Geometry for Computer Graphics

Fondamenti di Computer Graphics 2018/2019
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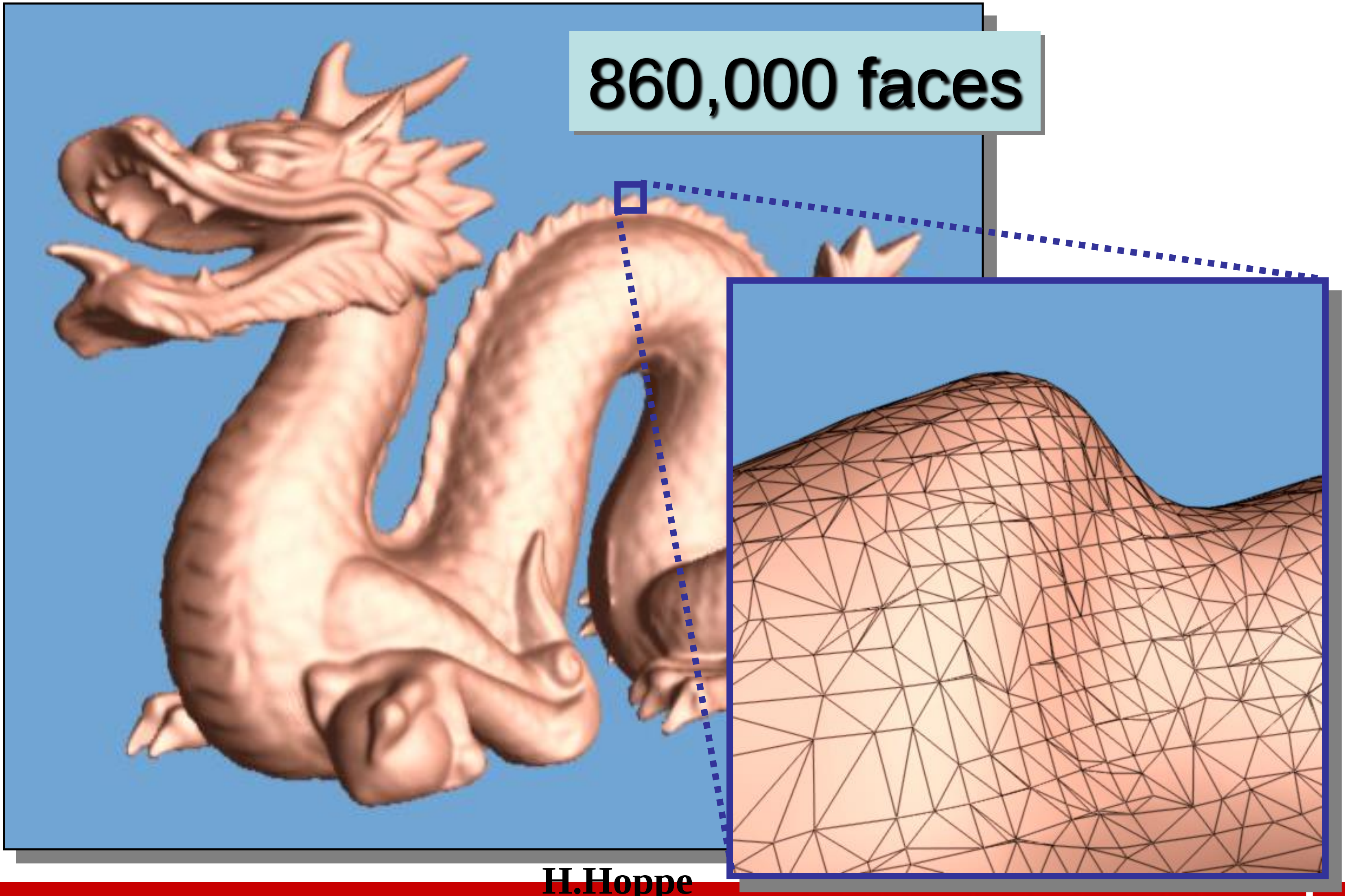
https://www.youtube.com/watch?v=vgnZ4PcqV98

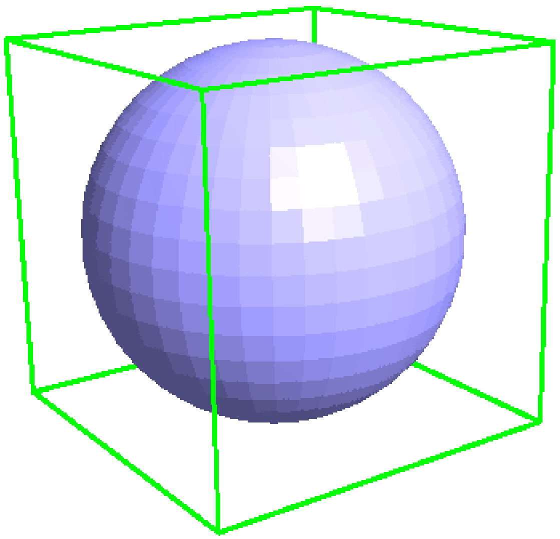


H.Hoppe

Geometric Objects and transformations

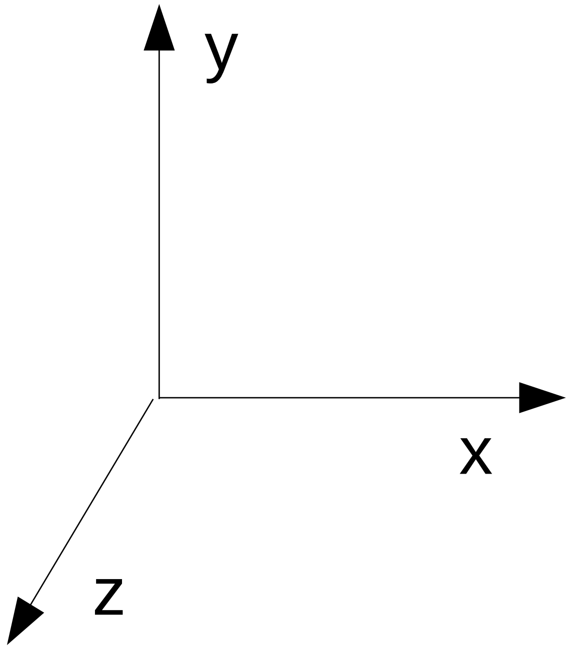
- We represent objects using mainly linear primitives:
  - points
  - lines, segments
  - planes, polygons

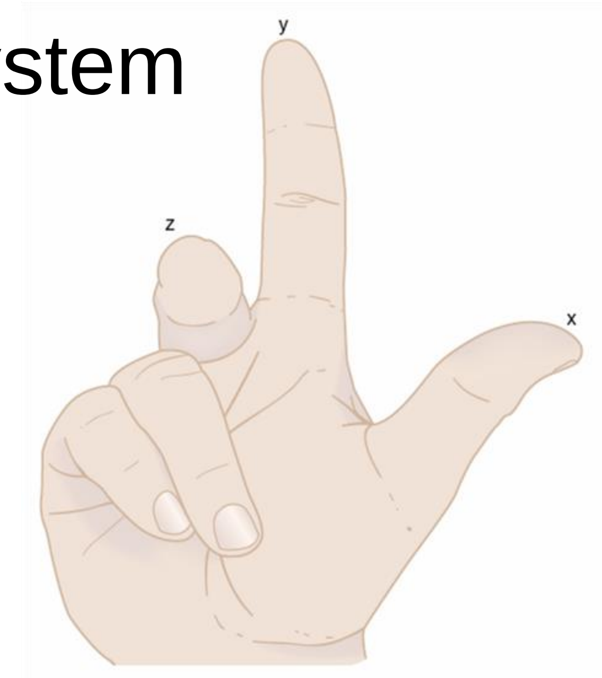
- Need to know how to transform objects, compute distances, change coordinate systems…



Coordinate Systems

- Right handed coordinate system

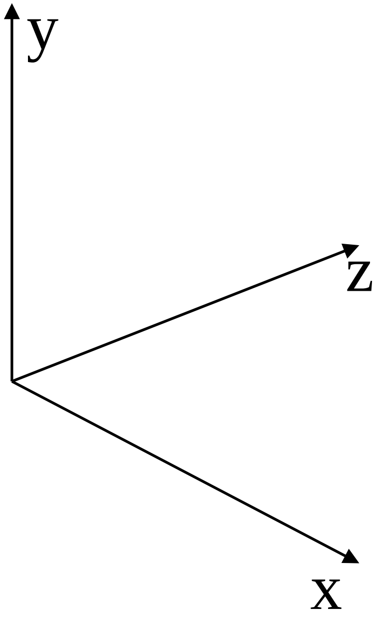




In CG we use the rule:
z axis perpendicular to the screen

Coordinate Systems

- Left handed coordinate system



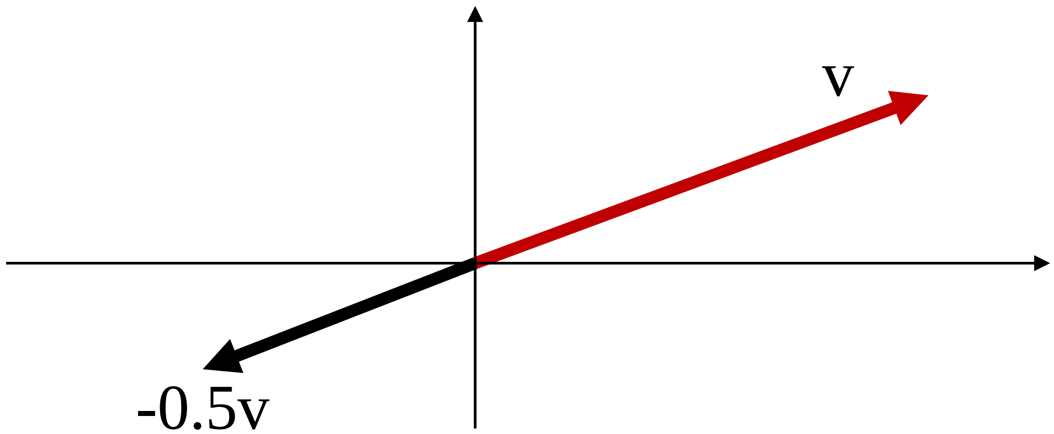
Scalars, Points and Vectors

- **Scalar** specifies ***magnitude*** (quantità)
- **Point** specifies ***location*** in space (or in the plane)
- **Vector** is a quantity with two attributes: ***magnitude*** and ***direction*** (direzione+verso). No location in space.

$$\boxed{\text{Points} \neq \text{Vectors}}$$

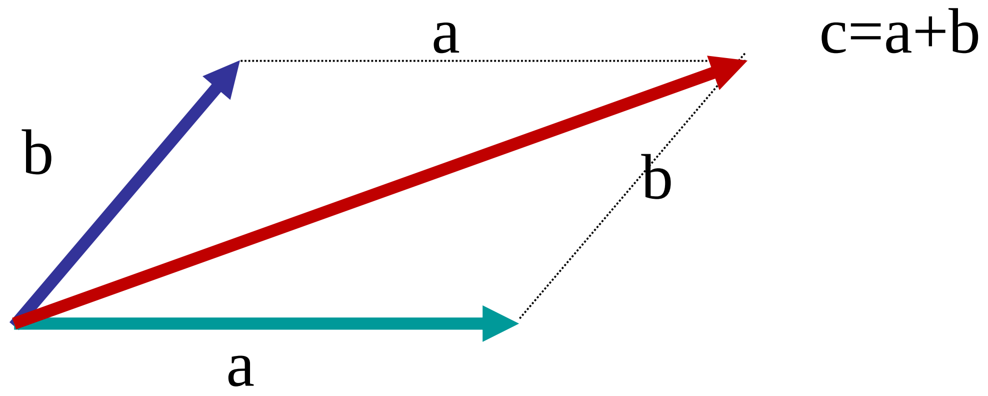
Linear/Vector Space

- Entities: VECTORS and SCALARS
- Operations:
  - Multiplication   scalar vector
  - Vector sum



vector + vector = vector

Parallelogram rule



$$\mathbf{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_x + b_x & a_y + b_y & a_z + b_z \end{bmatrix}$$

$$\mathbf{a} - \mathbf{b} = \begin{bmatrix} a_x - b_x & a_y - b_y & a_z - b_z \end{bmatrix}$$

$$-\mathbf{a} = \begin{bmatrix} -a_x & -a_y & -a_z \end{bmatrix}$$

$$s\mathbf{a} = \begin{bmatrix} sa_x & sa_y & sa_z \end{bmatrix}$$

Space Dimension

- In a vector space V, the maximum number of linearly independent vectors is fixed and is called the *dimension* of the space
- In an *n*-dimensional space, any set of n linearly independent vectors form a *basis* for the space
- Given a **basis** $a_1, a_2, \ldots, a_n$, any vector v in V can be written as the *linear combination*

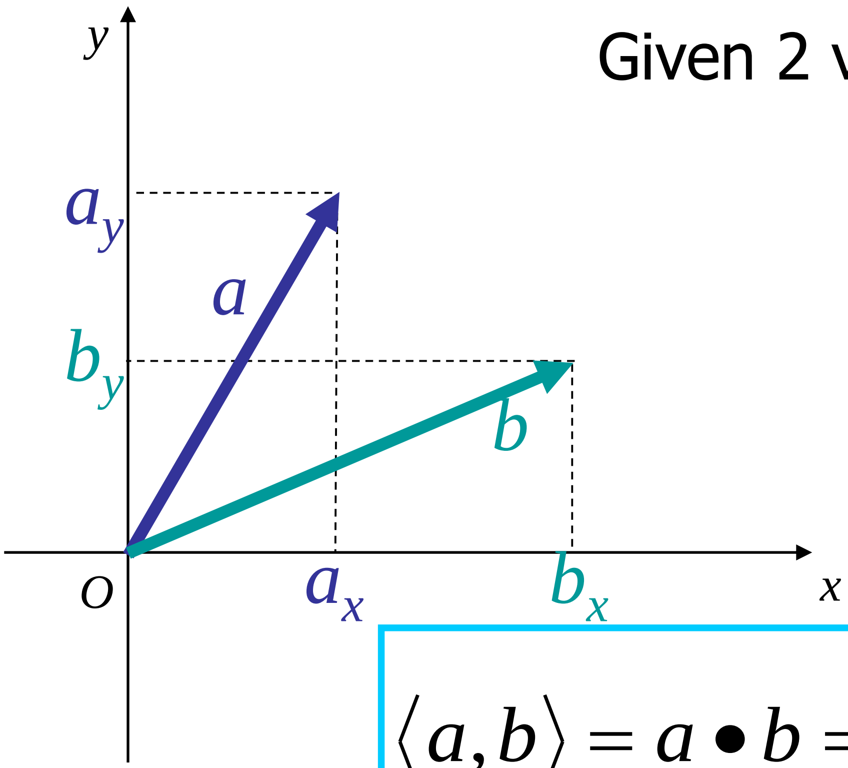
$$v = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n$$

where the $\{\alpha_i\}$ are unique

In $V=R^3$, the Euclidean vectors a_1, a_2, a_3 define a ***coordinate system***

Linear Space with **Dot Product** (or Inner Product)

Dot product in coordinates x,y:



Given 2 vectors get a scalar value

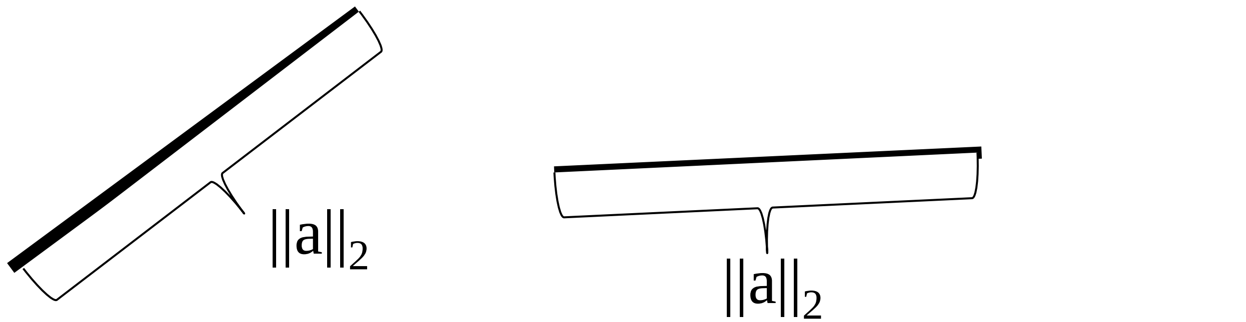
$$a = (a_x, a_y)$$
$$b = (b_x, b_y)$$
$$< a, b > = a_x b_x + a_y b_y$$

$$\langle a, b \rangle = a \bullet b = a^T b = \sum_{i=1}^{n} a_i b_i \quad in \quad \Re^n$$

2-norm or Euclidean norm

$$\|a\|_2 = \sqrt{\langle a, a \rangle} = \sqrt{a^T a} = \sqrt{\sum_{i=1}^{n} (a_i)^2}$$

Euclidean norm is a notion of length preserved by rotations/translations/reflections of space.



