low regularity assumptions on the coefficients of the vector fields in \mathcal{H} .

1. Introduction and main results

In this paper we discuss Carnot–Carathéodory balls and the Poincaré inequality for a family $\mathcal{H} = \{X_1, \dots, X_m\}$ of nonsmooth vector fields in \mathbb{R}^n satisfying a suitable involutivity condition of order $s \in \mathbb{N}$, which turns out to be a good substitute of the well known Hörmander’s rank hypothesis. Under our assumptions, control balls are not necessarily open sets in the ambient space \mathbb{R}^n , but each of them is contained in a suitable *orbit* associated with the vector fields of \mathcal{H} . In this setting we will prove a ball-box theorem and the related Poincaré inequality for control balls of the family \mathcal{H} . Our main tool consists of a class of *almost exponential maps* which are discussed below.

In the setting of Hörmander’s vector fields, control balls have been studied by Nagel, Stein and Wainger [NSW85], who proved the following fact: assume that the (smooth) vector fields X_j of the family \mathcal{H} together with their commutators of order at most s span the whole space \mathbb{R}^n at any point. Denote by $\mathcal{P} := \mathcal{P}_s := \{Y_1, \dots, Y_q\}$ the family of such commutators. Then, given the Carnot–Carathéodory ball $B_{cc}(x_0, r)$ associated with \mathcal{H} , there are commutators $Y_{i_1}, \dots, Y_{i_n} \in \mathcal{P}$ of lengths $\ell_{i_1}, \dots, \ell_{i_n} \leq s$ such that the *exponential map*

$$\Phi(u) := \exp\left(\sum_{1 \leq k \leq n} u_k r^{\ell_{i_k}} Y_{i_k}\right) x_0 \quad (1.1)$$

satisfies a “ball-box” double inclusion $\Phi(B_{\text{Euc}}(C^{-1})) \subseteq B_{cc}(x_0, r) \subseteq \Phi(B_{\text{Euc}}(C))$ where $B_{\text{Euc}}(C) := B_{\text{Euc}}(0, C) \subset \mathbb{R}^n$ denotes the Euclidean ball of radius $C > 0$ centered at the origin. Moreover, they showed that the Lebesgue measure of control balls is doubling.

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More recently, Tao and Wright [TW03] discovered that maps Φ could be manipulated without the Campbell–Baker–Hausdorff–Dynkin formula, using arguments more based on Gronwall’s inequality. Subsequently, Street [Str11] extended such approach showing that the Hörmander’s condition can be removed, provided that one assumes that for some $s \in \mathbb{N}$ the following s -integrability condition holds: for all $Y_i, Y_j \in \mathcal{P} = \mathcal{P}_s$, one can write

$$[Y_i, Y_j] = \sum_{1 \leq k \leq q} c_{ij}^k Y_k, \quad (1.2)$$

where the functions c_{ij}^k must have suitable regularity. This condition goes back to Hermann, [Her62] and it ensures that any Sussmann’s orbit $\mathcal{O}_{\mathcal{P}}$ of the family \mathcal{P} is an integral manifold of the distribution generated by \mathcal{P} . Under (1.2), control balls are contained in the orbits of the family \mathcal{P} and Street [Str11] has shown a complete generalization of the ball-box inclusion to such setting together with the doubling estimate for the pertinent measure of the control ball.

Given a family \mathcal{H} and its *Carnot–Carathéodory* distance d_{cc} , a remarkable estimate which embodies many properties of the metric space (\mathbb{R}^n, d_{cc}) is the associated Poincaré inequality. It is well known that such inequality plays a crucial role in several questions concerning analysis and geometry, consult the references [FL83, Jer86, SC92, GN96, Che99, HK00, KZ08], to see the Poincaré inequality in action.

The Poincaré inequality for Hörmander vector fields was proved first by Jerison in [Jer86]. It was observed in [Jer86], that the natural “exponential maps” to prove the Poincaré inequality should be factorizable as compositions of exponentials of the original vector fields of \mathcal{H} . However, in [Jer86] the Poincaré inequality was achieved for Hörmander vector fields with different techniques.¹ The program implicitly suggested by Jerison was carried out in the subsequent papers [LM00, MM04, MM12c]. Namely, in [MM12c], the present authors showed that, at least for Hörmander vector fields (even with quite rough coefficients), a “ball-box” double inclusion still holds if we change the map Φ in (1.1) with the *almost exponential map*

$$E(h) := \exp_{\text{ap}}(h_1 r^{\ell_{i_1}} Y_{i_1}) \circ \cdots \circ \exp_{\text{ap}}(h_n r^{\ell_{i_n}} Y_{i_n})(x),$$

where \exp_{ap} denote the approximate exponentials appearing in [NSW85, VSCC92, Mor00, MM12c]; see [MM12a, Section 2] for the precise definition.

In this paper, starting from some useful first order expansions of E obtained in [MM12a] (see Theorem 2.5 below) we discuss the structure of control balls for vector fields belonging to a regularity class which we call \mathcal{B}_s . We say that a family $\mathcal{H} = \{X_1, \dots, X_m\}$ belongs to the class \mathcal{B}_s if all $X_j \in \mathcal{H}$ belongs to C^s (this ensures that all commutators $Y_j \in \mathcal{P}$ are C^1); moreover, we require that (1.2) holds for the family \mathcal{P} and that the functions c_{ij}^k in (1.2) are C^1 smooth with respect to the differential structure of each orbit; see Definition 2.1.

To state our result we need the following notation. If $\mathcal{P} = \{Y_1, \dots, Y_q\}$ and $x \in \mathbb{R}^n$, then $P_x := \text{span}\{Y_j(x) : 1 \leq j \leq q\}$ and $p_x := \dim P_x$. Given $r > 0$ and $Y_{i_1}, \dots, Y_{i_p} \in \mathcal{P}$, let $\tilde{Y}_{i_k} = r^{\ell_{i_k}} Y_{i_k}$ be the scaled commutators and put

$$E_{I,x,r}(h) := \exp_{\text{ap}}(h_1 \tilde{Y}_{i_1}) \cdots \exp_{\text{ap}}(h_p \tilde{Y}_{i_p})x \quad (1.3)$$

¹It must be observed that Jerison’s paper also involves a study of some nontrivial global aspects of the Poincaré inequality which we do not discuss here.

for each h close to $0 \in \mathbb{R}^p$ (after passing to \tilde{Y}_{i_j} , the variable h lives at a unit scale). We also denote by σ^p the p -dimensional surface measure and by B_{cc} control balls. Finally B_ϱ denote balls with respect to the distance $\varrho \geq d_{\text{cc}}$ defined in (2.1).

Theorem 1.1. *Let \mathcal{H} be a family of \mathcal{B}_s vector fields. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then there is $C > 1$ such that the following holds. Let $x \in \Omega$ and take a positive radius $r < C^{-1}$. Then there is a family of $p_x =: p$ commutators Y_{i_1}, \dots, Y_{i_p} such that the map $E := E_{I,x,r}$ in (1.3) is C^1 smooth on the unit ball $B_{\text{Euc}}(1) \subset \mathbb{R}^p$ and satisfies*

$$C^{-1} \leq \frac{|\partial_1 E(h) \wedge \dots \wedge \partial_p E(h)|}{|\tilde{Y}_{i_1}(x) \wedge \dots \wedge \tilde{Y}_{i_p}(x)|} \leq C \quad \text{for all } h \in B_{\text{Euc}}(1), \text{ and} \quad (1.4)$$

$$E(B_{\text{Euc}}(1)) \supseteq B_\varrho(x, C^{-1}r). \quad (1.5)$$

Moreover, $E_{I,x,r}$ is one-to-one on $B_{\text{Euc}}(1)$ and we have the doubling property

$$\sigma^p(B_{\text{cc}}(x, 2r)) \leq C\sigma^p(B_{\text{cc}}(x, r)) \quad \text{for all } x \in \Omega \text{ and } 0 < r < C^{-1}. \quad (1.6)$$

Finally, for any C^1 function f , we have the Poincaré inequality

$$\int_{B_{\text{cc}}(x,r)} |f(y) - f_{B_{\text{cc}}(x,r)}| d\sigma^p(y) \leq C \sum_{j=1}^m \int_{B_{\text{cc}}(x,r)} |rX_j f(y)| d\sigma^p(y). \quad (1.7)$$

The constant C in Theorem 1.1 turns out to depend on an “admissible constant” L_1 which will be defined precisely in (2.7). Note that L_1 is defined in terms of the coefficients $c_{i_j}^k$ in (1.2) but does not involve any positive lower bound on the infimum $\nu(\Omega)$ in (2.10), which is allowed to vanish even on compact sets. This makes such result suitable in the perspective of multi-parameter distances studied in [Str11]. Observe that inequalities (1.7) and (1.6) can be proved in more regular settings using arguments from the papers [Jer86], [Str11] and [MM12c]. However such arguments do not provide optimal results; see the discussion in Section 5. Finally, since in (1.5) B_ϱ denotes the control ball defined by *all* commutators (with their degrees, see (2.1)), as a consequence we have the local inclusion $B_{\text{cc}}(x, r) \supseteq B_{\mathcal{O}}(x, C^{-1}r^s)$, where $B_{\mathcal{O}}$ is the geodesic ball on the orbit \mathcal{O} ; see Remark 3.3.

Let us mention that under our regularity assumptions, inclusion (1.5) is not completely trivial. Indeed, such inclusion implies in particular the following fact: a subunit path γ of the family $\mathcal{P} = \{Y_1, \dots, Y_q\}$ of commutators with $\gamma(0) =: x$, *cannot leave* the Sussmann’s orbit $\mathcal{O}_{\mathcal{H}}^x$ of the *horizontal* family \mathcal{H} .² This statement needs to be checked carefully. See the discussion in Remark 3.3 and see Lemma 3.5. In the Hörmander case, this issue does not appear, because $\mathcal{O}_{\mathcal{H}}^x = \mathbb{R}^n$, by Chow’s Theorem. Indeed, in [MM12b], under the Hörmander assumption, we are able to prove Theorem 1.1 under even lower regularity assumptions than those of the present paper: namely, we assume that higher order commutators are C^1 only along horizontal directions.

A further delicate part of our argument is the proof of the injectivity of maps E . Note that the clever argument by Tao and Wright, [TW03], [Str11], is peculiar of the standard exponential maps Φ and does not extend to our maps E . Since it does not

²Recall that given $\mathcal{H} = \{X_1, \dots, X_m\}$ and $x_0 \in \mathbb{R}^n$, then the Sussmann’s orbit $\mathcal{O}_{\mathcal{H}}^{x_0}$ is the set of points in \mathbb{R}^n which are reachable from x_0 via a path which is piecewise an integral curve of one among the vector fields of \mathcal{H} ; see [Sus73].

seem that any direct argument can be adopted, we will let to cooperate the maps E and Φ , which, although different, have analogous estimates on Jacobians. To accomplish this task, we need first to perform an accurate analysis of the standard exponential maps Φ . In particular we shall improve Street's ball-box theorem for maps Φ related to families of vector fields $\{Y_1, \dots, Y_q\}$ satisfying (1.2), where Y_j and $c_{ij}^k \in C^1$. This class is larger than the class originally studied in [Str11]; see especially the proof of Theorem 4.2-(ii). Then we show through a lifting argument that the map $E_{I,x,r}$ is one-to-one as a consequence of the injectivity of the map $\Phi_{I,x,r}$.

Before closing this introduction, we mention some more recent papers where nonsmooth vector fields are discussed. In [SW06], diagonal vector fields are discussed deeply. In the Hörmander case, in the model situation of equiregular families of vector fields, nonsmooth ball-box theorems have been studied by see [KV09, Gre10, Man10]. Finally, [BBP12] contains a nonsmooth lifting theorem.

The paper is organized as follows: In Section 2 we give some preliminaries. In Section 3 we prove the ball-box theorem for our almost exponential maps E . In Section 4 we discuss the ball-box theorem for maps Φ for vector fields in the class \mathcal{B}_s . In Section 5 we discuss an approach to the problem for more regular vector fields.

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2. Preliminaries

General notation about constants. We denote by $C, C_0, C_1, C_2 \dots$ large absolute constants. We denote instead by $t_0, \varepsilon_0, \varepsilon_1 r_0, \eta_0, \eta_1, \dots$ or C^{-1} small absolute constant. We will specify carefully along the paper what the constants we deal with depend on, i.e. what "absolute" means.

Vector fields, orbits and the control distance. Consider a family of vector fields $\mathcal{H} = \{X_1, \dots, X_m\}$ and assume that $X_j \in C^1(\mathbb{R}^n)$ for all j . Write $X_j =: f_j \cdot \nabla$, where $f_j: \mathbb{R}^n \rightarrow \mathbb{R}^n$. The vector field X_j , evaluated at a point $x \in \mathbb{R}^n$, will be denoted by $X_{j,x}$ or $X_j(x)$. All the vector fields in this paper are always defined on the whole space \mathbb{R}^n . Let

$$d_{cc}(x, y) := \inf \left\{ r > 0 : \text{there is } \gamma \in \text{Lip}((0, 1), \mathbb{R}^n) \text{ with } \gamma(0) = x, \gamma(1) = y \right. \\ \left. \text{and } \dot{\gamma}(t) \in \left\{ \sum_{1 \leq j \leq m} c_j r X_{j, \gamma(t)} : |c| \leq 1 \right\} \text{ for a.e. } t \in [0, 1] \right\}.$$

As usual, we call *Carnot–Carathéodory* or *control* distance the distance d_{cc} .

