Numerical Differentiation

PROBLEM

Estimate the derivatives (slope, curvature, etc.) of a function, given a set of function values at a discrete set of points.

→ Finite Difference Formulas

Numerical Differentiation

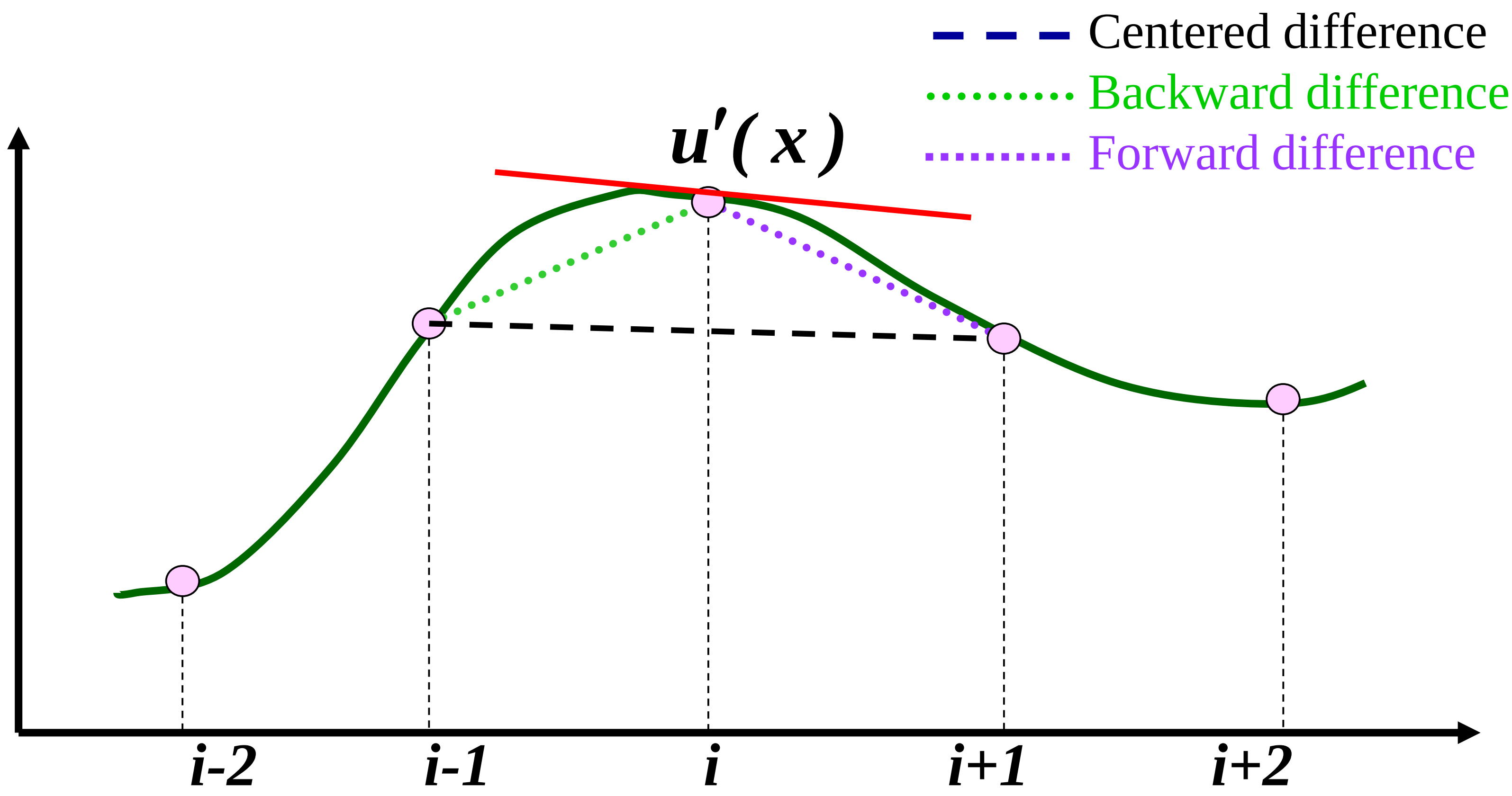
The simplest way to numerically compute a derivative is to mimic the formal definition:

$$u'(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$u'(x) \approx \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

For a linear function *u(x)=ax+b* the formula is exact.

First Derivative at a point: finite difference schemes



First Derivative

Centered difference

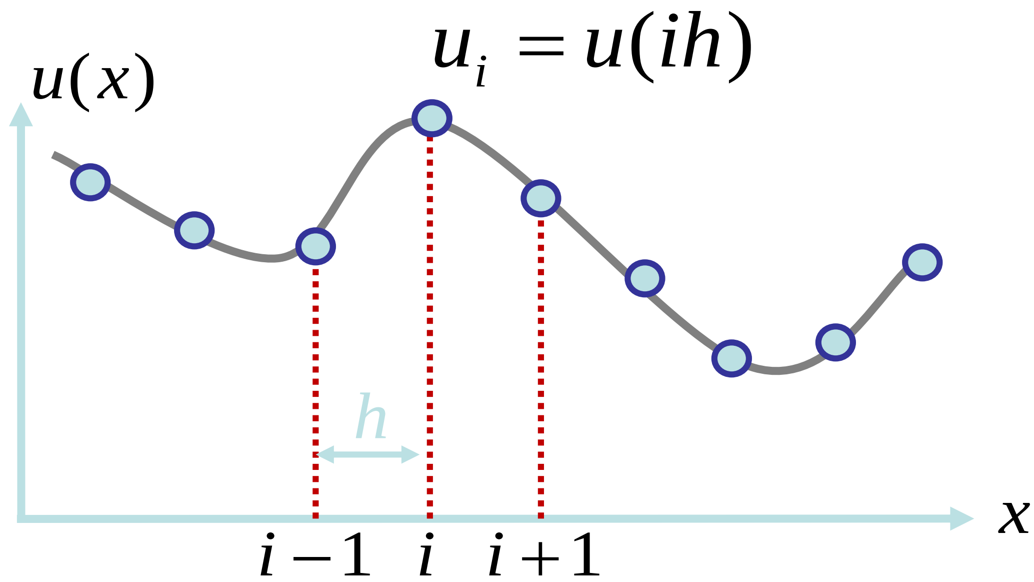
$$D_x u \equiv \frac{u_{i+1} - u_{i-1}}{2h}$$

Forward difference

$$D_x^+ u \equiv \frac{u_{i+1} - u_i}{h}$$

Backward difference

$$D_x^- u \equiv \frac{u_i - u_{i-1}}{h}$$



Local Truncation Error

We write a Taylor expansion of u(x) about $x=ih$

$$u_{i+1} = u(ih+h) = u(ih) + hu'(ih) + \tfrac{1}{2!}h^2 u''(ih) + O(h^3)$$

$$u_{i-1} = u(ih-h) = u(ih) - hu'(ih) + \tfrac{1}{2!}h^2 u''(ih) + O(h^3)$$

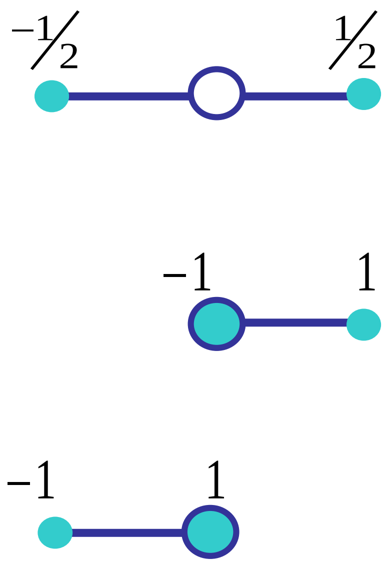
Second order error term

$$D_x u_i = u'(ih) + O(h^2)$$

First order error terms

$$D_x^+ u_i = u'(ih) + O(h)$$

$$D_x^- u_i = u'(ih) + O(h)$$



Stencils

Local Truncation Error

$$D_x u_i = u'(ih) + O(h^2)$$

Proof

$$u_{i+1} = u(ih+h) = u(ih) + hu'(ih) + \tfrac{1}{2!}h^2u''(ih) + \tfrac{1}{3!}h^3u'''(ih) + O(h^4)$$

$$u_{i-1} = u(ih-h) = u(ih) - hu'(ih) + \tfrac{1}{2!}h^2u''(ih) - \tfrac{1}{3!}h^3u'''(ih) + O(h^4)$$

Subtracting these two eqs:

$$u(ih+h) - u(ih-h) = 2hu'(ih) + 2\frac{h^3}{3!}u'''(ih) + O(h^4)$$

$$u'(ih) = \left(\frac{u(ih+h) - u(ih-h)}{2h}\right) + \frac{1}{3!}h^2u'''(ih) + O(h^3)$$

Second order error term

As the distance h tends to zero, we expect the approximation to improve.