Vector Magnitude

- The magnitude (length) of a vector a in R^3 is:

$$\|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

- A vector with length=1.0 is called a *unit vector*
- We can also *normalize* a vector to make it a unit vector:

$$\frac{a}{\|a\|_2}$$

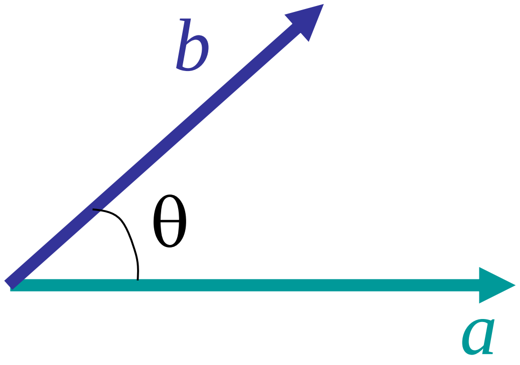
Inner (dot) product

Angle between 2 vectors

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

$$\cos\theta = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)$$



Properties

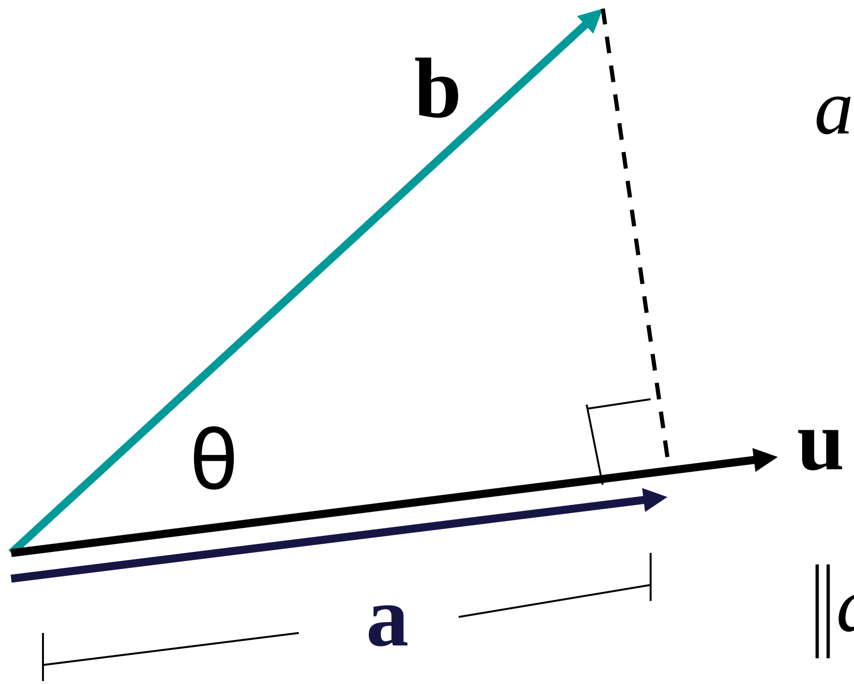
Commutative
<a,b>=<b,a>

Distributive
<a,b+c>=<a,b> + <a,c>

Bilinear, r scalar
<a,*r* b+c>=*r*<a,b>+<a,c>

Vector Projection (orthogonal projection)

- If $||\mathbf{u}||\neq 1.0$ then **||a||** is the length of the vector **a** which is the *projection* of **b** onto **u**



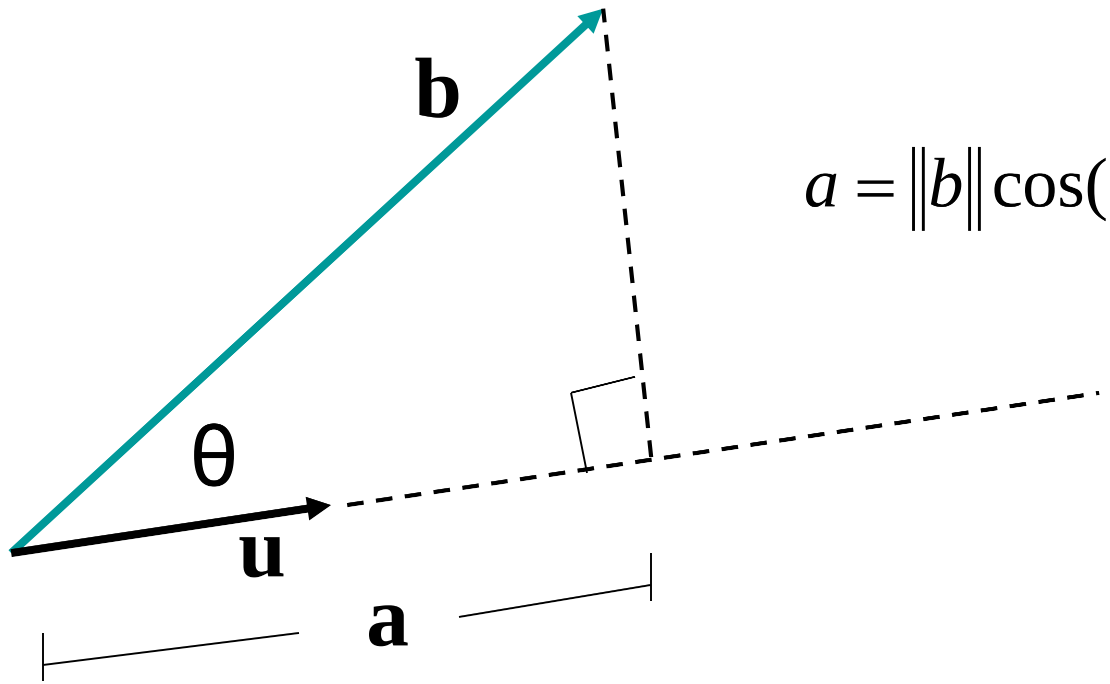
$$a = \|a\| \frac{u}{\|u\|} = \|b\| \cos(\theta) \frac{u}{\|u\|}$$

$$= \frac{<b,u>}{\|u\|} \frac{u}{\|u\|}$$

$$\|a\| = \frac{<b,u>}{\|u\|}$$

Vector Projection (orthogonal projection)

- If $||\mathbf{u}||=1.0$ then $<\mathbf{b},\mathbf{u}>$ is the length of the vector **a** which is the *projection* of **b** onto **u**



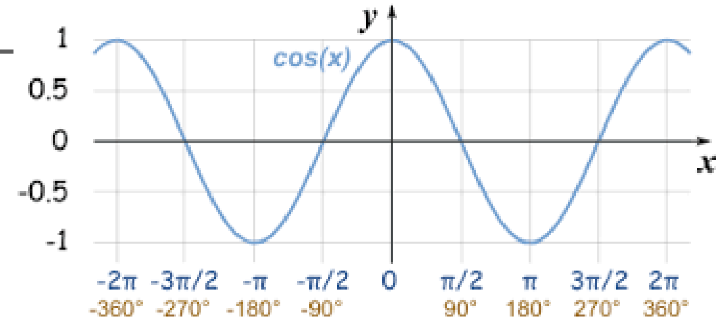
$$a = \|b\| \cos(\theta) \frac{u}{\|u\|} =< b,u > u$$

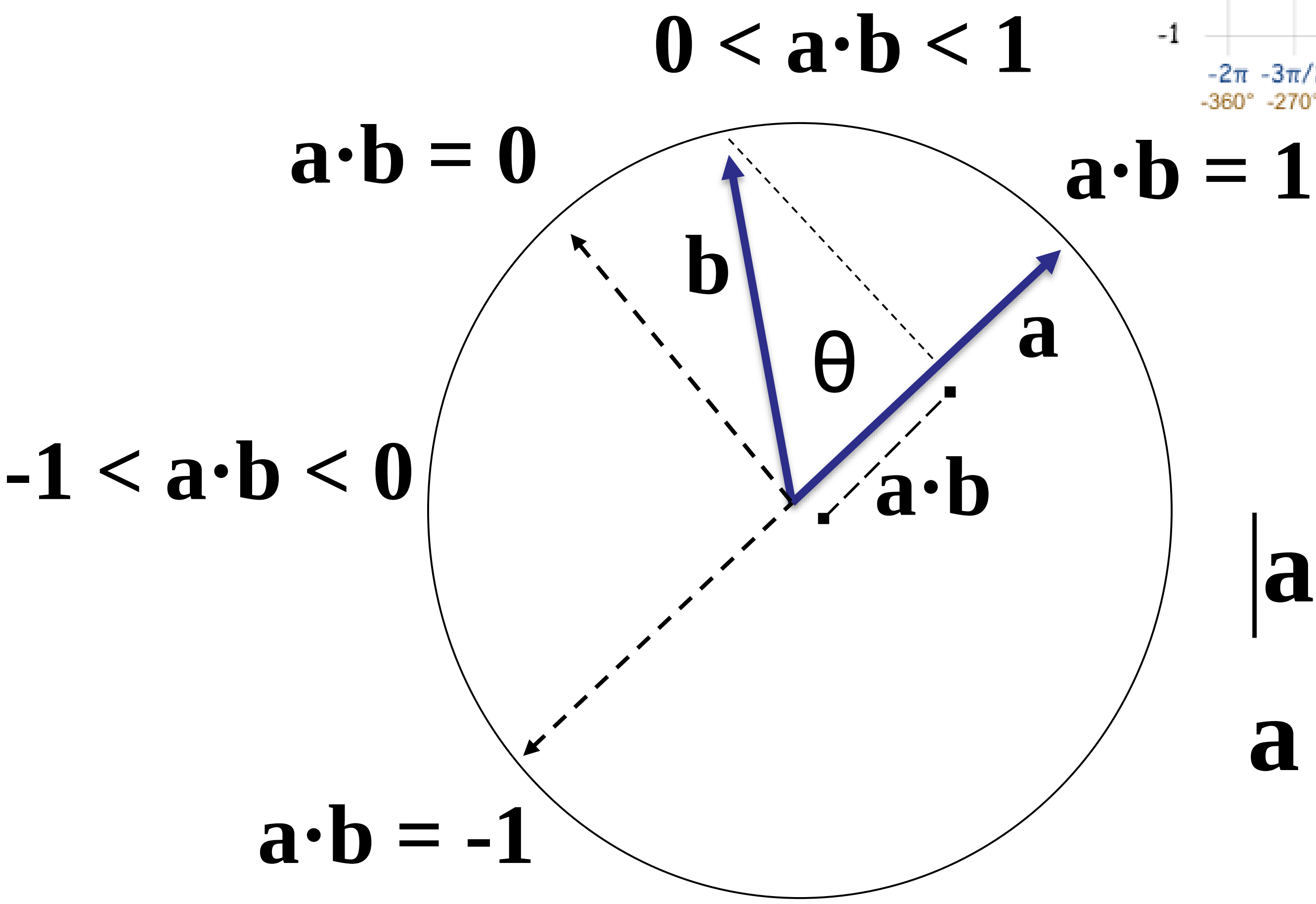
Dot Products with General Vectors

- The dot product is a scalar value that tells us something about the relationship between two vectors

  - If $\mathbf{a} \cdot \mathbf{b} > 0$ then $\theta < 90^\circ$
  - If $\mathbf{a} \cdot \mathbf{b} < 0$ then $\theta > 90^\circ$
  - If $\mathbf{a} \cdot \mathbf{b} = 0$ then $\theta = 90^\circ$ (or one or more of the vectors is degenerate (0,0,0))

Dot Products with unit Vectors



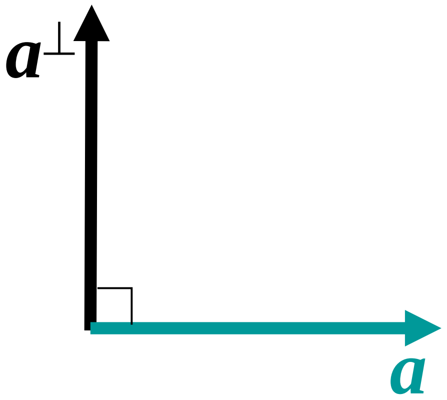


$$|\mathbf{a}| = |\mathbf{b}| = 1.0$$

$$\mathbf{a} \cdot \mathbf{b} = \cos(\theta)$$

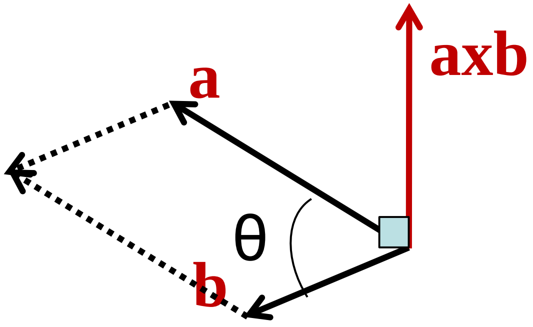
Perpendicular vectors

$$a = (a_x, a_y) \quad \Rightarrow \quad a^{\perp} = \pm(-a_y, a_x)$$



Cross Product : Properties

$\mathbf{a}\times\mathbf{b}$ is a *vector* perpendicular to both **a** and **b**, in the direction defined by the right hand rule



$$\left\|\mathbf{a}\times\mathbf{b}\right\| = \left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\sin\theta$$

$\left\|\mathbf{a}\times\mathbf{b}\right\| =$ Area of the parallelogram **ab**

$\left\|\mathbf{a}\times\mathbf{b}\right\| = 0$ if **a** and **b** are parallel

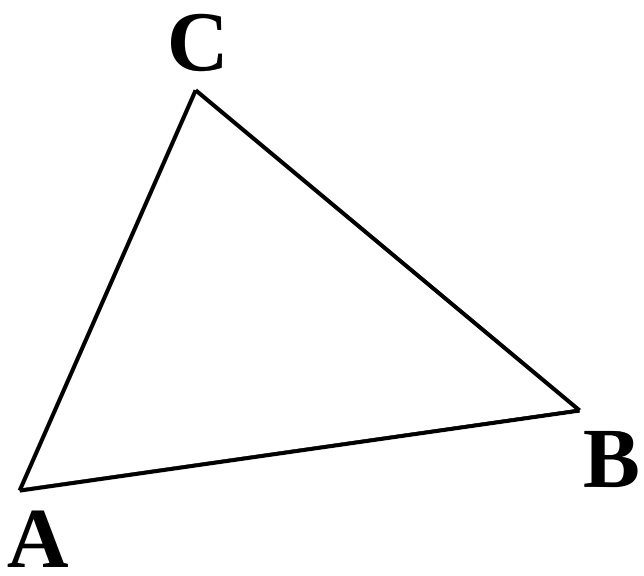
Cross Product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{bmatrix}$$

Example: Normal of a Triangle

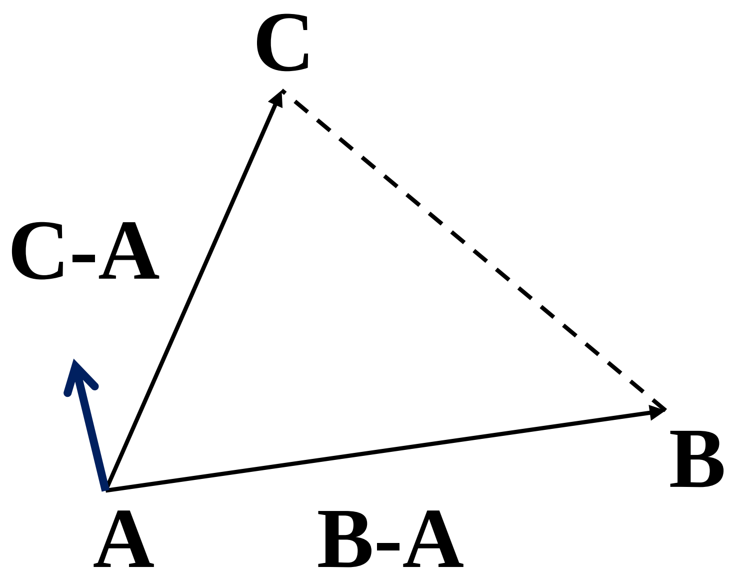
- Find the unit length normal of the triangle defined by 3D points **A**, **B**, and **C**