Given a fixed $s \geq 1$, denote by $\mathcal{P} := \{Y_1, \dots, Y_q\} = \{X_w : 1 \leq |w| \leq s\}$ the family of commutators of length at most s . Let $\ell_j \leq s$ be the length of Y_j and write $Y_j =: g_j \cdot \nabla$. The distance associated with \mathcal{P} (where each Y_j has degree ℓ_j) will be denoted by ϱ :

$$\varrho(x, y) := \inf \left\{ r \geq 0 : \text{there is } \gamma \in \text{Lip}((0, 1), \mathbb{R}^n) \text{ such that } \gamma(0) = x \right. \\ \left. \gamma(1) = y \text{ and } \dot{\gamma}(t) \in \left\{ \sum_{j=1}^q b_j r^{\ell_j} Y_j(\gamma(t)) : |b| \leq 1 \right\} \text{ for a.e. } t \in [0, 1] \right\}. \quad (2.1)$$

We denote by $B_\varrho(x, r)$, $B_{cc}(x, r)$ and $B_{\text{Euc}}(x, r)$ the balls of center x and radius r with respect to ϱ , d_{cc} and the Euclidean distance respectively. We also denote for brevity $B_{\text{Euc}}(r) := B_{\text{Euc}}(0, r)$.

Definition 2.1 (Vector fields of class \mathcal{B}_s). Let $\mathcal{H} = \{X_1, \dots, X_m\}$ be vector fields in \mathbb{R}^n . We say that \mathcal{H} is a family of class \mathcal{B}_s if $X_j \in C_{\text{Euc}}^s$ for $j \in \{1, \dots, m\}$ and moreover, given any open bounded set $\Omega_0 \subset \mathbb{R}^n$, there is $C_1 > 0$ such that we may write for suitable functions c_{ij}^k

$$[Y_i, Y_j] := (Y_i g_j - Y_j g_i) \cdot \nabla = \sum_{1 \leq k \leq q} c_{ij}^k Y_k \quad \text{where} \quad (2.2)$$

$$\sup_{x \in \Omega_0} |c_{ij}^k(x)| \leq C_1 \quad \text{for all } i, j, k \in \{1, \dots, q\}; \quad (2.3)$$

we require finally that for all $i, j, k \in \{1, \dots, q\}$, $\mu \leq n$, $x \in \mathbb{R}^n$ and $I = (i_1, \dots, i_\mu) \in \{1, \dots, q\}^\mu$, the map

$$\Omega_{I,x} \ni (u_1, \dots, u_\mu) \mapsto c_{ij}^k \left(\exp \left(\sum_{1 \leq \alpha \leq \mu} u_\alpha Y_{i_\alpha} \right) x \right) \quad (2.4)$$

is C_{Euc}^1 smooth on the open set $\Omega_{I,x} \subset \mathbb{R}^\mu$ where it is defined.

Remark 2.2. Class \mathcal{B}_s is a subclass of the class \mathcal{A}_s introduced in [MM12a]. More precisely, if a family \mathcal{H} belongs to \mathcal{B}_s , then it belongs to \mathcal{A}_s and the constants L_0 and C_0 in [MM12a] can be estimated by L_1 in (2.3).

Remark 2.3. (i) The assumption $X_j \in C_{\text{Euc}}^s$ ensures that all the vector fields Y_j are C_{Euc}^1 smooth. It is known that if (2.2) and (2.3) hold with c_{ij}^k locally bounded, then any subunit orbit

$$\mathcal{O}_{\mathcal{P}, \text{cc}}^{x_0} := \{y \in \mathbb{R}^n : d_{\text{cc}}(x, y) < \infty\} \quad (2.5)$$

with topology $\tau_{d_{\text{cc}}}$ is an immersed C^2 submanifold and it is an integral manifold of the distribution generated by \mathcal{P} . Charts are described in (3.6). In the paper [MM11] we show a more general statement involving Lipschitz vector fields.

- (ii) Hypothesis (2.4) leaves on the orbits $\mathcal{O} = \mathcal{O}_{\mathcal{P}}$ of the family $\mathcal{P} = \{Y_1, \dots, Y_q\}$ and it is ensured for instance by the assumption that $c_{ij}^k \in C_{\mathcal{O}}^1$, i.e. C^1 with respect to the differential structure of each orbit.
- (iii) Observe also that conditions (2.2) and (2.3) scale correctly. Indeed, take a family \mathcal{H} of \mathcal{B}_s vector fields, denote $\tilde{Y}_k := r^{\ell_k} Y_k$ for $k = 1, \dots, q$ and $r \in]0, 1]$. Then there are new C^1 functions $\tilde{c}_{jk}^i(x)$ and an algebraic constant $\hat{C}_1 > 0$ so that $|\tilde{Y}_h \tilde{c}_{jk}^i| \leq C_1 |\tilde{Y}_h c_{jk}^i|$ for all i, j, k, h and moreover for all $x \in \Omega_0$ we have

$$[\tilde{Y}_j, \tilde{Y}_k] := [r^{\ell_j} Y_j, r^{\ell_k} Y_k] = \sum_{i=1}^q \tilde{c}_{jk}^i \tilde{Y}_i \quad \text{and} \quad |\tilde{c}_{jk}^i| \leq C_1 + \hat{C}_1. \quad (2.6)$$

To see (2.6), if $\ell_j + \ell_k > s$, then let $\tilde{c}_{jk}^i(x) := r^{\ell_j + \ell_k - \ell_i} c_{jk}^i(x)$ and we are done. If instead $\ell_j + \ell_k \leq s$, then the Jacobi identity shows that there are algebraic constants a_{jk}^i such that $[Y_j, Y_k] = \sum_{i=\ell_j + \ell_k}^s a_{jk}^i Y_i$. Therefore (2.6) holds.

Given a family of \mathcal{B}_s vector fields in \mathbb{R}^n and $\Omega \Subset \Omega_0 \subset \mathbb{R}^n$ bounded sets, introduce the constant

$$L_1 := \sum_{j=1}^m \sum_{0 \leq |\alpha| \leq s} \sup_{\Omega_0} |D^\alpha f_j| + \sum_{i,j,k,\ell=1}^q \left(\sup_{\Omega_0} |c_{ij}^\ell| + \sup_{\Omega_0} |Y_k c_{ij}^\ell| \right). \quad (2.7)$$

In the remaining part of the paper we fix open bounded sets $\Omega \Subset \Omega_0 \Subset \mathbb{R}^n$ and we consider points $x \in \Omega$ and radii $r \leq r_0$ where r_0 is small enough to ensure that all balls $B_\rho(x, r_0)$ are contained in Ω_0 and that all points $E_{I,x,r}(h)$ and $\Phi_{I,x,r}(u)$ appearing in the paper belong to Ω_0 .

Wedge products and η -maximality conditions. Next, following [Str11], we define some algebraic quantities which we will use below. Define for any $p, \mu \in \mathbb{N}$, with $1 \leq p \leq \mu$, $\mathcal{I}(p, \mu) := \{I = (i_1, \dots, i_p) : 1 \leq i_1 < i_2 < \dots < i_p \leq \mu\}$. For each $x \in \mathbb{R}^n$ define $p_x := \dim \text{span}\{Y_{j,x} : 1 \leq j \leq q\}$. Obviously, $p_x \leq \min\{n, q\}$. Then for any $p \in \{1, \dots, \min\{n, q\}\}$, let

$$Y_{I,x} := Y_{i_1,x} \wedge \dots \wedge Y_{i_p,x} \in \bigwedge_p T_x \mathbb{R}^n \sim \bigwedge_p \mathbb{R}^n \quad \text{for all } I \in \mathcal{I}(p, q),$$

and, for all $K \in \mathcal{I}(p, n)$ and $I \in \mathcal{I}(p, q)$

$$Y_I^K(x) := dx^K(Y_{i_1}, \dots, Y_{i_p})(x) := \det(g_{i_\alpha}^{k_\beta})_{\alpha, \beta=1, \dots, p}. \quad (2.8)$$

Here we let $dx^K := dx^{k_1} \wedge \dots \wedge dx^{k_p}$ for any $K = (k_1, \dots, k_p) \in \mathcal{I}(p, n)$.

The family $e_K := e_{k_1} \wedge \dots \wedge e_{k_p}$, where $K \in \mathcal{I}(p, n)$, gives an orthonormal basis of $\bigwedge_p \mathbb{R}^n$, i.e. $\langle e_K, e_H \rangle = \delta_{K,H}$ for all K, H . Then we have the orthogonal decomposition $Y_I(x) = \sum_K Y_I^K(x) e_K \in \bigwedge_p \mathbb{R}^n$, so that the number $|Y_I(x)| := (\sum_{K \in \mathcal{I}(p, n)} Y_I^K(x)^2)^{1/2} = |Y_{i_1}(x) \wedge \dots \wedge Y_{i_p}(x)|$ gives the p -dimensional volume of the parallelepiped generated by $Y_{i_1}(x), \dots, Y_{i_p}(x)$.

Let $I = (i_1, \dots, i_p) \in \mathcal{I}(p, q)$ such that $|Y_I| \neq 0$. Consider the linear system $\sum_{k=1}^p \xi^k Y_{i_k} = W$, for some $W \in \text{span}\{Y_{i_1}, \dots, Y_{i_p}\}$. The Cramer's rule gives the unique solution

$$\xi^k = \frac{\langle Y_I, \iota^k(W) Y_I \rangle}{|Y_I|^2} \quad \text{for each } k = 1, \dots, p, \quad (2.9)$$

where we let $\iota_W^k Y_I := \iota^k(W) Y_I := Y_{(i_1, \dots, i_{k-1})} \wedge W \wedge Y_{(i_{k+1}, \dots, i_p)}$.

Let $r > 0$. Given $J \in \mathcal{I}(p, q)$, let $\ell(J) := \ell_{j_1} + \dots + \ell_{j_p}$. Introduce the vector-valued function

$$\Lambda_p(x, r) := (Y_J^K(x) r^{\ell(J)})_{J \in \mathcal{I}(p, q), K \in \mathcal{I}(p, n)} =: (\tilde{Y}_J^K(x))_{J \in \mathcal{I}(p, q), K \in \mathcal{I}(p, n)},$$

where we adopt the tilde notation $\tilde{Y}_k := r^{\ell_k} Y_k$ and its obvious generalization for wedge products. Note that $|\Lambda_p(x, r)|^2 = \sum_{J \in \mathcal{I}(p, q)} r^{2\ell(J)} |Y_J(x)|^2$.

Finally, for each $A \subset \mathbb{R}^n$, put

$$\nu(A) := \inf_{x \in A} |\Lambda_p(x, 1)|. \quad (2.10)$$

Definition 2.4 (η -maximality). *Let $x \in \mathbb{R}^n$, let $I \in \mathcal{I}(p_x, q)$ and $\eta \in (0, 1)$. We say that (I, x, r) is η -maximal if $|Y_I(x)| r^{\ell(I)} > \eta \max_{J \in \mathcal{I}(p_x, q)} |Y_J(x)| r^{\ell(J)}$.*

Note that, if (I, x, r) is a candidate to be η -maximal with $I \in \mathcal{I}(p, q)$, then by definition it *must* be $p = p_x$.

Approximate exponentials of commutators. Let $w_1, \dots, w_\ell \in \{1, \dots, m\}$. Given $t \in \mathbb{R}$, close to 0, define the approximate exponential $e_{\text{ap}}^{tX_{w_1 w_2 \dots w_\ell}} := \exp_{\text{ap}}(tX_{w_1 w_2 \dots w_\ell})$ as in [MM12a] and see also [NSW85, MM12c]. By standard ODE theory, there is t_0 depending on $\ell, \Omega, \Omega_0, \sup|f_j|$ and $\sup|\nabla f_j|$ such that $\exp_*(tX_{w_1 w_2 \dots w_\ell})x$ is well defined for any $x \in \Omega$ and $|t| \leq t_0$. Define, given $I = (i_1, \dots, i_p) \in \{1, \dots, q\}^p$, $x \in \Omega$ and $h \in \mathbb{R}^p$, with $|h| \leq C^{-1}$

$$\begin{aligned} E_{I,x}(h) &:= \exp_{\text{ap}}(h_1 Y_{i_1}) \cdots \exp_{\text{ap}}(h_p Y_{i_p})(x) \\ \|h\|_I &:= \max_{j=1, \dots, p} |h_j|^{1/\ell_{i_j}} \quad Q_I(r) := \{h \in \mathbb{R}^p : \|h\|_I < r\}. \end{aligned} \quad (2.11)$$

Recall the following result.