Consistency

For $h \to 0$ LTE approaches to zero

The "speed" in which the error goes to zero as h -> 0 is called the **rate of convergence**.

When the truncation error is of the order of O(h), we say that the method is a first order method. We refer to a method as a **pth-order method** if the truncation error is of the order of $O(h^p)$

Order of CONSISTENCY $O(h^p)$

Centered scheme ($O(h^2)$) is a more accurate formula than forward or backward (O(h)), that is the LTE decreases more rapidly.

Second Derivative

approximation of the second derivative by

Centered Difference formula

$$D_{xx}u \equiv \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$D_{xx}u_i = u''(ih) + O(h^2)$$

Second order error term

Proof

$$u_{i+1} = u(ih+h) = u(ih) + hu'(ih) + \tfrac{1}{2!}h^2u''(ih) + \tfrac{1}{3!}h^3u'''(ih) + O(h^4)$$

$$u_{i-1} = u(ih-h) = u(ih) - hu'(ih) + \tfrac{1}{2!}h^2u''(ih) - \tfrac{1}{3!}h^3u'''(ih) + O(h^4)$$

Sum of these two eqns:

$$u(ih+h) + u(ih-h) = 2u(ih) + 2\left(\frac{h^2}{2!}u''(ih)\right) + 2\frac{h^4}{4!}u^{iv}(ih) + \dots$$

$$u''(ih) = \left(\frac{u(ih+h) - 2u(ih) + u(ih-h)}{h^2}\right) - \frac{1}{12}h^2u^{(iv)}(ih) + \dots$$

Finite Difference Formulas for k>1

Linear Operators

$$\Delta^1 u(x) = u(ih+h) - u(ih)$$ Forward linear operator

$$\nabla^1 u(x) = u(ih) - u(ih-h)$$ Backward linear operator

$$\delta^1 u(x) = u(ih+\frac{h}{2}) - u(ih-\frac{h}{2})$$ Centered linear operator

$$\Delta^1 u_i = u_{i+1} - u_i \qquad \nabla^1 u_i = u_i - u_{i-1} \qquad \delta^1 u_i = u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}$$

Finite Difference Formulas for k>1

We define the linear operators of order k at $x_i = ih$ *as*

$$\Delta^k u_i = \Delta^1(\Delta^{k-1}u_i) = \Delta^{k-1}u_{i+1} - \Delta^{k-1}u_i$$
$$\nabla^k u_i = \nabla^1(\nabla^{k-1}u_i) = \nabla^{k-1}u_i - \nabla^{k-1}u_{i-1}$$
$$\delta^k u_i = \delta^1(\delta^{k-1}u_i) = \delta^{k-1}u_{i+\frac{1}{2}} - \delta^{k-1}u_{i-\frac{1}{2}}$$

Computing second order centered finite differencing

$$D_{xx}u_i = D^+D^-u_i$$
$$= D^-D^+u_i$$
$$= D_{1/2}D_{1/2}u_i$$

Finite Difference Formulas for k>1

Theorem

$$\text{Let } u(x) \in C^k[x_0, x_n], h > 0,\ x_i = x_0 + ih,$$

$$\text{then } \exists \eta \in [x_i, x_i + h]$$

$$u^k(\eta) = \frac{\Delta^k u_i}{h^k}$$

Compute...

$$\delta^4 u_i = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}$$

$$\Rightarrow u^{iv}(x_i) \approx \frac{u(x_{i+2}) - 4u(x_{i+1}) + 6u(x_i) - 4u(x_{i-1}) + u(x_{i-2})}{(h)^4}$$

Numerical problems

- Truncation Error:

  error due to the truncation of the Taylor expansion

- Rounding error:

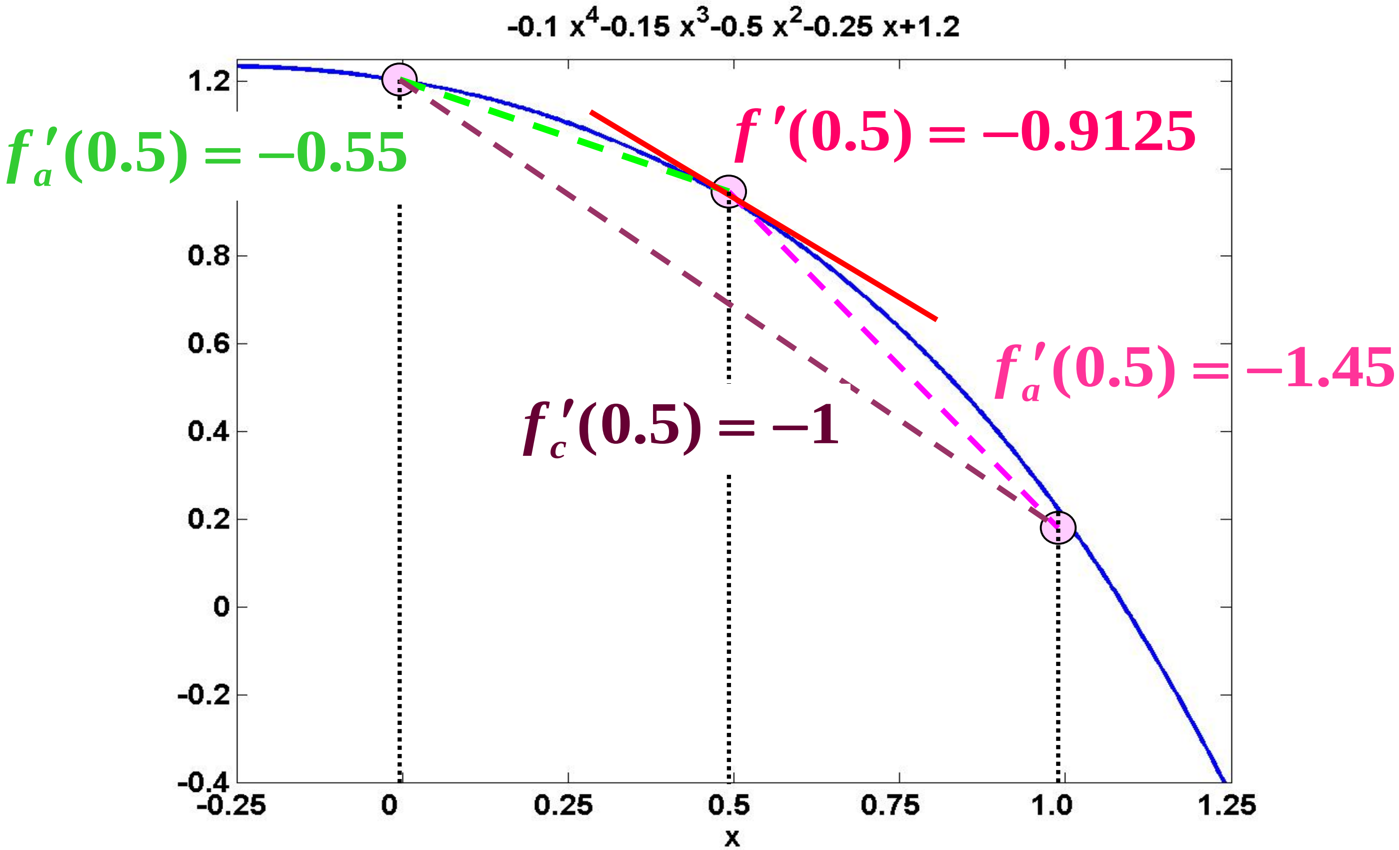
  approximation error in finite arithmetic

  In finite arithmetic a numerical evaluation which uses an arbitrarily small value of h does not lead to a reduction of total error.