Example: Normal of a Triangle

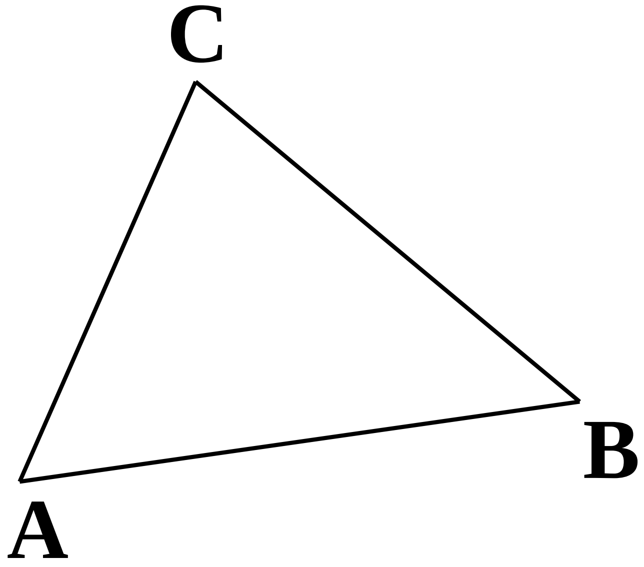
$$\mathbf{n}^* = (B - A) \times (C - A)$$

$$\mathbf{n} = \frac{\mathbf{n}^*}{\left\| \mathbf{n}^* \right\|}$$



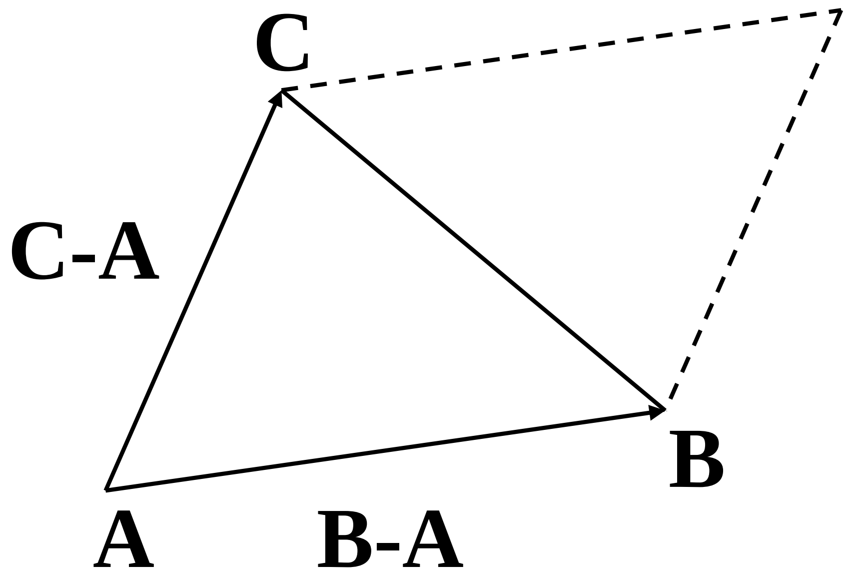
Example: Area of a Triangle

- Find the area of the triangle defined by 3D points **A**, **B**, and **C**



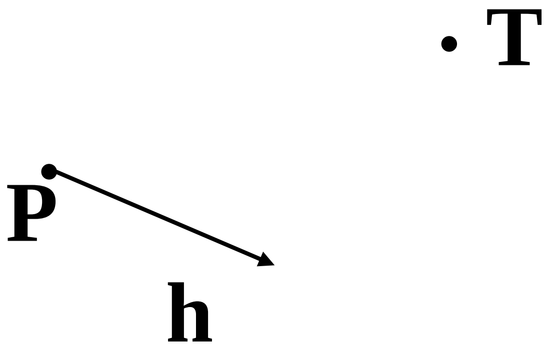
Example: Area of a Triangle

$$area = \frac{1}{2}\left\|\left(B - A\right)\times\left(C - A\right)\right\|$$



Example: Alignment to Target

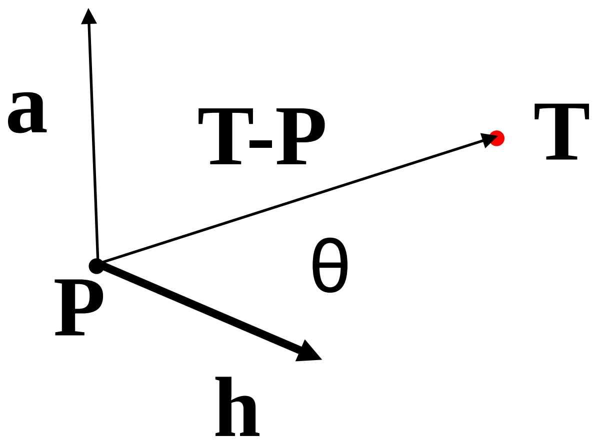
- An object is at position **P** with a unit length heading of **h**. We want to rotate it so that the heading is facing some target **T**.
- Find a unit axis **a** and an angle θ to rotate around.



Example: Alignment to Target

$$\mathbf{a} = \frac{\mathbf{h} \times (T - P)}{\left\| \mathbf{h} \times (T - P) \right\|}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{h} \cdot (T - P)}{\left\| (T - P) \right\|} \right)$$



Vector Class (C++)

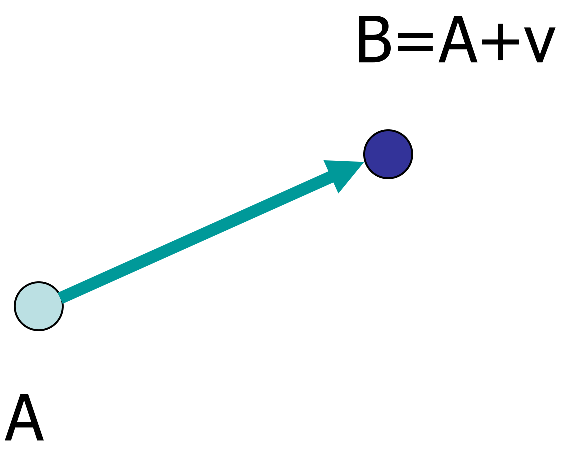
```
class Vector3 {
public:
    Vector3()                                  {x=0.0f; y=0.0f; z=0.0f;}
    Vector3(float x0,float y0,float z0)        {x=x0; y=y0; z=z0;}
    void Set(float x0,float y0,float z0)       {x=x0; y=y0; z=z0;}
    void Add(Vector3 &a)                       {x+=a.x; y+=a.y; z+=a.z;}
    void Add(Vector3 &a,Vector3 &b)            {x=a.x+b.x; y=a.y+b.y; z=a.z+b.z;}
    void Subtract(Vector3 &a)                  {x-=a.x; y-=a.y; z-=a.z;}
    void Subtract(Vector3 &a,Vector3 &b)       {x=a.x-b.x; y=a.y-b.y; z=a.z-b.z;}
    void Negate()                              {x=-x; y=-y; z=-z;}
    void Negate(Vector3 &a)                    {x=-a.x; y=-a.y; z=-a.z;}
    void Scale(float s)                        {x*=s; y*=s; z*=s;}
    void Scale(float s,Vector3 &a)             {x=s*a.x; y=s*a.y; z=s*a.z;}
    float Dot(Vector3 &a)                      {return x*a.x+y*a.y+z*a.z;}
    void Cross(Vector3 &a,Vector3 &b)
            {x=a.y*b.z-a.z*b.y; y=a.z*b.x-a.x*b.z; z=a.x*b.y-a.y*b.x;}
    float Magnitude()                          {return sqrtf(x*x+y*y+z*z);}
    void Normalize()                           {Scale(1.0f/Magnitude());}

    float x,y,z;
};
```

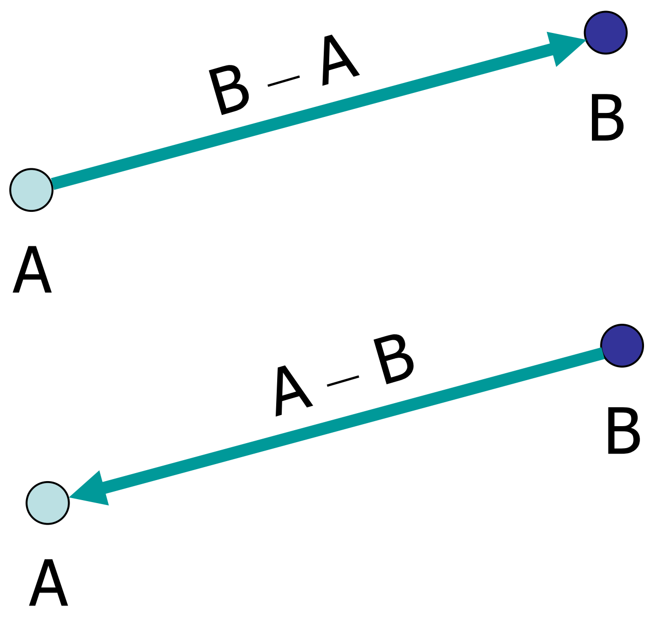
Affine Space

- In a linear space the concept of location is missing
- An affine space is an extension of a linear space which includes the **Point**
- New operations:
  - sum *point + vector*: defines a new point
  - difference *point-point*: defines a vector

Point + vector = point



point - point = vector



point + point: not defined!!

Map points to vectors

- If we have a coordinate system with origin at point O
- We can define correspondence between points and vectors:

$$\boldsymbol{P} \quad \rightarrow \quad \boldsymbol{v} = \boldsymbol{P} - \boldsymbol{O}$$
$$\boldsymbol{v} \quad \rightarrow \quad \boldsymbol{P} = \boldsymbol{O} + \boldsymbol{v}$$

Affine Combination

Affine Combination is a linear combination of points with coefficients that sum up to 1

$$P = a_1 P_1 + a_2 P_2 + \dots\dots + a_n P_n$$

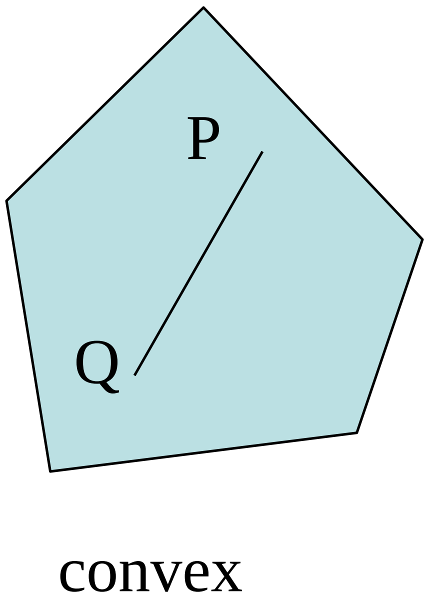
$$for \quad some \quad a_1 + a_2 + \dots\dots + a_n = 1$$

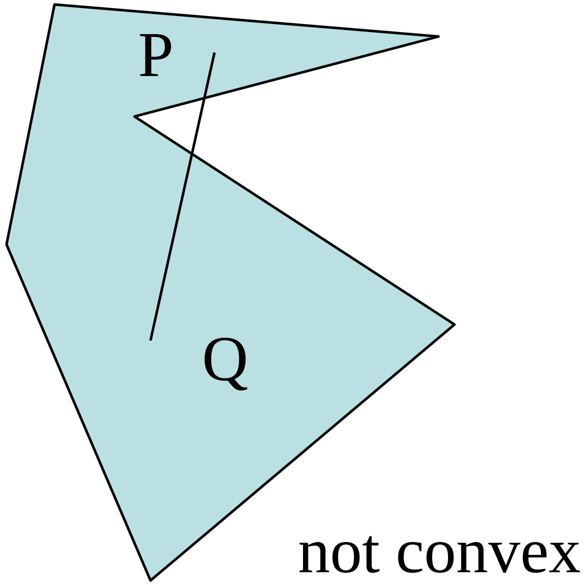
The coefficients $(a_1, a_2, \dots, a_n)$ are defined as **barycentric coordinates** of P in the affine space

Affine Combinations with coefficients $(a_1, a_2, \dots, a_n)$ in [0,1] are called **convex combinations**

Convexity

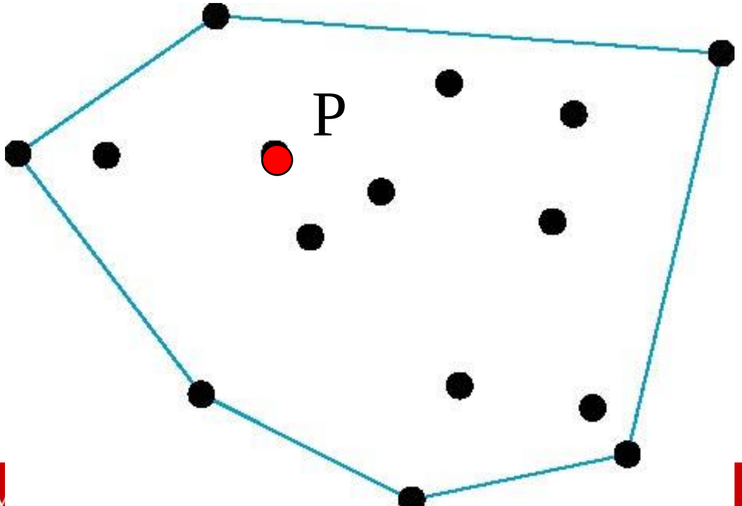
- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object





Convex hull

- Given a set of points $$P_1, P_2, \ldots\ldots, P_n$$
- The set of all the points **P** which can be represented as a convex combinations is called convex hull (guscio convesso) of the set
- Smallest convex object containing $P_1, P_2, \ldots\ldots P_n$



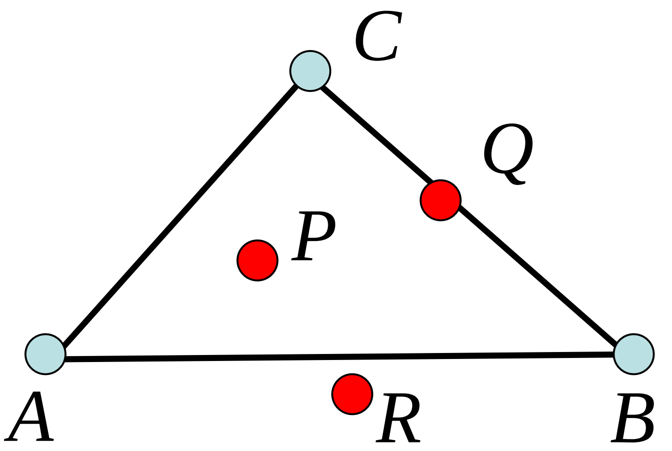
Barycentric coordinates (2D)

- Define a point's position relatively to some fixed points A,B,C

$$P = \alpha A + \beta B + \gamma C,$$

where A, B, C are not on one line, and $\alpha, \beta, \gamma \in \mathrm{R}$.