Theorem 2.5 ([MM12a, Theorem 3.11]). *Let \mathcal{H} be a \mathcal{B}_s family. Let $x \in \Omega$ and let $r \in (0, r_0)$. Fix $p \in \{1, \dots, q\}$ and $I \in \mathcal{I}(p, q)$. Then the function $E_{I,x,r}$ is C^1 smooth on $B_{\text{Euc}}(C^{-1})$. Moreover, for all $h \in B_{\text{Euc}}(C^{-1})$ and for any $k \in \{1, \dots, p\}$ we have $E_*(\partial_{h_k}) \in P_{E(h)}$ and we can write*

$$E_*(\partial_{h_k}) = \tilde{U}_{k,E(h)} + \sum_{\ell_j=d_k+1}^s a_k^j(h) \tilde{Y}_{j,E(h)} + \sum_{i=1}^q \omega_i^j(x, h) \tilde{Y}_{i,E(h)}, \quad (2.12)$$

where, for some $C > 1$ we have

$$|a_k^j(h)| \leq C \|h\|_I^{\ell_j - d_k} \quad \text{for all } h \in B_{\text{Euc}}(C^{-1}) \quad (2.13)$$

$$|\omega_i(x, h)| \leq C \|h\|_I^{s+1-d_k} \quad \text{for all } h \in B_{\text{Euc}}(C^{-1}) \quad x \in \Omega. \quad (2.14)$$

The proof of Theorem 2.5 in [MM12a] involves the more general class \mathcal{A}_s and constants in that paper depend on the data C_0 and L_0 there. Therefore, in view of Remark 2.2, constants in Theorem 2.5 depend quantitatively on L_1 in (2.7).

Gronwall's inequality. We shall refer several times to the following standard fact: for all $a \geq 0, b > 0, T > 0$ and f continuous on $[0, T]$,

$$0 \leq f(t) \leq at + b \int_0^t f(\tau) d\tau \quad \text{on } t \in [0, T] \quad \Rightarrow \quad f(t) \leq \frac{a}{b}(e^{bt} - 1) \quad \text{on } t \in [0, T]. \quad (2.15)$$

3. Ball-box theorem for almost exponential maps

In this section we prove the ball-box theorem for our almost exponential maps E associated with a family $\mathcal{H} = \{X_1, \dots, X_m\}$ of vector fields of class \mathcal{B}_s . Given $I \in \mathcal{I}(p, q)$ and $r > 0$, we denote as usual $\tilde{U}_j := r^{d_j} U_j := r^{\ell_{i_j}} Y_{i_j}$ and $E_{I,x,r}(h) := E(h) = e_{\text{ap}}^{h_1 \tilde{U}_1} \cdots e_{\text{ap}}^{h_p \tilde{U}_p} x$. Moreover, $Q_I(r)$ denotes the associated box (see (2.11)). Finally recall that we use the notation $|\chi|$ to denote the operator norm of any matrix χ with real elements.

Theorem 3.1. *Let \mathcal{H} be a family of \mathcal{B}_s vector fields. If (I, x, r) is $\frac{1}{2}$ -maximal, $x \in \Omega$ and $r < r_0$, then there are $C_2 > 1$ and $\varepsilon_0 < 1$ such that, for all $j = 1, \dots, p := p_x$,*

$$E_*(\partial_{h_j}) = \tilde{U}_{j,E(h)} + \sum_{1 \leq k \leq p} \chi_j^k(h) \tilde{U}_{k,E(h)} \quad \text{for all } h \in Q_I(\varepsilon_0), \quad (3.1)$$

where $\chi \in C_{\text{Euc}}^0(Q_I(\varepsilon_0), \mathbb{R}^{p \times p})$ satisfies

$$|\chi(h)| \leq C_2 \|h\|_I \quad \text{if } \|h\|_I \leq \varepsilon_0. \quad (3.2)$$

Note that Theorem 3.1 *does not require* any positive lower bound on the number ν defined in (2.10). Moreover it holds for the more general class \mathcal{A}_s in [MM12a] and the constants C_2 , ε_0 and r_0 depend quantitatively on L_0 and C_0 introduced in that paper, and ultimately—in view of Remark 2.2—on L_1 .

For future reference, we write (3.1) in matricial form as

$$dE(h) = [\tilde{Y}_{i_1, E(h)}, \dots, \tilde{Y}_{i_p, E(h)}] [I_p + \chi(h)] =: [\tilde{Y}_{I, E(h)}] [I_p + \chi(h)]. \quad (3.3)$$

Proof of Theorem 3.1. It follows immediately from Theorem 2.5. Indeed, starting from (2.12) and applying [MM12a, Remark 3.3] with $\eta = \frac{1}{2}$, we get

$$\begin{aligned} E_*(\partial_{h_k}) &= \tilde{U}_{k, E} + \sum_{\ell_k+1 \leq \ell_j \leq s} a_k^j(h) \sum_{\alpha=1}^p b_j^\alpha \tilde{U}_{\alpha, E} + \sum_{1 \leq i \leq q} \omega_k^i(x, h) \sum_{1 \leq \alpha \leq p} b_i^\alpha \tilde{U}_{\alpha, E} \\ &=: \tilde{U}_{k, E} + \sum_{1 \leq j \leq p} \chi_k^j(h) \tilde{U}_{j, E}, \end{aligned}$$

where $E = E(h)$ and, by [MM12a, Remark 3.3], we have $|b_i^\alpha| \leq C$. The coefficients χ_k^j are unique by the linear independence of the $\tilde{U}_{j, E}$. Moreover, since in Theorem [MM12a, Theorem 3.11] we have proved that $h \mapsto E_*(\partial_{h_k})$ is continuous and by assumption \mathcal{B}_s we know that the maps $h \mapsto U_j(E(h))$ are continuous, then, the Cramer's rule (2.9) shows that χ is continuous. Finally estimate (3.2) follows from the inequality $|a_k^j(h)| + |\omega_k^i(x, h)| \leq C \|h\|$; see (2.13) and (2.14). \square

Next we discuss our ball-box theorem in the class \mathcal{B}_s .

Theorem 3.2. *Let \mathcal{H} be a family of \mathcal{B}_s vector fields. Then there are $\varepsilon_0, \varepsilon_1 > 0$ and $C_2 > 0$ such that for any $\frac{1}{2}$ -maximal triple (I, x, r) with $x \in \Omega$, $I \in \mathcal{I}(p_x, q)$ and $r \in (0, r_0)$*

(a) *for any $\varepsilon \leq \varepsilon_0$ we have*

$$E_{I, x, r}(Q_I(\varepsilon)) \supset B_\rho(x, C_2^{-1} \varepsilon^s r); \quad (3.4)$$

(b) *the map $E_{I, x, r}$ is one-to-one on the set $Q_I(\varepsilon_1)$.*

The proof of inclusion (3.4) will be shown in Lemma 3.7. The proof of the injectivity statement will be given later, after some more work. See page 12. Note that in Theorem 3.2, all constants ε_0, C_2, r_0 only depend quantitatively on L_1 in (2.7) and there are no problems even if the infimum $\nu(\Omega)$ in (2.10) is zero.

Remark 3.3. *Concerning Theorem 3.2-(a) note the following aspects.*

(i) *Inclusion (3.4) ensures that $B_\rho(x, r) \subset \mathcal{O}_{\mathcal{H}}$. We have shown in [MM12a] that on the orbit $\mathcal{O}_{\mathcal{H}}$ there is a topology $\tau(\mathcal{U})$ with basis \mathcal{U} (see (3.5)), such that $(\mathcal{O}_{\mathcal{H}}, \tau(\mathcal{U}))$*

is a C^1 submanifold and $T_x\mathcal{O}_{\mathcal{H}} = P_x$ for all x .³ Therefore, to show inclusion (3.4) one must first give a rigorous proof of the fact that a subunit path of the family \mathcal{P} starting from $x \in \mathcal{O}_{\mathcal{H}}$ should stay in $\mathcal{O}_{\mathcal{H}}$ for t close to zero. We prove this statement in Lemma 3.5 where we show that the subunit orbit $\mathcal{O}_{\mathcal{P},\text{cc}}$ of the commutators (see (2.5)) coincides with the Sussmann's orbit $\mathcal{O}_{\mathcal{H}}$ of the original vector fields.

- (ii) Note also that (3.4) implies the Fefferman–Phong-type local inclusion $B_d(x, r) \supseteq B_{\mathcal{O}}(x, C^{-1}r^s)$, where $B_{\mathcal{O}}$ denotes the geodesic ball on \mathcal{O} . Here x belongs to a compact set and r is small enough: see [FP83]. Therefore the topology $\tau_{d_{\text{cc}}}$ on $\mathcal{O} := \mathcal{O}_{\mathcal{H}} = \mathcal{O}_{\mathcal{P},\text{cc}}$ is equivalent to the topology defined by the metric ϱ and both are equivalent to the topology associated with the geodesic Riemannian distance provided by the first fundamental form of \mathcal{O} .

The main application of the results in this section is the following.

Corollary 3.4. *Let $\mathcal{H} = \{X_1, \dots, X_m\}$ be a family of \mathcal{B}_s vector fields in \mathbb{R}^n . Then for any bounded open set $\Omega \subset \mathbb{R}^n$, there is $C > 1$ depending on L_1 in (2.7) such that for any $x \in \Omega$ and $r \in (0, C^{-1}]$, letting $p := p_x$, we have*

$$\sigma^p(B_{\text{cc}}(x, 2r)) \leq C\sigma^p(B_{\text{cc}}(x, r)) \quad \text{and}$$

$$\int_{B_{\text{cc}}(x, r)} |f(y) - f_{B_{\text{cc}}(x, r)}| d\sigma^p(y) \leq C \sum_{j=1}^m \int_{B_{\text{cc}}(x, r)} |rX_j f(y)| d\sigma^p(y).$$

The doubling property was already proved by Street [Str11] under more restrictive assumptions. At the author's knowledge, the Poincaré inequality in such setting, is new even in the smooth case.

Proof of Corollary 3.4. The proof of the doubling property is an immediate consequence of Theorems 3.1, 3.2 and of area formula. The proof of the Poincaré inequality can be obtained arguing as in [LM00]. We avoid here the repetition of the arguments. \square

Before starting the proof of Theorem 3.2-(a), recall that it was shown in [MM12a, Theorem 3.13] that maps of the form $E_{I,x}$ can be used to give to $\mathcal{O}_{\mathcal{H}}$ a structure of p -dimensional integral manifold of the distribution generated by \mathcal{P} . More precisely, one can introduce a topology $\tau(\mathcal{U})$ generated by the family

$$\mathcal{U} := \{E_{I,x}(O) : x \in \mathcal{O}, I \in \mathcal{I}(p, q), |Y_I(x)| \neq 0 \quad (3.5)$$

and $O \subset O_{I,x}$ is a open neighborhood of the origin\}

(here $O_{I,x}$ is a neighborhood of the origin such that $E_{I,x}(O_{I,x})$ is an embedded submanifold) and maps $E_{I,x}$ can be used as charts.

In order to prove (3.4), we need the following lemma. Let ϱ be the distance with respect to the family \mathcal{P} defined in (2.1). Let $\mathcal{O}_{\mathcal{P},\text{cc}}^x := \{y \in \mathbb{R}^n : \varrho(x, y) < \infty\}$ be the subunit orbit of the family \mathcal{P} (see (2.5)) and let τ_{ϱ} be the topology associated with ϱ .

³ Note that even if the vector fields are smooth, maps of the form $E_{I,x}$ are generically not much regular. For example, given the smooth vector fields $X_1 = \partial_1$ and $X_2 = (x_1 + x_1^2)\partial_2$, then the map $h \mapsto \exp_{\text{ap}}(h[X_1, X_2])(0, 0) = (0, h + h|h|^{1/2})$ is $C^{1,1/2}$ only; see [MM12c, Example 5.7].

Lemma 3.5. *Let \mathcal{H} be a family in \mathcal{B}_s for some s . Let $x_0 \in \mathbb{R}^n$. Then we have the following topologically continuous inclusions:*

$$(\mathcal{O}_{\mathcal{H}}^{x_0}, \tau(\mathcal{U})) \stackrel{(a)}{\subseteq} (\mathcal{O}_{\mathcal{P}, \text{cc}}^{x_0}, \tau_{\varrho}) \stackrel{(b)}{\subseteq} (\mathcal{O}_{\mathcal{H}}^{x_0}, \tau(\mathcal{U})).$$

Remark 3.6. *Note that on $\mathcal{O}^{x_0} := \mathcal{O}_{\mathcal{P}, \text{cc}}^{x_0} = \mathcal{O}_{\mathcal{H}}^{x_0}$ both inclusions $(\mathcal{O}, \tau(\mathcal{U})) \subseteq (\mathcal{O}, \tau_{\text{cc}}) \subseteq (\mathcal{O}, \tau_{\varrho})$ are trivially continuous. Therefore, Lemma 3.5 shows that all mentioned topologies are equivalent on \mathcal{O}^{x_0} .*

The proof of Lemma 3.5 relies on the following facts discussed in [MM11]. Let $\mathcal{P} = \{Y_1, \dots, Y_q\}$ be a family of C^1 vector fields satisfying (2.2) and (2.3). Fix a subunit orbit $\mathcal{O}_{\mathcal{P}, \text{cc}}$. Then $p_x =: p$ is constant as $x \in \mathcal{O}_{\mathcal{P}, \text{cc}}$ and moreover $(\mathcal{O}_{\mathcal{P}, \text{cc}}, \tau_{\varrho})$ is a C^2 integral manifold of the distribution spanned by \mathcal{P} . See [MM11]. Charts can be described as follows. For any $x \in \mathcal{O}_{\mathcal{P}, \text{cc}}$ and for each $I \in \mathcal{I}(p, q)$ such that $|Y_I(x)| \neq 0$ there are $\varepsilon, \delta > 0$ and $\beta \in C^1(B_{\text{Euc}}(x, \varepsilon), \mathbb{R}^{p \times p})$ such that the vector fields $V_j := \sum_{k=1}^p \beta_j^k Y_{i_k}$, where $j = 1, \dots, p$, are C^1 smooth of $B_{\text{Euc}}(x, \varepsilon)$ and satisfy $[V_j, V_k](\xi) = 0$ for all $\xi \in B_{\rho}(x, \delta) \subset B_{\text{Euc}}(x, \varepsilon)$ where ϱ is defined in (2.1). Moreover, the map

$$\Psi_{I,x}(u) := \exp\left(\sum_{1 \leq j \leq p} u_j V_j\right)x \quad (3.6)$$

is a C^2 full rank map from a neighborhood $O_{I,x}$ of the origin which parametrizes a C^2 embedded submanifold $\Psi_{I,x}(O_{I,x})$ which satisfies $T_{\Psi_{I,x}(h)}\Psi_{I,x}(O_{I,x}) = P_{\Psi_{I,x}(h)}$ for all $h \in O_{I,x}$. Furthermore, the family $\mathcal{S} := \{\Psi_{I,x}(O) : O \subset O_{I,x} \text{ is an open neighborhood of the origin}\}$ can be used as a base for a topology $\tau(\mathcal{S})$ on $\mathcal{O}_{\mathcal{P}, \text{cc}}$ which is equivalent to τ_{ϱ} .