Example

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Estimate the first derivative with backward, forward and centered differences at point x = 0.5 (with h = 0.5 and 0.25)



h=0.5

Example, h=0.5

Forward Difference

$$h = 0.5,\quad f'(0.5) = \frac{f(1) - f(0.5)}{1 - 0.5} = \frac{0.2 - 0.925}{0.5} = -1.45$$

$$\text{relative error} \quad \frac{-1.45 + 0.91250}{-0.91250} = 0.\ 58904$$

Backward Difference

$$h = 0.5,\quad f'(0.5) = \frac{f(0.5) - f(0)}{1 - 0.5} = \frac{0.925 - 1.2}{0.5} = -0.55$$

$$\text{relative error} \quad \frac{-0.55 + 0.91250}{-0.91250} = -0.39726$$

Centered Difference

$$h = 0.5,\quad f'(0.5) = \frac{f(1) - f(0)}{1 - 0} = \frac{0.2 - 1.2}{1} = -1.0$$

$$\text{relative error} \quad \frac{-1 + 0.91250}{-0.91250} = 0.09589$$

Example, h=0.25

Forward Difference

$$f_a'(0.5) = \frac{f(0.75) - f(0.5)}{0.75 - 0.5} = \frac{0.63632813 - 0.925}{0.25} = \text{-1.1547},$$

$$\text{relative error} \quad \frac{\text{-1.1547} + 0.91250}{-0.91250} = 0.26541$$

Backward Difference

$$f_i'(0.5) = \frac{f(0.5) - f(0.25)}{0.75 - 0.5} = \frac{0.925 - 1.10351563}{0.25} = \text{-0.71406}$$

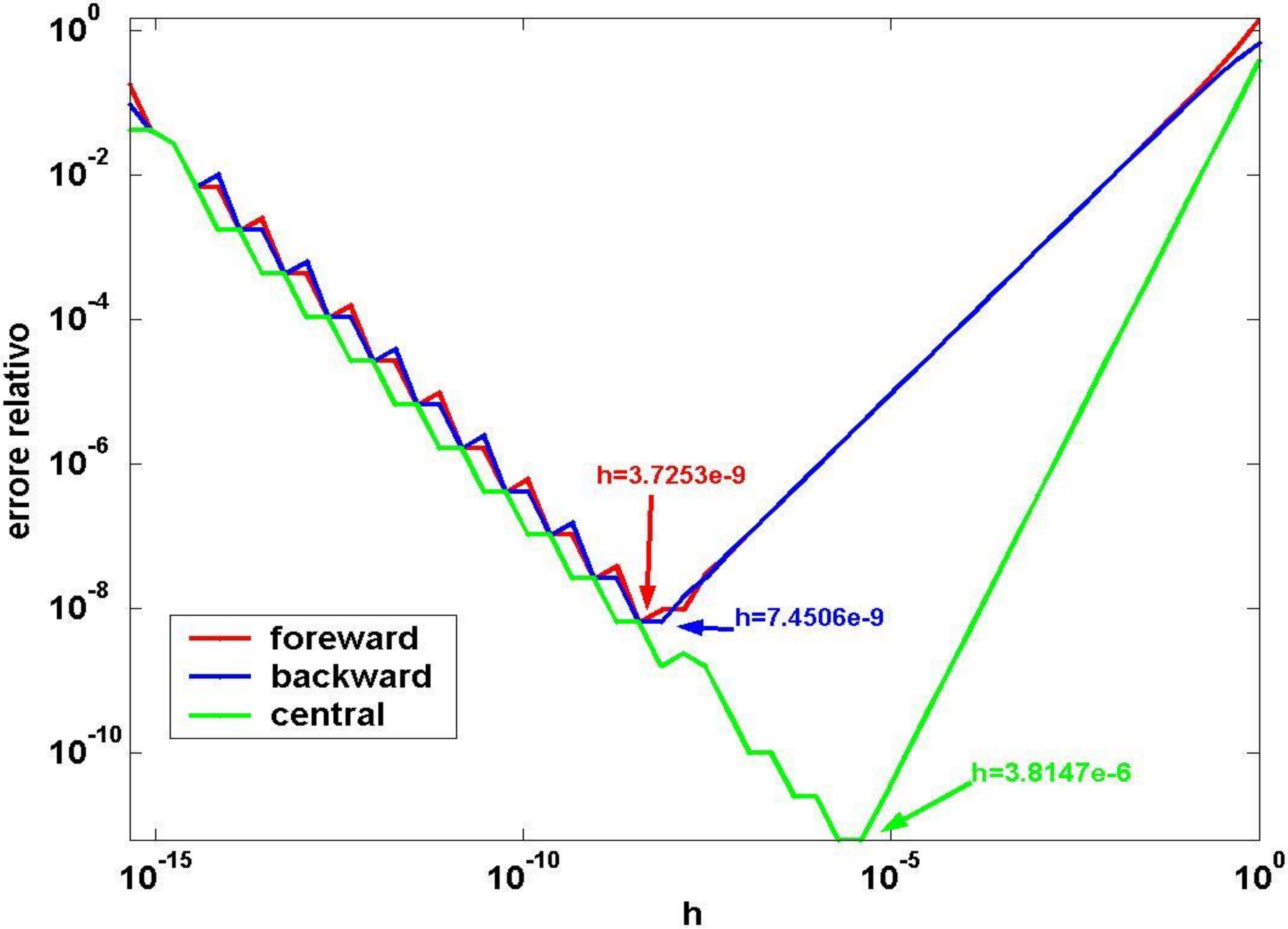
$$\text{relative error} \quad \frac{\text{-0.71406} + 0.91250}{-0.91250} = -0.21747$$

Centered Difference

$$f_c'(0.5) = \frac{f(0.75) - f(0.25)}{0.75 - 0.25} = \frac{0.63632813 - 1.10351563}{0.5} = \text{-0.93438}$$

$$\text{relative error} \quad \frac{\text{-0.93438} + 0.91250}{-0.91250} = 0.023973$$

Relative errors as a function of h



Remarks

- Rounding errors cause deterioration of the approximation for small values of h.
- The value of h which allows a correct evaluation of the formulas depends on the accuracy of the machine.
- If the terms $f(x_i \pm h)$ are calculated inaccurately then the errors are multiplied by a factor 1 / h, which grows very quickly for small values of h.

$$\tilde{f}(x_i + h) = f(x_i + h) + \delta \Rightarrow \hat{f}_a'(x_i) = \frac{\tilde{f}(x_i + h) - f(x_i)}{h} =$$

Computed value

$$= \frac{f(x_i + h) - f(x_i)}{h} + \frac{\delta}{h} = f_a'(x_i) + \frac{\delta}{h}$$

Approximation error

Differentiation Via Polynomial Interpolation

The first stage is to construct an interpolating polynomial from the data. An approximation of the derivative at any point can be then obtained by a direct differentiation of the interpolant.