- (α, β, γ) are called Barycentric coordinates of P with respect to A, B, C (unique!)
- If P is inside the triangle, $\alpha, \beta, \gamma \in [0, 1]$, $\alpha+\beta+\gamma = 1$

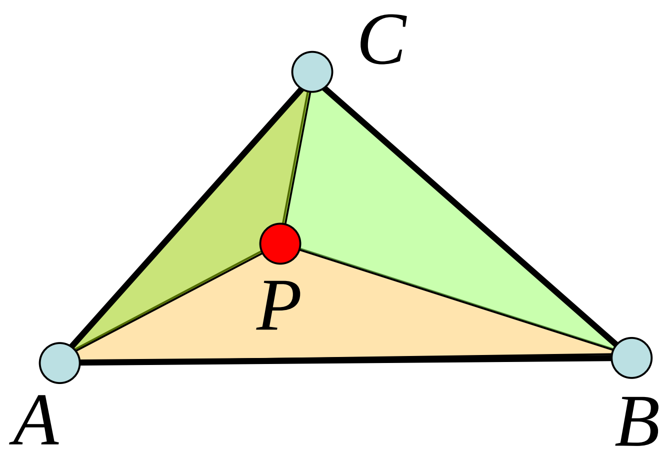


$Q, R??$

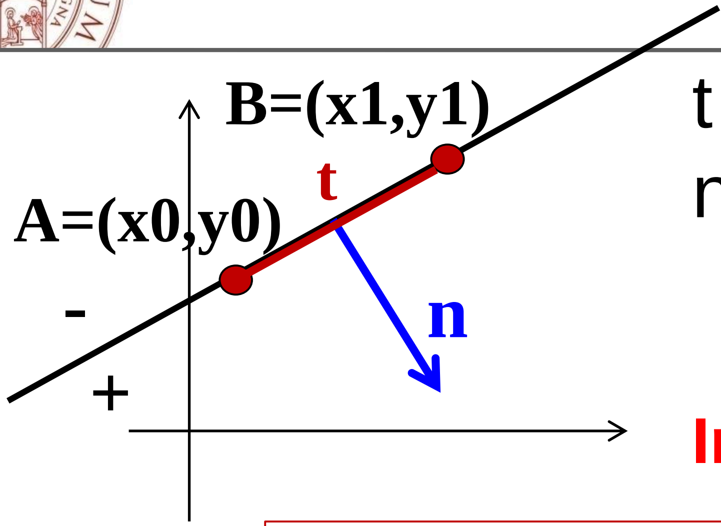
Barycentric coordinates (2D)

$$P = \frac{\langle P, B, C \rangle}{\langle A, B, C \rangle} A + \frac{\langle P, C, A \rangle}{\langle A, B, C \rangle} B + \frac{\langle P, A, B \rangle}{\langle A, B, C \rangle} C$$

$\langle \cdot, \cdot, \cdot \rangle$ denotes the area of the triangle



Compute BC



$t = (x_1\text{-}x_0, y_1\text{-}y_0)$

$n = \mathrm{Perp}(t) = (y_1\text{-}y_0, -(x_1\text{-}x_0))$

Normalize if desired

Implicit line equation E_{AB} :

$(P - A) \bullet n = 0$ for all points P on the line

$$\begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix} \begin{pmatrix} y_1 - y_0 \\ -(x_1 - x_0) \end{pmatrix} \begin{matrix} = 0 & on \\ < 0 & left \\ > 0 & right \end{matrix}$$

$$E_{AB}(x, y) = (x - x_0)(y_1 - y_0) - (y - y_0)(x_1 - x_0)$$

Compute BC

$$area_{ABC} = \frac{1}{2}\left\|(B-A)\times(C-A)\right\|$$

$$= \frac{1}{2}((x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(x_2 - x_0))$$

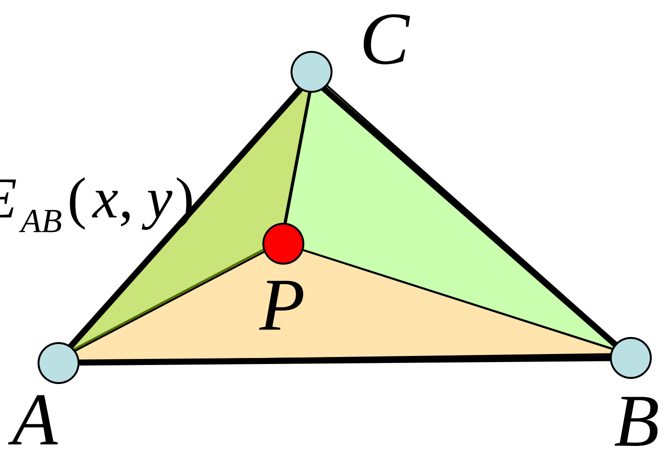
$$area_{ABC} = area_{APB} + area_{BPC} + area_{CPA}$$

$$area_{APB} = \frac{1}{2}\left\|(P-A)\times(B-A)\right\| =$$

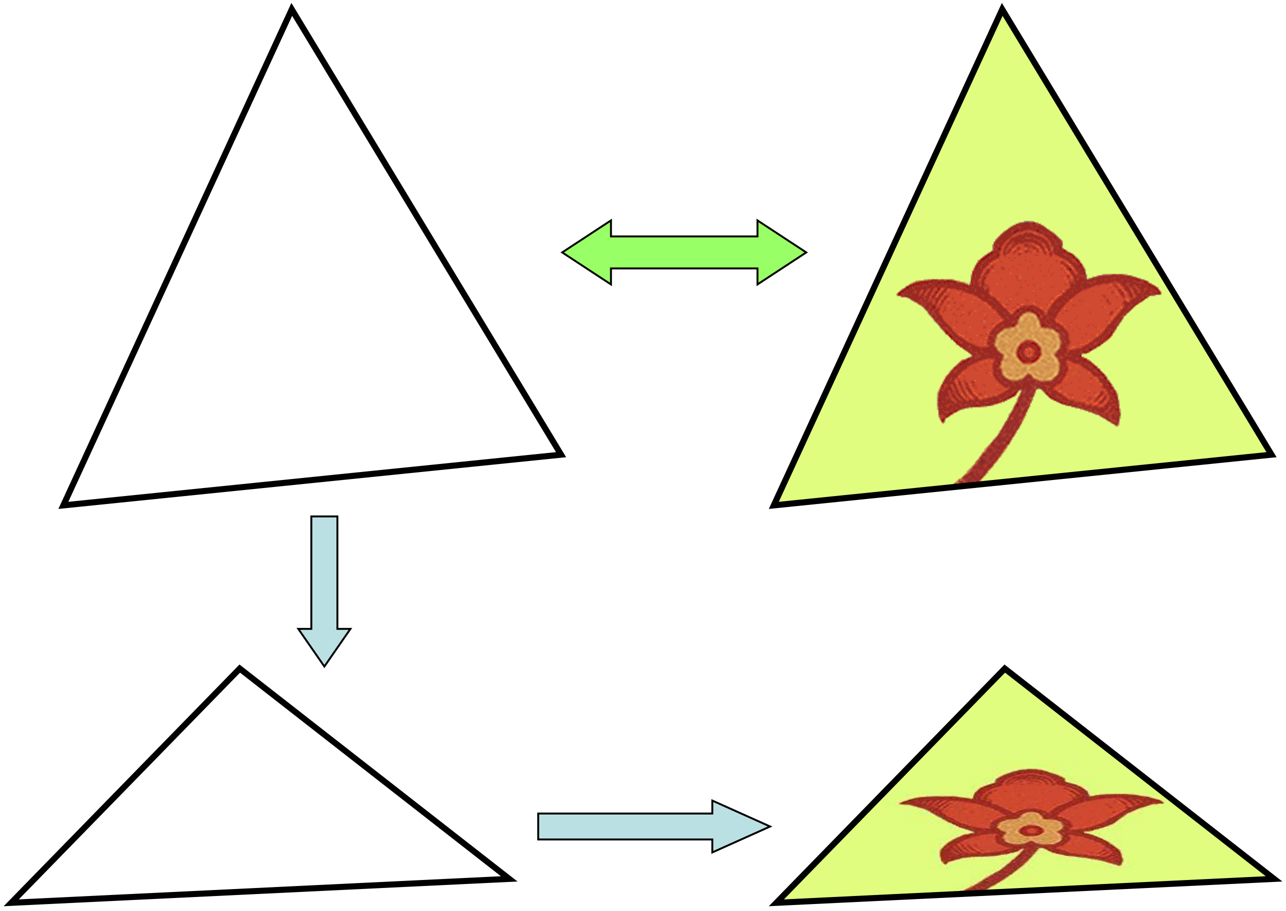
$$= \frac{1}{2}((x - x_0)(y_1 - y_0) - (y - y_0)(x_1 - x_0)) = \frac{1}{2}E_{AB}(x,y)$$

$$area_{BPC} = \frac{1}{2}E_{BC}(x,y)$$

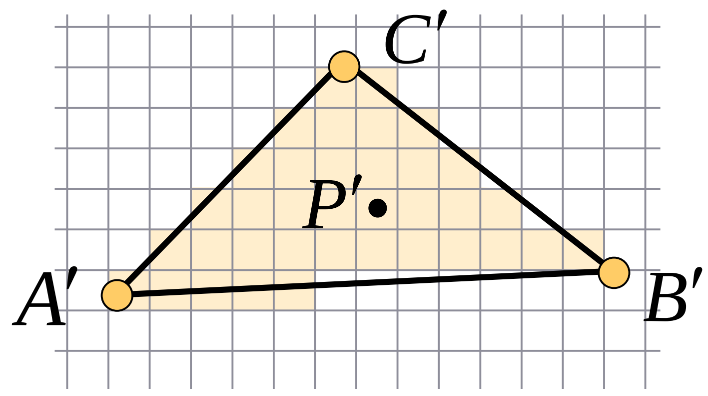
$$area_{CPA} = \frac{1}{2}E_{CA}(x,y)$$



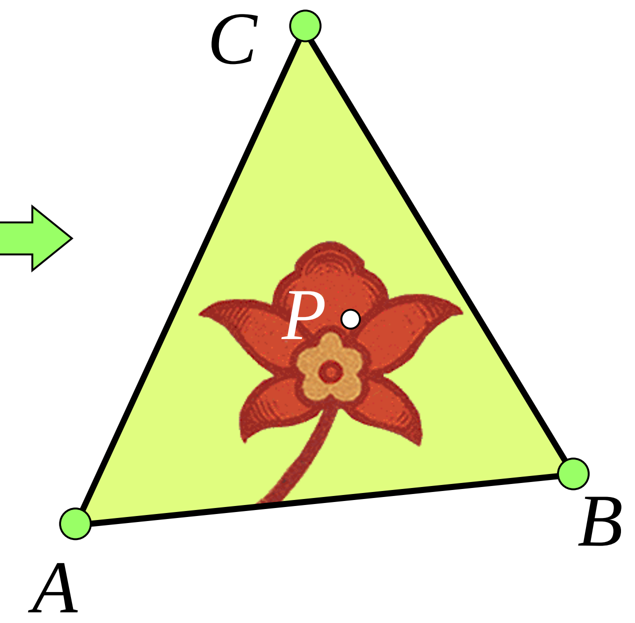
Example of usage: warping

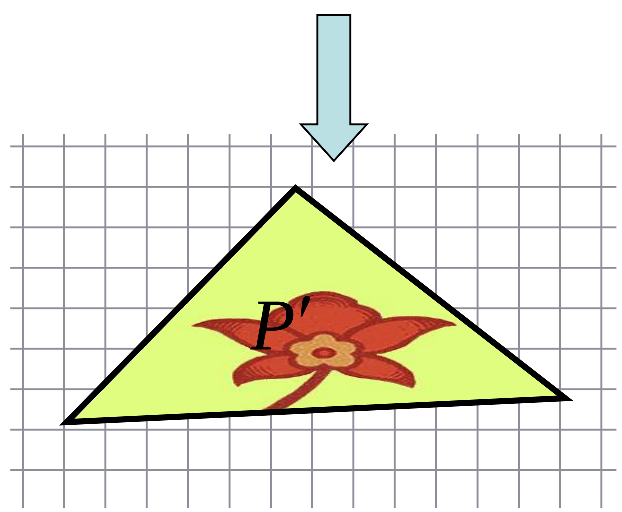


Example of usage: warping



TARGET





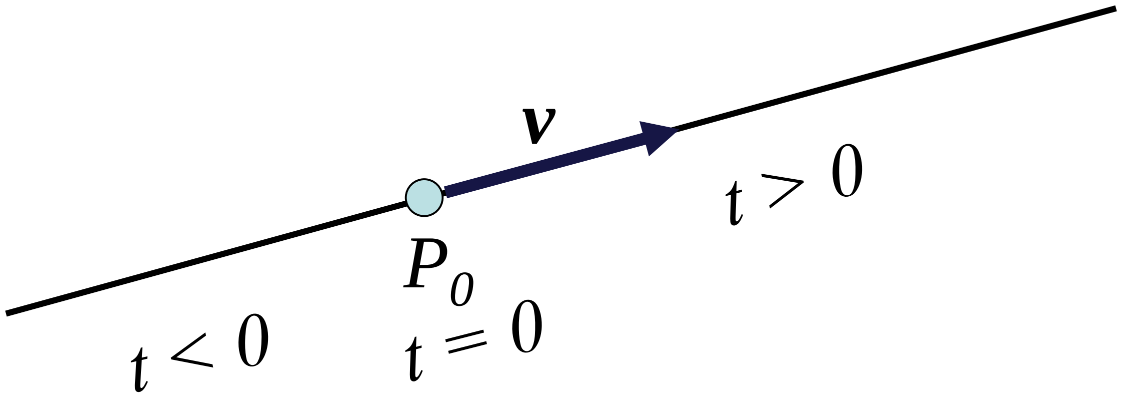
We take the barycentric coordinates α, β, γ of *P’* with respect to *A’*, *B’*, *C’*

Color(*P’*) = Color(α *A* + β *B* + γ *C*)

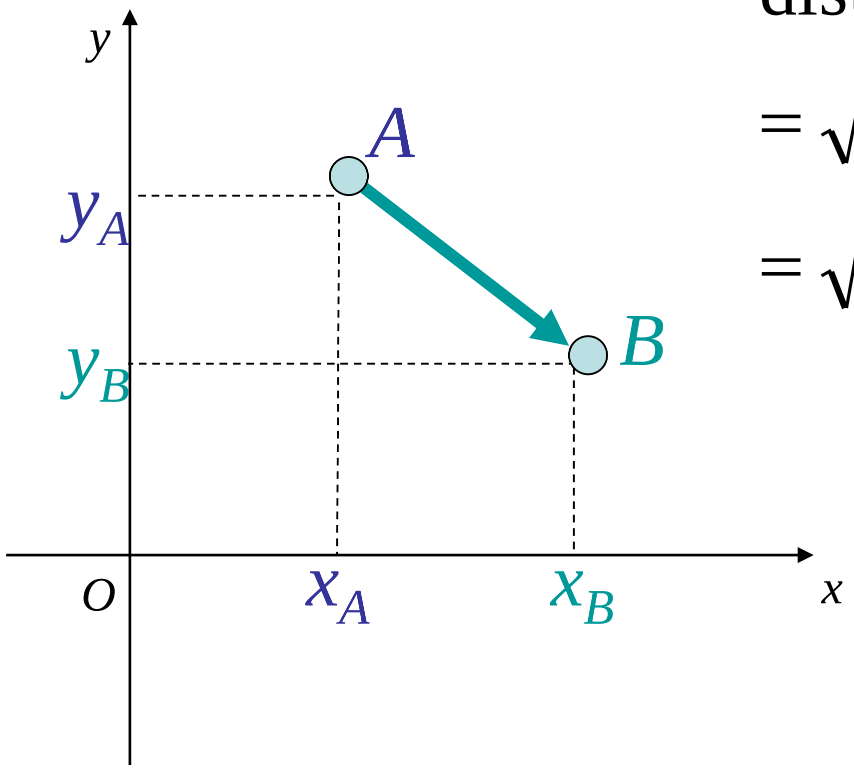
Parametric equation of a line

- Set of all points that pass through P_0 in the direction of the vector $\mathbf{v}$

$$\ell(t) = P_0 + t\mathbf{v}, \quad t \in (-\infty, \infty)$$



Distance between two points



$$\mathrm{dist}(A,B) = \| B - A \| =$$

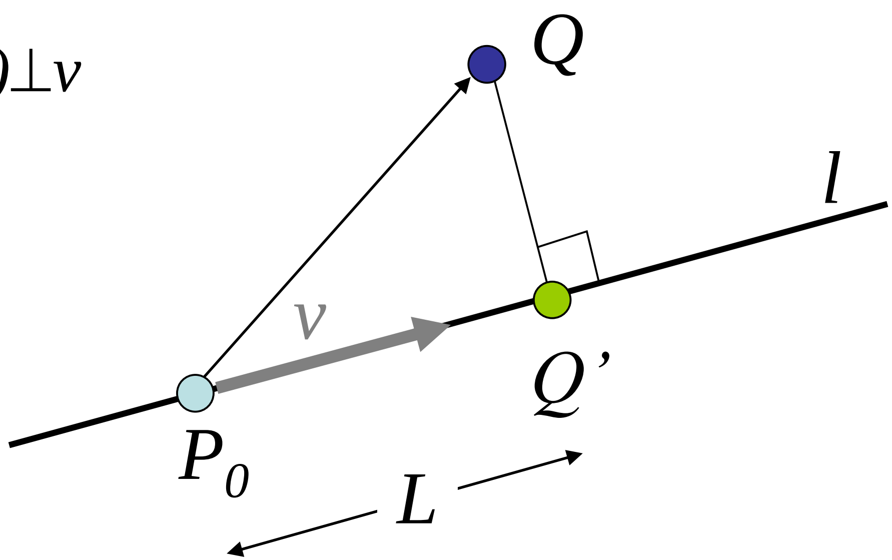
$$= \sqrt{< B - A, B - A >} =$$

$$= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Distance between point and line