All these facts have been proved in [MM11] for Lipschitz vector fields and in particular hold in our case.

Proof of Lemma 3.5. Inclusion (a) is obvious together with its continuity. Indeed, we always have $B_{\varrho}(x, r) \supset E_{I,x}(\{\|h\|_I < C^{-1}r\})$ for all x, r and for some universal C .

To prove (b), we use the topology $\tau(\mathcal{S})$ instead of τ_{ϱ} . Let Σ be a $\tau(\mathcal{U})$ -neighborhood of some fixed $x \in \mathcal{O}_{\mathcal{H}}$. Taking $I \in \mathcal{I}(p, q)$ such that $|Y_I(x)| \neq 0$, we may assume that for some neighborhood O of the origin $\Sigma \supset E_{I,x}(O)$, where $E_{I,x}(O)$ is a C^1 embedded p -dimensional submanifold. Possibly taking a smaller O , we may assume that $E_{I,x}(O) \cap B_{\text{Euc}}(x, \delta)$ is a C^1 graph. We claim that there is $\sigma > 0$ such that the inclusion $\Psi_{I,x}(B_{\text{Euc}}(\sigma)) \subset E_{I,x}(O)$ holds. This will conclude the proof. To show this claim, note that, given $u \in B_{\text{Euc}}(\sigma)$, we can write $\Psi_{I,x}(u) = \gamma(1)$, where γ is the integral curve of the C^1 vector field $\sum_j u_j V_j$. Since the vector fields V_j are C^1 , the required statement follows if σ is small enough by an application of Bony's theorem [Bon69, Theorem 2.1].⁴ \square

An alternative proof of (b) relies on the fact that if $|Y_I(x)| \neq 0$, then for all $O \subset O_{I,x}$ the map $E_{I,x}|_O$ with values into the C^2 manifold $\mathcal{O}_{\mathcal{P}, \text{cc}}$ is C^1 and nonsingular. Therefore it is open, because the dimensions of O and $\mathcal{O}_{\mathcal{P}, \text{cc}}$ are the same.

The following lifting lemma implies Theorem (3.2)-(a).

⁴Recall that an application of Bony's theorem states that, if $\Sigma \subset \mathbb{R}^n$ with a topology τ is a C^1 immersed submanifold of \mathbb{R}^n and V is a locally Lipschitz vector field such that $V(x) \in T_x \Sigma$ for all $x \in \Sigma$, then for all $x \in \Sigma$, $e^{tV}x \in \Sigma$ for t close to 0. More precisely, for all $\Omega \in \tau$ and $x \in \Omega$ there is t_0 such that $e^{tV}x \in \Omega$ if $|t| \leq t_0$.

Lemma 3.7. *Let \mathcal{H} be a family of \mathcal{B}_s vector fields. If (I, x, r) is $\frac{1}{2}$ -maximal, $x \in \Omega$ and $r < r_0$, then there are $C_2 > 1$ and $\varepsilon_0 < 1$ such that for all $\varepsilon \leq \varepsilon_0$ letting $C_\varepsilon := C_2 \varepsilon^{-s}$ the following holds: let γ be a Lipschitz path such that $\gamma(0) = x$, $\dot{\gamma} = \sum_{j=1}^q c_j (C_\varepsilon^{-1} r)^{\ell_j} Y_j(\gamma)$ a.e. on $[0, 1]$, where $|c| \leq 1$. Then there is a Lipschitz continuous path $\theta : [0, 1] \rightarrow \mathbb{R}^n$ such that $\theta(0) = 0$, $E_{I,x,r}(\theta(t)) = \gamma(t)$ and $\|\theta(t)\|_I \leq \varepsilon$ for all $t \in [0, 1]$.*

Before giving the proof of the lemma, recall that if $p \in \mathbb{N}$ and $\chi, b \in \mathbb{R}^{p \times p}$, then

$$|\chi| \leq \frac{1}{2} \quad \Rightarrow \quad |(I_p + \chi)^{-1}(I_p + b) - I_p| \leq 2(|\chi| + |b|) \quad \text{for all } b \in \mathbb{R}^{p \times p}. \quad (3.7)$$

This can be seen by writing $(I_p + \chi)^{-1}$ as a Neumann series.

Proof of Lemma 3.7. The argument of the proof is analogous to [NSW85, MM12c]. We include the argument because it will be used in Proposition 3.9.

First of all, by Lemma 3.5-(b), we know that γ belongs to $\mathcal{O}_{\mathcal{H}}$. Let $\varepsilon \leq \varepsilon_0$ and define $C_\varepsilon := C_2 \varepsilon^{-s}$, where the constant C_2 will be fixed soon. Let $\bar{t} \in [0, 1]$. We say that $\theta \in \text{Lip}_{\text{Euc}}([0, \bar{t}], \mathbb{R}^p)$ is an ε -lifting of γ on $[0, \bar{t}]$ if $\theta(0) = 0$, $E \circ \theta = \gamma$ on $[0, \bar{t}]$ and $\|\theta(t)\|_I \leq \varepsilon$ for all $t \in [0, \bar{t}]$. Let $t_0 := \sup\{\bar{t} \in [0, 1] : \text{there is a } \varepsilon\text{-lifting of } \gamma \text{ on } [0, \bar{t}]\}$. We already know that $t_0 > 0$. Our purpose is to show that $t_0 = 1$.

Next we claim that if θ is an ε -lifting of γ on $[0, \bar{t}]$, then it should be

$$\|\theta(t)\|_I \leq \frac{\varepsilon}{2} \quad \text{for all } t \in [0, \bar{t}]. \quad (3.8)$$

In order to prove (3.8), Let $t^* \in (0, \bar{t})$. In a neighborhood O^* of $\theta(t^*)$ the map $E : O^* \rightarrow E(O^*)$ is a C^1 diffeomorphism onto an open neighborhood $E(O^*)$ of $\gamma(t^*)$ in \mathcal{O} . Let F be its inverse. Then for a.e. t close to t^* we get for all $k \in \{1, \dots, p\}$

$$\frac{d}{dt} \theta^k(t) = \frac{d}{dt} F^k(\gamma_t) = \sum_{1 \leq \beta \leq q} c_\beta(t) C_\varepsilon^{-\ell_\beta} \tilde{Y}_\beta F^k(\gamma_t) = \sum_{1 \leq \beta \leq q} c_\beta(t) C_\varepsilon^{-\ell_\beta} \sum_{1 \leq j \leq p} b_\beta^j \tilde{Y}_{i_j} F^k(\gamma_t).$$

Here $\tilde{Y}_{i_j} := r^{\ell_{i_j}} Y_{i_j}$. Differentiating the identity $(F \circ E)(h) = h$ for $h \in O^*$, we also get $I_p = d(F \circ E) = dF(E)dE = dF(E)[\tilde{Y}_I(E)](I_p + \chi)$. Letting $I_p + \mu = (I_p + \chi)^{-1}$, we obtain $|\tilde{Y}_{i_j} F^k| = |\delta_j^k + \mu_j^k| \leq C$ for all $j, k = 1, \dots, p$. Observe that $|I_p + \mu| \leq C$, by (3.7) with $b = 0$. Therefore $|\frac{d}{dt} \theta^k(t)| \leq CC_\varepsilon^{-1}$ for all $t \in [0, \bar{t}]$.

Now we are in a position to prove estimate (3.8). Assume that it is false. Then, there is $\tilde{t} \in (0, \bar{t})$ such that for all $t \in [0, \tilde{t}]$ we have $\|\theta(t)\| < \frac{\varepsilon}{2} = \|\theta(\tilde{t})\|$. Therefore, we get for some $k \in \{1, \dots, p\}$,

$$\left(\frac{\varepsilon}{2}\right)^{d_k} = |\theta^k(\tilde{t})| = \left| \int_0^{\tilde{t}} \frac{d}{d\tau} \theta^k(\tau) d\tau \right| \leq CC_\varepsilon^{-s} = CC_2^{-1} \varepsilon^s.$$

Therefore, if C_2 is large enough to ensure that $CC_2^{-1} < \frac{1}{2^s}$, this chain of inequalities can not hold. This shows (3.8).

At this point, it is easy to check that an ε -lifting on $[0, \bar{t}]$ is unique. Indeed, if there were two different liftings θ_1, θ_2 , then the set $\{t \in [0, \bar{t}] : \theta_1(t) = \theta_2(t)\}$ would be nonempty, open and closed in $[0, \bar{t}]$. This implies that t_0 is actually a maximum. To conclude the argument, observe that it can not be $t_0 < 1$, because in this case we could extend the lifting on a small interval $[0, t_0 + \delta]$, for some $\delta > 0$. The proof is concluded. \square

Remark 3.8. Note that the constant C_2 depends quantitatively on the constant C_0 and L_0 in [MM12a]. See Remark 2.2. In the particular case where \mathcal{H} satisfies the Hörmander condition at step s , then we have $\mathcal{O}_{\mathcal{H}} = \mathcal{O}_{\mathcal{P},cc} = \mathbb{R}^n$ and Lemma 3.7 holds with C_2 depending on C_0 and L_0 .

We are left with the proof of Theorem 3.2-(b). To prove such statement, we need a multidimensional version of the lifting statement just proved and we also need an *ad hoc* version of Street's ball-box Theorem [Str11] (this will be discussed in Section 4).

Let η_1 be the constant in Theorem 4.2. Fix $\eta_2 \leq \eta_1$ small enough to ensure that

$$C_6 \eta_2^{1/s} \leq C_2^{-1} \varepsilon_0^s, \quad (3.9)$$

where C_6 and η_2 appear in (4.4), while C_2 and ε_0 denote the constants in the already proved Theorem 3.2-(a). Note that the constant C_6 in (4.4) is completely independent of the results of the present section. Therefore (3.4) and (3.9) give the inclusions

$$E(Q_I(\varepsilon_0)) \supset B_\rho(x, C_2^{-1} \varepsilon_0^s r) \supset B_\rho(x, C_6 \eta_2^{1/s} r) \supset \Phi(B_{\text{Euc}}(\eta_2)) \supset B_\rho(x, C_6^{-1} \eta_2^s r),$$

where we kept (4.4) into account in last inclusion. Here (I, x, r) is η -maximal, $E := E_{I,x,r}$ and $\Phi := \Phi_{I,x,r}$.

Here is our lifting result for the maps Φ .

Proposition 3.9 (lifting of standard exponential maps). *Let \mathcal{H} be a \mathcal{B}_s family. Let η_2 be a constant satisfying (3.9), let (I, x, r) be a $\frac{1}{2}$ -maximal triple and let $\Phi = \Phi_{I,x,r}$ and $E := E_{I,x,r}$ be the corresponding maps. Then there are $\eta_3 \leq \eta_2$, $C_3 > 1$ and $\theta \in C_{\text{Euc}}^1(B_{\text{Euc}}(\eta_3), Q_I(\varepsilon_0))$ such that $\theta(0) = 0$,*

$$E(\theta(u)) = \Phi(u) \quad \text{for all } u \in B_{\text{Euc}}(\eta_3) \quad (3.10)$$

and, letting $d\theta(u) =: I_p + \omega(u)$, we have

$$|\omega(u)| \leq C_3 |u|^{1/s} \leq \frac{1}{2} \quad \text{for all } u \in B_{\text{Euc}}(\eta_3). \quad (3.11)$$

The constants η_3 and C_3 depend on L_1 in (2.7).

From now on, we restrict the choice of ε_0 in Theorem 3.1 and Lemma 3.7 in order to ensure that

$$C_2 \varepsilon_0 \leq \frac{1}{4}, \quad (3.12)$$

where C_2 appears in (3.2).

Taking for a while Proposition 3.9 for granted, we are ready to prove the injectivity statement of Theorem 3.2.