Example The Lagrange form of the polynomial interpolation through 3 values (x_i,y_i) is:

$$p(x) = L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

$$= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_2)(x_3-x_1)}y_3$$

Differentiating the interpolant

$$p'(x) \cong \frac{2x - x_2 - x_3}{(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{2x - x_1 - x_3}{(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{2x - x_1 - x_2}{(x_3 - x_2)(x_3 - x_1)} y_3$$

Assuming uniform x points

$$p'(x) = \frac{2x - x_2 - x_3}{2\Delta x^2} y_1 + \frac{2x - x_1 - x_3}{-\Delta x^2} y_2 + \frac{2x - x_1 - x_2}{2\Delta x^2} y_3$$

First derivative of the Lagrange interpolant:

$$p'(x) = \frac{2x - x_2 - x_3}{2\Delta x^2} y_1 + \frac{2x - x_1 - x_3}{-\Delta x^2} y_2 + \frac{2x - x_1 - x_2}{2\Delta x^2} y_3$$

Evaluate the derivative at several points:

$$p'(x_1) = \frac{2x_1 - x_2 - x_3}{2\Delta x^2} y_1 + \frac{2x_1 - x_1 - x_3}{-\Delta x^2} y_2 + \frac{2x_1 - x_1 - x_2}{2\Delta x^2} y_3 = \frac{-3y_1 + 4y_2 - y_3}{2\Delta x}$$

$$p'(x_2) = \frac{2x_2 - x_2 - x_3}{2\Delta x^2} y_1 + \frac{2x_2 - x_1 - x_3}{-\Delta x^2} y_2 + \frac{2x_2 - x_1 - x_2}{2\Delta x^2} y_3 = \frac{y_3 - y_1}{2\Delta x}$$

We get the centered difference formula

$$p'(x_3) = \frac{2x_3 - x_2 - x_3}{2\Delta x^2} y_1 + \frac{2x_3 - x_1 - x_3}{-\Delta x^2} y_2 + \frac{2x_3 - x_1 - x_2}{2\Delta x^2} y_3 = \frac{y_1 - 4y_2 + 3y_3}{2\Delta x}$$

To calculate derivatives with higher order from the Lagrange interpolating polynomial,

$$p'(x) = \frac{2x - x_2 - x_3}{2\Delta x^2} y_1 + \frac{2x - x_1 - x_3}{-\Delta x^2} y_2 + \frac{2x - x_1 - x_2}{2\Delta x^2} y_3$$

differentiate

$$p''(x) = \frac{1}{\Delta x^2} y_1 + \frac{2}{-\Delta x^2} y_2 + \frac{1}{\Delta x^2} y_3 = \frac{y_1 - 2y_2 + y_3}{\Delta x^2}$$

To obtain derivatives of order n the interpolation polynomial must be of degree greater than or equal to n.

Truncation Error

We know that the interpolation error is

$$f(x) = p(x) + \frac{\Pi_k(x)}{(k+1)!} f^{(k+1)}(\xi) \quad ; \Pi_k(x) = \prod_{i=0}^{k} (x - x_i)$$

$$f'(x) = p'(x) + \frac{\Pi'_k(x)}{(k+1)!} f^{(k+1)}(\xi) + \frac{\Pi_k(x)}{(k+1)!} \frac{d}{dx} f^{(k+1)}(\xi)$$

$if \quad x = x_i \quad is\ one\ of\ the\ knots, then \quad \Pi_k(x_i) = 0:$

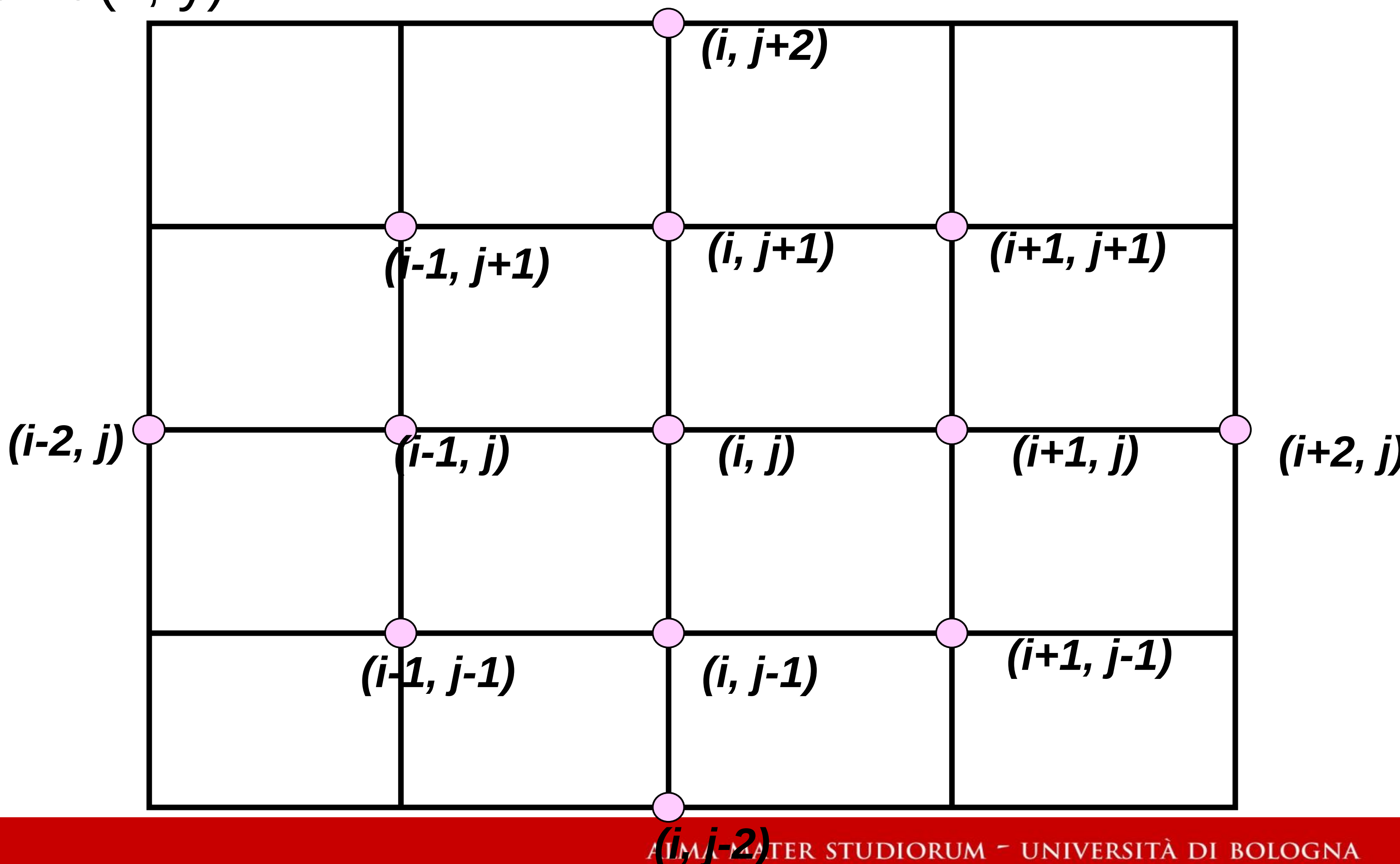
$$f'(x) = p'(x) + \frac{\Pi'_k(x_i)}{(k+1)!} f^{(k+1)}(\xi)$$

Truncation Error

Partial Derivatives

Extension of the one-dimensional case:

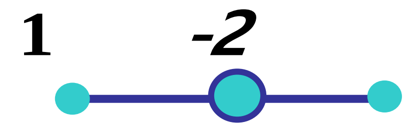
Finite difference formula to approximate partial derivatives of function u(x, y)