Find a point Q' such that $(Q - Q') \perp v$

dist(Q, l) = $\|Q - Q'\|$



Pitagora :

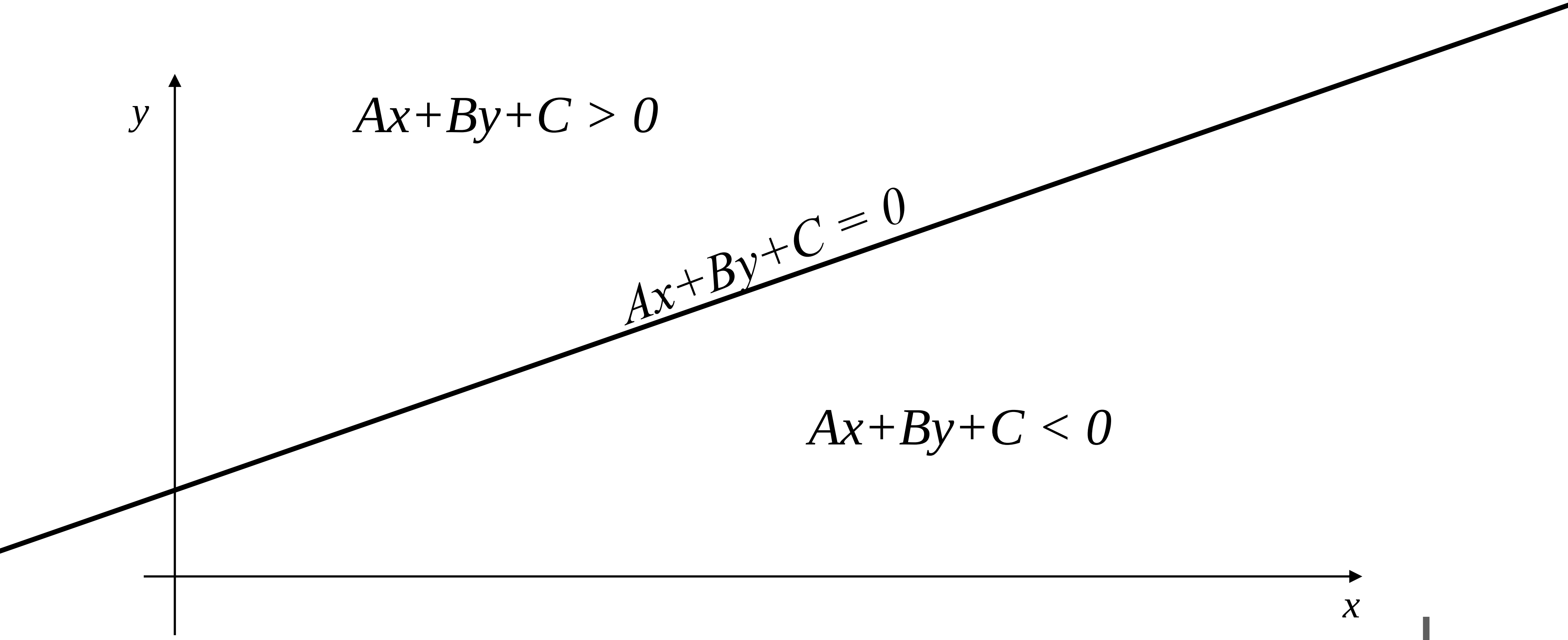
$$L^2 + \mathrm{dist}(Q, Q')^2 = \| Q - P_0 \|^2$$

$$L = \frac{< Q - P_0, v >}{\| v \|}$$

$$\Rightarrow \quad \mathrm{dist}(Q, Q')^2 = \| Q - P_0 \|^2 - L^2 = \| Q - P_0 \|^2 - \frac{< Q - P_0, v >^2}{\| v \|^2}.$$

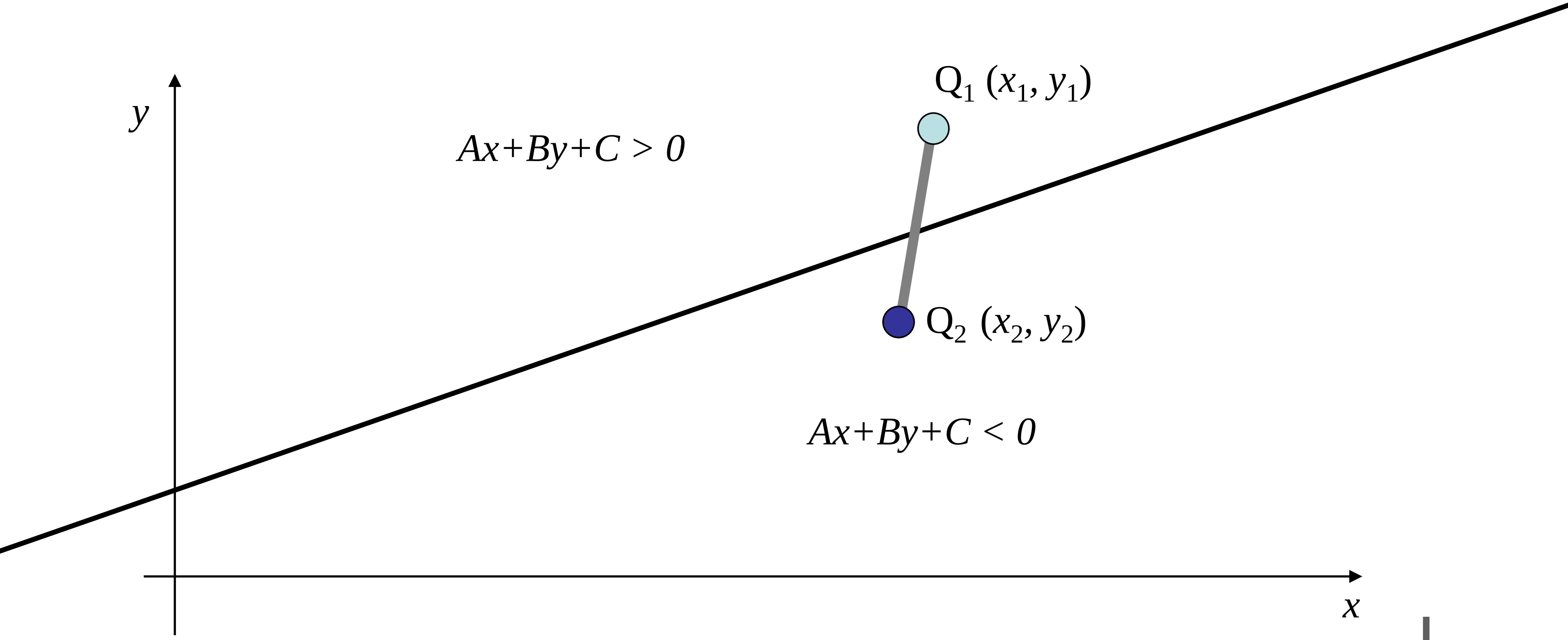
Implicit equation of a line in 2D

$$\boxed{Ax + By + C = 0, \quad A, B, C \in \mathbb{R},\ AB \neq 0}$$



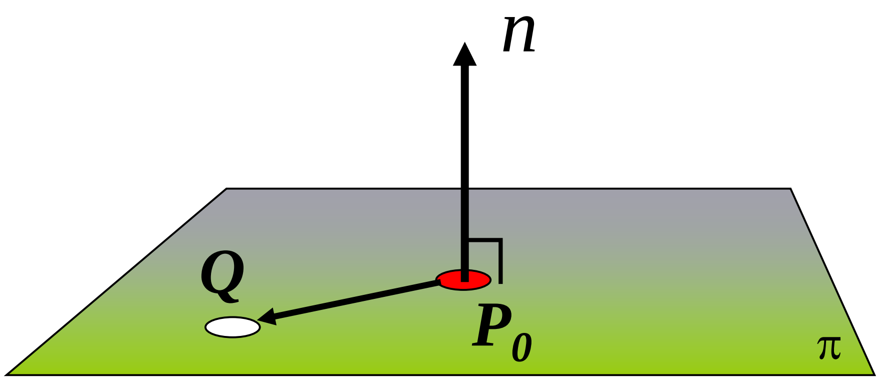
Line-segment intersection

$$\text{The segment } Q_1Q_2 \text{ intersects the line} \Leftrightarrow$$
$$\Leftrightarrow (Ax_1 + By_1 + C)(Ax_2 + By_2 + C) \le 0$$



Representation of a plane in 3D space

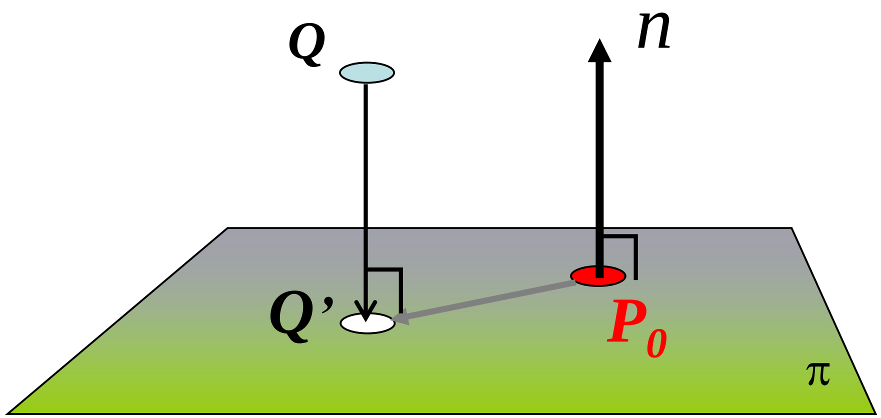
- The plane π is defined by a normal n and one point in the plane (P_0).
- A point Q belongs to the plane $\Leftrightarrow$ $\boxed{<Q - P_0 , n> = 0}$
- The normal n is perpendicular to all vectors in the plane



Distance between point and plane

- Project the point onto the plane in the direction of the normal:

$$\mathrm{dist}(Q, \pi) = ||Q' - Q||$$



Distance between point and plane

$$(Q' - Q) \parallel n \quad \Rightarrow \quad Q' - Q = sn, \quad s \in \mathrm{R} \quad \Rightarrow \quad Q' = Q + sn$$

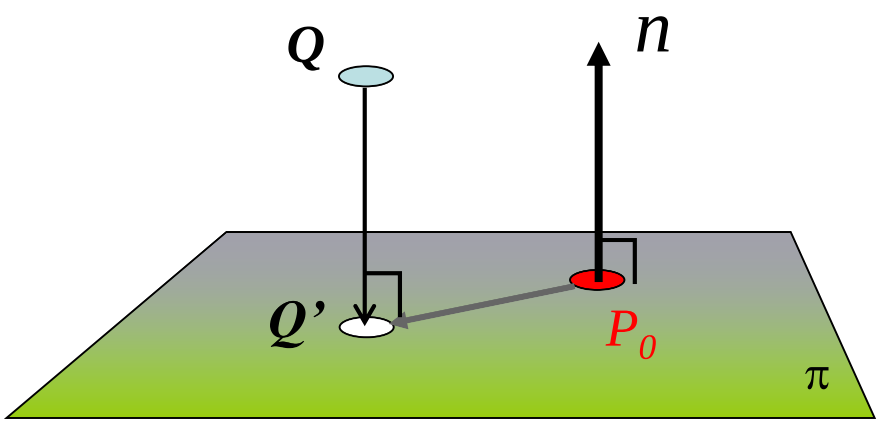
$$< Q' - P_0, n > = 0$$

$$< Q - P_0 + sn, n > = 0$$

$$< Q - P_0, n > + s < n, n > = 0$$

$$\boxed{s = -\frac{< Q - P_0, n >}{\| n \|^2}}$$

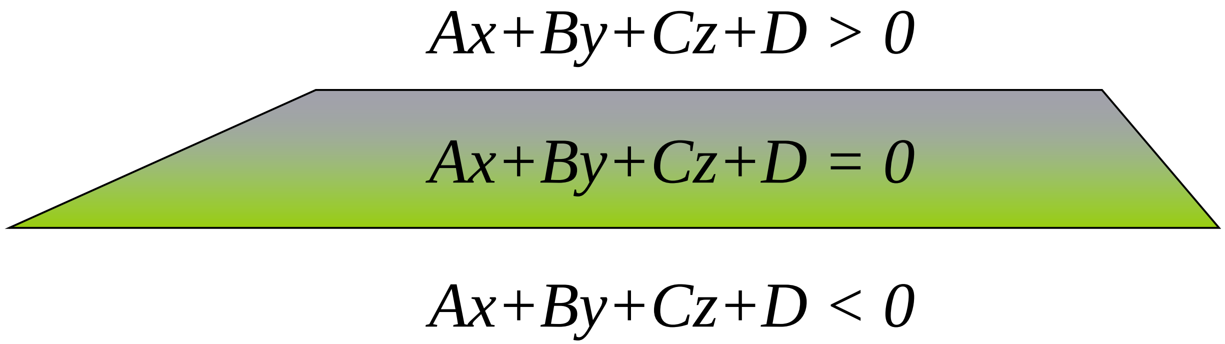
$$\mathrm{dist}^2(Q, \pi) = \| Q' - Q \|^2 = s^2 \| n \|^2 = \frac{< Q - P_0, n >^2}{\| n \|^2}$$



Implicit representation of planes in 3D

- (x, y, z) are coordinates of a point on the plane
- (A, B, C) are the coordinates of a normal vector to the plane

$$\boxed{Ax + By + Cz + D = 0, \quad A, B, C, D \in \mathrm{R},\ ABC \neq 0}$$



Coordinate Frame
(Sistemi di riferimento)

- A frame is defined by the quadruple

  F=(Po,v1,v2,v3)

  where
  - Po is a point (origin)
  - [v1,v2,v3] is a vector basis (orthonormal)

- In a frame F the coordinates (a_1, a_2, a_3) uniquely describe the point:

$$P = P_0 + a_1 v_1 + a_2 v_2 + a_3 v_3$$

Homogeneous Coordinates

- Representing both vectors and points using three scalar values is ambiguous..
- We consider a coordinate system which allows for a unique representation for points and vectors
- A **vector** is represented as

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3$$

- A **point** is represented as

$$P = P_0 + a_1 v_1 + a_2 v_2 + a_3 v_3$$

Homogeneous Coordinates

- Assume

  1*P=P   and   0*P=0   (zero vector)

- A **vector** is then given by

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3 + 0 \cdot P_0$$

- A **point** is then given by

$$P = a_1 v_1 + a_2 v_2 + a_3 v_3 + 1 \cdot P_0$$

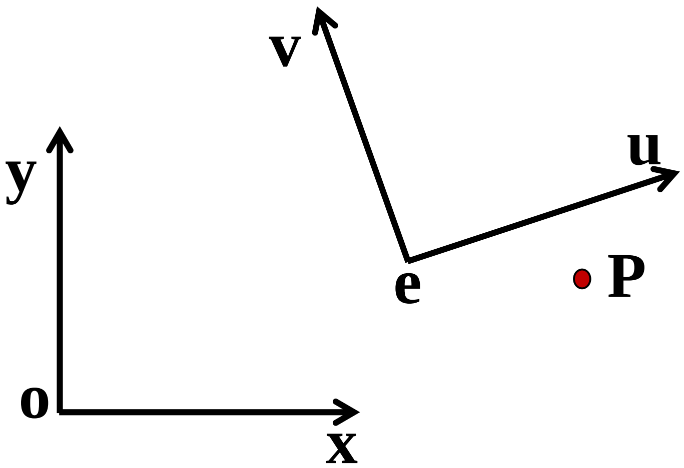
Homogeneous Coordinates

- Add an extra dimension, each point has an extra value, *0 or 1*
- Every point/vector is defined by 4 coordinates. In vector form:

Point coordinates: $\begin{bmatrix} a_1 & a_2 & a_3 & 1 \end{bmatrix}^T$

Vector coordinates: $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \end{bmatrix}^T$

Change of reference systems (frames) in 2D



Change the coordinates of a vector/point from one system to another

$$\begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_e \\ y_u & y_v & y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix}$$

$$P_{xy} = \begin{bmatrix} u & v & e \\ 0 & 0 & 1 \end{bmatrix} P_{uv}$$

Matrix of the coordinate change

Express P in the frame (o,x,y)

$$\mathbf{P} = (x_p, y_p) = \mathbf{o} + x_p\,\mathbf{x} + y_p\,\mathbf{y}$$

$$(x_p, y_p) = (2.5, 0.9)$$

Express P in the frame (e,u,v)

$$\mathbf{P} = (u_p, v_p) = \mathbf{e} + u_p\,\mathbf{u} + v_p\,\mathbf{v}$$

$$(u_p, v_p) = (0.5, -0.7)$$

Change of reference systems (frames) in 3D

Given the reference systems (frames) (o,x,y,z) and (e,u,v,w) represent P:

Coords of u
wrt (o,x,y,z)

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_w & x_e \\ y_u & y_v & y_w & y_e \\ z_u & z_v & z_w & z_e \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_p \\ v_p \\ w_p \\ 1 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} u & v & w & e \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{uvw}$$

$$P_{xyz} = M \; P_{uvw}$$

$$\begin{bmatrix} u_p \\ v_p \\ w_p \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_w & x_e \\ y_u & y_v & y_w & y_e \\ z_u & z_v & z_w & z_e \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix}$$

$$P_{uvw} = \begin{bmatrix} u & v & w & e \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} P_{xyz}$$

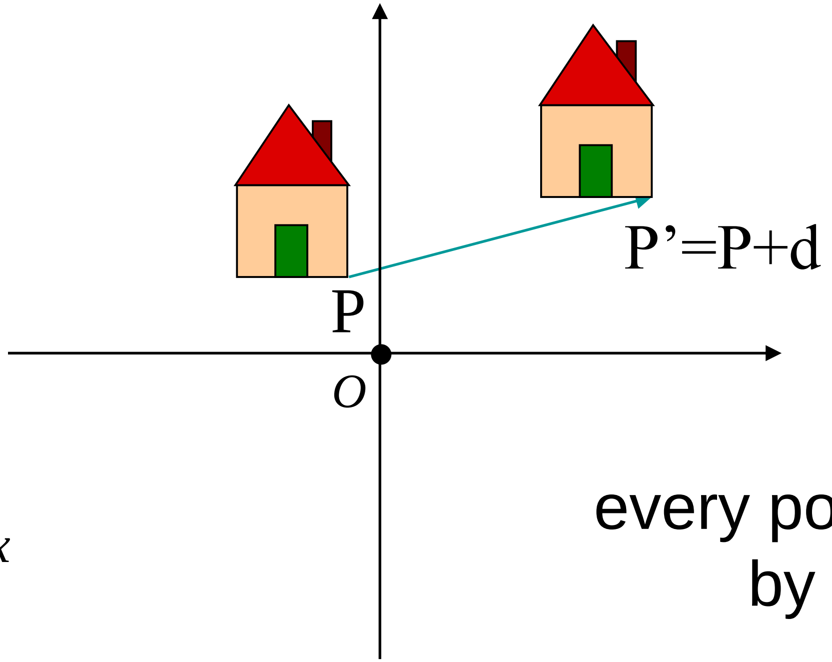
$$P_{uvw} = M^{-1} P_{xyz}$$

Geometric Transformations

- Transform the object coords in order to obtain a similar object which differs for **position, orientation and size**
- Modify the geometry but not the topology of the objects
- Transformations are used:
  - Position objects in a scene (modeling)
  - Change the shape of objects
  - Create multiple copies of objects
  - Projection for virtual cameras
  - Animations

2D Translation

- Move (translate, displace) a point P to a new location
- Displacement determined by a vector $d=(d_x,d_y)$

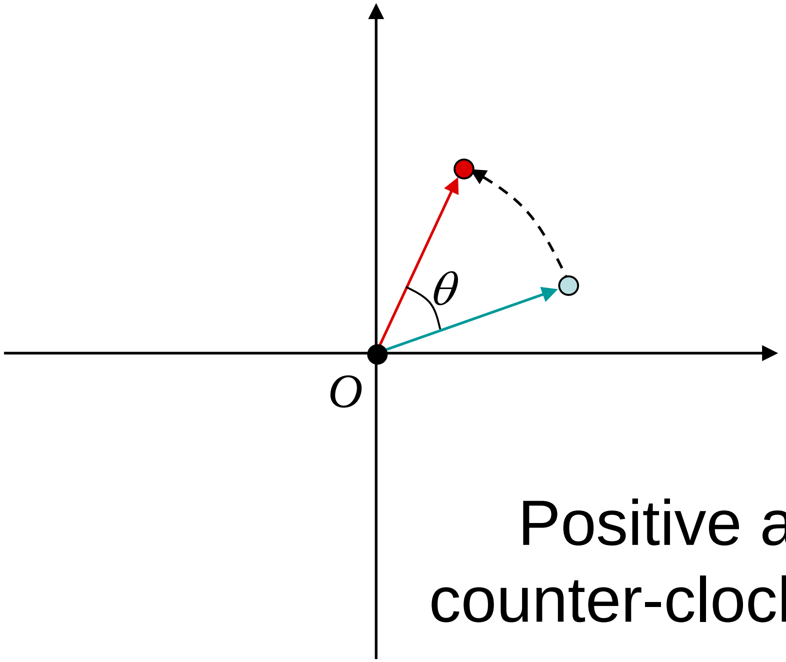


$$x' = x + d_x$$
$$y' = y + d_y$$

every point displaced by same vector

2D Rotation

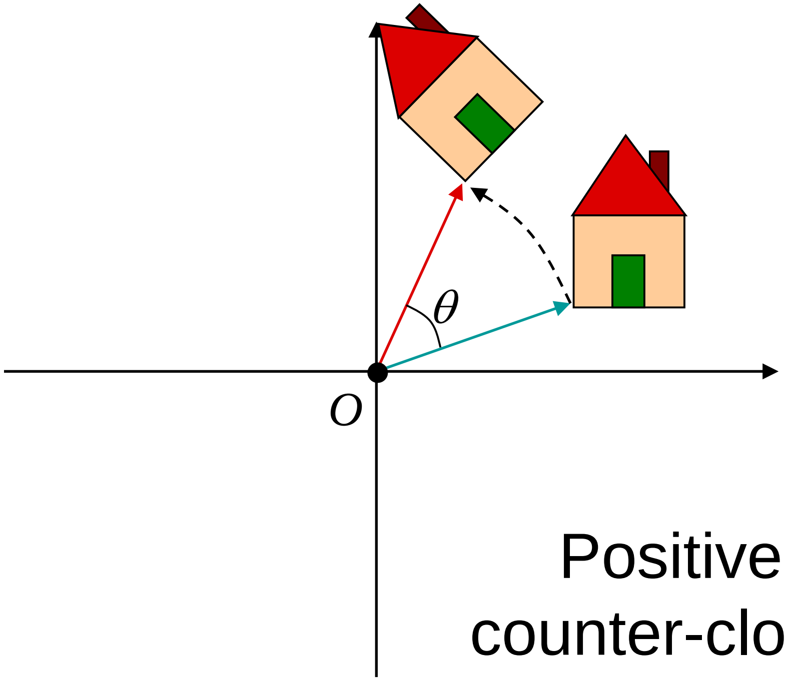
- Rotation about the origin by angle θ



Positive angle means counter-clockwise direction.

2D Rotation

- Rotation about the origin by angle θ



Positive angle means counter-clockwise direction.

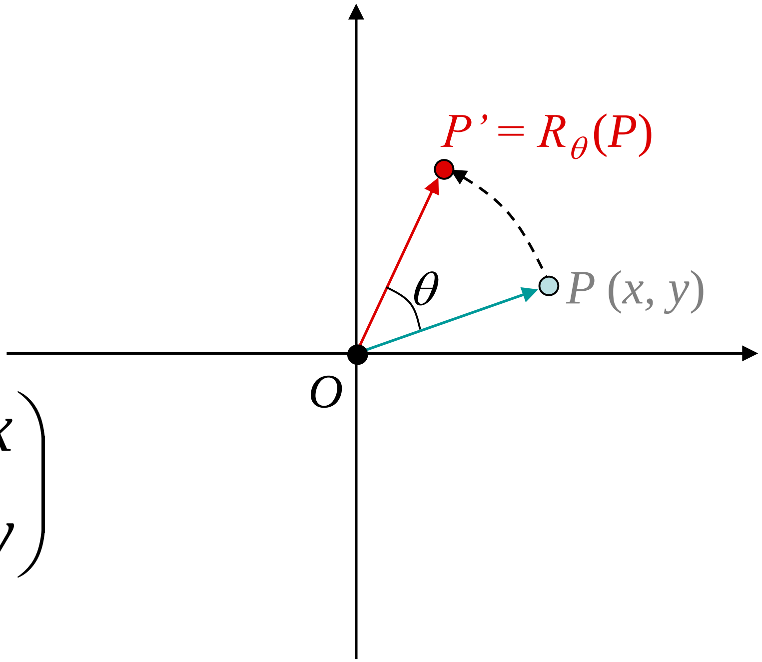
Rotation in 2D – matrix representation

- Multiply P=(x, y) by the rotation matrix:

$$P' = R_\theta(P)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = x\cos\theta \; - \; y\sin\theta$$
$$y' = x\sin\theta \; + \; y\cos\theta$$



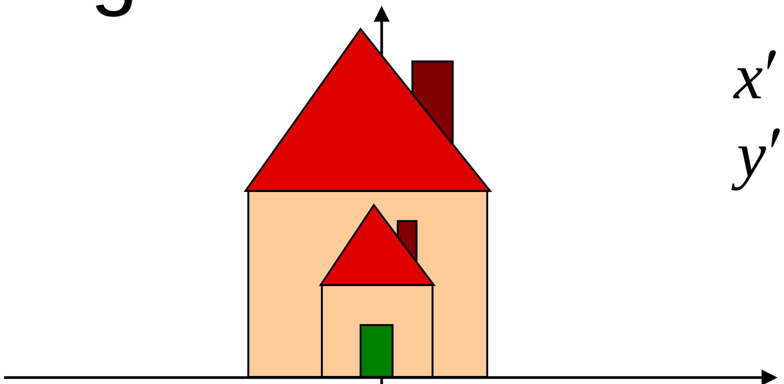
2D Scale

- Uniform sx=sy
- Non uniform sx≠sy
- About the origin

$$P' = S(P)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} sx & 0 \\ 0 & sy \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = sx * x$$
$$y' = sy * y$$



$0 \leq sx, sy < 1$ • the object will shrink

$sx, sy > 1$ • the object will grow by a factor of s in each dimension

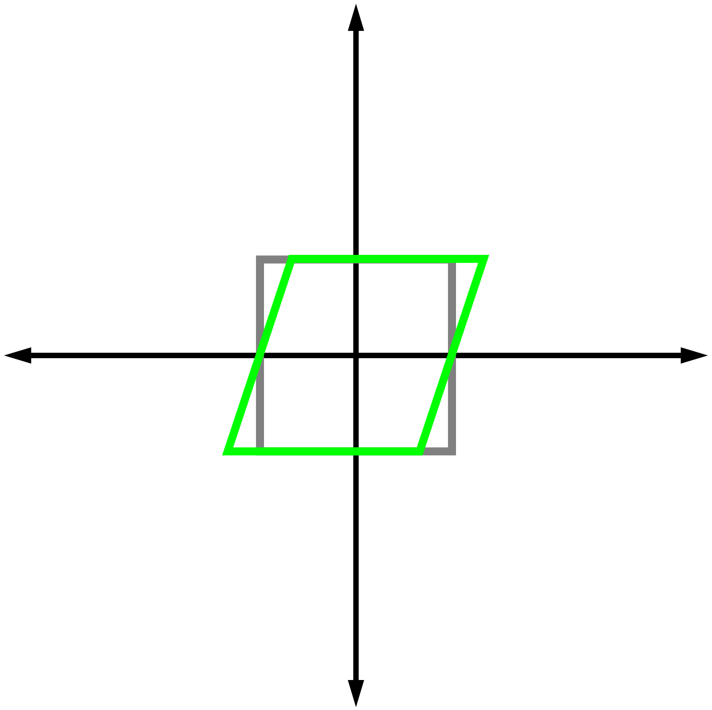
$sx < 0 \; o \; sy < 0$ • the object will be *reflected* across all two dimensions, leading to an object that is 'inside out'

2-D Shear (Horizontal)

Shear factor **s** (positive for the figure below)

$$P' = H_{xy}(s)P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$x' = x + sy$$

$$y' = y$$

Horizontal displacement proportional to vertical position

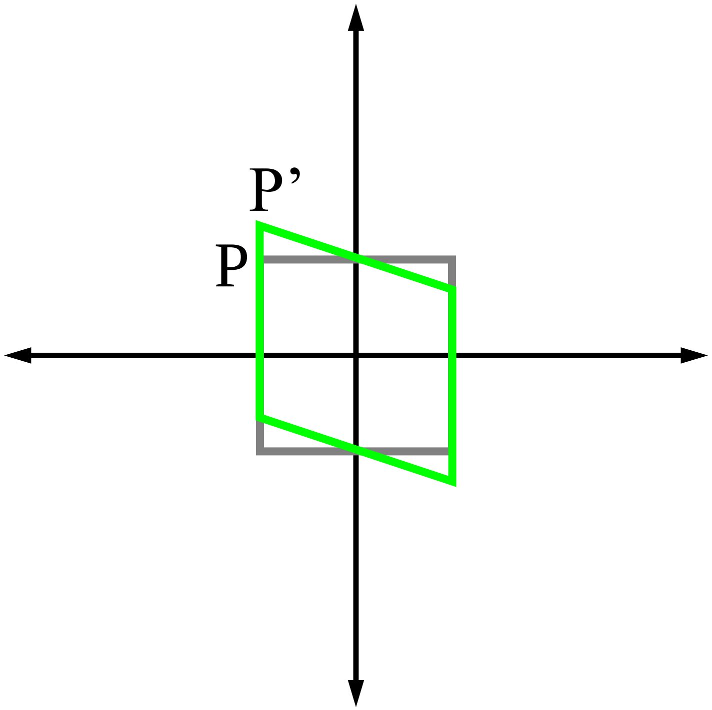
H_{ij} i coord. will change, j coord. will deform

2-D Shear (Vertical)

Shear factor **s** (negative for the figure below)

$$P' = H_{yx}(s)P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$x' = x$$

$$y' = y + sx$$

How are Invertible Linear Transformations Represented?

$$x' = ax + by$$

$$y' = dx + ey$$

An invertible linear transformation is represented by a non-singular matrix M

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p' = M p$$

Each non-singular matrix defines a linear transformation

$$\mathbf{L(p + q) = L(p) + L(q)}$$

$$\mathbf{L(ap) = a\, L(p)}$$

How are Affine Transformations Represented?

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

Compose linear transformation and translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = M p + t$$

Affine Transformations in Homogeneous Coordinates

Translations can be encoded in the matrix

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$p' = M p$$

M affine matrix

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

Affine Transformations in Homogeneous Coordinates

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Inverses

If **M** transforms **P** into **P'**, then **M**$^{-1}$ transforms **P'** back to **P**

$$MM^{-1} = I$$

$$P' = MP \quad -> \quad P = M^{-1}P'$$

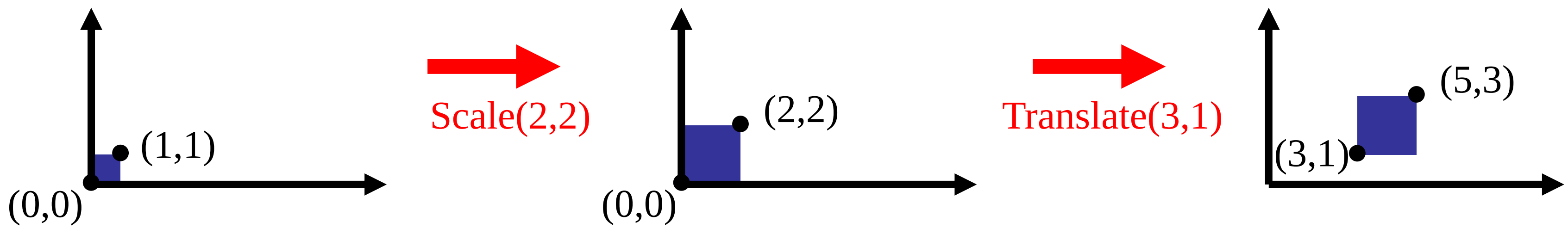
$$T^{-1}(d) = T(-d)$$

$$S^{-1}(s) = S(1/s_x, 1/s_y)$$

$$R^{-1}(\vartheta) = R^T(\vartheta) = R(-\vartheta)$$

How are transforms combined?