Proof of Theorem 3.2-(b). We combine the just stated proposition with Theorem 4.2. Let η_3 be the constant in Proposition 3.9. Since $\eta_3 \leq \eta_1$, where η_1 is the constant in Theorem 4.2, Φ must be one-to-one on $B_{\text{Euc}}(\eta_3)$. Thus, θ is one-to-one on the same set and E is one-to-one on $\theta(B_{\text{Euc}}(\eta_3))$. Clearly, estimate (3.11) implies that $\frac{1}{2}|u - \tilde{u}| \leq |\theta(u) - \theta(\tilde{u})| \leq 2|u - \tilde{u}|$, for all $u, \tilde{u} \in B_{\text{Euc}}(\eta_3)$. Therefore, $\theta(B_{\text{Euc}}(\eta_3)) \supseteq B_{\text{Euc}}(\eta_3/2)$. The proof is concluded taking $\varepsilon_1 = \eta_3/2$. \square

Proof of Proposition 3.9. The proof is articulated in three steps.

Step 1. Take η_3 so small that $\eta_3^{1/s} \leq C_2^{-1}\varepsilon_0^s$, where C_2 is the constant in Lemma 3.7. Then for any $\tilde{\eta} \leq \eta_3$ and for any $\theta \in C_{\text{Euc}}^1(B_{\text{Euc}}(\tilde{\eta}), \mathbb{R}^p)$ such that $\theta(0) = 0$ and $E(\theta) = \Phi$ on $B_{\text{Euc}}(\tilde{\eta})$, we have

$$\|\theta(u)\|_I \leq \frac{\varepsilon_0}{2} \quad \text{for all } u \in B_{\text{Euc}}(\tilde{\eta}). \quad (3.13)$$

To accomplish Step 1, assume that a lifting θ enjoying the described properties is given. Let $u \in B_{\text{Euc}}(\tilde{\eta})$ and look at the path $\gamma(t) = \Phi(tu)$, where $t \in [0, 1]$. Our choice of constants ensures that there is a unique lifting $\lambda \in \text{Lip}[0, 1]$, such that $\lambda(0) = 0$ and $E(\lambda(t)) = \gamma(t)$ on $[0, 1]$. (In fact here λ is C^1 smooth, because $\gamma \in C^1$.) Moreover, see estimate (3.8), we have $\|\lambda(1)\| \leq \frac{\varepsilon_0}{2}$. Since by uniqueness it must be $\theta(tu) = \lambda(t)$ for all t , Step 1 is accomplished.

Step 2. Let $\tilde{\eta} \leq \eta_3$ and let $\theta \in C^1(B_{\text{Euc}}(\tilde{\eta}))$ such that $\theta(0) = 0$ and $E \circ \theta = \Phi$ holds on $B_{\text{Euc}}(\tilde{\eta})$. Then we claim that (3.11) holds on $B_{\text{Euc}}(\tilde{\eta})$.

To prove the claim, observe that by Step 1 we know that $\|\theta(u)\| \leq \frac{\varepsilon_0}{2}$ for all $u \in B_{\text{Euc}}(\tilde{\eta})$. Therefore (3.3) gives

$$d\Phi(u) = dE(\theta(u))d\theta(u) = [\tilde{Y}_I(\Phi(u))] [I_p + \chi(\theta(u))]d\theta(u).$$

Combining with (4.5), which states that $d\Phi(u) = [\tilde{Y}_I(\Phi(u))] [I_p + b(u)]$, we conclude that $d\theta(u) = [I_p + \chi(\theta(u))]^{-1} [I_p + b(u)] =: I_p + \omega(u)$. To estimate $|\omega|$ observe that $\|\theta(u)\| \leq \varepsilon_0/2$, by Step 1. Therefore, (3.2) gives $|\chi(\theta(u))| \leq C_2 \|\theta(u)\| \leq \frac{1}{2} C_2 \varepsilon_0 \leq \frac{1}{8}$, by requirement (3.12) on ε_0 . Then (3.7) gives $|\omega(u)| \leq 2(|\chi(\theta(u))| + |b(u)|) \leq \frac{1}{4} + 2C_4\eta_3 \leq \frac{1}{2}$, if we choose η_3 small enough. Here C_4 is the constant appearing in (4.6). Thus $\text{Lip}_{\text{Euc}}(\theta; B_{\text{Euc}}(\tilde{\eta})) \leq 2$ and moreover

$$|\omega(u)| \leq 2(|\chi(\theta(u))| + |b(u)|) \leq 2(C_2 \|\theta(u)\| + C_4|u|) \leq C_3|u|^{1/s},$$

for some $C_3 > 1$ depending on L_1 only. Therefore (3.11) is completely proved and Step 2 is finished.

Step 3. Let $\Omega_1, \Omega_2 \subset B_{\text{Euc}}(\eta_3)$ be connected open sets. Assume that $\Omega_1 \cap \Omega_2$ is connected and that $0 \in \Omega_1$. Let also $\theta_i \in C_{\text{Euc}}^1(\Omega_i, \mathbb{R}^p)$ be such that $E \circ \theta_i = \Phi$, on Ω_i for $i = 1, 2$. Assume finally that $\theta_1(0) = 0$ and that $\theta_1(u_0) = \theta_2(u_0)$ for some $u_0 \in \Omega_1 \cap \Omega_2$. Then it must be $\theta_1 = \theta_2$ on $\Omega_1 \cap \Omega_2$.

To prove Step 3, let $A := \{u \in \Omega_1 \cap \Omega_2 : \theta_1(u) = \theta_2(u)\}$. Note that $A \neq \emptyset$ because $u_0 \in A$. We show that A is open and closed in $\Omega_1 \cap \Omega_2$. To see that A is open, let $\tilde{u} \in A$ and let $\tilde{h} = \theta_1(\tilde{u}) = \theta_2(\tilde{u})$. By Step 1 we know that $\|\tilde{h}\| \leq \frac{\varepsilon_0}{2}$. Since the map E is nonsingular, there is a neighborhood \tilde{O} of \tilde{h} such that $E|_{\tilde{O}} : \tilde{O} \rightarrow E(\tilde{O}) \subset \mathcal{O}$ is a C^1 diffeomorphism. Let \tilde{F} be its inverse. Note also that, since the maps θ_i are continuous, we may assume that for a small open set \tilde{V} containing \tilde{u} and contained in $\Omega_1 \cap \Omega_2$, we have $\theta_i(\tilde{V}) \subset \tilde{O}$. Therefore, starting from identity $E(\theta_1(u)) = E(\theta_2(u))$ for all $u \in \tilde{V}$, we can apply \tilde{F} and we get $\theta_1(u) = \theta_2(u)$ for all $u \in \tilde{V}$. This shows that A is open.

Finally, to show that A is closed, let $u_n \in A$ for all $n \in \mathbb{N}$, $u_n \rightarrow u \in \Omega_1 \cap \Omega_2$, as $n \rightarrow \infty$. Then, the continuity of θ_1 and θ_2 ensures that $\theta_1(u) = \theta_2(u)$, as desired.

Step 4. Finally, we show that the lifting exists. Let $\tilde{\eta} := \sup\{\eta \in (0, \eta_3] : \text{there is } \theta \in C^1(B_{\text{Euc}}(\tilde{\eta}), \mathbb{R}^p) \text{ such that } \theta(0) = 0 \text{ and } E(\theta) = \Phi \text{ on } B_{\text{Euc}}(\tilde{\eta})\}$. We will show that $\tilde{\eta} = \eta_3$.

To show Step 4, assume that $\tilde{\eta} < \eta_3$ strictly. Let (η_n) be a sequence with $\eta_n \nearrow \tilde{\eta}$. Then, there are $\theta_n \in C^1_{\text{Euc}}(B_{\text{Euc}}(\eta_n), \mathbb{R}^p)$ with $\theta_n(0) = 0$ and $E \circ \theta_n = \Phi$ on $B_{\text{Euc}}(\tilde{\eta})$. By Step 3, there is a unique $\tilde{\theta} \in C^1(B_{\text{Euc}}(\tilde{\eta}))$ which extends all the maps θ_n . Note that the map $\tilde{\theta}$ is 1/2-biLipschitz up to $\tilde{B}_{\text{Euc}}(\tilde{\eta}) =: \tilde{B}$, by Step 2. Now, fix a point $u_1 \in \partial\tilde{B}$. Let $B_{\text{Euc}}(u_1, \delta_1) \subset B_{\text{Euc}}(\eta_3)$ be a ball of sufficiently small radius δ_1 so that $\tilde{\theta}(B(u_1, \delta_1) \cap \tilde{B}) \subset O$, where O is a neighborhood of $\tilde{\theta}(u_1)$ such that $E|_O : O \rightarrow E(O) \subset \mathcal{O}$ is a C^1 -diffeomorphism (we can equip $\mathcal{O}_{\mathcal{H}} = \mathcal{O}_{\mathcal{P}, \text{cc}}$ with the C^2 differential structure on \mathcal{O} described by the family of charts of the form (3.6)). Let $F : E(O) \rightarrow O$ be its inverse. The set $\Phi^{-1}(E(O))$ contains the ball $B_{\text{Euc}}(u_1, \delta'_1)$ for some $\delta'_1 \leq \delta$. We can define the map $\theta_1(u) := F(\Phi(u))$ for all $u \in B(u_1, \delta'_1)$. Therefore, by Step 3, we have extended the lifting to the domain $\tilde{B} \cup B(u_1, \delta'_1)$. Iterating a finite number of times we discover that the map $\tilde{\theta}$ can be extended to a larger ball $B_{\text{Euc}}(\tilde{\eta} + \delta)$, for some small $\delta > 0$. Therefore it can not be $\tilde{\eta} < \eta_3$ strictly and the proof is concluded. \square

4. Ball-box theorem for standard exponential maps

Here we prove a ball-box theorem for the exponential maps Φ associated with a family $\mathcal{P} = \{Y_1, \dots, Y_q\}$ of vector fields. We use the methods introduced in [TW03] and [Str11]. However, since we assume less regularity than [Str11], we need to modify slightly some of the original techniques.

We keep our usual notation. Given a family $\mathcal{P} = \{Y_1, \dots, Y_q\}$ of C^1 vector fields, with degrees $\ell_1, \dots, \ell_q \leq s$, we write $Y_j = g_j \cdot \nabla$. Denote by B_ϱ balls with respect to the distance ϱ defined in (2.1). It is known that if there are locally bounded coefficients c_{ij}^k such that (1.2) holds, then any orbit $\mathcal{O}_{\mathcal{P}, \text{cc}}^{x_0} := \{y \in \mathbb{R}^n : d_{\text{cc}}(x, y) < \infty\}$ with topology $\tau_{d_{\text{cc}}}$ is an immersed C^2 submanifold and it is an integral manifold of the distribution generated by \mathcal{P} . (In the paper [MM11] we show a more general statement involving Lipschitz vector fields.) Here we assume that c_{ij}^k are C^1 -smooth on each orbit \mathcal{O} . Introduce the admissible constant

$$L_2 := \sum_{i,j,k,\ell=1}^q \sup_{\Omega_0} |c_{ij}^\ell| + \sup_{\Omega_0} |Y_k c_{ij}^\ell|. \quad (4.1)$$

Remark 4.1. *If a family $\mathcal{H} = \{X_1, \dots, X_m\}$ belongs to \mathcal{B}_s , then the constant L_2 associated with the family $\mathcal{P} = \{Y_1, \dots, Y_q\}$ satisfies obviously $L_2 \leq L_1$, see (2.7).*

Let $\Omega \Subset \Omega_0$ be the fixed sets introduced after (2.7). Fixed $x_0 \in \Omega$, $r > 0$ and $I \in \mathcal{I}(p_{x_0}, q)$, define for u close to the origin

$$\Phi(u) := \Phi_{I,x,r}(u) := \exp \left(\sum_{1 \leq j \leq p} u^j \tilde{Y}_{i_j} \right) (x_0) \quad (4.2)$$

where, for $k = 1, \dots, q$, we let $\tilde{Y}_k = r^{\ell_k} Y_k = \sum_{\alpha=1}^n \tilde{g}_k^\alpha \partial_\alpha$. If $|Y_I(x_0)| \neq 0$ and $\delta > 0$ is small enough, then the map $\Phi|_{B_{\text{Euc}}(\delta)} : B_{\text{Euc}}(\delta) \rightarrow \Phi(B_{\text{Euc}}(\delta)) \subset \mathcal{O}$ is a C^1 diffeomorphism. Here we equip \mathcal{O} with the C^2 differentiable structure given by charts of the form (3.6).

The inverse map $\Psi := (\Phi|_{B_{\text{Euc}}(\delta)})^{-1}$ is a C^1 chart on \mathcal{O} . Note that a map $f : \mathcal{O} \rightarrow \mathbb{R}$ is $C^1_{\mathcal{O}}$ if $f \circ \Phi$ is C^1_{Euc} for all charts of such family.