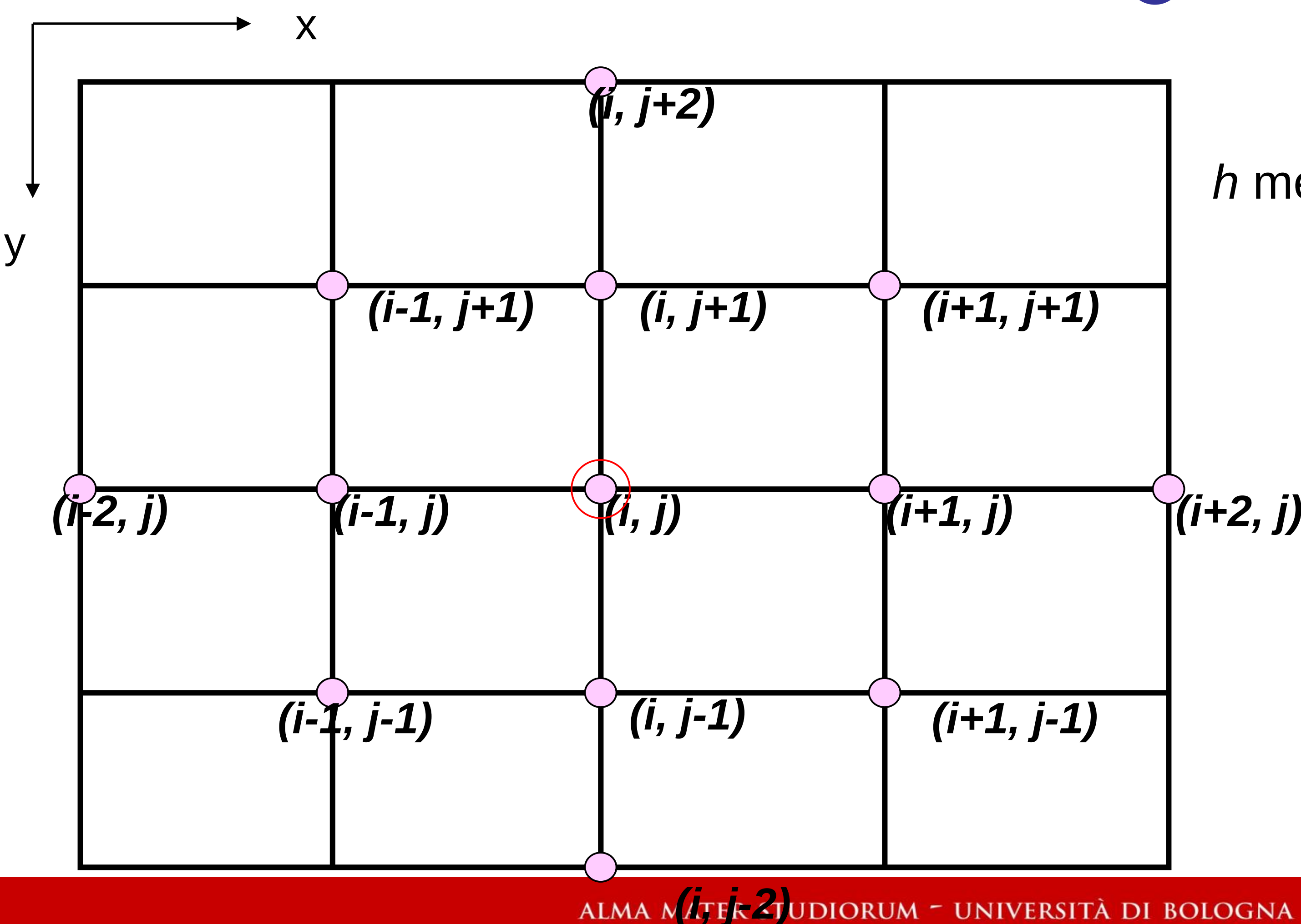


$$u_x = \frac{1}{2h}(-u(x_{i-1}, y_j) + u(x_{i+1}, y_j))$$

$$u_{xx} = \frac{1}{h^2}(u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j))$$

$$D_{xx}u \equiv \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

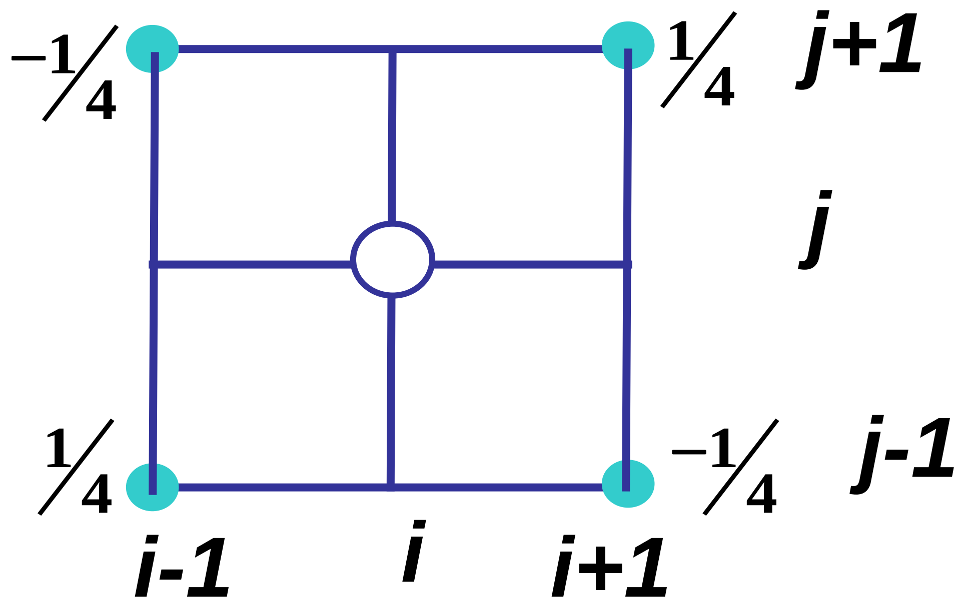




h mesh step

$$u_x = \frac{1}{2h}(u(x_{i+1}, y_j) - u(x_{i-1,} y_j))$$

$$u_{xy} = \frac{\partial}{\partial y}(u_x) = \frac{1}{4kh}(u(x_{i+1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) - u(x_{i-1,} y_{j+1}) + u(x_{i-1}, y_{j-1}))$$



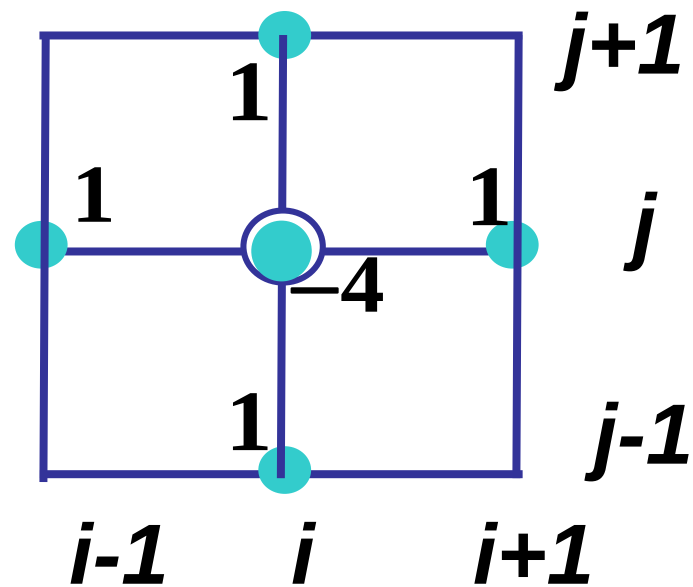
$$D_{xy}u \equiv \frac{u_{i+1,\,j+1} - u_{i-1,\,j+1} - u_{i+1,\,j-1} + u_{i-1,\,j-1}}{4h^2}$$

Laplacian Operator (5-points)

$$\nabla^2 u = u_{xx} + u_{yy} =$$

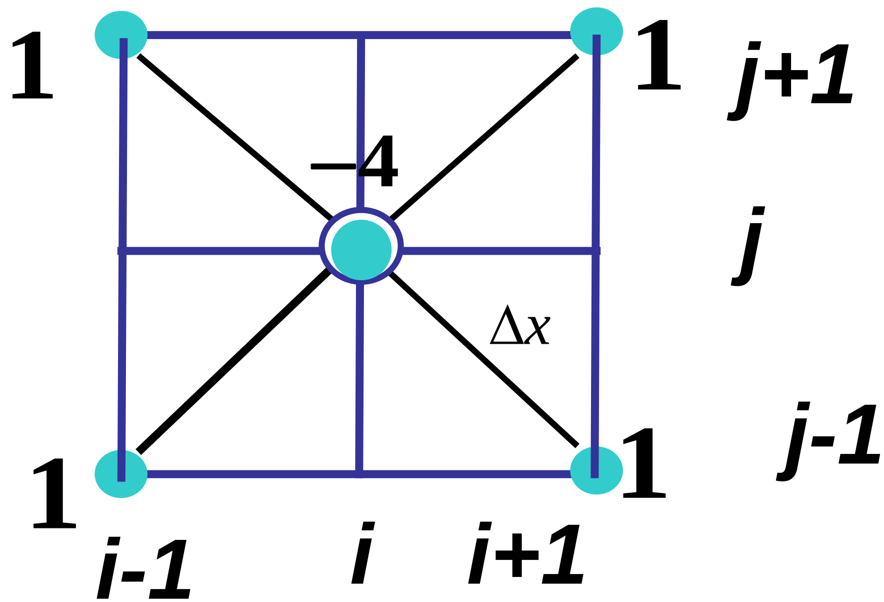
$$\approx \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2}$$