Scale then Translate



Use matrix multiplication:

p’ = S(p) p’’=T(p’)=T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiple Transformations

- v is transformed in v' by means of sequence of transformations:

$$\mathbf{v}' = \mathbf{M}_4 \cdot \left(\mathbf{M}_3 \cdot \left(\mathbf{M}_2 \cdot \left(\mathbf{M}_1 \cdot \mathbf{v}\right)\right)\right)$$

- Because matrix algebra obeys the *associative* law, we can regroup this as:

$$\mathbf{v}' = \left(\mathbf{M}_4 \cdot \mathbf{M}_3 \cdot \mathbf{M}_2 \cdot \mathbf{M}_1\right) \cdot \mathbf{v}$$

- This allows us to *concatenate* them into a single matrix:

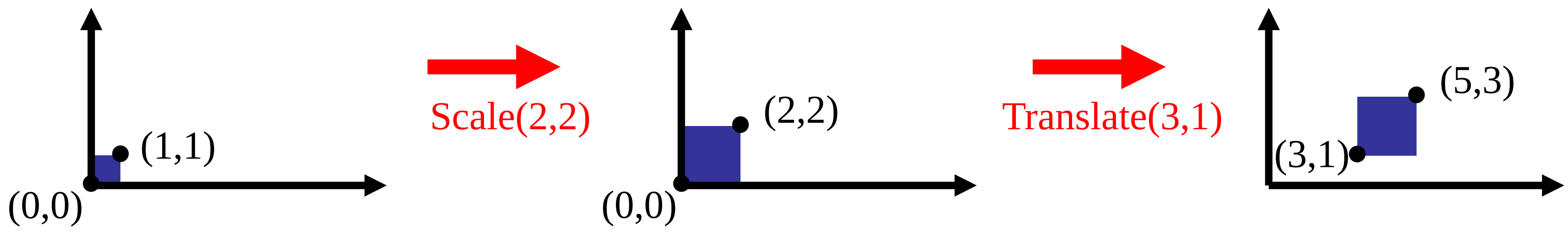
$$\mathbf{M}_{total} = \mathbf{M}_4 \cdot \mathbf{M}_3 \cdot \mathbf{M}_2 \cdot \mathbf{M}_1$$

$$\mathbf{v}' = \mathbf{M}_{total} \cdot \mathbf{v}$$

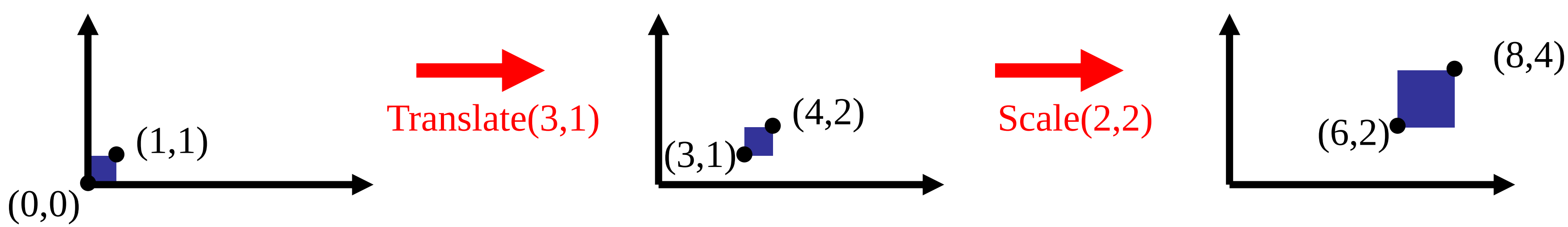
- Caution: matrix multiplication is NOT commutative! So the order of multiplications is important!

Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$



Translate then Scale: $p' = S(T\,p) = ST\,p$



Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Translate then Scale: $p' = S(T\,p) = ST\,p$

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Pivot Transformations

scaling around a point (dx,dy) that is not the origin

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate to the origin, **scale**, translate to the initial position

rotation around a point (dx,dy) that is not the origin

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate to the origin, **rotate**, translate to the initial position

3D Translate

$$x' = x + d_x$$

$$y' = y + d_y$$

$$z' = z + d_z$$

$$P' = P + \mathbf{d} \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

Homogeneous Transformations

- Affine 3D Transformations in homogeneous coordinates. In a general matrix form:

$$x' = a_1 x + b_1 y + c_1 z + d_1$$

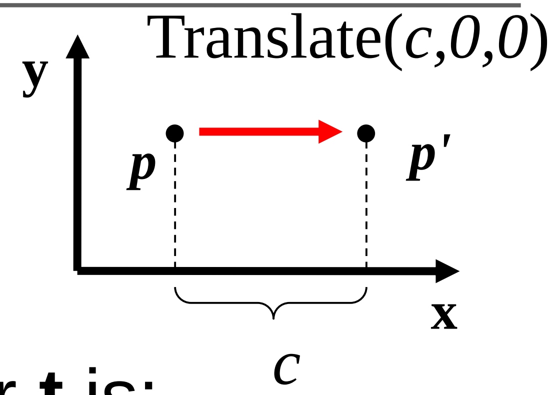
$$y' = a_2 x + b_2 y + c_2 z + d_2$$

$$z' = a_3 x + b_3 y + c_3 z + d_3$$

$$1 = 0x + 0y + 0z + 1$$

$$P' = \mathbf{M} \cdot P \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

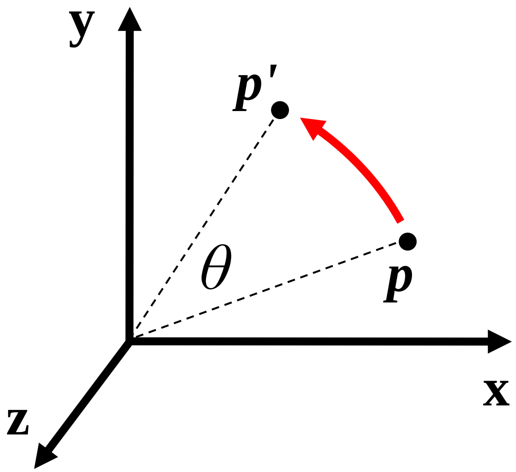
3D Translate (*tx, ty, tz*) in homogeneous coordinates



- Now translations
  can be encoded in the matrix!
- A 4x4 translation matrix that
  translates an object by the vector **t** is:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D Rotations



- About z axis

$$\mathbf{v}' = R_z(\vartheta) \cdot \mathbf{v}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation about z axis in three dimensions leaves all points with the same z

3D Rotations

- About x axis:

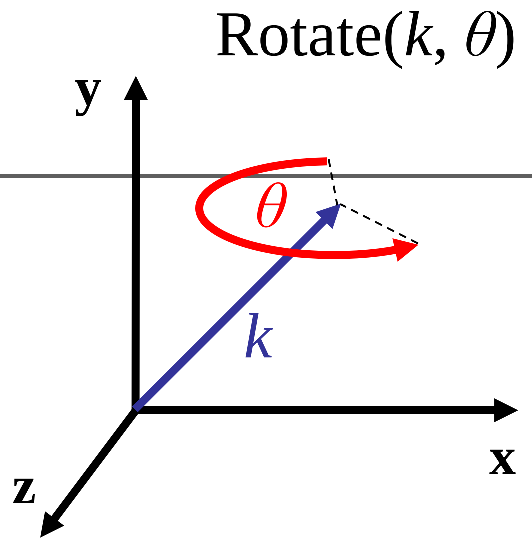
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- About y axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Multiple Rotations about an axis are commutative

Arbitrary Axis Rotation

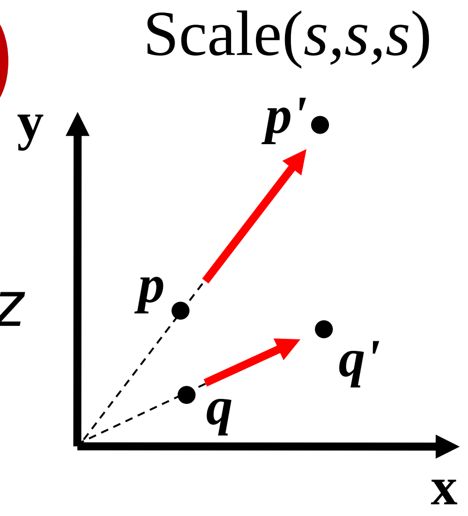


- About (k_x, k_y, k_z), a unit vector on an arbitrary axis (Rodrigues Formula)

$$
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} =
\begin{pmatrix}
k_x k_x(1-c)+c & k_z k_x(1-c)-k_z s & k_x k_z(1-c)+k_y s & 0 \\
k_y k_x(1-c)+k_z s & k_z k_x(1-c)+c & k_y k_z(1-c)-k_x s & 0 \\
k_z k_x(1-c)-k_y s & k_z k_x(1-c)-k_x s & k_z k_z(1-c)+c & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
$$

where $c = \cos\theta$ & $s = \sin\theta$

Scale (s_x, s_y, s_z)



- The uniform scaling matrix scales an entire object by scale factor $s = s_x = s_y = s_z$
- The non-uniform scaling matrix scales independently along the x, y, and z axes

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$x' = s_x x$$

$$y' = s_y y$$

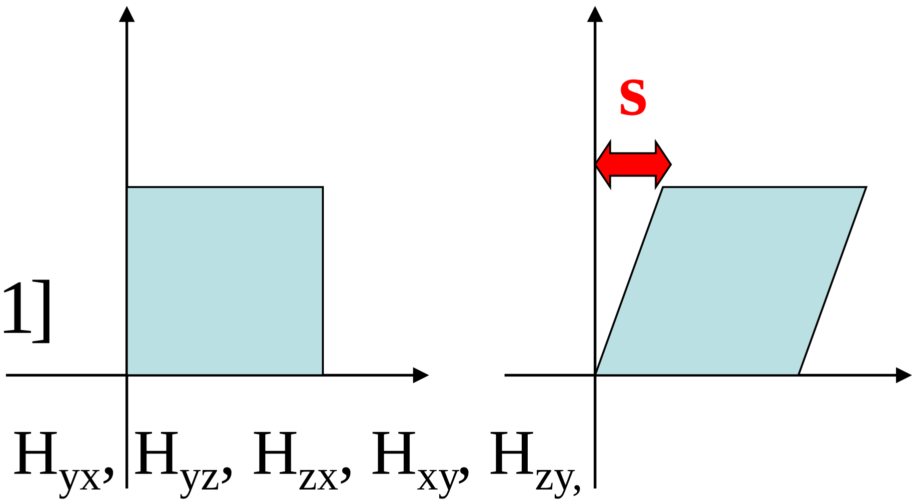
$$z' = s_z z$$

Shear Transformations

- Modify 2 or 3 vectors coords proportionally to the value of the other coords;
- H_{ij} i coord. will change, j coord. will deform

$$H_{xz}(s) = \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v' = [v_x + s v_z \quad v_y \quad v_z \quad 1]$$

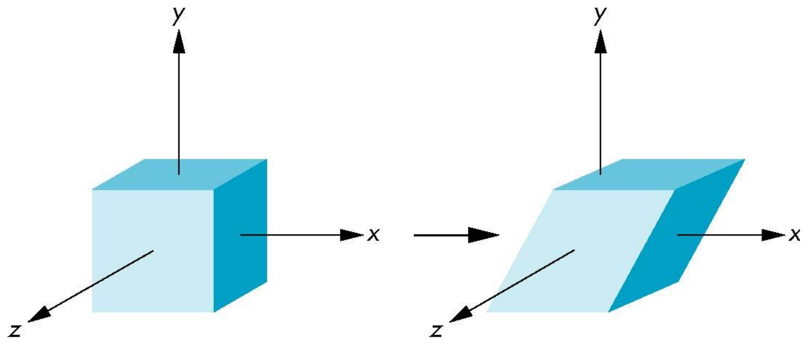


Analogously for H_{xy}, H_{yx}, H_{yz}, H_{zx}, H_{xy}, H_{zy},
$H(s)^{-1} = H(-s)$

Shear Transformations

- A *shear transformation* matrix looks something like this:

$$H(z_1 \ldots z_6) = \begin{bmatrix} 1 & z_1 & z_2 & 0 \\ z_3 & 1 & z_4 & 0 \\ z_5 & z_6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- With pure shears, only one of the constants is non-zero

- A shear can also be interpreted as a non-uniform scale along a rotated set of axes
- Shears are sometimes used in computer graphics for simple deformations or cartoon-like effects

Generalized 4 x 4 transformation matrix in homogeneous coordinates

$$P' = \mathbf{M} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

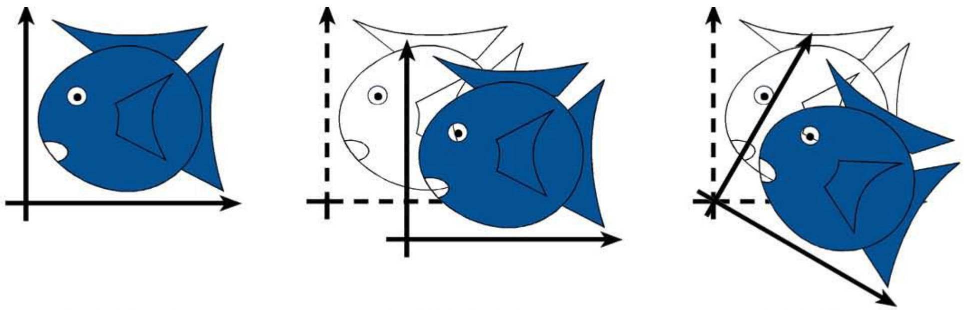
Linear Transformations

Translations

Perspective Projection

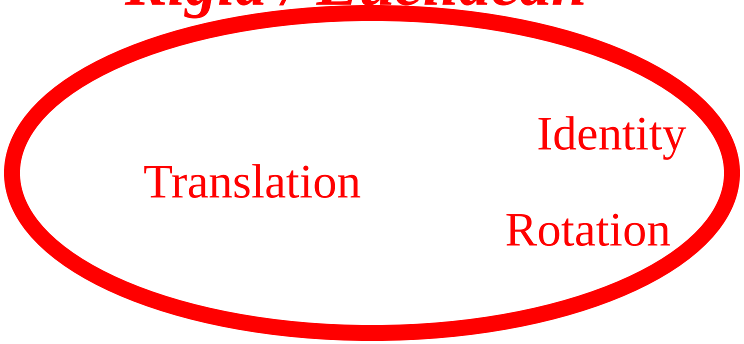
Rigid-Body / Euclidean Transformations

- Move the objects leaving shape and dimension unchanged
- Preserves distances
- Preserves angles