Theorem 4.2. *Let \mathcal{P} be a family of C^1 vector fields. Assume that there are functions c^k_{ij} locally bounded in \mathbb{R}^n such that the integrability condition (1.2) holds at any point. Assume also that $c^k_{ij} \in C^1(\mathcal{O})$. Let (I, x_0, r) be $\frac{1}{2}$ -maximal, where $x_0 \in \Omega$, $r \leq r_0$ and $I \in \mathcal{I}(p_{x_0}, q)$. Let $p_{x_0} =: p$ be the (constant on \mathcal{O}) dimension of P_{x_0} . Then there are constants $\eta_1, C_6, C_5 > 0$ depending on L_2 in (4.1) such that*

(i) *there is $A \in C^1_{\text{Euc}}(B_{\text{Euc}}(\eta_1), \mathbb{R}^{p \times p})$ such that the vector fields $Z_j = \partial_{u_j} + \sum_{k=1}^p a^k_j(u) \partial_{u_k}$ on $B_{\text{Euc}}(\eta_1)$, $j = 1, \dots, p$ satisfy $\Phi_* Z_j = \tilde{Y}_{i_j}$ and enjoy estimate*

$$\sup_{u \in B_{\text{Euc}}(\eta_1)} |\nabla A(u)| \leq C_5; \quad (4.3)$$

(ii) *the map $\Phi = \Phi_{I, x_0, r}$ is one-to-one on the Euclidean ball $B_{\text{Euc}}(\eta_1)$;*
(iii) *for all $\eta_2 \in]0, \eta_1]$ we have the inclusions*

$$B_\varrho(x_0, C_6 \eta_2^{1/s} r) \supseteq \Phi_{I, x_0, r}(B_{\text{Euc}}(\eta_2)) \supseteq B_\varrho(x_0, C_6^{-1} \eta_2^s r), \quad (4.4)$$

In Street [Str11], Theorem 4.2 was proved assuming that $Y_j \in C^2$ and that $c^k_{ij} \in C^2$. Here we improve the result to $Y_j \in C^1$ and $c^k_{ij} \in C^1_{\mathcal{O}}$, where $C^1_{\mathcal{O}}$ refers to C^1 regularity on the manifold \mathcal{O} described by charts of the form (3.6). The main novelty is in the proof of (ii). Namely, in Theorem 4.5, we use the Gronwall inequality instead of the uniform inverse map theorem used in [Str11, Proposition 3.20]. With Theorem 4.5 in hands, the proof of the injectivity of the map Φ is identical to the one contained in [TW03].

Since we are working with less regularity than [Str11], in order to keep constants under control in terms of our data, we give also a description of Street's arguments to show (i); see Lemma 4.3 and Theorem 4.4 below. Finally, we do not discuss the proof of (iii). Inclusion in the left-hand side is trivial, while the one in the right-hand side follows from a well known path-lifting argument (see [NSW85, MM12c, Str11]), which we already used in Section 3.

Note that under the hypotheses of Theorem 4.2, possibly shrinking η_1 , we get

$$\frac{\partial \Phi}{\partial u} = [\tilde{Y}_{i_1, \Phi}, \dots, \tilde{Y}_{i_p, \Phi}](I_p + b(u)) \quad (4.5)$$

where $I_p + b(u) := (I_p + a(u))^{-1}$ satisfies for some C_4 depending on L_2 in (4.1),

$$|b(u)| = \left| \sum_{k \geq 1} (-A(u))^k \right| \leq C |A(u)| \leq C_4 |u| \quad \text{for all } u \in B_{\text{Euc}}(\eta_1). \quad (4.6)$$

Before starting the proof of the theorem, we look at the behaviour of the ‘‘integrability coefficients’’ on a ball.

Lemma 4.3. *Let $I \in \mathcal{I}(p, q)$ be such that (I, x_0, r) is $\frac{1}{2}$ -maximal, with $x_0 \in \Omega$ and $r \leq r_0$, where r_0 is small enough to ensure that: $B_\varrho(x, r_0) \subset \Omega_0$. Then we may write*

$$[\tilde{Y}_i, \tilde{Y}_j]_x = \sum_{1 \leq k \leq p} \tilde{c}^k_{ij}(x) \tilde{Y}_{i_k, x} \quad \text{for all } i, j \leq q \quad x \in B_\varrho(x_0, \varepsilon_0 r), \quad (4.7)$$

where $\tilde{c}_{ij}^k \in C^1_{\mathcal{O}}(B_{\varrho}(x_0, \varepsilon_0 r))$ and

$$\max_{\substack{i,j=1,\dots,q \\ k=1,\dots,p}} \left(\sup_{B_{\varrho}(x_0, \varepsilon_0 r)} (|\tilde{c}_{ij}^k| + |\tilde{Y}_{\ell} \tilde{c}_{ij}^k|) \right) \leq C = C(L_2). \quad (4.8)$$

The constants $\varepsilon_0 < 1$ and $C(L_2) > 1$ depend on L_2 in (4.1) but not on $r \in (0, r_0)$.

Proof. Assume for simplicity that $I = (1, \dots, p)$. Let γ be a Lipschitz path satisfying, a.e. on $[0, 1]$, $\dot{\gamma} = \sum_{j=1}^q c_j \tilde{Y}_j(\gamma)$. Then, arguing as in [Str11, Section 4] (see also [MM12a, Proposition 3.2]), we get for a.e. $t \in [0, 1]$ the inequality $|\frac{d}{dt} \Lambda_p(\gamma(t), r)| \leq C |\Lambda_p(\gamma(t), r)|$. Therefore, the Gronwall's inequality (2.15) gives $|\Lambda_p(\gamma_t, r) - \Lambda_p(x, r)| \leq |\Lambda_p(x, r)| (e^{Ct} - 1)$. Moreover we have

$$|\tilde{Y}_I(x)| > C^{-1} \max_{H \in \mathcal{I}(p, q)} |\tilde{Y}_H(x)| \quad \text{for any } x \in B_{\varrho}(x_0, \varepsilon_0 r). \quad (4.9)$$

Thus, in the notation $I_{\ell}^k = (i_1, \dots, i_{k-1}, \ell, i_{k+1}, \dots, i_p)$, by the integrability (2.6) and the Cramer's rule (2.9) we have for all $x \in B_{\varrho}(x_0, \varepsilon_0 r)$

$$\begin{aligned} [\tilde{Y}_i, \tilde{Y}_j]_x &= \sum_{\ell=1}^q \tilde{c}_{ij}^{\ell}(x) \tilde{Y}_{\ell, x} = \sum_{\ell=1}^q \tilde{c}_{ij}^{\ell}(x) \sum_{k=1}^p \frac{\langle \tilde{Y}_{I_{\ell}^k, x}, \tilde{Y}_{I, x} \rangle}{|\tilde{Y}_{I, x}|^2} \tilde{Y}_{k, x} \\ &= \sum_{k=1}^p \left\{ \sum_{\ell=1}^q \tilde{c}_{ij}^{\ell}(x) \frac{\langle \tilde{Y}_{I_{\ell}^k, x}, \tilde{Y}_{I, x} \rangle}{|\tilde{Y}_{I, x}|^2} \right\} \tilde{Y}_{k, x} =: \sum_{k=1}^p \tilde{c}_{ij}^k(x) \tilde{Y}_{k, x}. \end{aligned}$$

Note that by our assumptions, we have $\tilde{c}_{ij}^k \in C^1(\mathcal{O})$. See the discussion after (4.2). Moreover, since $Y_j \in C^1_{\text{Euc}}$, for all j, ℓ , we have $\tilde{g}_j^{\ell} \in C^1_{\mathcal{O}}$. This ensures that $\tilde{c}_{ij}^k \in C^1_{\mathcal{O}}(B_{\varrho}(x_0, \varepsilon_0 r))$ and easily we have the estimate $|\tilde{c}_{ij}^k| \leq C$ on $B_{\varrho}(x_0, \varepsilon_0 r)$.

Next we need to estimate the derivatives of the coefficients \tilde{c}_{ij}^k . Note first that $\sup |\tilde{Y}_h \tilde{c}_{ij}^{\ell}| = r \sup |Y_h \tilde{c}_{ij}^{\ell}| \leq r L_2 \leq L_2$, see (4.1). Moreover, observe that for $x \in B_{\varrho}(x_0, \varepsilon_0 r)$, $h \in \{1, \dots, q\}$, $K \in \mathcal{I}(p, q)$ and $H \in \mathcal{I}(p, n)$, we have

$$|\tilde{Y}_h \tilde{Y}_K^H(x)| = \left| \frac{d}{dt} \tilde{Y}_K^H(e^{t \tilde{Y}_h} x) \Big|_{t=0} \right| \leq C |\Lambda_p(x, r)| \leq C |\tilde{Y}_I(x)|.$$

Here we used (4.9). This furnishes, on $B_{\varrho}(x_0, \varepsilon_0 r)$, the estimate $\left| \tilde{Y}_h \frac{\langle \tilde{Y}_K, \tilde{Y}_I \rangle}{|\tilde{Y}_I|^2} \right| \leq C$, for all $K \in \mathcal{I}(p, q)$ and $h \in \{1, \dots, q\}$. The proof of the lemma is easily concluded. \square

Let (I, x, r) be a $\frac{1}{2}$ -maximal triple for \mathcal{P} and let $\Phi = \Phi_{I, x, r}$ be the associated exponential. For small $\delta > 0$, the map $\Phi|_{B_{\text{Euc}}(\delta)} : B_{\text{Euc}}(\delta) \rightarrow \Phi(B_{\text{Euc}}(\delta)) \subset \mathcal{O}$ is a C^1 diffeomorphism. At this stage there is no control on δ in terms of the constant L_2 in (4.1). Following [TW03] and [Str11], for $j \in \{1, \dots, p\}$ let

$$\hat{Z}_j =: \sum_{k=1}^p \hat{h}_j^k(u) \partial_{u_k} =: \sum_{k=1}^p (\delta_j^k + \hat{a}_j^k(u)) \partial_{u_k} \quad (4.10)$$

be the pull-back of \tilde{Y}_{i_j} on the small Euclidean ball $B_{\text{Euc}}(\delta)$. Note that $\hat{a}_j^k(0) = 0$. Starting from identity $\sum_j u_j \partial_j = \sum_j u_j \hat{Z}_j$ on the ball $B_{\text{Euc}}(\delta)$ and commuting with \hat{Z}_i , one can show that the coefficients \hat{a}_j^k satisfy the ODE

$$\partial_\varrho(\varrho \hat{A}(\varrho\omega)) = -\{\hat{A}^2(\varrho\omega) + C(\varrho\omega)\hat{A}(\varrho\omega) + C(\varrho\omega)\} \quad (4.11)$$

for $0 < \varrho < \delta$ and $\omega \in \mathbb{S}^{p-1}$. Here

$$\begin{aligned} \hat{A}_{ik}(u) &:= \hat{a}_i^k(u) \quad \text{on } B_{\text{Euc}}(\delta) \quad \text{and} \\ C_{ik}(u) &:= \sum_{j=1}^p u^j (\tilde{c}_{ij}^k \circ \Phi)(u) \quad \text{on } B_{\text{Euc}}(\eta_1), \end{aligned} \quad (4.12)$$

where $\delta > 0$ is a possibly very small positive number, while we may choose $\eta_1 > 0$ depending ultimately on the admissible constant L_2 so that $\Phi(B_{\text{Euc}}(\eta_1)) \subseteq B_\varrho(x_0, \varepsilon_0 r)$. Equation (4.11) is obtained in [Str11], but some details are left to the reader. Unfortunately, we have not been able to derive (4.11) in a completely trivial way. Thus we decided to fill up the details in the appendix. In particular we shall discuss all the regularity issues related with the fact that our vector fields Y_j are C^1 smooth only.

Next we give a result, which is basically a restatement of [Str11, Theorem 3.10]. Since we are removing some of Street's regularity assumptions, we do not get estimates on derivatives of A of order greater than one.

Theorem 4.4. *Let $\mathcal{P} = \{Y_1, \dots, Y_q\}$ be a family as in the assumptions of Theorem 4.2. Let (I, x, r) be $\frac{1}{2}$ -maximal with $x \in \Omega$ and $r \leq r_0$. Denote by $C : B_{\text{Euc}}(\eta_1) \rightarrow \mathbb{R}^{p \times p}$ the matrix in (4.12). Then, possibly taking a smaller η_1 depending on L_2 in (4.1), there is a unique $A \in C^1(B_{\text{Euc}}(\eta_1), \mathbb{R}^{p \times p})$ which solves for all $\omega \in \mathbb{S}^{p-1}$*

$$\partial_\varrho(\varrho A(\varrho\omega)) = -\{A^2(\varrho\omega) + C(\varrho\omega)A(\varrho\omega) + C(\varrho\omega)\} \quad \text{if } 0 < \varrho < \eta_1, \quad (4.13)$$

satisfies $A(0) = 0$ and enjoys the global estimate

$$\sup_{|u| \leq \eta_1} |\nabla A(u)| \leq C_5, \quad (4.14)$$

where C_5 depends on L_2 . Moreover, on the small ball $B_{\text{Euc}}(\delta)$, we have $A_{jk} = \hat{A}_{jk}$, where \hat{A}_{jk} is defined in (4.12).

Proof. We recapitulate Street's arguments.

Step 1. By Lemma 4.3 there are $\tilde{c}_{ij}^k \in C^1_{\mathcal{O}}(B_\varrho(x_0, \varepsilon_0 r))$ such that (4.7) holds with estimate $\sup_{B_\varrho(x_0, \varepsilon_0 r)} |\tilde{c}_{ij}^k| \leq C$, see (4.8). Although at this stage, we do not have any estimate on the C^1 norm $\sup_u |\nabla_u (\tilde{c}_{ij}^k \circ \Phi)(u)|$, we may use the existence part of [Str11, Theorem 3.10] to obtain the existence of a unique $A \in C^0(B_{\text{Euc}}(\eta_1), \mathbb{R}^{p \times p})$ such that (4.13) holds and $|A(u)| \leq C|u|$ for all $u \in B_{\text{Euc}}(\eta_1)$.