$$\approx \frac{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}}{h^2} - 4\frac{u_{ij}}{h^2}$$



local truncation error $O(h^2)$

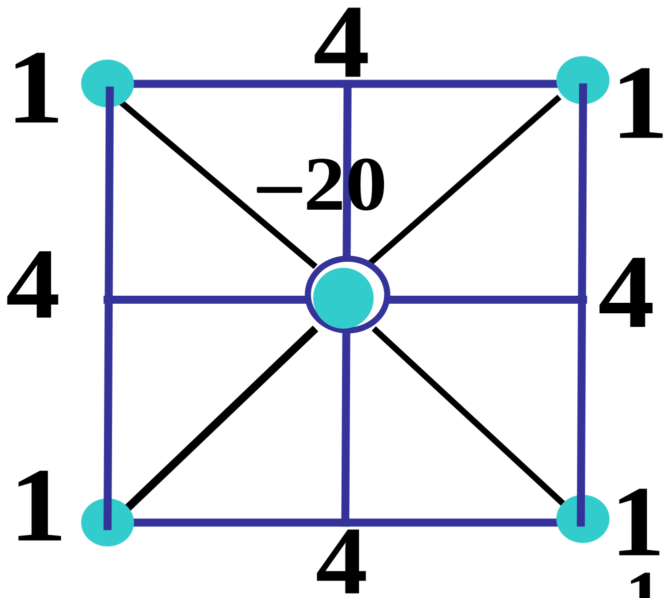
Laplacian Operator (5-points)



$$\nabla^2 u(x_i, y_j) \approx \frac{u_{i+1,j+1} + u_{i-1,j-1}}{2\Delta x^2} + \frac{u_{i+1,j-1} + u_{i-1,j+1}}{2\Delta x^2} - 4\frac{u_{ij}}{2\Delta x^2}$$

The local Truncation error for both the approximations is $O(h^2)$

Laplacian Operator (9-points)

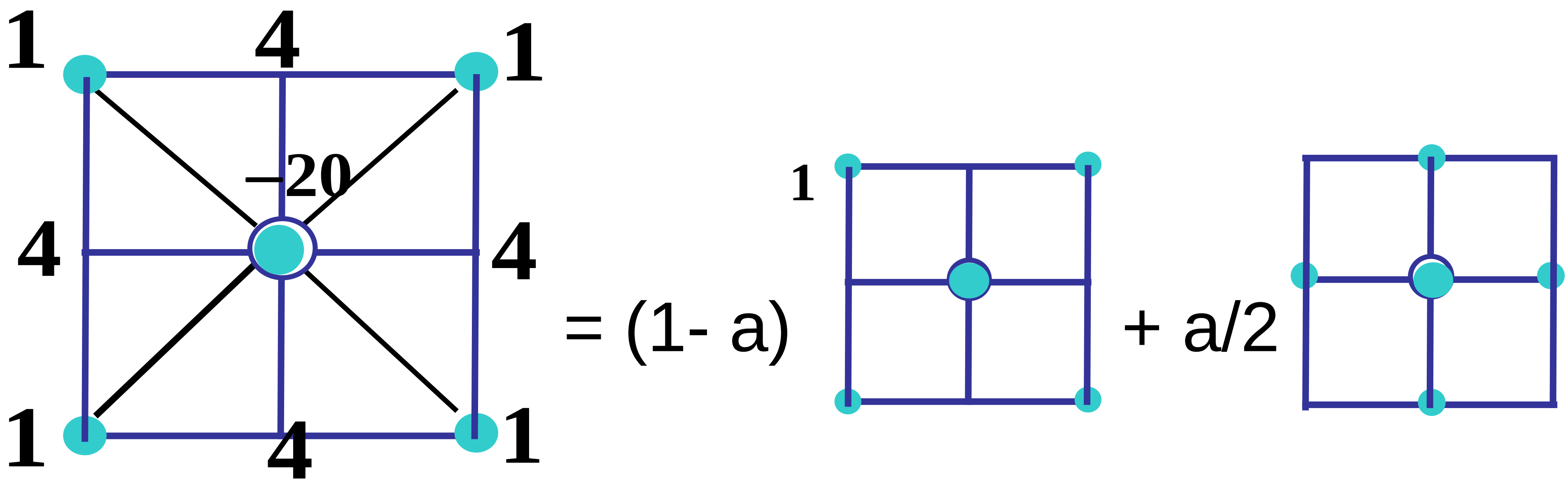


$$\nabla^2 u(x_i, y_j) \approx \frac{1}{6\Delta x^2}\Big[4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{ij}\Big]$$

The local Truncation Error is $O(h^2)$

$$\nabla_9^2 u(x_i, y_j) \approx \nabla^2 u + \frac{h^2}{2}(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) + O(h^4)$$

Laplacian Operator (9-points)



a=1/3 is the only value of a which yields a higher order of accuracy for the Laplacian

$$\nabla^2 u(x_i, y_j) \approx \frac{1}{6\Delta x^2}\Big[4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{ij}\Big]$$

The local Truncation Error is $O(h^2)$

Biharmonic operator

$$\nabla^4 u = (\nabla^2 u)^2 = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^2 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} \approx$$

$$\frac{1}{4h^4}\left\{\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ & & 2 & - & -8 & - & 2 & & \\ & & | & & | & & | & & \\ 1 & - & -8 & - & 20 & - & -8 & - & 1 \\ & & | & & | & & | & & \\ & & 2 & - & -8 & - & 2 & & \\ & & & & | & & & & \\ & & & & 1 & & & & \end{array}\right\} \begin{array}{c} j+2 \\ \\ j+1 \\ \\ j \\ \\ j-1 \\ \\ j-2 \end{array}$$

$$\begin{array}{ccccc} i-2 & i-1 & i & i+1 & i+2 \end{array}$$

Divergence Operator (1)

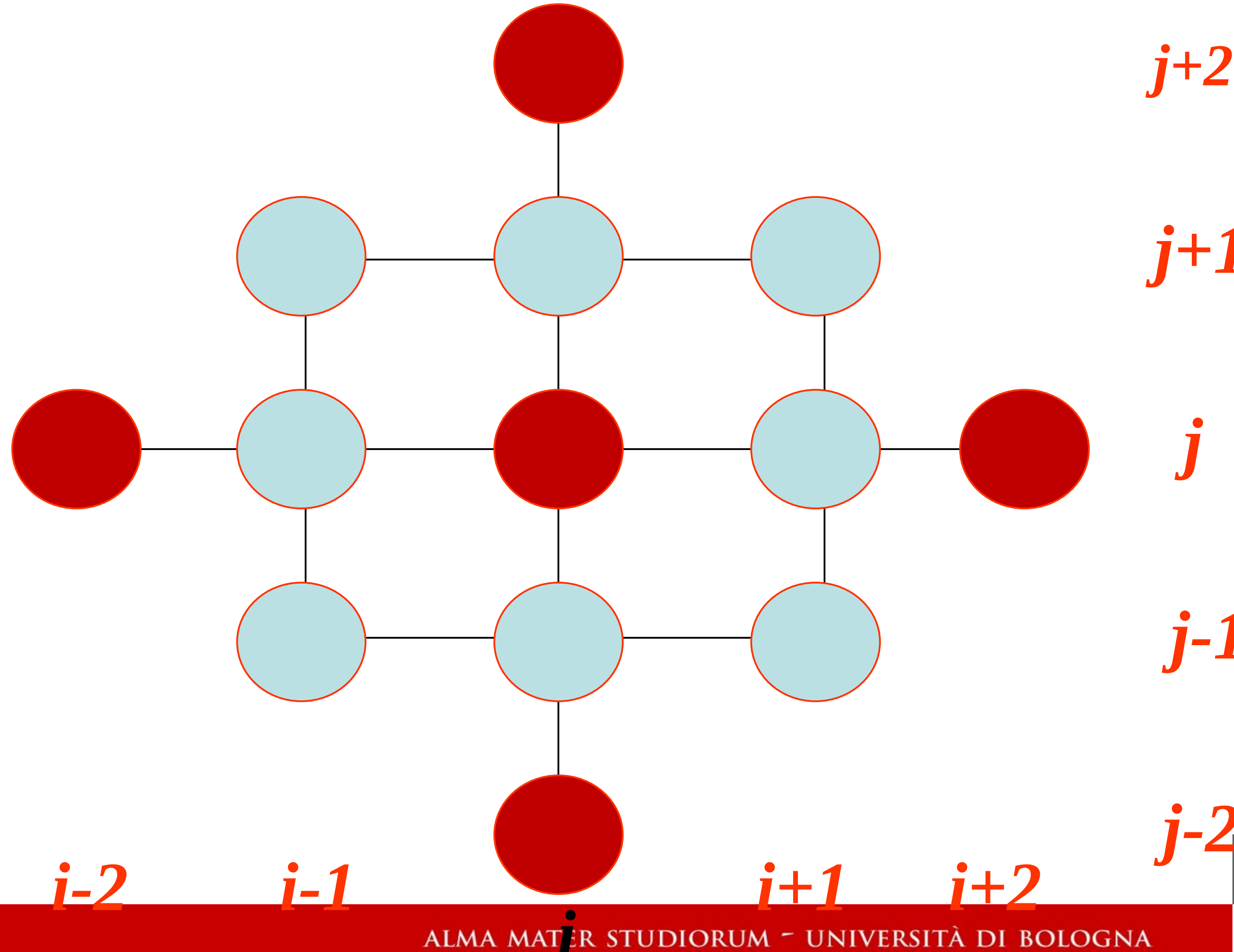
$$div(b\nabla u) = \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right)$$