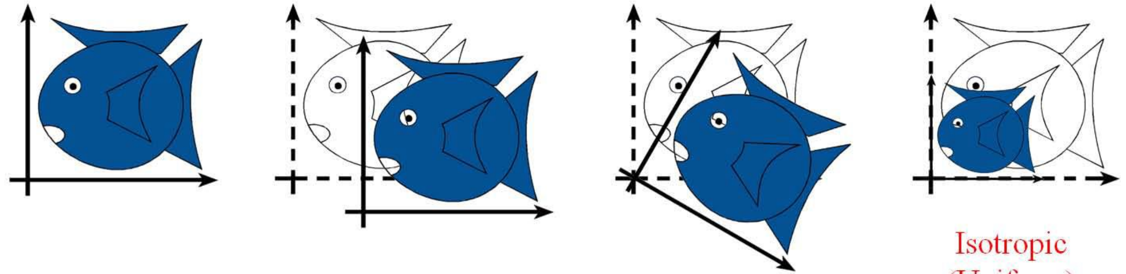
Identity Translation Rotation

Rigid / Euclidean

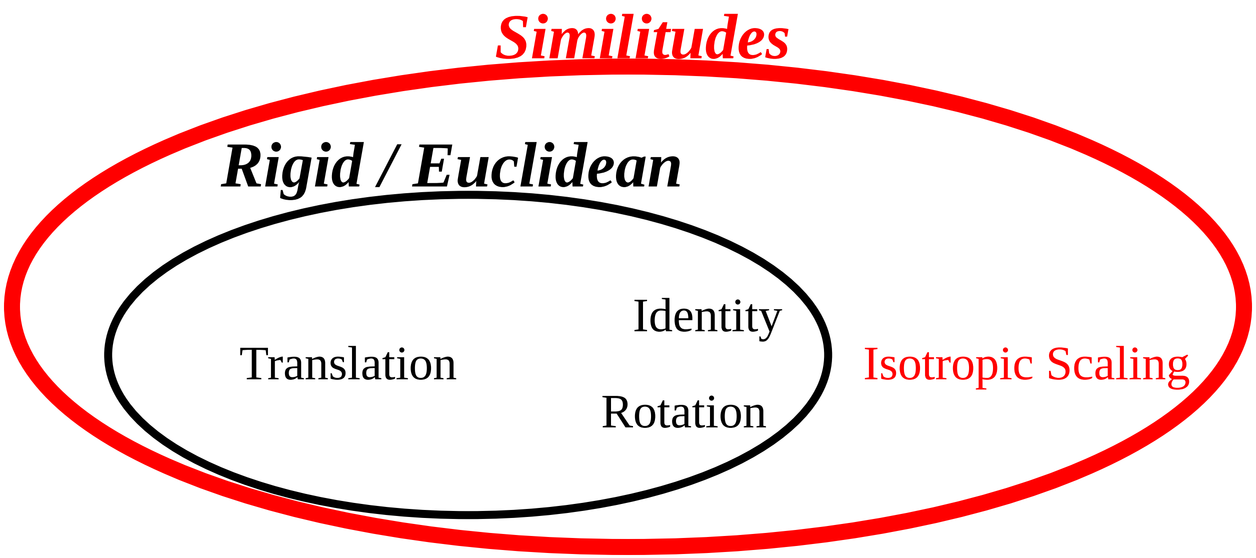


Similitudes / Similarity Transforms

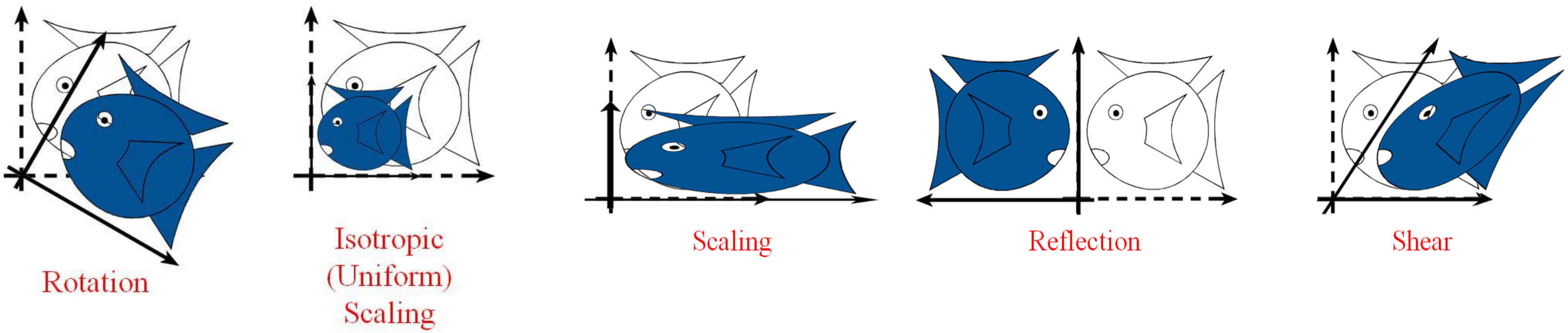
- Preserves angles

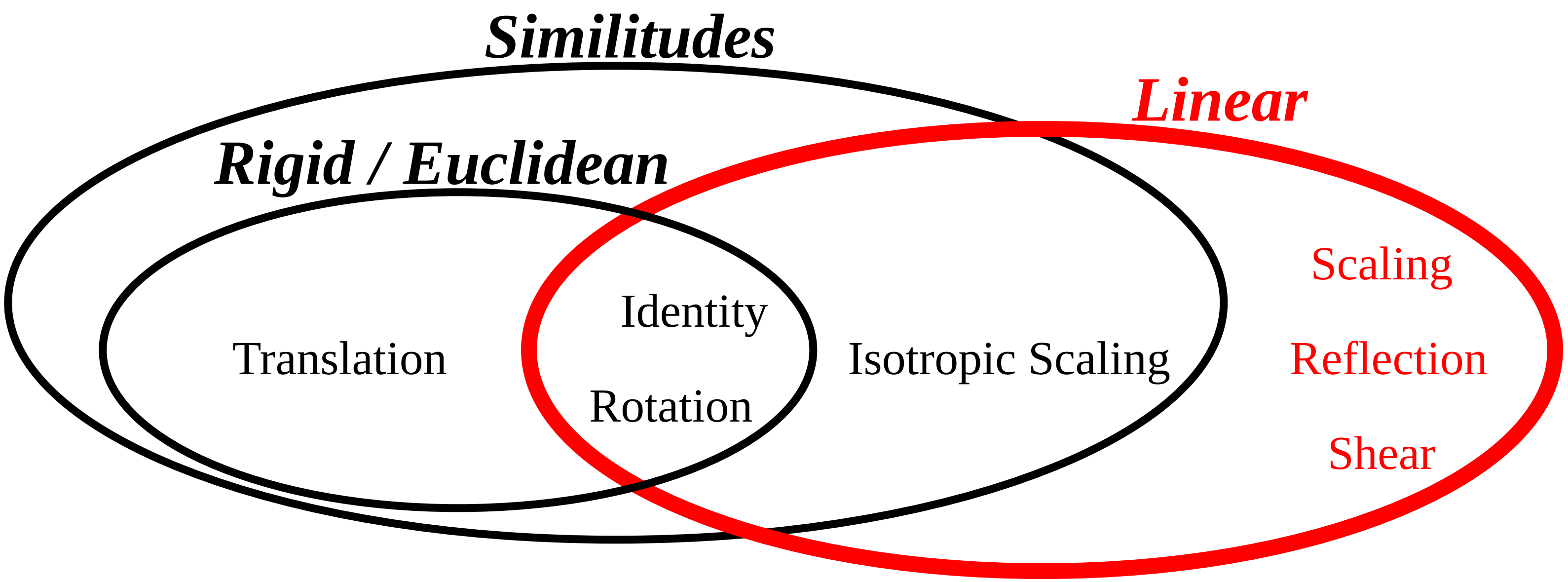


Identity — Translation — Rotation — Isotropic (Uniform) Scaling



Linear Transformations

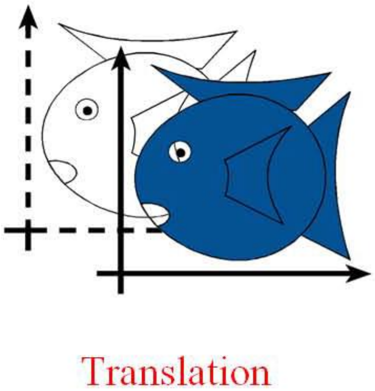


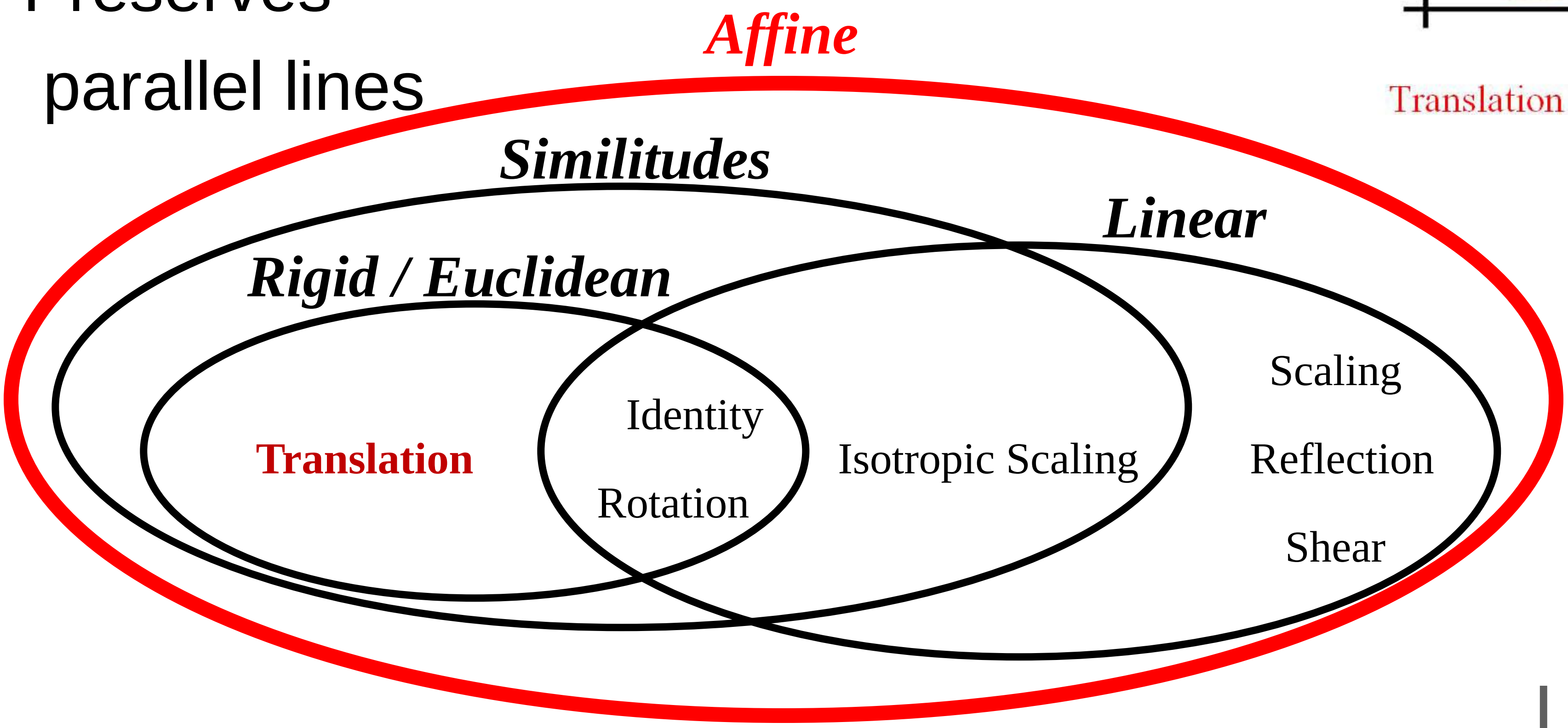


Courtesy of Prof. Fredo Durand

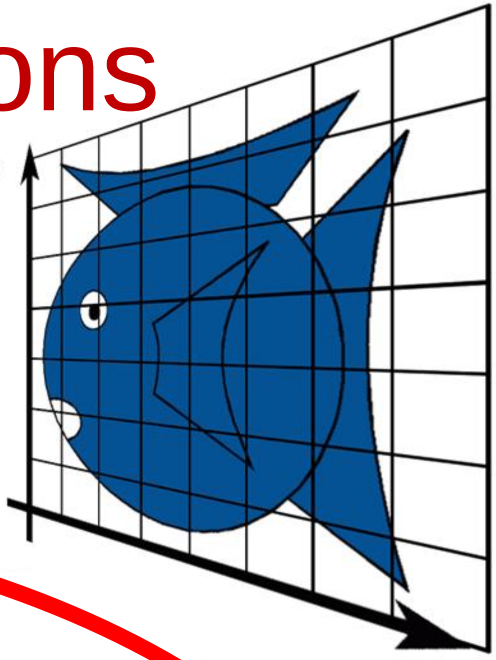
Affine Transformations

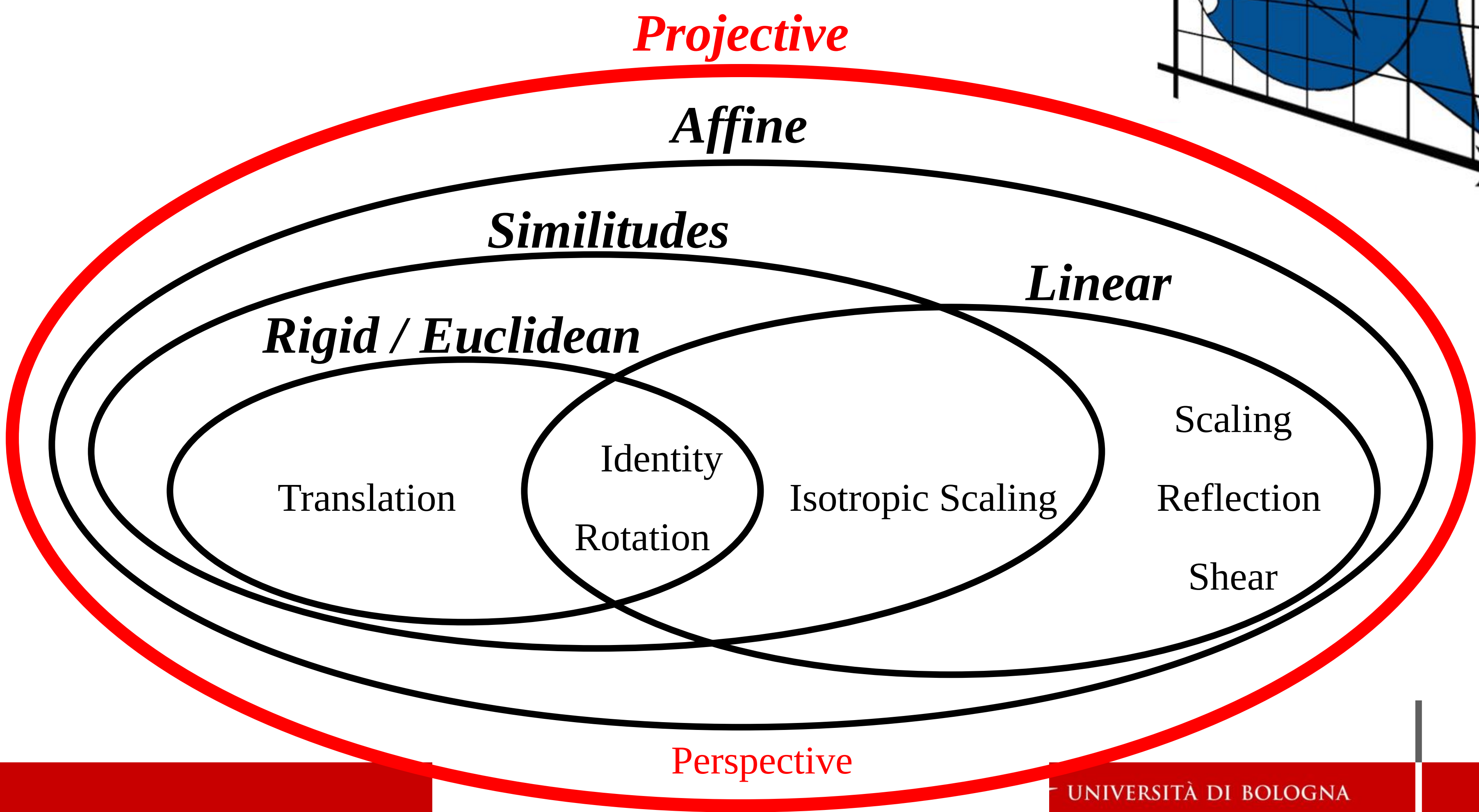
- Include all linear transformations plus translation
- Preserves parallel lines





Projective Transformations





Properties of Transformations

Type / Preserves	Rigid Body: Rotation & translation	Linear: General 3x3 matrix	Affine: Linear + translation	Projective: 4x4 matrix with last row $\neq(0,0,0,1)$
Lengths	Yes	No	No	No
Angles	Yes	No	No	No
Parallelness	Yes	Yes	Yes	No
Straight lines	Yes	Yes	Yes	Yes

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