Step 2. Now we use *Step 1* to estimate the C^1 norm of $(\tilde{c}_{ij}^k \circ \Phi)$. Note first that, since $\tilde{c}_{ij}^k \in C^1_{\mathcal{O}}(B_\varrho(x_0, \varepsilon_0 r))$ and $\Phi \in C^1(B_{\text{Euc}}(\eta_1), B_\varrho(x_0, \varepsilon_0 r))$, we have for all $1 \leq h, k \leq p$ and $1 \leq i, j \leq q$,

$$|Z_h(\tilde{c}_{ij}^k \circ \Phi)(u)| = |\tilde{Y}_h \tilde{c}_{ij}^k(\Phi(u))| \leq C \quad \text{for all } u \in B_{\text{Euc}}(\eta_1), \quad (4.15)$$

where the constant C depends on L_2 , see estimate (4.8). By *Step 1*, we can write for $h \in \{1, \dots, p\}$, $Z_h = \partial_{u_h} + \sum_{j=1}^p a_h^j(u) \partial_{u_j}$, where $|a_h^j|$ is very small. Therefore, estimate (4.15) is equivalent to $|\nabla_u(\tilde{c}_{ij}^k \circ \Phi)(u)| \leq C$ on $B_{\text{Euc}}(\eta_1)$ for some new constants C and η_1 depending on L_2 in (4.1).

Step 3. Here we use the hard work done in the regularity part of [Str11, Theorem 3.10] to deduce that $A \in C^1(B_{\text{Euc}}(\eta_1), \mathbb{R}^{p \times p})$ and satisfies estimate (4.14).

Step 4. As a last step, one shows that $A = \widehat{A}$ on $B_{\text{Euc}}(\delta)$. This can be done as in [Str11, Lemma 3.1]. \square

In order to show the injectivity, Theorem 4.2-(ii), given a $\frac{1}{2}$ -maximal triple (I, x_0, r) , for all $u_1 \in B_{\text{Euc}}(\eta_1)$, consider the exponential map

$$\Psi(v) := \Psi_{u_1}(v) := \exp\left(\sum_{1 \leq j \leq p} v_j Z_j\right)u_1, \quad (4.16)$$

where v belongs to a neighborhood of the origin in \mathbb{R}^p . The map is C^1 , because $Z_j \in C^1$.

Theorem 4.5. *Let (I, x_0, r) be $\frac{1}{2}$ -maximal, where $x_0 \in \Omega$, $I \in \mathcal{I}(p_{x_0}, q)$ and $r \leq r_0$. Then there is $\eta_2 > 0$ such that*

$$\frac{1}{2} \leq \frac{|\Psi_{u_1}(v) - \Psi_{u_1}(\bar{v})|}{|v - \bar{v}|} \leq 2 \quad \text{for all } u_1 \in B_{\text{Euc}}(\eta_2) \quad v, \bar{v} \in B_{\text{Euc}}(\eta_2). \quad (4.17)$$

Note that Theorem 4.5 implies that for all $u_1 \in B_{\text{Euc}}(\eta_2)$, the map Ψ_{u_1} is one-to-one on $B_{\text{Euc}}(\eta_2)$ and, by a standard path-lifting argument, it ensures the quantitative openness condition $\Psi_{u_1}(B_{\text{Euc}}(\eta_2)) \supset B_{\text{Euc}}(u_1, \frac{1}{2}\eta_2)$ for all $u_1 \in B_{\text{Euc}}(\eta_2)$.

Once Theorem 4.5 is proved, then the injectivity of the map Φ follows from the argument in [TW03, p. 622], or [Str11, Proposition 3.20]. We omit the proof.

Proof of Theorem 4.5. It suffices to show that there is $\eta_2 \leq \eta_1$ such that for all $u_1 \in B_{\text{Euc}}(\eta_2)$, the map $\Psi = \Psi_{u_1}$ satisfies

$$\sup_{v \in B_{\text{Euc}}(\eta_2)} |d\Psi_{u_1}(v) - I_p| \leq \frac{1}{2} \quad \text{for all } u_1 \in B_{\text{Euc}}(\eta_2),$$

where as usual $|\cdot|$ denotes the operator norm.

To show this estimate, recall that the vector fields $Z_j = \partial_j + \sum_k a_j^k(u) \partial_k$ on $B_{\text{Euc}}(\eta_1)$ satisfy (4.3). Therefore,

$$|a(u)| \leq C_5 |u| < \eta_1, \quad (4.18)$$

provided that $|u| < \eta_1/C_5$. Now we show that

$$|u_1| < \frac{\eta_1}{2C_5} \quad \text{and} \quad |v| < \frac{\eta_1}{4C_5} \quad \Rightarrow \quad |\Psi_{u_1}(v)| < \frac{\eta_1}{C_5}. \quad (4.19)$$

To prove (4.19) let $y = y(t, v) := \Psi_{u_1}(tv)$. Assume that for some $t_0 \leq 1$ we have

$$\frac{\eta_1}{C_5} = |y(t_0, v)| > |y(t, v)| \quad \text{for all } t \in [0, t_0[.$$

Then

$$\begin{aligned} \frac{\eta_1}{C_5} &= |y(t_0)| \leq |u_1| + \left| \int_0^{t_0} (I_p + a(y(\tau))) v d\tau \right| \leq |u_1| + |v|t_0 + \eta_1|v|t_0 \\ &< \frac{\eta_1}{2C_5} + \frac{\eta_1}{2C_5}t_0. \end{aligned}$$

But this can not hold unless $t_0 > 1$. Therefore, (4.19) is proved.

Let us look again at $y = y(t, v)$, note that $\frac{\partial y^k}{\partial v_j}(0, v) = 0$ and $|a(y(t, v))| < \eta_1$ for all $t \in [0, 1]$, if $|u| < \frac{\eta_1}{2C_5}$ and $|v| < \frac{\eta_1}{4C_5}$ (this follows from (4.18) and (4.19)). Write the variational equation

$$\frac{d}{dt} \frac{\partial y^k}{\partial v_j} = \frac{\partial}{\partial v_j} \left(\sum_{1 \leq \ell \leq p} (\delta_\ell^k + a_\ell^k(y)) v_\ell \right) = \delta_j^k + a_j^k(y) + \sum_{1 \leq \ell, h \leq p} \partial_h a_\ell^k(y) \frac{\partial y^h}{\partial v_j} v_\ell.$$

Denote $(\frac{\partial y^k}{\partial v_j}(t))_{j,k=1}^p =: w(t) \in \mathbb{R}^{p \times p}$ and $(L_v(t))_h^k := \sum_{\ell=1}^p \partial_h a_\ell^k(y(t)) v_\ell$. Note estimate $|L_v(t)| \leq C_5|v|$. Starting from the ODE $\dot{w}(t) = I_p + a(y(t)) + L_v(t)w(t)$ and integrating, we obtain

$$\begin{aligned} |w(t) - tI_p| &\leq \int_0^t \{C_5|v| |w(\tau) - \tau I_p| + \tau |L_v(\tau)| + |a(y(\tau))|\} d\tau \\ &\leq C_5|v| \int_0^t |w(\tau) - \tau I_p| d\tau + C_5 t |v| + t\eta_1, \end{aligned}$$

for all $t \in [0, 1]$. The Gronwall inequality (2.15) gives

$$|w(1) - I_p| \leq \frac{C_5|v| + \eta_1}{C_5|v|} (\exp(C_5|v|) - 1) \leq \frac{1}{2},$$

as soon as we assume without loss of generality that $\eta_1 < \frac{1}{4}$ and we take $|v| \leq \eta_2$ where η_2 is small enough, depending on C_5 and η_1 . \square

5. Some remarks in a more regular setting

In more regular situations than ours, the Poincaré inequality can be obtained putting together Jerison's result and Street's theorem, [Jer86] and [Str11]. In this section we briefly discuss this idea and we explain why our regularity classes fall out of this approach. The discussion included here has been suggested by the referee.

Let $\mathcal{H} := \{X_1, \dots, X_m\}$ be a smooth family and assume that the family $\mathcal{P} = \mathcal{P}_s = \{Y_1, \dots, Y_q\}$ satisfies (1.2) for smooth coefficients c_{ij}^k . By [Str11], there is $C_b > 1$ such that if (I, x, r) is $\frac{1}{2}$ -maximal ($I \in \mathcal{I}(p, q)$), then the associated map $\Phi := \Phi_{I,x,r}$ is one-to-one on $B_{\text{Euc}}(C_b^{-1}) \subset \mathbb{R}^p$ and satisfies

$$\begin{aligned} C_b^{-1} |\Lambda_p(x, r)| &\leq |J_\Phi(u)| \leq C_b |\Lambda_p(x, r)| \quad \text{for all } u \in B_{\text{Euc}}(C_b^{-1}), \\ B_\rho(x, C_b^{-2}r) &\subset \Phi(B_{\text{Euc}}(C_b^{-1})). \end{aligned}$$

Moreover, letting $Z_j := \Phi^*(rX_j)$ be the pullbacks of the vector fields and taking $f \in C^1(\mathbb{R}^n, \mathbb{R})$, we have for any $a \in \mathbb{R}$,

$$\int_{B_{\text{cc}}(x, C_b^{-2}r)} |f(y) - a| d\sigma^p(y) \leq C_b |\Lambda_p(x, r)| \int_{B_Z(0, C_b^{-2})} |g(u) - a| du.$$

where $g = f \circ \Phi$. By the Jerison's Poincaré inequality, since Z_1, \dots, Z_m are smooth Hörmander vector fields in $B_{\text{Euc}}(C_b^{-1})$, there is $C_{\sharp} > 1$ so that

$$\inf_a \int_{B_Z(0, C_b^{-2})} |g(u) - a| du \leq C_{\sharp} \int_{B_Z(0, C_b^{-2})} \sum_{1 \leq j \leq m} |Z_j g(u)| du. \quad (5.1)$$

Going back to the variable y , one gets the Poincaré inequality for the original smooth vector fields X_j .

Let us make some comments on the constants C_b, C_{\sharp} appearing in the computations above. By Street's theorem [Str11], we have $C_b = C_b(\|Y_j\|_{C^2}, \|c_{ij}^k\|_{C^2})$. The constant C_{\sharp} appears in Jerison's paper. It is known that $C_{\sharp} = C_{\sharp}(\|X_j\|_{C^M})$, where M is a rather large number for which there is no precise estimate.

Regularity requirements in the argument above can be improved avoiding the Jerison's proof of the Poincaré inequality and using the approach in [MM12c]. Indeed, in [MM12c], using almost exponential maps, it was proved that (5.1) holds with

$$C_{\sharp} = C_{\sharp} \left(\|Z_j\|_{C^{s-1,1}}, \inf_{B_{\text{Euc}}(C_b^{-1})} \max_{|w_1|, \dots, |w_p| \leq s} |\det(Z_{w_1}, \dots, Z_{w_p})| \right),$$

where Z_{w_j} is a commutator of Z_1, \dots, Z_m of length $|w_j|$. By [Str11], the infimum can be controlled similarly to C_b , in terms of $\|Y_j\|_{C^2}, \|c_{ij}^k\|_{C^2}$. In order to ensure that $Z_j := \Phi^*(rX_j) \in C^{s-1,1}$, it suffices to assume that $\Phi \in C^{s,1}$ and $X_j \in C^{s-1,1}$. In view of the explicit form $\Phi(u) = \exp(\sum_{k=1}^p u_k Y_{i_k})(x)$, a concrete sufficient condition on the original vector fields X_1, \dots, X_m to ensure that $\Phi \in C^{s,1}$ is the assumption $Y_1, \dots, Y_q \in C^{s,1}$.⁵ This is a stronger assumption than ours, which requires that only the original vector fields X_1, \dots, X_m belong to C^s .

A. Appendix

Here we discuss a detailed derivation of (4.11), in which we use the fact that any orbit $\mathcal{O}_{\mathcal{P}}$ associated with $\mathcal{P} = \{Y_1, \dots, Y_q\}$, is a p -dimensional C^2 immersed submanifold of \mathbb{R}^n . Since we are discussing a regularity issue, without loss of generality we may assume that $r = 1$ so that no tilde symbols appear.

Recall that given a C^2 manifold \mathcal{O} , we say that U is a C^1 vector field on \mathcal{O} if in any C^2 coordinate system $\mathcal{O} \supset \Omega \ni x \mapsto \alpha(x) = \xi \in \alpha(\Omega) \subset \mathbb{R}^p$, we have $U_x = \sum U^j(x) \left(\frac{\partial}{\partial \xi_j} \right)_x$ for all $x \in \Omega$, where $U^j = U \alpha^j$ is a C^1 function on Ω .

Remark A.1. We recall some known facts about C^2 manifolds.

- (a) The notion of C^1 vector field is well defined (coordinate invariant) provided that \mathcal{O} is at least C^2 .
- (b) Integral curves of a C^1 vector field on a C^2 manifold \mathcal{O} are unique and the map $x \mapsto e^{tU} x$ is C^1 smooth. Indeed, a path $t \mapsto \gamma(t) \in \Omega$ is an integral curve of U if and only if $\alpha \circ \gamma$ is an integral curve of the vector field $\sum_k (U^k \circ \alpha^{-1})(\xi) \frac{\partial}{\partial \xi_k}$ which is a C^1 vector field in $\alpha(\Omega)$.