$$div(b\nabla u) \approx \delta_x(b_{i,j}\delta_x u_{i,j}) + \delta_y(b_{i,j}\delta_y u_{i,j})$$

$$\approx \frac{b_{i+1,j}u_{i+2,j} + b_{i-1,j}u_{i-2,j} + b_{i,j+1}u_{i,j+2} + b_{i,j-1}u_{i,j-2}}{4h^2}$$

$$- \frac{(b_{i+1,j} + b_{i-1,j} + b_{i,j+1} + b_{i,j-1})u_{ij}}{4h^2}$$

Divergence Operator



It’s not robust!

Divergence Operator (2)

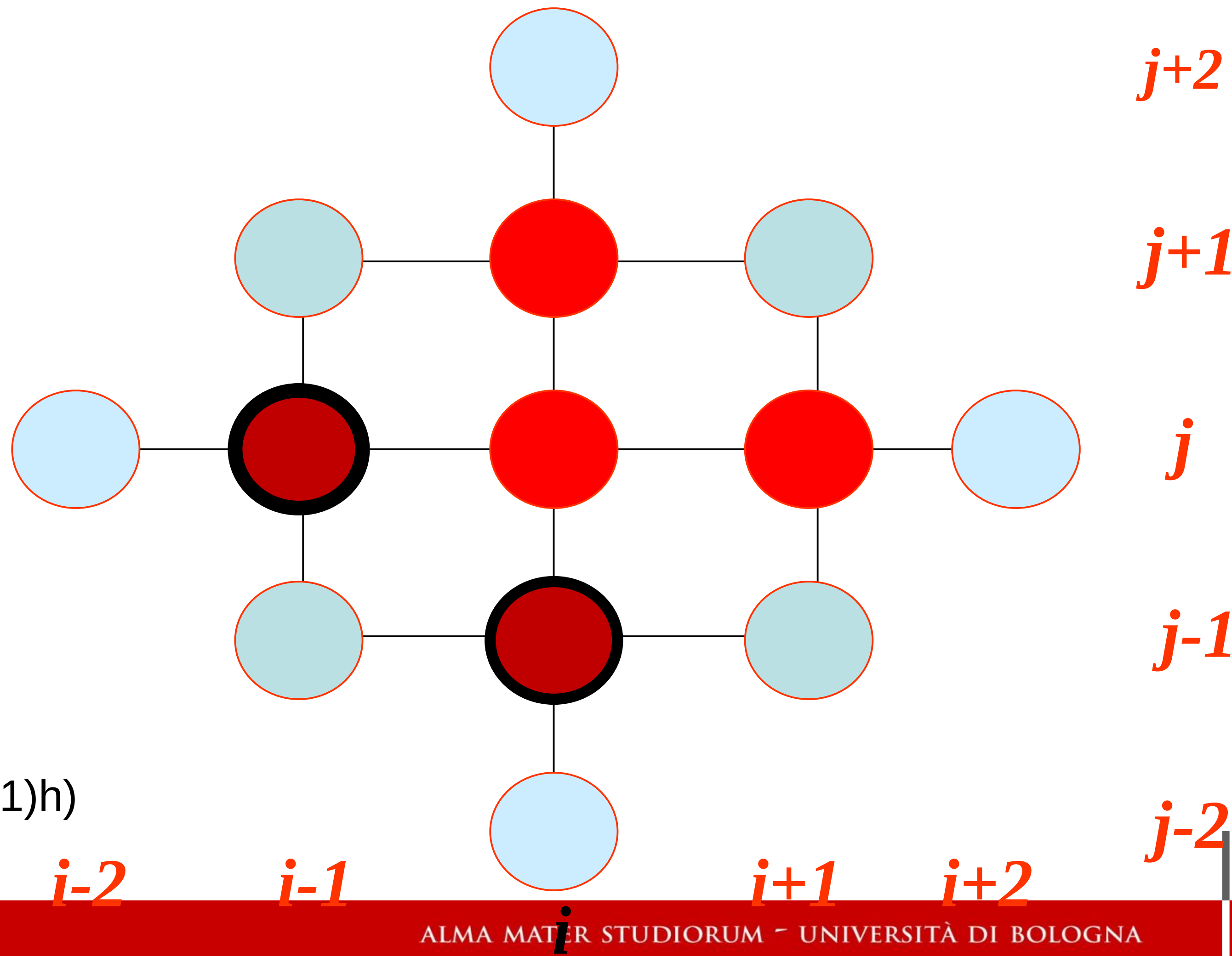
$$div(b\nabla u) = \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right)$$

$$div(b\nabla u) \approx D_x^{+}(b_{i,j}D_x^{-}u_{i,j}) + D_y^{+}(b_{i,j}D_y^{-}u_{i,j})$$

$$\approx \frac{b_{i+1,j}u_{i+1,j} + b_{i,j}u_{i-1,j} + b_{i,j+1}u_{i,j+1} + b_{i,j}u_{i,j-1}}{h^2}$$

$$- \frac{(b_{i+1,j} + b_{i,j+1} + 2b_{i,j})u_{ij}}{h^2}$$

Divergence Operator



It does not use
b((i-1)h,jh) and b(ih,(j-1)h)

Divergence Operator (3)

$$div(b\nabla u) = \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right)$$

$$\delta_x^* = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{h} \qquad \delta_y^* = \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{h}$$

$$div(b\nabla u) \approx \delta_x^*(b_{i,j}\delta_x^* u_{i,j}) + \delta_y^*(b_{i,j}\delta_y^* u_{i,j})$$

$$\approx \frac{b_{+0,j}u_{i+1,j} + b_{-0}u_{i-1,j} + b_{0+}u_{i,j+1} + b_{0-}u_{i,j-1}}{h^2} - \frac{(b_{+0} + b_{-0} + b_{0+} + b_{0-})u_{ij}}{h^2}$$

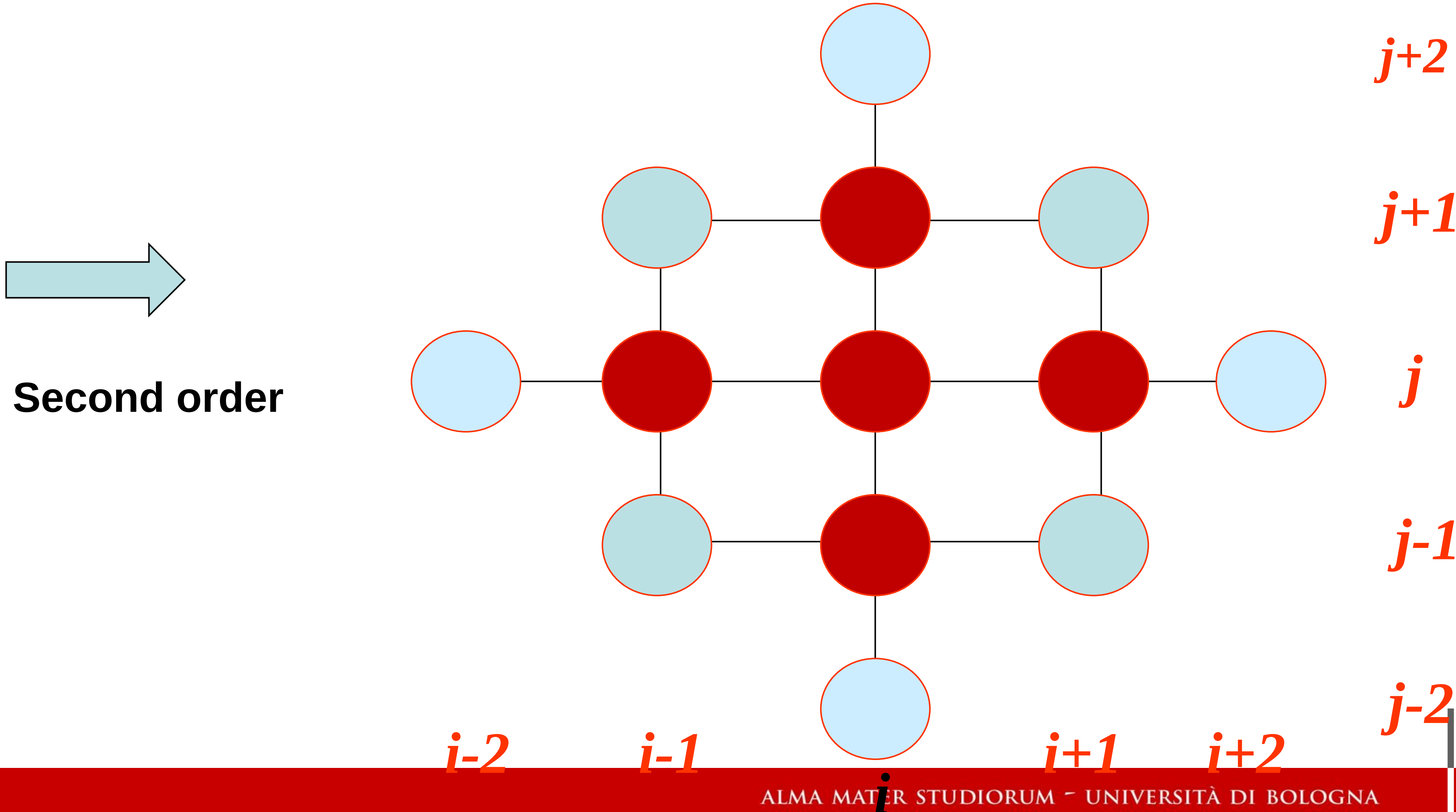
where

$$b_{\pm 0} = b_{i\pm\frac{1}{2},j} \quad e \quad b_{0\pm} = b_{i,j\pm\frac{1}{2}}$$

Interpolated Values

Divergence Operator

Second order



Remarks

- To increase the order of accuracy of the formulas is necessary to increase the number of points involved in the calculation.

- Higher precision is equivalent to a higher computational complexity.

- To achieve greater accuracy without increasing the order of the formulas we can use extrapolation techniques such as that of Richardson.

Richardson's Extrapolation

This technique uses the concept of grids with variable amplitude step procedure for improving the accuracy of approximations.

Example in which we show how to turn a **second-order approximation of the second derivative** into a fourth order approximation of the same quantity

$$f''(x_i) = \left(\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{\Delta x^2} \right) + a_1 \Delta x^2 + a_2 \Delta x^4 + \ldots$$

Richardson's Extrapolation

second-order approximation of the second-derivative:

$$f''(x_i) = \left(\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{\Delta x^2} \right) + \quad a_1 \Delta x^2 + a_2 \Delta x^4 + \quad \ldots$$

We write the equation for grids of different sizes

$$f''(x_i) = F(\Delta x) + \quad a_1 \Delta x^2 + a_2 \Delta x^4 + \quad \ldots$$

$$f''(x_i) = F\left(\frac{\Delta x}{2}\right) + \quad a_1 \left(\frac{\Delta x}{2}\right)^2 + a_2 \left(\frac{\Delta x}{2}\right)^4 + \quad \ldots$$

Richardson's Extrapolation

This, of course, is still a second-order approximation of the derivative. However, the idea is to combine [1] with [2] such that the Δx^2 term in the error vanishes.

$$f''(x_i) = F(\Delta x) + a_1 \Delta x^2 + a_2 \Delta x^4 + \ldots \quad [1]$$

$$f''(x_i) = F\left(\frac{\Delta x}{2}\right) + a_1 \left(\frac{\Delta x^2}{4}\right) + a_2 \left(\frac{\Delta x^4}{16}\right) + \ldots \quad [2]$$

Richardson's Extrapolation

Indeed, multiplying eq.[2] by 4 and subtracting [1] from [2]

$$4f''(x_i) = 4F\left(\frac{\Delta x}{2}\right) + 4a_1\left(\frac{\Delta x^2}{4}\right) + 4a_2\left(\frac{\Delta x^4}{16}\right) + \ldots$$

$$-f''(x_i) = -F(\Delta x) - a_1\Delta x^2 - a_2\Delta x^4 + \ldots$$

$$3f''(x_i) = 4F\left(\frac{\Delta x}{2}\right) - F(\Delta x) - 12a_2\left(\frac{\Delta x^4}{16}\right) + \ldots$$

Richardson's Extrapolation

The equation can be rewritten as:

$$f''(x_i) = \left(\frac{4F\left(\frac{\Delta x}{2}\right) - F(\Delta x)}{3} \right) - a_2 \left(\frac{\Delta x^4}{4} \right) + \quad \ldots$$

The accuracy of the new estimation of $f''(x_i)$ is

$O(\Delta x^4)$ instead of $O(\Delta x^2)$

Richardson's Extrapolation

We can continue to eliminate higher order terms of the error using a grid even finer:

$$f''(x_i) = B(\Delta x) + b_1 \Delta x^4 + b_2 \Delta x^6 + \ldots$$

$$f''(x_i) = \left(\frac{16B\left(\frac{\Delta x}{2}\right) - B(\Delta x)}{15} \right) + O\left(\Delta x^6\right) + \ldots$$

Richardson's Extrapolation example

Given the function

$$f(x) = x^3 - 2x^2 + 4x - 8$$

Find the first derivative at x=1.25 using a centered difference formula and step Δh = 0.25.

$$f'(x) = 3x^2 - 4x + 4 = 3(1.25)^2 - 4(1.25) + 4 = 3.6875$$

Richardson's Extrapolation example

Given the discrete data set:

x	f(x)
1	-5
1.125	-4.607
1.25	-4.172
1.375	-3.682
1.5	-3.125

Compute first derivatives using centered differencing

$$F(\Delta x) \approx f'(1.25) = \frac{f(1.5) - f(1.0)}{2(0.25)} = \frac{-3.125 + 5}{0.5} = 3.75$$

$$F(\Delta x/2) \approx f'(1.25) = \frac{f(1.375) - f(1.125)}{2(0.125)} = \frac{-3.6816 + 4.6074}{0.25} = 3.7032$$

Richardson's Extrapolation example

Compute the error with the exact first derivatives:

$$F(\Delta x) \approx f'(1.25) = 3.75 \qquad \text{Error} = 1.69\%$$

$$F\left(\frac{\Delta x}{2}\right) \approx f'(1.25) = 3.7032 \qquad \text{Error} = 0.425\%$$

Apply Richardson's extrapolation using these results to find a better solution

$$f'(1.25) = \frac{4F\left(\frac{\Delta x}{2}\right) - F(\Delta x)}{3} = \frac{4(3.7032) - 3.75}{3} = 3.6876 \qquad \text{Error} = 0.003\%$$

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