⁵Note that the Y_j 's involve derivatives up to order $s - 1$. Therefore an easy-to-state assumption to ensure that $Y_j \in C^{s,1}$ is $X_j \in C^{2s-1,1}$.

(c) If U and V are C^1 vector fields in $\Omega \subseteq \mathcal{O}$, one can check that the commutator $[U, V]_x := \sum_j (UV^j(x) - VU^j(x))(\partial_{\xi^j})_x$ is well defined independently on the coordinate system and it turns out that $[U, V] = \mathcal{L}_U V$. Finally, if U, V are C^1 vector fields and $\Psi \in C^1(\Omega)$, then

$$\begin{aligned} \mathcal{L}_u V &= [U, V] = \sum_j \{UV^j(x) - VU^j(x)\} \left(\frac{\partial}{\partial \xi^j} \right)_x \quad \text{and} \\ [U, \Psi V] &= U\Psi V + \Psi[U, V] \end{aligned} \quad (\text{A.1})$$

(d) If \mathcal{O} is a C^2 submanifold of \mathbb{R}^n and Y is a C^1 vector field in \mathbb{R}^n which satisfies $Y_x \in T_x \mathcal{O}$ for all $x \in \mathcal{O}$, then any integral curve of Y starting from \mathcal{O} can not leave \mathcal{O} for small times.

All items (a),(b) and (c) can be checked relying on the fact that the coordinate versions of a C^1 vector field on a C^2 manifold \mathcal{O} are C^1 . Statement (d) is related with the embedding of \mathcal{O} in \mathbb{R}^n and can be checked for instance by writing a C^2 local change of coordinates in \mathbb{R}^n which makes \mathcal{O} of the form $\mathbb{R}^p \times \{0\}$. This standard argument works well as soon as the manifold is $C^{1,1}$ at least. In less regular cases one can use Bony's theorem.

Next we come to the derivation of (4.11). Let $W := \sum_{j=1}^p u_j \partial_{u_j}$. Start from identity

$$\Phi_* \left(\sum_{j=1}^p u_j \partial_{u_j} \right) = \frac{d}{d\varepsilon} \Phi((1 + \varepsilon)u) \Big|_{\varepsilon=0} = \sum_{j=1}^p u_j Y_{j, \Phi(u)}. \quad (\text{A.2})$$

Since \mathcal{O} is a C^2 manifold and $T_x \mathcal{O} = \text{span}\{Y_{j,x} : j = 1, \dots, p\}$ for all $x = \Phi(u)$, Remark A.1, (d) ensures that $\Phi(B_{\text{Euc}}(\delta)) \subset \mathcal{O}$. The map $\Phi|_{B_{\text{Euc}}(\delta)} : B_{\text{Euc}}(\delta) \rightarrow \Phi(B_{\text{Euc}}(\delta)) \subseteq \mathcal{O}$ is a C^1 diffeomorphism and its inverse $\Psi = \Phi^{-1}$ can be used as a C^1 chart. Then, at any $x = \Phi(u)$ with $|u| \leq \delta$ we have

$$(\Phi_* W)_x = \sum_j \Psi^j(x) Y_{j,x} \quad (\text{A.3})$$

Observe that the a priori continuous vector field $\Phi_* W$ is actually C^1 . This follows looking at the right-hand side of (A.3). Indeed, Ψ^j is a C^1 function and, by Remark A.1-(d), Y_j is a C^1 vector field (in both statements C^1 refers to the C^2 differential structure of \mathcal{O} described in (3.6)). Thus its integral curves are unique and the flow $x \mapsto e^{-t\Phi_* W} x$ is a C^1 local diffeomorphism on \mathcal{O} . See Remark A.1-(b).

Next, note that, if $\delta > 0$ is small enough and $|u| \leq \delta$, then the linear system $D\Phi(u)h_j(u) = g_j(\Phi(u))$ has a unique solution $h_j(u) \in \mathbb{R}^p$ (here $D\Phi(u) \in \mathbb{R}^{n \times p}$ denotes the Jacobian matrix). The solution $h_j(u)$ is given by the Cramer's rule (2.9). Let $Z_{j,u} := h_j(u) \cdot \nabla$ be the corresponding continuous vector field. Pulling back (A.2), we get $\sum_j u_j \partial_j = \sum_j u_j Z_j$, see [TW03, Str11].

We claim that $u \mapsto W^\sharp h_j(u) := \mathcal{L}_W Z_j(u)$ is a continuous function in $B_{\text{Euc}}(\delta)$. By the Cramer's rule (2.9), this claim follows from the continuity of $u \mapsto W^\sharp(\partial_j \Phi)(u)$, which will be checked in Lemma A.2 below, and from the continuity of $g_j \circ \Phi$.

Since $W = \sum_{j=1}^p u_j \partial_{u_j}$ then $e^{tW} u = e^t u$ and this gives the expansion $h_i(e^t u) = h_i(u) + W^\sharp h_i(u)t(1 + o(1))$, as $t \rightarrow 0$ and

$$\mathcal{L}_W Z_i(u) = \lim_{t \rightarrow 0} \frac{1}{t} \{e^{-t} h_i(e^t u) - h_i(u)\} = -h_i(u) + W^\sharp h_i(u).$$

Let now $\eta = \sum_{k=1}^p \eta_k(u) du_k$ be a smooth C_c^∞ one form. Testing $\mathcal{L}_W Z_j$ against η , we get

$$\begin{aligned} \langle \mathcal{L}_W Z_i, \eta \rangle &= \int \sum_k \left(W^\sharp h_i^k(u) \eta_k(u) - h_i^k(u) \eta_k(u) \right) du \\ &= \left\langle \sum_{k,j} u_j (D_{u_j} h_i^k) \partial_k, \eta \right\rangle - \left\langle \sum_k h_i^k(u) \partial_k, \eta \right\rangle, \end{aligned} \quad (\text{A.4})$$

where $D_{u_j} h_i^k \in \mathcal{D}'$ denotes the distributional derivative of the continuous function h_i^k . Equality (A.4) can be checked using the definition of $W^\sharp := \mathcal{L}_W$ and integrating by parts.

Next we want to write $\mathcal{L}_W Z_i$ in a different way, in order to use the integrability condition. To this aim, we calculate its push forward. Let $\Phi_* : TB_{\text{Euc}}(\delta) \rightarrow T\mathcal{O}$ be the tangent map. Fix $u \in B_{\text{Euc}}(\delta)$ and let $x = \Phi(u)$. Then,

$$\begin{aligned} \Phi_*(\mathcal{L}_W Z_i)_u &:= \Phi_* \lim_{t \rightarrow 0} \frac{1}{t} \{ e_*^{-tW} (Z_{i,e^t W u}) - Z_{i,u} \} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \{ (\Phi \circ e^{-tW})_* (Z_{i,e^t W u}) - Y_{i,x} \}, \end{aligned}$$

because $\Phi : B_{\text{Euc}}(\delta) \rightarrow \mathcal{O}$ is C^1 , so that $\Phi_* e_*^{-tW} = (\Phi \circ e^{-tW})_*$. Since the function $e^{-t\Phi_* W}$ is the flow of a $C^1_{\mathcal{O}}$ vector field, it is $C^1_{\mathcal{O}}$, see Remark A.1-(b). Therefore, $\Phi \circ e^{-tW} = e^{-t\Phi_* W} \circ \Phi$ and we have

$$(\Phi \circ e^{-tW})_* Z_{i,e^t W u} = (e^{-t\Phi_* W} \circ \Phi)_* Z_{i,e^t W u} = e_*^{-t\Phi_* W} \Phi_* Z_{i,e^t W u} = e_*^{-t\Phi_* W} Y_{i,e^t \Phi_* W \Phi(u)}.$$

We have shown that $\Phi_* \mathcal{L}_W Z_i = \mathcal{L}_{\Phi_* W} \Phi_* Z_i = \mathcal{L}_{\sum_j \Psi^j Y_j} Y_i$ under our regularity assumptions (this is a well known fact for smooth vector fields). Since the vector field $\sum \Psi^j Y_j$ is C^1 on \mathcal{O} , by (A.1), we may write

$$\begin{aligned} \Phi_* \mathcal{L}_W Z_i &= \mathcal{L}_{\sum_j \Psi^j Y_j} Y_i = \left[\sum_j \Psi^j Y_j, Y_i \right] = - \sum_j Y_i \Psi^j Y_j - \sum_j \Psi^j [Y_i, Y_j] \\ &= - \sum_j Y_i \Psi^j Y_j - \sum_{j,k} \Psi^j c_{ij}^k Y_k. \end{aligned}$$

Pulling back, we get

$$\mathcal{L}_W Z_i = \Phi_*^{-1} \Phi_* \mathcal{L}_W Z_i = - \sum_j h_i^j(u) Z_j - \sum_{j,k} u_j (c_{ij}^k \circ \Phi) Z_k. \quad (\text{A.5})$$

Here we used the equality $Y_i \Psi^j = h_i^j(u)$.⁶

We have obtained two different expressions for $\mathcal{L}_W Z_i$, namely (A.4) and (A.5). In order to compare them, it suffices to test (A.5) against η . This gives the distributional identity

$$\sum_{k,j} u_j D_j h_i^k \partial_k - \sum_j Z_i u_j \partial_j = - \sum_j Z_i u_j Z_j - \sum_{j,k} u_j (c_{ij}^k \circ \Phi) Z_k,$$

⁶This can be proved as follows. Possibly choosing a smaller δ , we may extend Ψ to a C^1 function $\bar{\Psi}$ defined in a open set in \mathbb{R}^n containing $\Phi(B_{\text{Euc}}(\delta))$. Then,

$$Y_i \Psi^j(x) = \sum_{\alpha=1}^n \partial_\alpha \bar{\Psi}^j(\Phi(u)) g_i^\alpha(\Phi(u)) = \sum_{\alpha=1}^n \partial_\alpha \bar{\Psi}^j(\Phi(u)) \sum_k h_i^k(u) \partial_k \Phi^\alpha(u) = h_i^j(u),$$

because $\bar{\Psi} \circ \Phi(u) = u$ for all u .

where $Z_i u_j = h_i(u) \cdot \nabla u_j = h_i^j(u)$. Last equality is exactly formula (3.5) in [Str11]. From now on, it suffices to follow Street's calculations and we get the ODE (4.11).

Lemma A.2. *Let $Y_j := g_j \cdot \nabla$, where $g_j \in C_{\text{Euc}}^1$ for $j = 1, \dots, p$. Let Φ be the exponential map in (4.2) and let $\Phi_j := \partial_j \Phi$. Then the map $u \mapsto W^\# \Phi_j(u) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\Phi_j(e^\varepsilon u) - \Phi_j(u))$ is continuous in a neighbourhood of the origin.*

Proof. Let $\eta(t, u) = \Phi(tu)$ be the solution of $\frac{\partial \eta}{\partial t}(t, u) = \sum_{j=1}^p u_j g_j(\eta(t, u))$ with $\eta(0, u) = x_0$. Since $e^\varepsilon W = 0$ for all ε , we have $W^\# \Phi_j(0) = 0$.

In order to calculate $W^\# \Phi_j(u)$ for $u \neq 0$, note that $\frac{\partial \eta}{\partial u_j}(t, u) = t \Phi_j(tu)$, for any t and u close to 0. Therefore, if $u \neq 0$, we have

$$\begin{aligned} W^\# \Phi_j(u) &:= \lim_{t \rightarrow 1} \frac{1}{t-1} (\Phi_j(tu) - \Phi_j(u)) \\ &= \lim_{t \rightarrow 1} \frac{1}{t-1} \left(\frac{1}{t} \frac{\partial \eta}{\partial u_j}(t, u) - \frac{\partial \eta}{\partial u_j}(1, u) \right). \end{aligned} \quad (\text{A.6})$$

But the definition of partial derivative and the variational equation give

$$\lim_{t \rightarrow 1} \frac{1}{t-1} \left(\frac{\partial \eta}{\partial u_j}(t, u) - \frac{\partial \eta}{\partial u_j}(1, u) \right) = \frac{\partial^2 \eta}{\partial t \partial u_j}(1, u) = \frac{\partial}{\partial u_j} \sum_k u_k g_k(\eta(1, u)).$$

In other words, since $\eta(1, u) = \Phi(u)$,

$$\frac{\partial \eta}{\partial u_j}(t, u) = \frac{\partial \eta}{\partial u_j}(1, u) + (t-1) \left(g_j(\Phi(u)) + \sum_{k,i} u_k \partial_i g_k(\Phi(u)) \Phi_j^i(u) \right) (1 + o(1)),$$

which, inserted into (A.6), gives $W^\# \Phi_j(u) = -\Phi_j(u) + g_j(\Phi(u)) + \sum_{k,i} u_k \partial_i g_k(\Phi(u)) \Phi_j^i(u)$. This shows that $W^\# \Phi_j$ is a continuous function at any $u \neq 0$. Moreover, $W^\# \Phi_j(u) \rightarrow -\Phi_j(0) + g_j(\Phi(0)) = 0$, as $u \rightarrow 0$. Since we already claimed that $W^\# \Phi_j(0) = 0$, the proof is concluded. \square

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